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Selection and Screening Procedures to Determine Optimal Product Designs

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Abstract

To compare several promising product designs, manufacturers must measure their performance under *multiple* environmental conditions. In many applications, a product design is considered to be seriously flawed if its performance is poor for any level of the environmental factor. For example, if a particular automobile battery design does not function well under temperature extremes, then a manufacturer may not want to put this design into production. Thus this paper considers the measure of a product's quality to be its *worst* performance over the levels of the environmental factor. We develop statistical procedures to identify (a near) optimal product design among a given set of product designs, i.e., the manufacturing design that maximizes the worst product performance over the levels of the environmental variable. We accomplish this for intuitive procedures based on the split-plot experimental design (and the randomized complete block design as a special case); split-plot designs have the essential structure of a product array and the practical convenience of local randomization. Two classes of statistical procedures are provided. In the first, the δ-*best formulation* of selection problems, we determine the number of replications of the basic split-plot design that are needed to guarantee, with a given confidence level, the selection of a product design whose minimum performance is within a specified amount, δ, of the performance of the optimal product design. In particular, if the difference between the quality of the best and $2nd$ best manufacturing designs is δ or more, then the procedure guarantees that the best design will be selected with specified probability. For applications where a split-plot experiment that involves several product designs has been completed without the planning required of the δ-best formulation, we provide procedures to construct a "confidence subset" of the manufacturing designs; the selected subset contains the optimal product design with a prespecified confidence level. The latter is called the *subset selection formulation* of selection problems. Examples are provided to illustrate the procedures.

Keywords: Indifference-zone selection; Least favorable configuration; Optimal product design; Restricted randomization; Robust design; Statistical screening; Subset selection.

1 Introduction

The performance of most products varies, sometimes considerably, under different enviromental ("noise") conditions. Any comparison of potentially promising product designs must account for the multiplicity of potential operating conditions. In many applications it is appropriate to use the worst possible performance of a product under the different environments as a performance or quality index. This criterion is natural in situations where a low response at *any* level of the noise factor can have potentially serious consequence. Seat belts or heart valves that fail catastrophically under rare, though non-negligable, sets of operating conditions must be identified early in the product design cycle.

We call the criterion adopted by this paper the *maxmin* criterion for the choice of an optimal product design. In contrast, Taguchi used the signal-to-noise ratio over the level of the environmental variable as a criterion for choosing an optimal product design. However, as shown in Box (1988), this quantity can be problematic for the analysis of experiments in which a larger (smaller) response is better. Thus we adopt the maxmin criterion as natural in applications involving the comparision of product designs that are to be used under numerous environmental or noise factors.

A typical application of these ideas is an experiment comparing several engineering prototypes of a prosthetic heart value that was initially reported in Beeson (1965) and is also described by Anderson and McLean (1974). Beeson wished to evaluate the performance of four prosthetic cardiac valves designs. He tested each design at the six pulse rates 60, 80, ..., 160 beats per minute in a tank of fluid that mimicked the circulatory system. Since it was relatively difficult and time-consuming to change the valve once the apparatus was set up, Beeson randomly selected a valve and measured its flow gradient at all six pulse rates (using a separate random order for each valve). Thus his experiment involved a split-plot design in which *prosthesis design* formed the whole-plot factor and *pulse rate* formed the split-plot factor.

This paper proposes statistical procedures for assessing the performance of several product designs under multiple environmental conditions using a split-plot experiment (and the randomized complete block experiment as a special case by assuming appropriate variances in the split-plot models to be zero). We study split-plot designs because they have the essential structure of a product array and allow the practical convenience of local randomization. Following Box and Jones (1992) we assign product designs as the whole-plot factor and en-

viromental conditions as the split-plot factor; this arrangement is often the most logistically convenient (or necessary). The opposite assignment is discussed in Section 5.

Implicitly, our quality criterion—the minimum performance over the levels of the noise factor—assumes that larger responses are considered better for the application at hand. However, the opposite can be the case and the procedures proposed herein can be modified in an obvious manner. Also notice that when the response is bounded above, an engineering design that is optimal by the minimum response criterion has the *smallest maximum range.* Thus this criterion conforms to the spirit of modern quality control in minimizing product variability.

Frequently, each of the design and noise factors are themselves the set of treatment combinations of two or more variables. For example, Box and Jones (1992) present a study in which the goal is to formulate an optimal cake recipe, according the results of test taste, when the ingredients of flour, shortening and egg powder are each set at two levels; their example has $8 = 2^3$ levels for the manufacturing design variable. In such cases, we regard the set of treatment combinations corresponding to the manufacturing variables as the levels of a *single* product design factor; we similarly construct a *single* environmental factor.

We consider two formulations of the selection problem, each suitable for different practical situations. The first, *δ-correct selection*, is appropriate when it is possible to design the experiment by choosing the number b of replications of the split-plot (or RCBD) design. Its objective is to determine the number of blocks required so that with prespecified confidence level, the "natural" selection procedure identifies a product design whose minimum performance over the levels of the noise factor is within a given value $\delta > 0$ of the mimimum for the optimal product design. In practice, the quantity δ is the minimum quality difference worth detecting. The δ-correct selection formulation thus guarantees that *if* the difference in quality between the best and $2nd$ best product is of practical importance, i.e. greater than δ, then the experiment has been designed on such a scale as to identify *the* optimal product design with given probability. Conversely, *if* there are several "good" product designs whose quality is trivially different from the best product design, then the procedure will select either the optimal product design or one of the essentially equivalent product designs with given probability. Lastly, notice that when quality of the $2nd$ best design is very close to that of the best design, any reasonable statistical procedure will select the $2nd$ best as frequently as the true best. This intuition shows that it is *impossible* to identify the true optimal product

with nontrivial prespecified confidence, i.e., to take $\delta = 0$. Thus δ-correct selection is the most natural formulation at the design stage of the experiment.

The second formulation, *subset selection*, is an analysis tool; subset selection is appropriate when the number of replications b has already been chosen by external considerations such as fiscal or time constraints. Its objective is screening; a random subset of the designs is chosen so that the optimal product design is contained in this subset with a prespecified confidence level. Bechhofer, Santner, and Goldsman (1995) give an introduction to these, and other, selection formulations.

In this paper we propose both δ-correct selection and subset selection procedures for identifying the product design having the greatest value of the minimum mean performance over the levels the environmental factor. Section 2 describes the model and the two formulations of the selection problem. Section 3 describes an δ-correct selection procedure and its properties while Section 4 describes a screening procedure. An example is given to illustrate the selection procedure.

2 Models and Problem Formulations

We assume that r product designs are to be evaluated in c environments. In practice, it would usually be the case that the (product) design factor would itself have a factorial structure—thus r would denote the number of factorial product designs and the same would be true of c. A total of b replicates of the entire experiment with $r \times c$ treatment combinations is to be conducted. With the exception of the discussion in Section 5, we assume throughout that the design factor is assigned to the whole-plots and the noise factor is assigned to the subplots in a split-plot design.

Let Y_{ijk} denote the response when the ith level of the design factor and the jth level of the noise factor are used in the kth replication of the experiment. Let μ_{ij} denote the mean of Y_{ijk}. We make no assumption about the structure of the μ_{ij}. As usual for split-plot data, we assume that

$$Y_{ijk} = \beta_k + \mu_{ij} + \omega_{ik} + \epsilon_{ijk} \quad (i = 1, \ldots, r; \ j = 1, \ldots, c; \ k = 1 \ldots b) \qquad (2.1)$$

where the block effects, $\{\beta_k\}_k$, the (potential) confounding effect, $\{\omega_{ik}\}_{i,k}$, and the measurement errors $\{\epsilon_{ijk}\}_{i,j,k}$, are mutually independent with

$$\beta_k \quad \text{i.i.d.} \quad N(0, \sigma_\beta^2) \ \ (k = 1 \ldots b)$$

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$$\omega_{ik} \quad \text{i.i.d.} \quad N(0, \sigma_\omega^2) \quad (i = 1, \ldots, r; \ k = 1 \ldots b)$$

$$\epsilon_{ijk} \quad \text{i.i.d.} \quad N(0, \sigma_\epsilon^2) \quad (i = 1, \ldots, r; \ j = 1, \ldots, c; \ k = 1 \ldots b)$$

(see Milliken and Johnson 1984, for example). The Randomized Complete Block Design corresponds to $\sigma_\omega^2 = 0$.

As motivated in Section 1, we adopt the minimum response across the levels of the noise factor,

$$\eta_i = \min\{\mu_{i1}, \ldots, \mu_{ic}\} \quad (i = 1, \ldots, r), \tag{2.2}$$

as the measure of quality of the ith product design. Denote the ordered η_i's by

$$\eta_{[1]} \leq \cdots \leq \eta_{[r]}.$$

Any product design for which $\eta_i > \eta_{[r]} - \delta$ is called a *δ-best product design*. We call the selection of a δ-best product design a *δ-correct selection*.

First we provide statistical procedures that select a δ-best product design with given confidence level and then we give procedures that select a subset of product designs that contains the optimal design with given confidence level. The probability requirements for the two cases are as follows.

Design Requirement $\mathcal{R}_\delta$: For specified α $(0 < \alpha < 1)$ we require that

$$P_{\boldsymbol{\mu}}\{\text{CS-}\delta\} \geq 1 - \alpha \tag{2.3}$$

for all $\boldsymbol{\mu}$ where CS-δ denotes the event that a δ-best product design has been selected.

If an experiment has been completed with b fixed by economic considerations, intuition, or another crerition, we can still adopt the screening goal of selecting, with a prespecified confidence level, a subset of product designs so that the subset includes the optimal design. That is, we wish to identify a subset of $\{1, \ldots, r\}$ so that the following holds.

Confidence Requirement $\mathcal{R}_\mathcal{S}$: For specified α $(0 < \alpha < 1)$ we require that

$$P_{\boldsymbol{\mu}}\{\text{CS-}\mathcal{G}\} \geq 1 - \alpha \tag{2.4}$$

for all $\boldsymbol{\mu}$ where CS-$\mathcal{G}$ occurs when the selected subset contains the design associated with $\eta_{[r]}$.

When $\eta_{[r]} - \eta_{[r-1]} \geq \delta$ there is but one product design satisfying $\eta_i > \eta_{[r]} - \delta$, namely the product design associated with $\eta_{[r]}$ itself. Thus (2.3) implies that the best design is

selected whenever $\eta_{[r]} - \eta_{[r-1]} \geq \delta$. A procedure that selects the best design whenever the best design is sufficiently better than the $2nd$ best design is said to satisify an *indifference-zone design requirement.* Bechhofer (1954) introduced the indifference-zone formulation of selection problems for the one-way layout. Fabian (1962) proved that Bechhofer's procedure satisfied a strengthened version, corresponding to our $\mathcal{R}_\delta$, of the indifference-zone design requirement.

Sections 2 and 3 provide statistical procedures that satisfy the δ-correct and subset selection probability requirements, respectively. Throughout the remainder of the paper we let $\overline{Y}_{ij.}$ denote the sample mean of all observations having mean μ_{ij} for $1 \leq i \leq r$ and $1 \leq j \leq c$.

3   Procedures for Selecting a δ-best Optimal Design

In this section we analyse the following "natural" selection procedure based on the sample means.

Procedure $\mathcal{N}$: *Compute the estimate $\widehat{\eta}_i = \min\{\overline{Y}_{i1.}, \ldots, \overline{Y}_{ic.}\}$ of η_i $(i = 1, \ldots, r)$. Denote the ordered $\widehat{\eta}_i$'s by*

$$\widehat{\eta}_{[1]} \leq \cdots \leq \widehat{\eta}_{[r]}. \tag{3.1}$$

Select the product design corresponding to $\widehat{\eta}_{[r]}$ as the optimal design.

In practice, the number of product designs, r, and the number of environmental conditions, c, will be specified by the experimenter. The procedure $\mathcal{N}$ is completely defined when the amount of replication of the basic design, b, is determined. Subsection 3.1 determines b so that $\mathcal{N}$ correctly selects a δ-best design with a specified probability when the confounding variance, σ_ω^2, and the measurement error variance, σ_ϵ^2, are known. Subsection 3.2 finds b for the case when the *relative* size of the confounding variance to the measurement error variance is *known* with the magnitudes of the *individual* variances being *unknown*; this corresponds to assuming that $\gamma = \sigma_\omega^2/\sigma_\epsilon^2$ is known but the individual σ_ω^2 and σ_ϵ^2 are unknown.

3.1   A Selection Procedure When $(\sigma_\omega^2, \sigma_\epsilon^2)$ is Known

We determine the minimum number of blocks b required by $\mathcal{N}$ to achieve the guarantee (2.3) of selecting a δ-best product design. To compute this value, we find a configuration of means $\boldsymbol{\mu}$ for which the probability of correct selection over $\boldsymbol{\mu}$ is minimum. The number of blocks

required to achieve probability $1 - \alpha$ at this, so-called, least favorable configuration is the mimimum number of blocks needed to plan the experiment. Theorem 3.1 identifies such a least favorable parameter configuration for $\mathcal{N}$.

Theorem 3.1 The probability of δ-correct selection for $\mathcal{N}$, $P_{\boldsymbol{\mu}}\{\text{CS-}\delta\}$, is minimized over the set of all $\boldsymbol{\mu}$ when

$$\boldsymbol{\mu}_O = \begin{pmatrix} -\delta & +\infty & \cdots & +\infty \\ \vdots & \vdots & \cdots & \vdots \\ -\delta & +\infty & \cdots & +\infty \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2}$$

(and the PCS is calculated as the probability that $\mathcal{N}$ chooses the design r).

The proof the Theorem 3.1 is given in the Appendix. It shows that configuration $\boldsymbol{\mu}_0$ is not the unique least favorable parameter configuration. The same value as $P_{\boldsymbol{\mu}_0}\{\text{CS-}\delta\}$ occurs for any $\boldsymbol{\mu}$ obtained by permutating the μ_{ij}'s in $\boldsymbol{\mu}_0$ within columns (whole-plots) or rows (split-plots). However, for the purpose of determining b, the least favorable parameter configuration in Theorem 3.1 is the simplest to use.

According to Theorem 3.1, we need only set the probability of correct selection at $\boldsymbol{\mu}_0$ (greater than or) equal to $1 - \alpha$. It can be shown that $P_{\boldsymbol{\mu}_0}\{\text{CS-}\delta\}$ is equal to

$$P\left\{\min_{1 \leq j \leq c} U_j + V \geq Q_i - \delta \ \ (1 \leq i < r)\right\}$$

where all random variables are mutually independent, $U_1, \ldots, U_c$ are distributed $N\left(0, \sigma_\epsilon^2/b\right)$, V is distributed $N\left(0, \sigma_\omega^2/b\right)$, and $Q_1, \ldots, Q_{r-1}$ are distributed $N\left(0, (\sigma_\epsilon^2 + \sigma_\omega^2)/b\right)$. This probability can be expressed as

$$P\left\{\frac{\sigma_\epsilon}{\sqrt{b}} \min_{1 \leq j \leq c} Z_j + \frac{\sigma_\omega}{\sqrt{b}} Z_{r+c} \geq \frac{\sqrt{\sigma_\omega^2 + \sigma_\epsilon^2}}{\sqrt{b}} Z_{i+c} - \delta \ \ (1 \leq i < r)\right\}$$

where $Z_1, \ldots, Z_{r+c}$ are mutually independent $N(0,1)$ random variables or as

$$P\left\{\min_{1 \leq j \leq c} Z_j + \sqrt{\gamma}\, Z_{r+c} \geq \sqrt{1+\gamma}\, Z_{i+c} - \frac{\sqrt{b}}{\sigma_\epsilon}\delta \ \ (1 \leq i < r)\right\} \tag{3.3}$$

where $\gamma = \sigma_\omega^2/\sigma_\epsilon^2$.

We summarize these results and give two equivalent expressions, (3.4) and (3.5)/(3.6), that can be used to determine the required number of blocks. The first is a representation in terms of standard normal random variables and can be used to calculate the b via simulation. The second, obtained by conditioning on $\max_{1 \leq i < r} Z_{i+c}$ and $\min_{1 \leq j \leq c} Z_j$, is a double integral that can be used to evaluate b by quadrature.

Theorem 3.2 The mimimum number of blocks required for Procedure $\mathcal{N}$ to satisfy Probability Requirement $\mathcal{R}_\delta$ is the smallest integer b for which $b \geq q^2 \sigma_\epsilon^2 / \delta^2$ where q solves the equation

$$P\left\{ \min_{1 \leq j \leq c} Z_j \geq \sqrt{1 + \gamma} \max_{1 \leq i < r} Z_{i+c} - \sqrt{\gamma}\, Z_{r+c} - q \right\} = 1 - \alpha, \tag{3.4}$$

with $Z_1, \ldots Z_{r+c}$ being mutually independent $N(0,1)$ random variables. Equivalently, the left-hand side probability in (3.4) is

$$(r-1)c \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi(x)\phi(y) \left[\Phi(x)\right]^{r-2} \left[1 - \Phi(y)\right]^{c-1} \left[1 - \Phi\left(\frac{x\sqrt{1+\gamma} - y - q}{\sqrt{\gamma}}\right)\right] dxdy \tag{3.5}$$

when $\gamma \neq 0$ and

$$(r-1)c \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \phi(x)\phi(y + x - q) \left[\Phi(x)\right]^{r-2} \left[1 - \Phi(y + x - q)\right]^{c-1} dxdy \tag{3.6}$$

when $\gamma = 0$.

We have used both simulation and quadrature to evaluate $P_{\boldsymbol{\mu}_0}\{\text{CS-}\delta\}$; our experience is that both techniques work well with small r and c ($r \leq 4$ and $c \leq 5$). Figure 1 presents plots of the left-hand side of (3.4) versus q for a selected set of (r, c, γ). These values can be used to determine, approximately, the associated number of blocks for a given experiment. The interested reader can also obtain a FORTRAN program from the first author that uses quadrature routines from the International Mathematics and Statistics Library (IMSL) to evaluate the left-hand side of (3.5)/(3.6) for arbitrary (r, c, γ, q).

Example 1 Montgomery (Ch. 7, 1991) applies ANOVA methods to compare three engineering designs for automobile batteries. Batteries of each engineering design are tested at the three operating temperatures: 15^oF, 70^oF, and 125^oF. The response is the total number of hours of battery life. The manufacturer has, of course, no control over the environmental temperature in which the battery will be used once it is purchased. Consider applying the methods proposed in this paper to design an experiment to select a battery design that has (near) the longest minimum mean life over the three operating temperatures.

Based on data from a factorial experiment conducted using a randomized complete block design, Montgomery tests for interaction between the battery designs and operating temperatures, and their main effects. Treating Montgomery's data as pilot information to suggest parameter values, we design an experiment to select a δ-best design with $\delta = 24$ hours

when $\sigma_\epsilon = \sqrt{675}$ hours. Thus we treat any battery design as "nearly" optimal if its minimum mean life is no more than 24 hours worse than the best battery design. Suppose that we specify that the probability in $\mathcal{R}_\delta$ that $\mathcal{N}$ selects such a design be 80%. If we follow Montgomery and construct a randomized complete block experiment to compare the $r = 3$ designs at the $c = 3$ temperatures, then $\gamma = 0.0$ and $q = 2.35$ solves (3.4); thus $(q^2 \times \sigma_\epsilon^2)/\delta^2 = (2.35^2 \times 675)/24^2 = 6.47$ or $b = 7$ blocks should be used. If a split-plot experiment is conducted in which it is assumed the relative variability due to confounding is $\gamma = .2$ then $q = 2.55$ solves (3.4) and $(q^2 \times \sigma_\epsilon^2)/\delta^2 = (2.55^2 \times 675)/24^2 = 7.62$ so that $b = 8$ blocks must be used. $\square$

3.2   Selection When $(\sigma_\omega^2, \sigma_\epsilon^2)$ is Unknown

If the variance σ_ϵ^2 of the experiemental errors is unknown, the Design Requirement $\mathcal{R}_\delta$ cannot be satisfied based on a one-stage experiment, even if the ratio γ is known. Intuitively, the number of blocks required to guarantee δ-correct selection with probability $1 - \alpha$ increases to infinity as the measurement error σ_ϵ^2 increases. This paper will not discuss multi-stage procedures.

There are at least two cases where a single-stage experimental design is possible. The first is when γ *and* an upper bound for σ_ϵ^2 are known; then use of the upper bound in place of σ_ϵ^2 in Theorem 3.2 leads to a conservative solution for b. A second case where a single-stage solution exists is when the experimenter is willing to adopt the following modified version of $\mathcal{R}_\delta$ that is stated in terms of the relative difference of the treatment means.

Design Requirement $\mathcal{R}_{\delta rel}$: For a prespecified α $(0 < \alpha < 1)$ we require that

$$P_{\boldsymbol{\mu}, \sigma_\epsilon}\{\text{CS-}(\delta, \sigma_\epsilon)\} \geq 1 - \alpha \tag{3.7}$$

for all $\boldsymbol{\mu}$ and $\sigma_\epsilon > 0$; here $[\text{CS-}(\delta, \sigma_\epsilon)]$ denotes the event that a product design i has been selected for which $\eta_i > \eta_{[r]} - \delta\sigma_\epsilon$.

Defining a δ-best product design in terms of its mean divided by measurement standard deviation in (3.7) is analogous to a power requirement of an $\mathcal{F}$ test stated in terms of an appropriate non-centrality parameter. We determine the minimum number of blocks b required for procedure $\mathcal{N}$ to attain $\mathcal{R}_{\delta rel}$.

It is straightforward to mimic the arguments in Appendix and show that, for $\mathcal{N}$, the probability of δ-correct selection is minimized over the set of all $(\boldsymbol{\mu}, \sigma_\epsilon)$ when

$$\boldsymbol{\mu}_O = \begin{pmatrix} -\delta\sigma_\epsilon & +\infty & \cdots & +\infty \\ \vdots & \vdots & \cdots & \vdots \\ -\delta\sigma_\epsilon & +\infty & \cdots & +\infty \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.8}$$

(and the PCS is calculated as the probability that $\mathcal{N}$ chooses design r). The value of $P_{\boldsymbol{\mu}_0,\sigma_\epsilon}\{\text{CS-}(\delta,\sigma_\epsilon)\}$ is independent of σ_ϵ when (3.8) holds; an analytic expression for this minimum probability is obtained by setting $\sigma_\epsilon = 1$ in either Equation (3.4) or (3.5)/(3.6). Thus the number of blocks required to satisfy Design Requirement $\mathcal{R}_{\mathcal{I}rel}$ is the smallest integer b greater than or equal to q^2/δ^2 where q solves the Equation (3.4), say.

4 Procedures for Selecting a Subset Containing the Optimal Design

In many applications, the number of blocks used in an experiment is determined by external time or cost considerations rather than a design criteria such as $\mathcal{R}_\delta$ or $\mathcal{R}_{\delta rel}$. In such a situation, the investigator might still elect to use the selection procedure $\mathcal{N}$ to select the optimal design after conducting a sensitivity assessment of the operating characteristics of this procedure for the *number of blocks used in the experiment*. For example, the investigator might compute the probability of correct selection in Equation (3.4) for a range of δ's based on the ANOVA estimates of σ_ω^2 and σ_ϵ^2. A plot of the pairs of δ values versus the corresponding minimum probability of correct selection provides a basis to interpret the degree of confidence about the selection.

This section considers the alternative approach of using a screening procedure that satisfies the confidence requirement $\mathcal{R}_\mathcal{S}$ as the inference tool for identifying the optimal design. If the data strongly indicate a single design as being best, the proposed procedure selects one (or a few) designs; if the data show high variability or the $\widehat{\eta}_i$ are very close, the procedure chooses a larger subset. Subsection 4.1 considers the case when $(\sigma_\omega^2, \sigma_\epsilon^2)$ is known while Subsection 4.2 studies the case when $(\sigma_\omega^2, \sigma_\epsilon^2)$ is unknown. Throughout, we let $\widehat{\eta}_i = \min\{\overline{Y}_{i1\cdot}, \ldots, \overline{Y}_{ic\cdot}\}$ denote the estimator of η_i $(i = 1, \ldots, r)$ and let

$$\widehat{\eta}_{[1]} \leq \ldots \leq \widehat{\eta}_{[r]} \tag{4.1}$$

denote the ordered $\widehat{\eta}_i$'s

4.1 A Screening Procedure When $(\sigma_\omega^2, \sigma_\epsilon^2)$ is Known

Of course, applications in which σ_ω^2 and σ_ϵ^2 are known do not occur as frequently as those in which one or more of these variances is unknown. However, the known variance case provides the basis for the analysis of the unknown variance case.

Procedure $\mathcal{S}_\mathcal{K}$: *Include design i in the subset if and only if*

$$\widehat{\eta}_i \geq \widehat{\eta}_{[r]} - d\,\sigma_\epsilon, \tag{4.2}$$

where $d = q/\sqrt{b}$ and q is the solution of (3.4); equivalently the left-hand probability in (3.4) can computed from (3.5)/(3.6).

In principle, to determine d we require an expression for the minimum, over all $\boldsymbol{\mu}$, of the probability of correct selection. The following theorem describes the least favorable configuration for the procedure $\mathcal{S}_\mathcal{K}$ which then allows us to compute the minimum probability.

Theorem 4.1 The probability that the selection procedure $\mathcal{S}_\mathcal{K}$ contains the optimum design is minimized for the configuration

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 0 & +\infty & \cdots & +\infty \\ \vdots & \vdots & \cdots & \vdots \\ 0 & +\infty & \cdots & +\infty \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.3}$$

and the PCS is calculated as the probability that $\mathcal{S}_\mathcal{K}$ includes design r.

The value of the probability of correct selection at $\boldsymbol{\mu}_1$ can be computed as

$$P_{\boldsymbol{\mu}_1}\{\text{CS-}\mathcal{G}\} = P_{\boldsymbol{\mu}_1}\left\{\min_{1 \leq j \leq c} \overline{Y}_{rj\cdot} \geq \overline{Y}_{i1\cdot} - d\,\sigma_\epsilon \ \ (1 \leq i < r)\right\}$$

or equivalently it is

$$P\left\{\min_{1 \leq j \leq c} Z_j + \sqrt{\gamma}\,Z_{r+c} \geq \sqrt{1+\gamma}\,Z_{i+c} - \sqrt{b}\,d \ \ (1 \leq i < r)\right\} \tag{4.4}$$

where Z_j, $j = 1, \ldots, r+c$ are *iid* $N(0,1)$ random variables and $\gamma = \sigma_\omega^2/\sigma_\epsilon^2$. With the identification $q = \sqrt{b}d$, this probability is exactly the left-hand side of (3.4).

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4.2 A Screening Procedure When σ_ω^2 and σ_ϵ^2 are Unknown

We propose a screening procedure that selects a random-sized subset of the designs in such a way as to contain the optimal design with prespecified probability $1 - \alpha$ no matter what the $\boldsymbol{\mu}$ and the variances $\sigma_\omega^2 \geq 0$ and $\sigma_\epsilon^2 > 0$. First we provide an exact solution for the situation when the relative ratio $\gamma = \sigma_\omega^2 / \sigma_\epsilon^2$ is assumed to be known and then an approximate solution when γ is unknown.

When γ is known, we propose the following subset selection procedure which satisfies the confidence requirement

$$P_{\boldsymbol{\mu}, \sigma_\epsilon} \{\text{CS-}\mathcal{G}\} \geq 1 - \alpha \tag{4.5}$$

for all $\boldsymbol{\mu}$ and all $\sigma_\epsilon > 0$.

Procedure $\mathcal{S}_U$: *Include design i in the subset if and only if*

$$\widehat{\eta}_i \geq \widehat{\eta}_{[r]} - d \, s_\epsilon,$$

where $s_\epsilon^2 = \sum\limits_{i=1}^{r} \sum\limits_{j=1}^{c} \sum\limits_{k=1}^{b} (Y_{ijk} - \overline{Y}_{ij} - \overline{Y}_{..k} + \overline{Y}_{...})^2 / \nu_\epsilon$ is based on $\nu_\epsilon = (c-1)(rb-1)$ d.o.f., $d = q/\sqrt{b}$, and q is the solution of (4.6); equivalently the left-hand probability in (4.6) can be computed from (4.7)/(4.8).

The yardstick d must make the procedure achieve (4.5) for the parameter configuration $(\boldsymbol{\mu}, \sigma_\epsilon)$ at which the probability of correct selection using $\mathcal{S}_U$ is a minimum. Theorem 4.2 identifies global minimizer of the PCS and is the analog of Theorem 4.1.

Theorem 4.2 The minimum of the probability that procedure $\mathcal{S}_U$ contains the optimal design occurs at the configuration $\boldsymbol{\mu}_1$ defined by (4.3).

The PCS evaluated at $\boldsymbol{\mu}_1$ is independent of σ_ϵ^2. To obtain d, we need only set the probability of correct selection computed at $\boldsymbol{\mu}_1$ equal to $1 - \alpha$. This probability, set equal to the desired confidence level, is

$$P_{\boldsymbol{\mu}_1, \sigma_\epsilon} \left\{ \min_{1 \leq j \leq c} \overline{Y}_{rj\cdot} \geq \overline{Y}_{i1\cdot} - d \, s_\epsilon, \ (1 \leq i < r) \right\} = 1 - \alpha.$$

Using an argument similar to that in Subsection 4.1, we can reexpress this equation as

$$P \left\{ \min_{1 \leq j \leq c} Z_j \geq \sqrt{1 + \gamma} \max_{1 \leq j < r} Z_{i+c} - \sqrt{\gamma} \, Z_{r+c} - \sqrt{W} \, q \ (1 \leq i < r) \right\} = 1 - \alpha \tag{4.6}$$

where $Z_1, \ldots, Z_{r+c}$ are *iid* $N(0,1)$ variables and $\nu_\epsilon W \sim \chi^2_{\nu_\epsilon}$ is independent of the Z_js (recall $\nu_\epsilon = (c-1)(rb-1)$). Alternatively, if $f_\nu(t)$ denotes the density function of a chi-square random variable with ν d.o.f., the left-hand probability in (4.6) is

$$(r-1)c \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{+\infty} \phi(x)\phi(y) \left[\Phi\left(x\right)\right]^{r-2} \left[1-\Phi\left(y\right)\right]^{c-1} \left[1-\Phi\left(\frac{\sqrt{1+\gamma}x - y - q\sqrt{\frac{t}{\nu}}}{\sqrt{\gamma}}\right)\right] f_{\nu_\epsilon}(t) \, dt\,dx\,dy \tag{4.7}$$

for $\gamma \neq 0$ and

$$(r-1)c \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{+\infty} \phi(x)\phi(x-y-q\sqrt{\frac{t}{\nu}}) \left[\Phi\left(x\right)\right]^{r-2} \left[1-\Phi\left(x-y-q\sqrt{\frac{t}{\nu}}\right)\right]^{c-1} f_{\nu_\epsilon}(t) \, dt\,dx\,dy \tag{4.8}$$

when $\gamma = 0$.

As in Section 3, both simulation and quadrature can be used to compute the minimum probability (4.6). However, quadrature involving the three-fold integral (4.7)/(4.8) is substantially slower than simulation. In fact, we found that quadrature was not feasible to perform the simulation study described two paragraphs below.

When γ is unknown, an exact procedure can be developed by replacing s_ϵ with $s_{c\omega+\epsilon}$ in Procedure $\mathcal{S}_U$. However, the resulting procedure can be conservative and will be very complicated to analyse. An approximate procedure results when one uses $\mathcal{S}_U$ with γ replaced by the moment estimator

$$\hat\gamma = \max\left\{0, \left[\left(s^2_{c\omega+\epsilon}/s^2_\epsilon\right) - 1\right]/c\right\}$$

where

$$s^2_{c\omega+\epsilon} = \sum_{i=1}^r \sum_{j=1}^c \sum_{k=1}^b \left(Y_{ijk} - \overline{Y}_{\cdot j \cdot} - \overline{Y}_{\cdot\cdot k} + \overline{Y}_{\cdots}\right)^2/\nu_{c\omega+\epsilon}$$

is the ANOVA estimator of the sum $c\sigma_\omega^2 + \sigma_\epsilon^2$ with $\nu_{c\omega+\epsilon} = (r-1)(b-1)$ d.o.f. Thus q, and hence d, depends on the data through $\hat\gamma$. When the degrees of freedom associated with $\hat\gamma$ are moderate to large, even for experiments with a few blocks, the approximation will be reasonably good. Intuitively, the reason is that the value of the minimum probability of correct selection is fairly flat when viewed as a function of q. Thus "small" errors resulting from the estimation of σ_ϵ^2 in the definition the selection rule will not have a large effect on the achieved PCS.

To illustrate, we present the results of a simulation study in which the achieved PCS of the approximate rule was estimated at the (least favorable) configuration $\boldsymbol{\mu}_1$ for (σ_ϵ, r, c)

$= (1, 3, 4)$. These estimates, based on 1000 simulation trials, are for a variety of true $(\gamma, b, \sigma_\omega^2)$ values. For each of the 1000 trials, the value of q corresponding to the sample $\hat{\gamma}$ for that trial was determined from (4.6) based on a secondary simulation of 500 replicates. The standard error of the estimated PCS is about .005.

Table 1: Estimated Achieved Probability of Correct Selection for $\mathcal{S}_U$ at $(\mu, \sigma_\epsilon) = (\mu_1, 1)$ when $(r, c) = (3, 4)$ and the Nominal Confidence Level is 80%

σ_ω^2	b	ν_ϵ	$\nu_{c\omega+\epsilon}$	$P_{\boldsymbol{\mu}_1,1}\{\text{CS-}\mathcal{G}\}$
0.0	3	4	24	0.812
0.0	6	10	51	0.810
0.0	9	16	78	0.794
0.0	12	22	105	0.818
0.2	3	4	24	0.795
0.2	6	10	51	0.811
0.2	9	16	78	0.792
0.2	12	22	105	0.811
0.4	3	4	24	0.785
0.4	6	10	51	0.814
0.4	9	16	78	0.786
0.4	12	22	105	0.816

The performance of the procedure depends primarily on the degrees of freedom for the the two mean squared errors which are $\nu_\epsilon = 3(3b - 1)$ and $\nu_{c\omega+\epsilon} = 2(b - 1)$ for this example. Even for the smallest values of these degrees of freedom, 4 and 24, the achieved performance characteristics are nearly nominal for a wide range of γ values.

5   Discussion

We comment on three issues—the minimum of the PCS of the proposed procedures compared with certain no-data procedures, the use of less conservative bounds for the means, and the symmetric assignment of whole-plots and split-plots to the noise and product design factors.

Consider the probability of correct selection using $\mathcal{N}$. In the case of a one-factor selection problem one can use the no-data procedure that picks a level of the factor at random. If there are r levels of the factor, this procedure achieves probability of correct selection $1/r$. In the asymmetric two-factor problem discussed in this paper we also wish to select the levels of one factor; the same no-data procedure achieves probability of correct selection equal to $1/r$

in the present problem. However, if one uses $\mathcal{N}$ and the true $\boldsymbol{\mu}$ is "near" the least favorable configuration, then the achieved PCS can less than $1/r$ if the number of blocks, b, is too few. The reason for this can be seen by examining (3.4) for the simplest possible case, $\gamma = 0$ ($\sigma_\omega^2 = 0$: the randomized complete block design). This probability is

$$P \left\{ \min_{1 \leq j \leq c} Z_j \geq \max_{1 \leq i < r} Z_{i+c} - q \right\}.$$

The best level of the design variable is r; in order for level r to be selected as best the minimum of the column variables, $\min_{1 \leq j \leq c} Z_j$, must exceed the minimum for each of the other rows. However, in the least favorable configuration, only the first column comes into play because the remaining columns all have mean $+\infty$. Thus there is an 'order statistic effect' for this case that increases with c.

To illustrate this effect consider (3.4) when $q = 0$. Table 2 illustrates this phenomenon. First notice that all values of (3.4) are less than $1/r$ (.5 or .25 for $r = 2$ and $r = 4$,

Table 2: Probability (3.4) for $q = 0$ and Selected (r, c, γ)

r	c	γ	(3.4) with $q = 0$
2	2	0.0	0.333
2	4	0.0	0.200
2	2	0.2	0.349
2	4	0.2	0.236
2	2	0.4	0.361
2	4	0.4	0.249
4	2	0.0	0.102
4	4	0.0	0.003
4	2	0.2	0.116
4	4	0.2	0.005
4	2	0.4	0.126
4	4	0.4	0.006

respectively). The lower bound decreases as c increases (r and γ fixed) and the lower bound increases as γ increases (r and c fixed). However, for any fixed (r, c, γ), (3.4) $\rightarrow 1$ as $q \rightarrow +\infty$ showing that it is always possible to design an experiment achieving $\mathcal{R}_\delta$ for any (δ, α).

The argument in the previous paragraph also shows that if an experimenter knows that the relative spread of the means among the operating conditions is less than the $+\infty$ that occur in the least favorable configuration (3.2), then it is desirable to improve the lower

bound (3.4). For example, if it is known that $L \leq \mu_{ij} \leq U$ for all i, j, then

$$\begin{pmatrix} L & U & \vdots & U \\ \vdots & \vdots & \cdots & \vdots \\ L & U & \cdots & U \\ L+\delta & L+\delta & \cdots & L+\delta \end{pmatrix} \text{ and } \begin{pmatrix} L & U & \vdots & U \\ \vdots & \vdots & \cdots & \vdots \\ L & U & \cdots & U \\ L & L & \cdots & L \end{pmatrix}$$

are the least favorable configurations for δ-correct selection using $\mathcal{N}$ and $\mathcal{G}$-correct selection using $\mathcal{S}_{\mathcal{K}}$, respectively. However the value of the corresponding PCS at these $\boldsymbol{\mu}$ is more complicated to calculate than those at either $\boldsymbol{\mu}_0$ or $\boldsymbol{\mu}_1$.

Suppose that one considers the (symmetric) alternative design in which the experimenter assigns the levels of the *noise factor* to the *whole-plots* and then the levels of the *product design factor* to the *split-plots* using, as usual, separate randomizations for each whole-plot and split-plot. The following shows that such a design is much more difficult to implement than the assignment of product design and noise factors as advocated in Section 3.

When the noise factor is assigned to whole-plots,

$$Y_{ijk} = \beta_k + \mu_{ij} + \omega_{jk} + \epsilon_{ijk} \tag{5.1}$$

is the appropriate analog of Model (2.1) where ω_{jk} is the potential confounding effect of the jth level of the noise factor in the kth block, and β_k, μ_{ij} and ϵ_{ijk} are as in Model (2.1). If, aside from the means, all terms have independent normal distributions then it can be shown that if q is determined by

$$\min_{(j_1,\ldots,j_{r-1})\in\{1,\ldots,c\}^{r-1}} P\left\{\min_{1\leq j\leq c}\left\{\sqrt{\gamma}Z_j + Z_j^\star\right\} \geq \max_{1\leq i<r}\left\{\sqrt{\gamma}Z_{j_i} + Z_{i+c}\right\} - q\right\} = 1-\alpha \tag{5.2}$$

where $Z_1, \ldots, Z_{r+c-1}$, and $Z_1^\star, \ldots, Z_c^\star$ are *iid* $N(0,1)$ random variables, then procedure $\mathcal{N}$ satisfies $\mathcal{R}_\delta$ when b is the smallest integer for which $b \geq q^2\sigma_\epsilon^2/\delta^2$ and that procedure $\mathcal{S}_{\mathcal{K}}$ satisfies $\mathcal{R}_{\mathcal{S}}$ when d is defined by $d = q/\sqrt{b}$.

Unfortunately, the ordered $(r-1)$-tuple $(j_1,\ldots,j_{r-1})$ that minimizes the probability in (5.2) depends on r, c, and γ. A lower bound for the left-hand side of (5.2) is

$$P\left\{\min_{1\leq j\leq c}\left\{\sqrt{\gamma}Z_j + Z_j^\star\right\} \geq \max_{1\leq i\leq m}\sqrt{\gamma}Z_i + \max_{1\leq i<r}Z_{i+c} - q\right\} \tag{5.3}$$

$$= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{j=1}^{c}\left[1 - \Phi\left(\sqrt{\gamma}\max_{1\leq i\leq m}z_i + \max_{1\leq i<r}z_{i+c} - \sqrt{\gamma}\,z_j - q\right)\right]\prod_{i=1}^{r+c-1}\phi(z_i)dz_i$$

where $m = \min\{r-1, c\}$, and the Z_j and $Z_j^\star$ are as in (5.2). Setting (5.3) equal to $1-\alpha$ and defining b or d as in the previous paragraph gives conservative procedures satisfying

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the corresponding probability guarantee. Clearly, it is much simpler to use the alternative randomization recommended in Section 3 than attempting to implement the exact procedure via (5.2) or the conservative procedure defined by (5.3).

In some situations, cost and time may demand the use of a fractional factorial design instead of a split-plot design. Selection and screening procedures for fractional factorial experiments based on the minimum mean performance over the levels of the noise factor are being investigated by the authors. Interested readers are referred to Santner and Pan (1997) for details.

Acknowledgement

A   Appendix

Proof of Theorem 3.1

Proof: Let $\widehat{\eta}_{(i)}$ be the estimator corresponding to the design with $\eta_{[i]}$, $\overline{Y}_{(i)1.}, \ldots \overline{Y}_{(i)c.}$ be the sample means associated with $\widehat{\eta}_{(i)}$, and let $\{\mu_{[i]j}\}_{j=1}^{c}$ denote the means associated with $\eta_{[i]}$. Thus $\widehat{\eta}_{(i)} = \min\{\overline{Y}_{(i)1.}, \ldots, \overline{Y}_{(i)c.}\}$ and $\eta_{[i]} = \min\{\mu_{[i]1}, \ldots, \mu_{[i]c}\}$ for $i = 1, \ldots, r$. For simplicity, let $s = r - 1$ throughout.

The idea of the proof is to first show that the procedure selects the (unique) best treatment when $\eta_{[r]} - \eta_{[r-1]} \geq \delta$ and then prove that this implies (2.3) for any $\boldsymbol{\mu}$. Suppose first that $\boldsymbol{\mu}$ satisfies $\eta_{[r]} - \eta_{[s]} \geq \delta$. Then, by definition, the probability of correct selection is

$$
\begin{aligned}
P_{\boldsymbol{\mu}}\{\text{CS-}\delta\} &= P_{\boldsymbol{\mu}}\{\widehat{\eta}_{(r)} \geq \widehat{\eta}_{(i)} \ \ (1 \leq i < r)\} \\
&= P_{\boldsymbol{\mu}}\{\widehat{\eta}_{(r)} - \eta_{[r]} \geq (\widehat{\eta}_{(i)} - \eta_{[i]}) + (\eta_{[i]} - \eta_{[r]}) \ \ (1 \leq i < r)\} \\
&\geq P_{\boldsymbol{\mu}}\{\widehat{\eta}_{(r)} - \eta_{[r]} \geq (\widehat{\eta}_{(i)} - \eta_{[i]}) - \delta \ \ (1 \leq i < r)\}, \qquad (A.4)
\end{aligned}
$$

since $\eta_{[i]} - \eta_{[r]} \leq -\delta$,

$$
\begin{aligned}
(A.4) &= P_{\boldsymbol{\mu}}\left\{\min_{1 \leq j \leq c}\left\{(\overline{Y}_{(r)j.} - \mu_{[r]j}) + (\mu_{[r]j} - \eta_{[r]})\right\} \geq (\widehat{\eta}_{(i)} - \eta_{[i]}) - \delta \ \ (1 \leq i < r)\right\} \\
&\geq P_{\boldsymbol{\mu}}\left\{\min_{1 \leq j \leq c}\{\overline{Y}_{(r)j.} - \mu_{[r]j}\} \geq (\widehat{\eta}_{(i)} - \eta_{[i]}) - \delta \ \ (1 \leq i < r)\right\}, \qquad (A.5)
\end{aligned}
$$

because $\mu_{[r]j} - \eta_{[r]} \geq 0$. Thus

$$
(A.5) \geq P_{\boldsymbol{\mu}}\left\{\min_{1 \leq j \leq c}\{\overline{Y}_{(r)j.} - \mu_{[r]j}\} \geq \overline{Y}_{(i)j_i.} - \mu_{[i]j_i} - \delta \ \ (1 \leq i < r)\right\}, \qquad (A.6)
$$

where j_i is the index for which $\eta_{[i]} = \mu_{[i]j_i}$ since $\overline{Y}_{(i)j_i.} - \mu_{[i]j_i} \geq \widehat{\eta}_{(i)} - \eta_{[i]}$. We have

$$
(A.6) = P_{\boldsymbol{\mu}}\left\{\min_{1 \leq j \leq c}\{\overline{Y}_{(r)j.} - \mu_{[r]j}\} \geq \overline{Y}_{(i)1.} - \mu_{[i]1} - \delta \ \ (1 \leq i < r)\right\}, \qquad (A.7)
$$

because the distribution of $\left(\overline{Y}_{(1)j_1.} - \mu_{[1]j_1}, \ldots, \overline{Y}_{(s)j_s.} - \mu_{[s]j_s}, \min_{1 \leq j \leq c}\left\{\overline{Y}_{(r)j.} - \mu_{[r]j}\right\}\right)$ is the same as that of $\left(\overline{Y}_{(1)1.} - \mu_{[1]1}, \ldots, \overline{Y}_{(s)1.} - \mu_{[s]1}, \min_{1 \leq j \leq c}\left\{\overline{Y}_{(r)j.} - \mu_{[r]j}\right\}\right)$. This gives

$$
(A.7) = P_{\boldsymbol{\mu}}\left\{\min_{1 \leq j \leq c}\{\overline{Y}_{rj.} - \mu_{rj}\} \geq \overline{Y}_{i1.} - \mu_{i1} - \delta \ \ (1 \leq i < r)\right\}, \qquad (A.8)
$$

because the distribution of $\left(\overline{Y}_{(1)1.} - \mu_{[1]1}, \ldots, \overline{Y}_{(s)1.} - \mu_{[s]1}, \min_{1 \leq j \leq c}\left\{\overline{Y}_{(r)j.} - \mu_{[r]j}\right\}\right)$ is identical to that of $\left(\overline{Y}_{11.} - \mu_{11}, \ldots, \overline{Y}_{s1.} - \mu_{s1}, \min_{1 \leq j \leq c}\left\{\overline{Y}_{rj.} - \mu_{rj}\right\}\right)$. Finally,

$$
\begin{aligned}
(A.8) &= P_{\boldsymbol{\mu}_0}\left\{\min_{1 \leq j \leq c}\{\overline{Y}_{rj.}\} \geq \overline{Y}_{i1.} \ \ (1 \leq i < r)\right\} \qquad &(A.9) \\
&= P_{\boldsymbol{\mu}_0}\left\{\min_{1 \leq j \leq c}\{\overline{Y}_{rj.}\} \geq \min_{1 \leq j \leq c}\{\overline{Y}_{ij.}\} \ \ (1 \leq i < r)\right\} \qquad &(A.10) \\
&= P_{\boldsymbol{\mu}_0}\left\{\widehat{\eta}_{(r)} \geq \widehat{\eta}_{(i)} \ \ (1 \leq i < r)\right\} = P_{\boldsymbol{\mu}_0}\{\text{CS-}\delta\} \qquad &(A.11)
\end{aligned}
$$

where (A.9) holds because the distribution of $\left(\overline{Y}_{11\cdot} - \mu_{11}, \ldots, \overline{Y}_{s1\cdot} - \mu_{s1}, \min_{1 \le j \le c}\left\{\overline{Y}_{rj\cdot} - \mu_{rj}\right\}\right)$ under $\boldsymbol{\mu}$ is the same as that of $\left(\overline{Y}_{11\cdot}, \ldots, \overline{Y}_{s1\cdot}, \min_{1 \le j \le c}\left\{\overline{Y}_{rj\cdot}\right\}\right)$ under $\boldsymbol{\mu}_0$, and (A.10) holds because $\mu_{ij} = +\infty$ for $i = 1, \ldots, s$ and $j = 2, \ldots, c$.

For the second part of the proof, assume that $\boldsymbol{\mu}$ is such that $\eta_{[r]} - \eta_{[s]} < \delta$. Let $S_{\mathcal{N}}$ denote the index of the design selected by $\mathcal{N}$, i.e., $\widehat{\eta}_{S_{\mathcal{N}}}$ is the maximum $\widehat{\eta}_i$. By definition

$$
\begin{aligned}
P_{\boldsymbol{\mu}}\{\text{CS-}\delta\} &= P_{\boldsymbol{\mu}}\left\{\eta_{[r]} - \delta \le \eta_{S_{\mathcal{N}}}\right\} \\
&\ge P_{\boldsymbol{\mu}}\left\{\eta_{S_{\mathcal{N}}} \ge \eta_{[r]} - \delta + (\widehat{\eta}_{S_{\mathcal{N}}} - \widehat{\eta}_{(r)})\right\} \quad\quad (A.12) \\
&= P_{\boldsymbol{\mu}}\left\{(\widehat{\eta}_{(r)} - \eta_{[r]}) \ge (\widehat{\eta}_{S_{\mathcal{N}}} - \eta_{S_{\mathcal{N}}}) - \delta\right\} \quad\quad (A.13)
\end{aligned}
$$

where (A.12) holds because $\widehat{\eta}_{S_{\mathcal{N}}} \ge \widehat{\eta}_{(r)}$. Thus

$$
\begin{aligned}
(A.13) &\ge P_{\boldsymbol{\mu}}\left\{(\widehat{\eta}_{(r)} - \eta_{[r]}) \ge (\widehat{\eta}_{(i)} - \eta_{[i]}) - \delta \ (1 \le i < r)\right\} \\
&= P_{\boldsymbol{\mu}}\left\{(\widehat{\eta}_{(r)} + \delta) - \eta_{[r]} \ge (\widehat{\eta}_{(i)} - \eta_{[i]}) \ (1 \le i < r)\right\} \quad\quad (A.14)
\end{aligned}
$$

Let $\boldsymbol{\mu}^{\delta} = (\mu_{ij}^{\delta})$ be defined

$$
\mu_{ij}^{\delta} = \begin{cases} \mu_{ij} - \eta_{[i]} & \text{for } i \ne [r] \text{ and } 1 \le j \le c \\ \mu_{[r]j} + \delta - \eta_{[r]} & \text{for } i = [r] \text{ and } 1 \le j \le c \end{cases}
$$

For $i = 1, \ldots, r$, let η_i^{δ} correspond to $\boldsymbol{\mu}^{\delta}$. Then $\eta_i^{\delta} = \min\{\mu_{ij} - \eta_i\} = 0$ for $i \ne [r]$ and $\eta_{[r]}^{\delta} = \min\{\mu_{[r]j} + \delta - \eta_{[r]}\} = \delta$. Thus we obtain,

$$
\begin{aligned}
(A.14) &= P_{\boldsymbol{\mu}^{\delta}}\left\{\widehat{\eta}_{(r)} \ge \widehat{\eta}_{(i)} \ (1 \le i < r)\right\} \\
&\ge P_{\boldsymbol{\mu}_0}\{\text{CS-}\delta\} \quad\quad (A.15)
\end{aligned}
$$

where (A.15) holds by (A.11) because $\boldsymbol{\mu}^{\delta}$ has an associated $\boldsymbol{\eta}^{\delta}$ that satisfies the condition of Part I. $\quad\square$

Proof of Theorem 4.1

Proof: Using the same notation as in the proof of Theorem 3.1, the probability that the optimum design is contained in the selected subset is

$$
\begin{aligned}
P_{\boldsymbol{\mu}}\{\text{CS-}\mathcal{G}\} &= P_{\boldsymbol{\mu}}\{\widehat{\eta}_{(r)} \ge \widehat{\eta}_{[r]} - d\,\sigma_{\epsilon}\} \\
&= P_{\boldsymbol{\mu}}\{\widehat{\eta}_{(r)} \ge \widehat{\eta}_{(i)} - d\,\sigma_{\epsilon} \ (1 \le i < r)\} \\
&= P_{\boldsymbol{\mu}}\left\{\min_{1 \le j \le c}\{\overline{Y}_{(r)j\cdot}\} \ge \min_{1 \le j \le c}\{\overline{Y}_{(i)j\cdot}\} - d\,\sigma_{\epsilon} \ (1 \le i < r)\right\} \\
&= P_{\boldsymbol{\mu}}\left\{\min_{1 \le j \le c}\{\overline{Y}_{(r)j\cdot} - \eta_{[r]}\} \ge \min_{1 \le j \le c}\{\overline{Y}_{(i)j\cdot} - \eta_{[i]}\} + (\eta_{[i]} - \eta_{[r]}) - d\,\sigma_{\epsilon} \ (1 \le i < r)\right\} \\
&\ge P_{\boldsymbol{\mu}}\left\{\min_{1 \le j \le c}\{\overline{Y}_{(r)j\cdot} - \eta_{[r]}\} \ge \min_{1 \le j \le c}\{\overline{Y}_{(i)j\cdot} - \eta_{[i]}\} - d\,\sigma_{\epsilon} \ (1 \le i < r)\right\} \quad (A.16)
\end{aligned}
$$

since $\eta_{[i]} - \eta_{[r]} \leq 0$ for $1 \leq i < r$. Thus

$$(A.16) \quad = \quad P_{\boldsymbol{\mu}} \left\{ \min_{1 \leq j \leq c} \{ \overline{Y}_{(r)j\cdot} - \mu_{[r]j} + \mu_{[r]j} - \eta_{[r]} \} \geq \min_{1 \leq j \leq c} \{ \overline{Y}_{(i)j\cdot} - \eta_{[i]} \} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\}$$

$$\geq \quad P_{\boldsymbol{\mu}} \left\{ \min_{1 \leq j \leq c} \{ \overline{Y}_{(r)j\cdot} - \mu_{[r]j} \} \geq \min_{1 \leq j \leq c} \{ \overline{Y}_{(i)j\cdot} - \eta_{[i]} \} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\} \qquad (A.17)$$

since $(\mu_{[r]j} - \eta_{[r]}) \geq 0$ for $j = 1, \ldots, c$. For $1 \leq i < r$, let j_i denote the index for which $\overline{Y}_{(i)j_i\cdot} = \min_{1 \leq j \leq c} \overline{Y}_{(i)j\cdot}$; we obtain

$$(A.17) \quad \geq \quad P_{\boldsymbol{\mu}} \left\{ \min_{1 \leq j \leq c} \{ (\overline{Y}_{(r)j\cdot} - \mu_{[r]j}) \} \geq \overline{Y}_{(i)j_i\cdot} - \eta_{[i]} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\}$$

$$\geq \quad P_{\boldsymbol{\mu}} \left\{ \min_{1 \leq j \leq c} \{ (\overline{Y}_{(r)j\cdot} - \mu_{[r]j}) \} \geq \overline{Y}_{(i)j_i\cdot} - \mu_{[i]j_i} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\} \quad (A.18)$$

$$= \quad P_{\boldsymbol{\mu}} \left\{ \min_{1 \leq j \leq c} \{ (\overline{Y}_{(r)j\cdot} - \mu_{[r]j}) \} \geq \overline{Y}_{(i)1\cdot} - \mu_{[i]1} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\} \quad (A.19)$$

where (A.18) holds because $\mu_{[i]j_i} - \eta_{[i]} \geq 0$ and equality holds in (A.19) because $(\overline{Y}_{(i)j_i\cdot} - \mu_{[i]j_i})$, for $1 \leq i < r$, and $\min_{1 \leq j \leq c} \{ \overline{Y}_{(r)j} - \mu_{(r)j} \}$ are mutually independent and their distributions do not depend on $\boldsymbol{\mu}$. Thus

$$(A.19) \quad = \quad P_{\boldsymbol{\mu}} \left\{ \min_{1 \leq j \leq c} \{ \overline{Y}_{rj\cdot} - \mu_{rj} \} \geq \overline{Y}_{i1\cdot} - \mu_{i1} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\}$$

$$= \quad P_{\boldsymbol{\mu}_1} \left\{ \min_{1 \leq j \leq c} \{ \overline{Y}_{rj\cdot} \} \geq \overline{Y}_{i1\cdot} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\}, \qquad (A.20)$$

where $\boldsymbol{\mu}_1$ is given in Theorem 4.1 because the joint distribution of $(\overline{Y}_{rj} - \mu_{rj})$, for $j = 1, \ldots, c$, and $(Y_{i1\cdot} - \mu_{i1})$, for $1 \leq i < r$, under $\boldsymbol{\mu}$, is identical to the joint distribution of $\overline{Y}_{rj\cdot}$, for $j = 1, \ldots, c$, and $Y_{i1\cdot}$, for $1 \leq i < r$ under $\boldsymbol{\mu}_1$. Thus

$$(A.20) \quad = \quad P_{\boldsymbol{\mu}_1} \left\{ \min_{1 \leq j \leq c} \overline{Y}_{rj\cdot} \geq \min_{1 \leq j \leq c} Y_{ij\cdot} - d \, \sigma_\epsilon \ \ (1 \leq i < r) \right\} \qquad (A.21)$$

$$= \quad P_{\boldsymbol{\mu}_1} \{ \text{CS-}\mathcal{G} \}$$

where (A.21) holds because $\mu_{ij} = +\infty$ for $1 \leq i < r$ and $j = 2, \ldots, c$ in $\boldsymbol{\mu}_1$ $\qquad \square$

References

Anderson, V.L. and McLean, R.A. (1974) *Design of Experiments: A Realistic Approach*. Marcel Dekker, Inc, New York.

Bechhofer, R.E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. *Annals of Mathematical Statistics* **25**, 16-39.

Bechhofer, R.E., Santner, T.J. and Goldsman, D.M. (1995). *Designing Experiments for Statistical Selection, Screening, and Multiple Comparisons*, J. Wiley and Sons, New York.

Beeson, J. R. (1965). *A Simulator for Evaluating Prosthetic Cardiac Valves*. Unpublished M.S. Thesis, Purdue University Library, West Lafayette, IN.

Box, G.E.P. (1988). Signal-to-Noise Ratios, Performance Criteria, and Transformations. *Technometrics*, **30**, 1-17.

Box, G.E.P. and Jones, S. (1992). Split-Plot Designs for Robust Product Experimentation. *Jour. Applied Statistics*, **19**, 3-26.

Fabian, V. (1962). On Multiple Decision Methods for Ranking Population Means. *Ann. Math. Statist.* **33**, 248-254.

Milliken, G. and Johnson, D. (1984). *The Analysis of Messy Data: Volume 1, Experimental Data*. Van Nostrand Reinhold Co., New York.

Montgomery, D.C. (1991). *Design and Analysis of Experiments, Ed. 3*. J. Wiley & Sons, New York.

Santner, T. J. and Pan, G. (1997). The Use of Subset Selection in Combined-Array Experiments to Determine Optimal Product or Process Designs. *Advances in Statistical Decision Theory and Methodology*, Editors: Panchapakesan, S. and Balakrishnan, N., to appear.
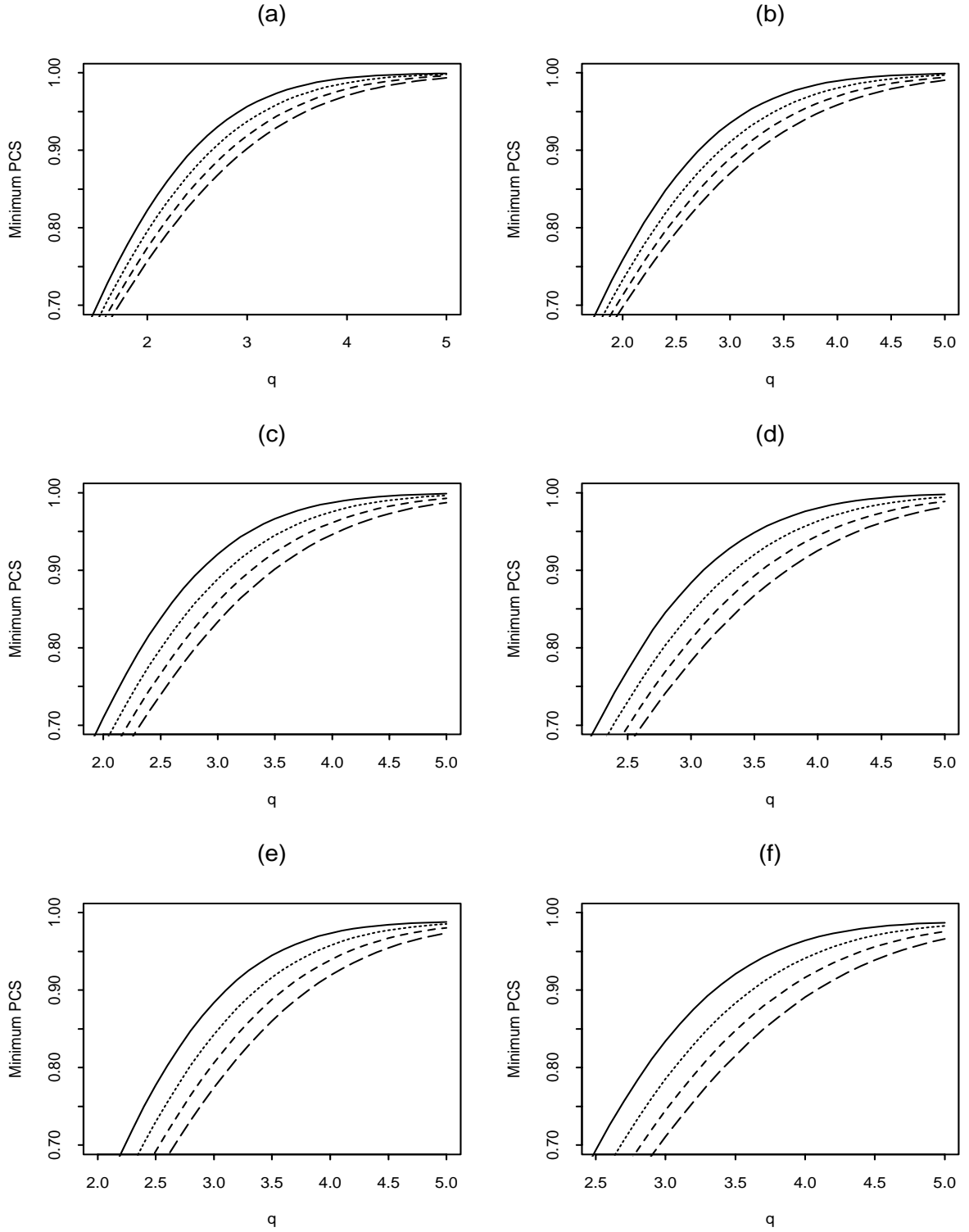
Figure 1: Plot of $\left(q, P\left\{\min_{1 \le j \le c} Z_j + \sqrt{\gamma}\, Z_{r+c} \ge \sqrt{1+\gamma}\, \max_{1 \le i < r} Z_{i+c} - q\right\}\right)$ for $(r, c, \gamma) \in \{2, 3, 4\} \times \{3, 5\} \times \{0.0, 0.2, 0.4, 0.6\}$. Here $Z_1, \ldots, Z_{r+c}$ are *iid* $N(0, 1)$ random variables. The solid line (———) is $\gamma = 0.0$, dotted line ($\cdots$) is $\gamma = 0.2$, the short-dashed line (– –) is $\gamma = 0.4$, and the long-dashed line (— —) is $\gamma = 0.6$. Panel (a) is $(r, c) = (2, 3)$, panel (b) is $(r, c) = (2, 5)$, panel (c) is $(r, c) = (3, 3)$, panel (d) is $(r, c) = (3, 5)$, panel (e) is $(r, c) = (4, 3)$, and panel (f) is $(r, c) = (4, 5)$.