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Renewal Theory for Functionals of a Markov Chain with Compact State Space

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Renewal theory for functionals of a Markov chain with compact state space *

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Abstract

Motivated by multivariate random recurrence equations we prove a new analogue of the Key Renewal Theorem for functionals of a Markov chain with compact state space in the spirit of Kesten [14]. We simplify and modify Kesten's proof.

Key words: key renewal theorem, Markov chain, random recurrence equation, Riemann integrability

AMS (1991) Subject Classification: primary 60J05, 60K05, 60K15, 60H25.

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1 Introduction

"We prove an analogue of Blackwell's Renewal Theorem or the Key Renewal Theorem in the following setup: $(x_n)_{n\geq 0}$ is a Markov chain with separable metric state space S and $(u_n)_{n\geq 0}$ is a sequence of random variables such that the conditional distribution of u_i, given all the x_j and u_j, $j \neq i$, depends on x_i and x_{i+1} only. Here the $v_n = \sum_{i=1}^n u_i$, $n \in \mathbb{N}$, take the role of the partial sums of iid random variables in ordinary renewal theory. The Key Renewal Theorem in this setup states that $\lim_{t\to\infty} E_x \sum_{n=0}^\infty g(x_n, t - v_n)$ exists for suitable functions g and is independent of x."

This is quoted from the abstract of Kesten's famous paper [14], which has attracted vast attention in particular in the area of random recurrence equations; see e.g. Verwaat [21], Goldie [8] and Goldie and Maller [9]. Such equations play an important role in many applications as e.g. in queueing; see Brandt, Franken and Lisek [3] and in financial time series; see Engle [5]. The Key Renewal Theory is used in such models to derive the tail behaviour and study extreme value theory of a stationary version of $(x_n)_{n\geq 0}$. Some special examples have been worked out as ARCH(1) and GARCH(1,1); see Goldie [8], de Haan et al. [10] and Mikosch and Starica [16].

In this paper we review and modify Kesten's paper [14] motivated by examples more general than the above. We consider multivariate random recurrence equations of the type

$$Y_n = A_n Y_{n-1} + \zeta_n\,, \quad n \in \mathbb{N}\,, \tag{1.1}$$

where the Y_n and ζ_n are column vectors of size q and A_n are $(q \times q)$ matrices. Moreover, we assume that $((A_n, \zeta_n))$ are iid.

Under appropriate stability conditions (see Goldie and Maller [9]) equation (1.1) has a stationary distribution defined by

$$Y = \zeta_1 + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1} \zeta_k\,. \tag{1.2}$$

Questions of interest concern the tail behaviour

$$\mathbf{P}(x'Y > t) \quad \text{as} \quad t \to \infty \tag{1.3}$$

for every $x \in S = \{z \in \mathbb{R}^q \,:\, |z| = 1\}$ and the extremal behaviour of the corresponding stationary distribution and process respectively. Here $|\cdot|$ denotes any norm in $\mathbb{R}^q$.

In the one-dimensional case ($q = 1$) Goldie [8] has solved the problem in a very elegant way and found the tail behaviour (1.3). But for the multivariate model ($q > 1$) renewal theory is called for. One can show (see, for example, Kesten [13] and Le Page [18]) that the function $\mathbf{P}(x'Y > t)$ is asymptotically equivalent to a renewal function, that is

$$\mathbf{P}(x'Y > t) \sim G(x,t) = \mathbf{E}_x \sum_{i=0}^{\infty} g(x_n, t - v_n)\,, \quad t \to \infty\,, \tag{1.4}$$

where $\sim$ means that the quotient of both sides tends to 1. Here $g(\cdot,\cdot)$ is some continuous function satisfying condition (2.4). In the context of model (1.1) the processes $(x_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are defined as

$$x_0 = x \in S\,, \qquad x_n = \frac{A_n x_{n-1}}{|A_n x_{n-1}|} = \frac{A_n \cdots A_1 x}{|A_n \cdots A_1 x|}\,, \quad n \geq 1, \tag{1.5}$$

and $v_0 = 0$ and for $n \in \mathbb{N}$,

$$v_n = \sum_{i=1}^{n} u_i = \log|A_n \cdots A_1 x|\,, \qquad \text{with } u_n = \log|A_n x_{n-1}|. \tag{1.6}$$

To obtain the asymptotic behaviour of $G(x,t)$ we apply the Key Renewal Theorem to (1.4). Unfortunately, to apply this theorem one has to check a "direct Riemann integrability" condition for the function $g(\cdot,\cdot)$; see Kesten [14], equation (1.11). This is a difficult task because it requires the explicit form of the infinite distributions of the processes (1.5) and (1.6). For matrices with non-negative elements Kesten [13] proved that his notion of "direct Riemann integrablity" is equivalent to our condition (2.4) below, which is in general weaker than Kesten's condition. Since models like ARCH(1) and GARCH(1,1) play a prominent role as volatility models in finance, which are by nature positive, Kesten's results apply. When we consider more general models like autoregressive models with GARCH errors or random coefficient autoregressive models, elements of A_n are often normally distributed, and this means model (1.1) falls outside the scope of Kesten's work. The tail behaviour and extreme value theory of an AR(1) model with ARCH(1) errors was investigated in Borkovec and Klüppelberg (2001) by different (purely analytic) methods. It seems to be difficult, if not impossible, to extend these methods to higher order processes of this kind. For this reason we come back to Kesten's methods as an appropriate remedy. Our generalisation in this respect goes in the same direction as le Page [18].

On the other hand, all models we want to consider have compact state space; indeed, our models have state space $S = \{z \in \mathbb{R}^q : |z| = 1\}$. Kesten and le Page, however, work with Markov chains with general state space (which can be unbounded). Hence, in our context, Kesten's conditions and also proofs can be simplified considerably.

The result of this paper is applied to various models in the accompanying paper Klüppelberg and Pergamenchtchikov [15].

Our paper is organised as follows. In Section 2 we state the conditions and the Key Renewal Theorem, which is our main result. In Section 3 we prove the for us necessary version of the Choquet-Deny lemma similar to the one used already in Feller [6] for the proof of the classical Key Renewal Theorem. Section 4 ensures the existence of a limit for a time changed version of the Markov chain $(x_n)_{n\geq 0}$ under investigation and the overshoot of the corresponding time changed process $(v_n)_{n\geq 0}$. Some properties of the renewal function are investigated in Section 5 and, finally, the Key Renewal Theorem is proved in Section 6. Some technical results are summarized in the Appendix as not to disturb the flow of arguments in the paper.

2 Main result

We consider a filtered probability space $(\Omega,\, \mathcal{F},\, (\mathcal{F}_n)_{n\geq 0},\, \mathbf{P})$; i.e. $(\mathcal{F}_n)_{n\geq 0}$ is a non-decreasing family of sub-σ-fields of $\mathcal{F}$. Let $(x_n)_{n\geq 0}$ be a homogeneous Markov chain with compact state space $S \subset \mathbb{R}^q$, on which a σ-field $\mathcal{G}$ is given. We suppose that $(x_n)_{n\geq 0}$ is an $\mathcal{F}_n$-adapted process; i.e. x_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}_0$.

For this Markov process we denote the transition probabilities

$$\mathbf{P}_x(\Gamma) = \mathbf{P}(x_1 \in \Gamma | x_0 = x), \qquad \mathbf{P}_x^{(n)}(\Gamma) = \mathbf{P}(x_{n+k} \in \Gamma | x_k = x), \quad k \in \mathbb{N},$$

for every $x \in S$ and every measurable set $\Gamma \subseteq S$.

We also consider the $(\mathcal{F}_n)_{n\geq 0}$-adapted stochastic process $(u_n)_{n\in\mathbb{N}}$. We request certain further conditions on the processes $(x_n)_{n\geq 0}$ and $(u_n)_{n\geq 0}$. These are the following.

$\mathbf{C}_1$) *For every bounded measurable function* $f\,:\,\mathbb{R}\times\Pi_{i=0}^{\infty}(S\times\mathbb{R})\to\mathbb{R}$ *and for every* $\mathcal{F}_n$*-measurable random variable* η

$$\begin{aligned} &\mathbf{E}(f(\eta, x_{n+1}, u_{n+1}, \ldots, x_{n+l}, u_{n+l}, \ldots)|\mathcal{F}_n) \\ &\quad = \mathbf{E}_{x_n} f(\eta, x_{n+1}, u_{n+1}, \ldots, x_{n+l}, u_{n+l}, \ldots) \\ &\quad = \Phi(\eta, x_n)\,, \end{aligned} \tag{2.1}$$

where $\Phi(a, x) = \mathbf{E}_x f(a, x_1, u_1, \ldots, x_l, u_l, \ldots)$ *for every* $a \in \mathbb{R}$ *and* $x \in S$.

We assume further that

$$m^* = \sup_{x\in S} \mathbf{E}_x |u_1| < \infty\,. \tag{2.2}$$

Now consider the sequence

$$v_0 = 0 \quad \text{and} \quad v_n = \sum_{i=1}^{n} u_i\,, \quad n \geq 1\,. \tag{2.3}$$

Property (2.1) implies that the bivariate process $(x_n, v_n)_{n\geq 0}$ is a Markov process.

We study the asymptotic properties of the renewal function

$$\mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k),$$

where the function $g : S \times \mathbb{R} \to \mathbb{R}$ satisfies the following uniform direct Riemann integrability condition

$$\sum_{l=-\infty}^{\infty} \sup_{x \in S} \sup_{l \le t \le l+1} |g(x,t)| < \infty \,. \tag{2.4}$$

With this notation we can formulate the following conditions:

$\mathbf{C}_2$) *There exists a probability measure $\pi(\cdot)$ on S, which is equivalent to Lebesgue measure such that*

$$\|\mathbf{P}_x^{(n)}(\cdot) - \pi(\cdot)\| \to 0\,, \quad n \to \infty\,, \tag{2.5}$$

for all $x \in S$, where $\|\cdot\|$ denotes total variation of measures on S. Note that this implies that $(x_n)_{n\ge 0}$ is recurrent.
Moreover, there exists a constant $\beta > 0$ such that for all $x \in S$

$$\lim_{n\to\infty} \frac{v_n}{n} = \beta \qquad \mathbf{P}_x - a.s.$$

$\mathbf{C}_3$) *There exists a number $m \in \mathbb{N}$ such that for all $\nu \in \mathbb{R}$ and for all $\delta > 0$ there exist $y_{\nu,\delta} \in S$ and $\varepsilon_0 = \varepsilon_0(\nu, \delta) > 0$ such that $\forall\ 0 < \varepsilon < \varepsilon_0$*

$$\inf_{x \in B_{\varepsilon,\delta,\nu}} \mathbf{P}_x(|x_m - y_{\nu,\delta}| < \varepsilon,\ |v_m - \nu| < \delta) = \rho_* = \rho_{\varepsilon,\delta,\nu} > 0\,, \tag{2.6}$$

where $B_{\varepsilon,\delta,\nu} = \{x \in S :\ |x - y_{\nu,\delta}| < \varepsilon\}$.

$\mathbf{C}_4$) *Let $\Phi\ :\ S \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Then there exists some $l \in \mathbb{N}$ such that the function $\Phi_1(x,t) = \mathbf{E}_x \Phi(x_l, v_l, t)$ satisfies the following property:*

$$\sup_{|x-y|<\varepsilon} \sup_{t \in \mathbb{R}} |\Phi_1(x,t) - \Phi_1(y,t)| \to 0\,, \quad \varepsilon \to 0\,.$$

The following is the main result of this paper.

Theorem 2.1. (Key Renewal Theorem for Markov chains with compact state space) *Assume that conditions $\mathbf{C}_1 - \mathbf{C}_4$ are satisfied. Then for every continuous bounded function g satisfying condition (2.4)*

$$\lim_{t\to\infty} \mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k) = \frac{1}{\beta} \int_S \pi(dx) \int_{-\infty}^{\infty} g(x,t)\, dt\,, \quad \forall x \in S\,. \tag{2.7}$$

Remark 2.2. (a) For a non-negative sequence $(u_n)_{n\ge 0}$, this theorem follows directly from Kesten [14]. This case was also considered in Shurenkov [20] by analytic methods, and he proved this result for g satisfying a weaker condition than (2.4).

(b) This type of result can also be obtained by regeneration methods for Markov chains as developed in Athreya, McDonald and Ney [1] and Alsmeyer [2]. In these papers almost sure convergence of (2.7) with respect to the stationary distribution π was shown. Fuh and Lai [7] finally found the rate of this a.s. convergence. Unfortunately, we cannot use these results, since we want to apply Theorem 2.1 for a single value of $x \in S$ to obtain the tail behaviour of certain models; see Klüppelberg and Pergamenchtchikov [15] for details.

3 The Choquet-Deny lemma

In this section we prove an analogue of the Choquet-Deny lemma for our situation under conditions $\mathbf{C}_1 - \mathbf{C}_4$. Such a lemma is prominent in proofs of renewal theorems as in Kesten [14], Section 2, but also already in the classical case; see Feller [6], Lemma XI.2.1 and Corollary to Lemma XI.9.1.

Define for $t \geq 0$,

$$N(t) = \inf\{n > 0 : \ v_n > t\}\,, \quad N(0) = 0\,, \tag{3.1}$$

and

$$Z(t) = x_{N(t)}\,, \quad W(t) = v_{N(t)} - t\,. \tag{3.2}$$

By (2.5) we have immediately for all $t \geq 0$ that $N(t) < \infty$ $\mathbf{P}_x$ - a.s. for all $x \in S$ and that $\lim_{t\to\infty} N(t) = \infty$.

Let $f \ : \ S \times \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous bounded function. Define H_0 and H by

$$H_0(x,t) = \mathbf{E}_x f(Z(t), W(t))\chi_{\{t\geq 0\}}\,, \tag{3.3}$$

where χ_B denotes the indicator function of a set B, and

$$H(x,t) = \int_{-\infty}^{\infty} H_0(x, t+s)\theta(s)\mathrm{d}s, \tag{3.4}$$

where $\theta(\cdot)$ is some continuous function with compact support.

Lemma 3.1. *Assume that conditions $\mathbf{C}_1 - \mathbf{C}_4$ are satisfied. Then every sequence $(t_n)_{n\in\mathbb{N}}$ which tends to infinity as $n \to \infty$ contains a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that for every $s \in \mathbb{R}$,*

$$\lim_{k\to\infty} H(x, t_{n_k} + s)$$

exists and is independent of x and s.

Proof. Take $l \in \mathbb{N}$ fix. We first investigate for $x \in S$ and $t \geq 0$ the function

$$H_l(x,t) \;=\; \int_{-\infty}^{\infty} H_{0,l}(x,t+s)\theta(s)\mathrm{d}s$$

with

$$\begin{aligned}
H_{0,l}(x,t) &= \mathbf{E}_x f(Z(t),W(t))\chi_{\{N(t)>l\}}\chi_{\{t\geq 0\}} \\
&= \mathbf{E}_x \sum_{j=l+1}^{\infty} f(x_j, v_j - t)\chi_{\{N(t)=j\}} \\
&= \mathbf{E}_x \sum_{j=l+1}^{\infty} f(x_j, v_j - t)\chi_{\{v_1\leq t,\ldots,v_{j-1}\leq t, v_j>t\}} \\
&= \mathbf{E}_x \chi_{\{v_1\leq t,\ldots,v_{l-1}\leq t\}}\mathbf{E}(\tilde{f}(v_l, x_{l+1}, u_{l+1},\ldots)|\mathcal{F}_l)\,,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}(v_l, x_{l+1}, u_{l+1},\ldots) &= \sum_{j=1}^{\infty} f(x_{j+l}, v_{j+l} - v_l - (t-v_l))\chi_{\left\{v_l\leq t,\ldots,v_{j+l-1}-v_l\leq t-v_l, v_{j+l}-v_l>t-v_l\right\}} \\
&= \sum_{j=1}^{\infty} f\left(x_{j+l}, \sum_{i=l+1}^{j+l} u_i - (t-v_l)\right)\chi_{\left\{v_l\leq t,\ldots,\sum_{i=l+1}^{j+l-1}u_i\leq t-v_l, \sum_{i=l+1}^{j+l}u_i>t-v_l\right\}}\,.
\end{aligned}$$

By (2.1) we obtain that

$$H_{0,l}(x,t) = \mathbf{E}_x\chi_{\{v_1\leq t,\ldots,v_{l-1}\leq t\}}\Phi(x_l, v_l, t)\,,$$

where

$$\begin{aligned}
\Phi(x,v,t) &= \mathbf{E}_x\tilde{f}(v, x_1, u_1, \ldots) \\
&= \chi_{\{v\leq t\}}\sum_{j=1}^{\infty} f\left(x_j, v_j - (t-v)\right)\chi_{\{v_1\leq t-v,\ldots,v_{j-1}\leq t-v, v_j>t-v\}} \\
&= \chi_{\{v\leq t\}}\sum_{j=1}^{\infty} f\left(x_j, v_j - (t-v)\right)\chi_{\{N(t-v)=j\}} \\
&= \chi_{\{v\leq t\}}\mathbf{E}_x f(Z(t-v), W(t-v)) \qquad (3.5) \\
&= H_0(x, t-v)\,.
\end{aligned}$$

Hence,

$$H_{0,l}(x,t) \;=\; \mathbf{E}_x H_0(x_l, t-v_l) - \delta(x,t)$$

with

$$\delta(x,t) = \mathbf{E}_x(1-\chi_{\{v_1\leq t,\cdots,v_{l-1}\leq t\}})H_0(x_l, t-v_l)\,.$$

By the dominated convergence theorem $\delta(x,t) \to 0$ as $t \to \infty$ for all $x \in S$. Therefore

$$H_l(x,t) = \int_0^\infty H_{0,l}(x,u)\theta(u-t)\mathrm{d}u = H_{1,l}(x,t) - \Delta(x,t)\,,$$

where

$$H_{1,l}(x,t) = \int_0^\infty \mathbf{E}_x H_0(x_l, u - v_l)\theta(u-t)\mathrm{d}u\,, \qquad \Delta(x,t) = \int_0^\infty \delta(x,u)\theta(u-t)\mathrm{d}u\,.$$

Notice that the function $H_{1,l}(x,t)$ (by condition $\mathbf{C}_4$) satisfies the property:

$$\sup_{|x-y|<\varepsilon}\ \sup_{|t'-t''|<\delta} |H_{1,l}(x,t') - H_{1,l}(y,t'')| \to 0\,, \quad \delta \to 0\,, \quad \varepsilon \to 0$$

and $\Delta(x,t) \to 0$ as $t \to \infty$ for every $x \in S$. Therefore we can (by standard diagonal selection methods (Rudin [19], Theorem 7.23)) find a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ for which

$$\lim_{k\to\infty} H_l(x, t_{n_k} + s)$$

exists for every $x \in S$ and $s \in \mathbb{R}$.

Since f is bounded we get for every $l \in \mathbb{N}$ and $x \in S$

$$H_0(x,t) - H_{0,l}(x,t) \to 0\,, \quad t \to \infty\,. \tag{3.6}$$

But this means that $H(x,t) - H_l(x,t) \to 0$ as $t \to \infty$. Therefore, for every $(x,s) \in S \times \mathbb{R}$, there exists the limit

$$G(x,s) = \lim_{k\to\infty} H(x, t_{n_k} + s)\,.$$

By (3.5), since $u_1 = v_1$,

$$\begin{aligned} &H_0(x,t) - \mathbf{E}_x H_0(x_1, t - u_1) \\ &\quad= \quad H_0(x,t) - H_{0,1}(x,t) + H_{0,1}(x,t) - \mathbf{E}_x H_0(x_1, t-u_1) \\ &\quad= \quad o(1) - \delta(x,t) \ \to \ 0\,, \quad t \to \infty\,. \end{aligned}$$

From this we conclude for every $x \in S$,

$$\lim_{t\to\infty} (H(x,t) - \mathbf{E}_x H(x_1, t-u_1)) = 0$$

and we have that

$$G(x,s) = \mathbf{E}_x G(x_1, s-u_1) = \mathbf{E}_x G(x_2, s - v_2) = \cdots = \mathbf{E}_x G(x_l, s - v_l)\,, \quad l \geq 1\,. \tag{3.7}$$

By condition $\mathbf{C}_4$ this function satisfies the following continuity condition:

$$\sup_{|x-y|<\varepsilon}\ \sup_{|t'-t''|<\delta} |G(x,t') - G(y,t'')| \to 0\,, \quad \delta \to 0\,, \quad \varepsilon \to 0\,. \tag{3.8}$$

We show now that for each $x \in \mathbb{R}$,

$$G(x,s) = G(x, s+\nu)\,, \quad \nu \in \mathbb{R}\,. \tag{3.9}$$

Suppose that there exist $x_0 \in S\,, s_0 \in \mathbb{R}, \nu \in \mathbb{R}$ such that

$$G(x_0, s_0) < G(x_0, s_0+\nu)\,.$$

Set $\tilde{G}(x,s) = G(x, s+\nu)$. Notice that the sequences

$$(G(x_n, s_0 - v_n))_{n\in\mathbb{N}} \quad \text{and} \quad (\tilde{G}(x_n, s_0 - v_n))_{n\in\mathbb{N}}$$

are bounded martingales, which converge $\mathbf{P}_{x_0}$-a.s. to random variables G_∞ and $\tilde{G}_\infty$ such that for all $n \in \mathbb{N}$

$$G(x_0,s_0) = \mathbf{E}_{x_0} G(x_n, s_0 - v_n) = \mathbf{E}_{x_0} G_\infty, \quad \tilde{G}(x_0,s_0) = \mathbf{E}_{x_0} \tilde{G}(x_n, s_0 - v_n) = \mathbf{E}_{x_0} \tilde{G}_\infty\,.$$

Since $G_\infty < \tilde{G}_\infty$, there exists $a < b$ such that

$$\mathbf{P}_{x_0}(G_\infty < a < b < \tilde{G}_\infty) = r > 0. \tag{3.10}$$

Further define

$$\begin{aligned} A &= \{(x,v)\ :\ G(x, s_0 - v) < a\} \subset S \times \mathbb{R}\,,\\ B &= \{(x,v)\ :\ \tilde{G}(x, s_0 - v) > b\} \subset S \times \mathbb{R}\,, \end{aligned}$$

and $C = A \cap B$. Denote by $z_n = (x_n, v_n)$ the bivariate Markov chain on $S \times \mathbb{R}$ with initial value $z_0 = (x_0, 0)$.

We shall use the following probabilistic fact.

Lemma 3.2. (Jacod [12], p. 89) *Let $(\Gamma_n)_{n\geq 0}$ be a sequence of the measurable sets of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$. Then*

$$\lim_{n\to\infty} \mathbf{P}\left(\bigcap_{k\geq n} \Gamma_k \mid \mathcal{F}_n \right) = \chi_{\Gamma_*} \qquad a.s.,$$

where χ_{Γ_*} denotes the indicator function of the set $\Gamma_* = \liminf_{n\to\infty} \Gamma_n := \cup_{j\geq 1} \cap_{k\geq j} \Gamma_k$.

Setting $\mathcal{F}_n = \sigma\{z_0, \ldots, z_n\}$, $\Gamma_n = \{z_n \in C\}$, $n \geq 0$, and taking into account that $(z_n)_{n\geq 0}$ is a Markov chain we obtain

$$\lim_{n\to\infty} \mathbf{P}_{z_n}\left(\bigcap_{k=0}^{\infty} \{z_k \in C\} \right) = \chi_{\liminf_{n\to\infty}\{z_n \in C\}} \quad \mathbf{P}_{z_0} - \text{a.s.}$$

From this and (3.10) we conclude

$$\mathbf{P}_{z_0}\left(\lim_{n\to\infty}\mathbf{P}_{z_n}\left(\bigcap_{k=0}^{\infty}\{z_k\in C\}\right)=1\right)\geq r>0. \tag{3.11}$$

Let $\nu\in\mathbb{R}$ be as before. Taking (3.8) into account, we can fix $\delta>0$ and $\varepsilon>0$ such that $y_{\nu,\delta}\in S$ is as in $\mathbf{C}_3$ and

$$\sup_{|x-y|<\varepsilon}\ \sup_{|t'-t''|<\delta}|G(x,t')-G(y,t'')|\leq\frac{b-a}{4}. \tag{3.12}$$

By condition $\mathbf{C}_2$ we have $\pi(B_{\varepsilon,\nu,\delta})>0$ ($(x_n)_{n\geq 0}$ is recurrent), therefore for every $x\in S$,

$$\mathbf{P}_x\left(\bigcap_{l\geq 1}\bigcup_{n\geq l}\{x_n\in B_{\varepsilon,\nu,\delta}\}\right)=1\,. \tag{3.13}$$

By (3.11) there exist $z_n^*=(x_n^*,v_n^*)$ such that $\lim_{n\to\infty}P_{z_n^*}\left(\bigcap_{k=0}^{\infty}\{z_k\in C\}\right)=1$. Hence for all $\rho>0$ there exists some $n_0\in\mathbb{N}$ such that $P_{z_n^*}\left(\bigcap_{k=0}^{\infty}\{z_k\in C\}\right)>1-\rho/2$ for all $n\geq n_0$. Combining this with (3.13) there exists $z_*=(x_*,v_*)$ with $x_*\in B_{\varepsilon,\nu,\delta}$ such that

$$\mathbf{P}_{z_*}\left(\bigcap_{k=0}^{\infty}\{z_k\in C\}\right)\geq 1-\rho/2\,.$$

This means that $z_*\in C=A\cap B$; i.e.

$$G(x_*,s_1)<a\,,\qquad\text{and}\qquad\tilde{G}(x_*,s_1)>b\,,$$

and for every $k\in\mathbb{N}$ either

$$\mathbf{P}_{z_*}\Big(G(x_k,s_0-v_k)<a\,,\ \tilde{G}(x_k,s_0-v_k)>b\Big)>1-\frac{\rho}{2}$$

or

$$\mathbf{P}_{x_*}\Big(G(x_k,s_1-v_k)<a\,,\ \tilde{G}(x_k,s_1-v_k)>b\Big)>1-\frac{\rho}{2}\,,$$

where $s_1=s_0-v_*$. Setting $k=m$, $\rho=\rho_*>0$ as in (2.6) and taking into account that $x_*\in B_{\varepsilon,\delta,\nu}$ we get

$$\begin{aligned}\mathbf{P}_{x_*}\Big(G(x_m,s_1-v_m)<a,\ \tilde{G}(x_m,s_1-v_m)>b\Big)\ &>\ 1-\frac{\rho_*}{2}\,,\\ P_{x_*}(|x_m-y_{\nu,\delta}|<\varepsilon,|v_m-\nu|<\delta)\ &>\ \rho_*\,.\end{aligned}$$

Since the event in the following probability is the intersection of two events, one with probability $1-\rho_*/2$ and the other with probability ρ_*, we conclude

$$\mathbf{P}_{x_*}\left(G(x_m,s_1-v_m+\nu)>\beta,\ |x_m-y_{\nu,\delta}|<\varepsilon,\ |v_m-\nu|<\delta\right)>\rho_*/2>0\,.$$

On this event we have that

$$\begin{aligned} b-a &< G(x_m, s_1 - v_m + \nu) - G(x_*, s_1) \\ &\le |G(x_m, s_1 - v_m + \nu) - G(y_{\nu,\delta}, s_1 - v_m + \nu)| \\ &\quad + |G(y_{\nu,\delta}, s_1 - v_m + \nu) - G(x_*, s_1)| \\ &\le 2 \sup_{|x-y|<\varepsilon,\ x,y\in B_{\varepsilon,\delta,\nu}} \sup_{|t'-t''|<\delta} |G(x,t') - G(y,t'')| \\ &\le \frac{b-a}{2}\,. \end{aligned}$$

By means of this contraction we obtain for every $x \in S$ and $s \in \mathbb{R}$,

$$G(x,s) = G(x,0)$$

and therefore, by condition $\mathbf{C}_2$, for each $x \in S$ and $s \in \mathbb{R}$,

$$\begin{aligned} G(x,s) &= G(x,0) = \mathbf{E}_x G(x_n, -v_n) = \mathbf{E}_x G(x_n, 0) \\ &= \lim_{n\to\infty} \mathbf{E}_x G(x_n, 0) = \int_S G(z,0)\pi(\mathrm{d}z)\,. \end{aligned}$$

□

Corollary 3.3. *Assume that conditions $\mathbf{C}_1 - \mathbf{C}_4$ are satisfied. If there exists a measure μ on $S \times \mathbb{R}$ of finite total mass m_0 such that*

$$\lim_{t\to\infty} \int_{S\times\mathbb{R}} H(z, t-w)\mu(dz, dw) = \gamma\,,$$

then

$$\lim_{t\to\infty} H(x,t) = \gamma/m_0\,.$$

Proof. Let $x_0 \in S$ and $(t_n)_{n\in\mathbb{N}}$ a sequence tending to infinity as $n \to \infty$ such that

$$\gamma^* = \lim_{n\to\infty} H(x_0, t_n)\,.$$

By Lemma 3.1 there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} H(z, t_{n_k} - w) = \gamma^*\,.$$

Hence, by the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{S\times R} H(z, t_{t_{n_k}} - w)\mu(\mathrm{d}z, \mathrm{d}w) = m_0\gamma^*$$

and $\gamma^* = \gamma/m_0$. □

4 Change of time theorem

In this section we obtain a limit theorem for the process $(Z(t), W(t))_{t\geq 0}$ as defined in (3.2).

Theorem 4.1. *Assume that conditions $\mathbf{C}_1 - \mathbf{C}_4$ are satisfied. Then there exists a random vector $(z_\infty,\ w_\infty)$ such that for every initial value $x \in S$*

$$(Z(t),\ W(t)) \Rightarrow\ (z_\infty,\ w_\infty)\,, \quad t \to \infty\,, \tag{4.1}$$

where $\Rightarrow$ denotes weak convergence with respect to the measure $\mathbf{P}_x$.

Proof. Let $f\ :\ S \times (0,\infty) \to \mathbb{R}$ be bounded and uniformly continuous. Define $H_0(x,t)$ and $H(x,t)$ as in (3.3) and (3.4) with continuous $\theta(\cdot)$ satisfying for some $\eta > 0$

$$\theta(s) \geq 0\,, \quad \theta(s) = 0\,,\ |s| > \eta\,, \quad \text{and} \quad \int_{-\infty}^{\infty} \theta(s)\mathrm{d}s = 1\,. \tag{4.2}$$

As we show in the Appendix, there exists a probability measure μ on $S\times\mathbb{R}_+$ (the invariant measure of $(Z(t), W(t))_{t\geq 0}$) such that

$$\begin{aligned}\lim_{t\to\infty}\int_{S\times\mathbb{R}_+} H(z, t-w)\mu(\mathrm{d}z,\mathrm{d}w) &= \int_{S\times\mathbb{R}_+} f(z,w)\mu(\mathrm{d}z,\mathrm{d}w)\\ &= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y\int_0^{v_{\nu_1}} f(x_{\nu_1}, w)\mathrm{d}w\,,\end{aligned} \tag{4.3}$$

where

$$\nu_1 = \inf\{n > 0\ :\ v_n > 0\}$$

and $\sigma(\cdot)$ is some measure on $(S,\mathcal{G})$ with $0 < \sigma(S) < \infty$. Thus, if f is bounded and uniformly continuous, then Corollary 3.3 applies and

$$\lim_{t\to\infty} H(x,t) = \int_{S\times\mathbb{R}_+} f(z,w)\mu(\mathrm{d}z,\mathrm{d}w)\,, \quad x \in S\,. \tag{4.4}$$

We apply (4.4) first with $f(z,w) = k(w)$ for some uniformly continuous function $k(\cdot)$ satisfying $0 \leq k(w) \leq 1$ and for some $\eta > 0$,

$$k(w) = 1 \ \text{ if } \ \ 0 \leq w \leq 4\eta \quad \text{and} \quad k(w) = 0 \quad \text{if} \quad w \geq 5\eta\,.$$

For $t > \eta$ the corresponding H satisfies

$$\begin{aligned} H(x,t) &= \int_{-\infty}^{\infty} \theta(s)\mathbf{E}_x k(W(t+s))\mathrm{d}s\\ &\geq \int_{-\infty}^{\infty} \theta(s)\mathbf{P}_x(W(t+s) \leq 4\eta)\mathrm{d}s\\ &\geq \mathbf{P}_x(W(t+\eta) \leq 2\eta)\,.\end{aligned}$$

Thus by (4.4),

$$\begin{aligned}
\limsup_{t\to\infty}\mathbf{P}_x(W(t-\eta)\le 2\eta) &\le \lim_{t\to\infty} H(x,t-2\eta)\\
=\int_{S\times\mathbb{R}_+} k(w)\mu(\mathrm{d}z\,,\mathrm{d}w) &= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y\int_0^{v_{\nu_1}} k(w)\mathrm{d}w \le 5\eta\sigma(S)/\beta\\
=\varepsilon_1(\eta)\to 0\,,\quad \eta\to 0\,.&
\end{aligned}$$

Observe now that there is no ladder hight of $(v_n)_{n\ge 0}$ in $(t-\eta,t+\eta)$ and hence we have for $|s|<\eta$ on the set

$$\{W(t-\eta)>2\eta\}=\{v_{N(t-\eta)}>t+\eta\}$$

the following identities

$$N(t+s)=N(t)=N(t-\eta),\quad Z(t+s)=Z(t)=Z(t-\eta),\quad W(t+s)=W(t)-s\,.$$

Thus

$$\begin{aligned}
H(x,t) &= \mathbf{E}_x\int_{-\infty}^{\infty} f(Z(t+s),W(t+s))\theta(s)\mathrm{d}s\\
&= \mathbf{E}_x\int_{-\infty}^{\infty} \chi_{\{W(t-\eta)>2\eta\}} f(Z(t),W(t)-s)\theta(s)\mathrm{d}s\\
&\quad+\mathbf{E}_x\int_{-\infty}^{\infty} \chi_{\{W(t-\eta)\le 2\eta\}} f(Z(t+s),W(t+s))\theta(s)\mathrm{d}s\,.
\end{aligned}$$

From this we obtain

$$|\mathbf{E}_x f(Z(t),W(t))-H(x,t)|\le \sup_{z\in S}\sup_{|t-s|\le\eta}|f(z,t)-f(z,s)|+2\sup_{z\in S,\ t\in\mathbb{R}}|f(z,t)|\varepsilon_1(\eta).$$

Taking (4.4) into account and letting $\eta\to 0$, the right-hand side converges to 0 and we obtain (4.1) for every random vector with distribution μ. $\square$

5 Properties of the renewal function

In this section we study the properties of the renewal function

$$G(x,t)=\mathbf{E}_x\sum_{n=0}^{\infty} g(x_n,t-v_n)\,. \tag{5.1}$$

Proposition 5.1. *Let $g : S\times\mathbb{R}\to\mathbb{R}$ be a bounded (by $g^*\in\mathbb{R}$) jointly continuous function such that $g(x,t)=0$ for $|t|>L$ for some $L>0$. If conditions $\mathbf{C}_2$ and $\mathbf{C}_4$ hold, then the function (5.1) is bounded and jointly continuous on $S\times\mathbb{R}$.*

Proof. For $l \in \mathbb{N}$ define the sets

$$C_j = \left\{ x \in S \ : \ \mathbf{P}_x \left(\bigcap_{m \geq j} \{ v_{m+l} - v_l \geq m/j \} \right) > 1/4 \right\} , \quad j \in \mathbb{N} , \tag{5.2}$$

which constitute an increasing sequence, and by condition $\mathbf{C}_1$,

$$\mathbf{P}_x \left(\bigcap_{m \geq j} \{ v_{m+l} - v_l \geq m/j \} \right) \ = \ \mathbf{E}_x \mathbf{P} \left(\bigcap_{m \geq j} \{ v_{m+l} - v_l \geq m/j \} | \mathcal{F}_l \right) = \mathbf{E}_x \Phi_j(x_l) \, ,$$

where $\Phi_j(x) = \mathbf{P}_x(\cap_{m \geq j} \{ v_m \geq m/j \})$. By $\mathbf{C}_4$ the integer $l \in \mathbb{N}$ can be chosen such that the function $\mathbf{P}_x(\bigcap_{m \geq j} \{ v_{m+l} - v_l > m/j \})$ is continuous in x and therefore the C_j are open sets (in the topology on S). By condition $\mathbf{C}_2$ we have that

$$S = \bigcup_{j \geq 1} C_j \, .$$

Since S is compact, there exists some $k \in \mathbb{N}$ such that

$$S = \bigcup_{j=1}^{k} C_j = C_k \, . \tag{5.3}$$

As a first step we show for each $b > 0$

$$\sup_{t \in \mathbb{R}} \sup_{x \in S} \mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{ t \leq v_n \leq t+b \}} \ \leq \ 4k \, (1 + l + b + l \, m^*) + 4l \ =: \ M \, , \tag{5.4}$$

where the constant m^* is defined in (2.2). By (5.3) every $x \in S$ belongs to C_k, i.e. we have for every $n \in \mathbb{N}$ by (5.2), invoking $\mathbf{C}_1$,

$$\mathbf{P} \left(\bigcap_{m \geq k} \{ v_{m+l+n} - v_{l+n} \geq m/k \} \, | \mathcal{F}_n \right) = \mathbf{P}_{x_n} \left(\bigcap_{m \geq k} \{ v_{m+l} - v_l \geq m/k \} \right) > 1/4$$

Define the stopping times

$$\tau_0 = \min \{ n \geq 0 \ : \ v_n \in [t, t+b] \}$$

and for $j \geq 0$ (recall that $k, l \in \mathbb{N}$ are fixed by the construction),

$$\tau_{j+1} = \min \left\{ n \geq \tau_j + k + l : \ v_n \in [t, t+b] \, , \ v_n - v_{\tau_j + l} < \frac{n - \tau_j - l}{k} \right\} .$$

Further

$$\begin{aligned}
\mathbf{P}(\tau_{j+1} < \infty \, | \mathcal{F}_{\tau_j}) \ &= \ \chi_{\{ \tau_j < \infty \}} \mathbf{P}(\tau_{j+1} < \infty \, | \mathcal{F}_{\tau_j}) \\
&\leq \ \sum_{n=0}^{\infty} \chi_{\{ \tau_j = n \}} \mathbf{P} \left(v_{\tau_{j+1}} - v_{n+l} < \frac{\tau_{j+1} - n - l}{k} \, | \mathcal{F}_n \right) \\
&\leq \ \sum_{n=0}^{\infty} \chi_{\{ \tau_j = n \}} \left(1 - \mathbf{P} \left(\bigcap_{m \geq k} \{ v_{m+n+l} - v_{n+l} \geq m/k \} \, | \mathcal{F}_n \right) \right) \\
&\leq \ \frac{3}{4} \chi_{\{ \tau_j < \infty \}}
\end{aligned}$$

and therefore for $x \in S$

$$\mathbf{P}_x(\tau_{j+1} < \infty) = \mathbf{E}_x \chi_{\{\tau_j < \infty\}} \mathbf{P}(\tau_{j+1} < \infty | \mathcal{F}_{\tau_j}) \leq \frac{3}{4} \mathbf{P}_x(\tau_j < \infty)\,.$$

We obtained that for all $j \in \mathbb{N}$ and $x \in S$

$$\mathbf{P}_x(\tau_j < \infty) \leq \left(\frac{3}{4}\right)^j\,. \tag{5.5}$$

Further we have

$$\sum_{n=0}^{\infty} \chi_{\{t \leq v_n \leq t+b\}} \leq \sum_{j=0}^{\infty} \chi_{\{\tau_j < \infty\}} \sum_{\tau_j \leq n < \tau_{j+1}} \chi_{\{t \leq v_n \leq t+b\}}.$$

If $t \leq v_n \leq t+b$ and $\tau_j + k + l \leq n < \tau_{j+1}$ then

$$t + b \geq v_n \geq v_{\tau_j + l} + \frac{n - \tau_j - l}{k} \geq \Delta_j(l) + t + \frac{n - \tau_j - l}{k}\,,$$

that is

$$n - \tau_j \leq l + (b - \Delta_j(l))k,$$

where $\Delta_j(l) = v_{\tau_j + l} - v_{\tau_j}$. Thus

$$\sum_{\tau_j \leq n < \tau_{j+1}} \chi_{\{t \leq v_n \leq t+b\}} \leq k + l + \sum_{\tau_j + k + l \leq n < \tau_{j+1}} \chi_{\{t \leq v_n \leq t+b\}} \leq k + l + (b + |\Delta_j(l)|)k$$

and we obtain

$$\mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{t \leq v_n \leq t+b\}} \leq M_0 \sum_{j=0}^{\infty} \mathbf{P}_x(\tau_j < \infty) + k \sum_{j=0}^{\infty} \mathbf{E}_x \chi_{\{\tau_j < \infty\}} |\Delta_j(l)|,$$

where $M_0 = k + l + bk$. Taking into account that for every $j \in \mathbb{N}$

$$\mathbf{E}_x \chi_{\{\tau_j < \infty\}} |\Delta_j(l)| \leq \mathbf{P}_x(\tau_j < \infty) \sup_{z \in S} \mathbf{E}_z v_l \leq \mathbf{P}_x(\tau_j < \infty)\, l \sup_{z \in S} \mathbf{E}_z |u_1|\,,$$

by (2.2) and inequality (5.5) we obtain (5.4).
Since the function $g(x,t) = 0$ for $|t| > L$, we have $g(x, t - v_n) = 0$ for $|t - v_n| > L$. From this we conclude that

$$|g(x, t - v_n)| < g^* \chi_{\{|t - v_n| \leq L\}} = g^* \chi_{\{t - L \leq v_n \leq t + L\}}$$

which implies that $|G(x,t)| < g^* \sum_{n \geq 1} \chi_{\{t - L \leq v_n \leq t + L\}}$. By (5.4) this sum is bounded uniformly in x and t by M. From this we conclude that the function $G(x,t)$ is bounded.

We show next that the function $G(x,t)$ is continuous. For $n \in \mathbb{N}$ and $L > 0$ (given by the assumption) set

$$\tau_N = \inf\{n \geq N \;:\; t - L \leq v_n \leq t + L\}\,. \tag{5.6}$$

By this definition and the fact that $g(x,t)=0$ for $|t|>L$ we have

$$\left|\mathbf{E}_x \sum_{n=N}^{\infty} g(x_n, t-v_n)\right| = \left|\mathbf{E}_x \sum_{n=\tau_N}^{\infty} g(x_n, t-v_n)\right| \leq E_x \chi_{\{\tau_N<\infty\}} \Phi(x_{\tau_N}, v_{\tau_N}),$$

where

$$\Phi(x,v) = E_x \sum_{n=0}^{+\infty} g(x_n, t-v-v_n).$$

Taking again into account that g has bounded t-support,

$$|\Phi(x,v)| \leq E_x \sum_{n=0} |g(x_n, t-v-v_n)| \leq g^* E_x \sum_{n=0} \chi_{\{|t-v-v_n|\leq L\}} \leq g^* E_x \sum_{n=0} \chi_{\{-2L\leq v_n\leq 2L\}},$$

where the last inequality holds for $|t-v|\leq L$. We use this estimate for $v=v_{\tau_N}$; i.e. $|t-v|=|t-v_{\tau_N}|\leq L$. This yields finally the estimate

$$\begin{aligned}
\left|\mathbf{E}_x \sum_{n=N}^{\infty} g(x_n, t-v_n)\right| &\leq \mathbf{E}_x \chi_{\{\tau_N<\infty\}} \mathbf{E}_{x_{\tau_N}} \left|\sum_{n=0}^{\infty} g(x_n, t-v_{\tau_N}-v_n)\right| \\
&\leq g^* \mathbf{E}_x \chi_{\{\tau_N<\infty\}} \mathbf{E}_{x_{\tau_N}} \sum_{n=0}^{\infty} \chi_{\{-2L\leq v_n\leq 2L\}} \\
&\leq g^* M \mathbf{P}_x(\tau_N<\infty),
\end{aligned}$$

where we have used inequality (5.4) in the last step.

Further, for some $0<\epsilon\leq\beta$,

$$\mathbf{P}_x(\tau_N<\infty) \leq \mathbf{P}_x(\tau_N<\infty,\ \sup_{n\geq N}|\delta_n|\leq\epsilon) + \mathbf{P}_x(\tau_N<\infty,\ \sup_{n\geq N}|\delta_n|>\epsilon),$$

where

$$\delta_n = \frac{v_n}{n} - \beta.$$

Therefore, for the first probability on the right hand side we have for $N>\frac{t+L}{\beta-\epsilon}$ that $v_{\tau_N}\geq N(\beta-\varepsilon)\geq t+L$, hence

$$\mathbf{P}_x(\tau_N<\infty,\ \sup_{n\geq N}|\delta_n|\leq\epsilon) = \mathbf{P}_x(t-L<v_{\tau_N}<t+L,\ \sup_{n\geq N}|\delta_n|\leq\epsilon) = 0.$$

This means that for this N

$$\left|\mathbf{E}_x \sum_{n=N}^{\infty} g(x_n, t-v_n)\right| \leq g^* M \mathbf{P}_x(\sup_{n\geq N}|\delta_n|>\epsilon). \tag{5.7}$$

By $\mathbf{C}_2$ the probability on the right hand side tends to zero as $N\to\infty$ for each $x\in S$. Further, notice that for $N>l$ by $\mathbf{C}_1$,

$$\mathbf{P}_x(\sup_{n\geq N}|\frac{v_n}{n}-\beta|>\varepsilon) = \mathbf{P}_x(\sup_{n\geq N}|\delta_n|>\epsilon) = \mathbf{E}_x\Phi_1(x_l, v_l),$$

where

$$\Phi_1(x,v) = \mathbf{P}_x \left(\sup_{n \geq N-l} \left| \frac{v_n + v}{n+l} - \beta \right| > \epsilon \right).$$

Therefore, by $\mathbf{C}_4$ the function $\mathbf{P}_x(\sup_{n\geq N} |\delta_n| > \epsilon)$ is continuous for each $N > l$. Since the function $g(x,t)$ is jointly continuous, for every $N < \infty$ the function

$$G_N(x,t) = \mathbf{E}_x \sum_{n=0}^{N} g(x_n, t - v_n)$$

is jointly continuous on $S \times \mathbb{R}$. Now we have that for $N > (2|t_0| + L)/(\beta - \epsilon) + l$

$$\begin{aligned} |G(x,t) - G(x_0,t_0)| \quad &\leq \quad |G_N(x,t) - G_N(x_0,t_0)| + g^* M |\mathbf{P}_x(\sup_{n\geq N} |\delta_n| > \epsilon) - \mathbf{P}_{x_0}(\sup_{n\geq N} |\delta_n| > \epsilon)| \\ &\quad\quad + 2g^* M \mathbf{P}_{x_0}(\sup_{n\geq N} |\delta_n| > \epsilon). \end{aligned}$$

By letting first $x \to x_0$ and $t \to t_0$, and then $N \to \infty$ we obtain joint continuity for $G(x,t)$. $\square$

6 Proof of the renewal theorem 2.1

First we prove this theorem for a function g satisfying the conditions of Proposition 5.1. In this case, using (2.1),

$$\begin{aligned} G(x,t) &= \mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v_n) \\ &= \mathbf{E}_x \sum_{n=N(t-L)}^{\infty} g(x_n, t - v_n) \\ &= \mathbf{E}_x \, \mathbf{E}\big[\sum_{n=N(t-L)}^{\infty} g(x_n, t - v_n) | \mathcal{F}_{N(t-L)} \big] \\ &= \mathbf{E}_x G\left(Z(t-L), t - v_{N(t-L)} \right) \\ &= \mathbf{E}_x f\left(Z(t-L), W(t-L) \right) \end{aligned}$$

where $f(z,w) = G(z, L-w)$. By Theorem 4.1 and Proposition 5.1 the following limit exists and is independent of x:

$$\lim_{t\to\infty} G(x,t) = \lim_{t\to\infty} \mathbf{E}_x f\left(Z(t-L), W(t-L) \right) = G_\infty\,. \tag{6.1}$$

Then by the boundedness of $G(\cdot,\cdot)$ and the dominated convergence theorem,

$$G_\infty = \lim_{T\to\infty} \int_S \pi(\mathrm{d}x) \frac{1}{T} \int_0^T G(x,t) \mathrm{d}t\,.$$

Further we have

$$
\begin{aligned}
\frac{1}{T}\int_0^T G(x,t)\mathrm{d}t &= \mathbf{E}_x \sum_{n=0}^{\infty} \frac{1}{T}\int_0^T g(x_n, t-v_n)\mathrm{d}t \\
&= \frac{1}{T}\mathbf{E}_x \sum_{n=M_1(T)}^{M_2(T)} \int_0^T g(x_n, t-v_n)\mathrm{d}t + \mathbf{E}_x\Delta_1(T) \\
&= \frac{1}{T}\mathbf{E}_x \sum_{n=M_1(T)}^{M_2(T)} \int_{-\infty}^{\infty} g(x_n, t)\mathrm{d}t + \mathbf{E}_x\Delta_1(T) - \mathbf{E}_x\Delta_2(T)\,, \qquad (6.2)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1(T) &= \frac{1}{T}\sum_{n=0}^{M_1(T)-1} \int_0^T g(x_n, t-v_n)\mathrm{d}t + \frac{1}{T}\sum_{n>M_2(T)} \int_0^T g(x_n, t-v_n)\mathrm{d}t\,, \\
\Delta_2(T) &= \frac{1}{T}\sum_{n=M_1(T)}^{M_2(T)} \left(\int_T^{\infty} g(x_n, t-v_n)\mathrm{d}t + \int_{-\infty}^{0} g(x_n, t-v_n)\mathrm{d}t\right)\,.
\end{aligned}
$$

We set

$$
M_1(T) = \left[\frac{\varepsilon}{\beta}T\right], \quad M_2(T) = \left[\frac{1-\varepsilon}{\beta}T\right],
$$

where $[a]$ denotes the integer part of a. By substituting the first term in (6.2) in the integral with respect to the stationary measure $\pi(\cdot)$ we have

$$
\begin{aligned}
&\int_S \pi(\mathrm{d}x)\frac{1}{T}\mathbf{E}_x \sum_{n=M_1(T)}^{M_2(T)} \int_{-\infty}^{\infty} g(x_n,t)\mathrm{d}t \\
&= \int_S \pi(\mathrm{d}x)\int_{-\infty}^{\infty} g(x,t)\mathrm{d}t \frac{M_2(T)-M_1(T)}{T} \\
&\to \frac{1-2\varepsilon}{\beta}\int_S \pi(\mathrm{d}x)\int_{-\infty}^{\infty} g(x,t)\mathrm{d}t\,, \quad T\to\infty\,.
\end{aligned}
$$

Further, since $g(x,t)=0$ for $|t|\geq L$ the last term in (6.2) is bounded by

$$
\mathbf{E}_x|\Delta_2(T)| \leq g_1^*\frac{1}{T}\sum_{n=M_1(T)}^{M_2(T)} \left(\mathbf{P}_x(v_n > T-L) + \mathbf{P}_x(v_n < L)\right)\,,
$$

where $g_1^* = \sup_{x\in S}\int_{-\infty}^{\infty}|g(x,t)|\mathrm{d}t$. By condition $\mathbf{C}_2$, for every $\varepsilon>0$,

$$
\lim_{T\to\infty}\mathbf{E}_x|\Delta_2(T)| = 0\,, \quad x\in S\,. \qquad (6.3)
$$

Moreover, concerning $E_x\Delta_1(T)$ we have

$$
\mathbf{E}_x\left|\frac{1}{T}\sum_{n=0}^{M_1(T)-1}\int_0^T g(x_n, t-v_n)\mathrm{d}t\right| \leq g_1^*\varepsilon/\beta.
$$

and

$$\mathbf{E}_x\left|\frac{1}{T}\sum_{n>M_2(T)}\int_0^T g(x_n,t-v_n)\mathrm{d}t\right| \le \frac{2\varepsilon}{\beta}g_1^* + \mathbf{E}_x\left|\frac{1}{T}\sum_{n>M_3(T)}\int_0^T g(x_n,t-v_n)\mathrm{d}t\right| ,$$

where $M_3(T) = [(1+\varepsilon)T/\beta]$. Now by (5.7),

$$\frac{1}{T}\int_0^T \mathbf{E}_x \sum_{n>M_3(T)} |g(x_n,t-v_n)|\mathrm{d}t \le g^* M \mathbf{P}_x\left(\sup_{n\ge M_3(T)}|\delta_n| > \rho\right)$$

for $\rho < \beta\varepsilon/(1+\varepsilon)$ and sufficiently large $T\to\infty$. By condition $\mathbf{C}_2$ we obtain that

$$\limsup_{T\to\infty}\int_S \pi(\mathrm{d}x)\mathbf{E}_x|\Delta_1(T)| \le \frac{2g_1^*}{\beta}\varepsilon\,.$$

Since $\mathbf{E}_x|\Delta_1(T)|$ is bounded, and recalling convergence (6.3), we find upon letting $\varepsilon\to 0$

$$\lim_{t\to\infty} G(x,t) = \frac{1}{\beta}\int_S \pi(\mathrm{d}x)\int_{-\infty}^{\infty} g(x,t)\mathrm{d}t.$$

Now let g be an arbitrary continuous bounded function on $S\times\mathbb{R}$, satisfying condition (2.4) and let $\lambda : \mathbb{R}\to[0,1]$ be a continuous function such that $\lambda(t)=1$ for $|t|\le L-1$ and $\lambda(t)=0$ for $|t|>L$. Then, making use of inequality (5.4), we obtain that

$$\begin{aligned}
&\left|\mathbf{E}_x\sum_{n=0}^{\infty} g(x_n,t-v_n) - \mathbf{E}_x\sum_{n=0}^{\infty} g(x_n,t-v_n)\lambda(t-v_n)\right| \\
&\le \sum_{|j|\ge L-2}\mathbf{E}_x\sum_{n=0}^{\infty}|g(x_n,t-v_n)|\chi_{\{t+j\le v_n\le t+j+1\}} \\
&\le \sum_{|j|\ge L-2}\sup_{z\in S,\ j\le t\le j+1}|g(z,t)|\mathbf{E}_x\sum_{n=0}^{\infty}\chi_{\{t+j\le v_n\le t+j+1\}} \\
&\le M\sum_{|j|\ge L-2}\sup_{z\in S}\sup_{j\le t\le j+1}|g(z,t)|\,.
\end{aligned}$$

This last expression tends to zero as $L\to\infty$, because g satisfies inequality (2.4). $\square$

APPENDIX

In this section we prove the relationship (4.3) by a modification of the proof of Lemma 2 in Kesten [14]. First we need to construct a measure σ on $\mathcal{G}$. We imitate the construction of Kesten [14] simplifying at the appropriate places for our special situation of a compact state space.

A.1) Construction of the measure σ.

Consider the measurable space $(X,\,\mathcal{X})$, where

$$X=\Pi_{-\infty}^{\infty}(S\times\mathbb{R})\quad\text{ and }\quad\mathcal{X}=\Pi_{-\infty}^{\infty}(\mathcal{G}\times\mathcal{B})\,,$$

and $\mathcal{B}$ is the Borel σ-field in $\mathbb{R}$. Denote $(\tilde{x}_n,\,\tilde{u}_n)_{n\in\mathbb{Z}}$ the coordinate process in this space. We also define $\tilde{\mathcal{F}}_k=\sigma\{\tilde{x}_i\,,\tilde{u}_i\,:\,-\infty<i\le k\}$

By Kolmogorov's theorem we can construct the stationary measure on this space, which has the following finite dimensional distributions

$$\tilde{\mathbf{P}}(\tilde{x}_{k+1}\in\Gamma_1,\tilde{u}_{k+1}\in A_1,\ldots,\tilde{x}_{k+m}\in\Gamma_m,\tilde{u}_{k+m}\in A_m)$$

$$=\int_S\pi(\mathrm{d}y)\mathbf{P}_y(x_1\in\Gamma_1,\,u_1\in A_1,\,\ldots,\,x_m\in\Gamma_m,\,u_m\in A_m)\tag{A.1}$$

for every $-\infty<k<\infty$.

It follows from (A.1) that for every bounded measurable function $f\,:\,\Pi_{i=0}^{\infty}(S\times\mathbb{R})\ \to\ \mathbb{R}$

$$\tilde{\mathbf{E}}(f(\tilde{x}_k,\tilde{u}_k,\ldots,\tilde{x}_{k+m},\tilde{u}_{k+m},\ldots|\tilde{\mathcal{F}}_k)=f_1(\tilde{x}_k,\tilde{u}_k)\,,\tag{A.2}$$

where

$$f_1(x,u)=\mathbf{E}_x f(x,u,x_1,u_1,\ldots,x_m,u_m,\ldots)\,.$$

To construct the functions of this process we set

$$\tilde{v}_n=\begin{cases}\sum_{i=1}^n\tilde{u}_i\,, & \text{if}\ \ n\ge 1\,;\\ 0\,, & \text{if}\ \ n=0\,;\\ -\sum_{i=0}^{n+1}\tilde{u}_i\,, & \text{if}\ \ n<0\,.\end{cases}\tag{A.3}$$

Set

$$\tilde{\nu}_0=\max\{n\le 0\,:\,\tilde{v}_n>\sup_{j<n}\tilde{v}_j\}\quad(=-\infty\qquad\text{if no such } n \text{ exists})\,.$$

Now we define the measure σ on $\mathcal{G}$ by

$$\sigma(\Gamma)=\tilde{\mathbf{P}}(\tilde{\nu}_0=0,\,\tilde{x}_0\in\Gamma)\,,\quad\Gamma\in\mathcal{G}\,.\tag{A.4}$$

Further we need to introduce a sequence of stopping times by

$$\nu_0=0\,,\quad\nu_j=\inf\{i>\nu_{j-1}\,:\,v_n>v_{\nu_{j-1}}\}\,,\quad j\in\mathbb{N},\quad(=\infty\ \ \text{if no such } n \text{ exists})\,.\tag{A.5}$$

Notice that condition $\mathbf{C}_1$ implies that for every bounded measurable function $f\,:\,\mathbb{R}\times\mathbb{R}\times S\to\mathbb{R}$ and for $i\ge 0$

$$\mathbf{E}(f(\nu_{i+1}-\nu_i,\,v_{\nu_{i+1}}-v_{\nu_i},\,x_{\nu_{i+1}})|\mathcal{F}_{\nu_i})=\pi(f)(x_{\nu_i}),\tag{A.6}$$

where $\pi(f)(x) = \mathbf{E}_x f(\nu_1, v_{\nu_1}, x_{\nu_1})$. This means that the process $(x_{\nu_i})_{i\in\mathbb{N}}$ is a Markov chain. Now we need the following lemma, which is proved in Kesten [14], p. 368, under different conditions.

Lemma A.1. *If conditions* $\mathbf{C}_1 - \mathbf{C}_2$ *hold, then*

$$\mathbf{P}_x(\nu_i < \infty) = 1 \qquad \forall x \in S, \quad \forall i \geq 1, \tag{A.7}$$

and

$$\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0) > 0. \tag{A.8}$$

Further, σ *is an invariant measure of the Markov chain* $(x_{\nu_i})_{i\geq 0}$, *i.e.*

$$\int_S \sigma(dy)\mathbf{E}_y f(x_{\nu_i}) = \int_S f(y)\sigma(dy) \tag{A.9}$$

for every bounded measurable function f *on* S. *Finally,*

$$\int_S \sigma(dy)\mathbf{E}_y v_{\nu_1} = \beta \tag{A.10}$$

Proof. First notice that condition $\mathbf{C}_2$ implies (A.7). Further, from $\mathbf{C}_2$ and (A.3) it follows that $\tilde{\mathbf{P}}(\tilde{\nu}_0 > -\infty) = 1$; that is, taking (A.1) into account, we obtain

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \tilde{\mathbf{P}}(\tilde{\nu}_0 = -n) &\leq \sum_{n=0}^{\infty} \tilde{\mathbf{P}}(\tilde{v}_{-n} > \sup_{j<n} \tilde{v}_j) \\ &= \sum_{n=0}^{\infty} \tilde{\mathbf{P}}(\tilde{v}_0 > \sup_{j<0} \tilde{v}_j) &= \sum_{n=0}^{\infty} \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\,. \end{aligned}$$

From this we get (A.8).

Further, from (A.4) we obtain that for every bounded measurable function $f : S \to \mathbb{R}$

$$\int_S \sigma(\mathrm{d}y)\mathbf{E}_y f(x_{\nu_1}) = \tilde{\mathbf{E}}\chi_{\{\tilde{\nu}_0=0\}}\Phi(\tilde{x}_0)\,,$$

where $\Phi(y) = \mathbf{E}_y f(x_{\nu_1}) = \mathbf{E}_y f_1(x_1, u_1, \ldots, x_m, u_m, \ldots)$,

$$f_1(x_1, u_1, \ldots, x_m, u_m, \ldots) = \sum_{k=1}^{\infty} f(x_k)\chi_{\{v_1 \leq 0, \ldots, v_{k-1} \leq 0, v_k > 0\}}.$$

Therefore, by (A.2) we have that

$$\begin{aligned} \int_S \sigma(\mathrm{d}y)\mathbf{E}_y f(x_{\nu_1}) &= \sum_{k=1}^{\infty} \tilde{\mathbf{E}} f(\tilde{x}_k)\chi_{\{\sup_{j<0}\tilde{v}_j<0\}}\chi_{\{\tilde{v}_1\leq 0,\ldots,\tilde{v}_{k-1}\leq 0,\tilde{v}_k>0\}} \\ &= \sum_{k=1}^{\infty} \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\sup_{j<-k}\tilde{v}_j<v_{-k}\}}\chi_{\{\tilde{v}_{-k+1}\leq\tilde{v}_{-k},\ldots,\tilde{v}_{-1}\leq\tilde{v}_{-k}<0,\tilde{v}_{-k}>0\}} \\ &= \sum_{k=1}^{\infty} \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\tilde{\nu}_0=0\}}\chi_{\{\lambda=-k\}}\,, \end{aligned}$$

where $\lambda = \inf\{l < 0 : \tilde{v}_l = \sup_{j<0} \tilde{v}_j\}$. Notice that by condition $\mathbf{C}_2$ we know that $\tilde{\mathbf{P}}(\lambda > -\infty) = 1$, therefore we obtain

$$\int_S \sigma(\mathrm{d}y)\mathbf{E}_y f(x_{\nu_1}) = \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\tilde{\nu}_0=0\}} \sum_{k=1}^{\infty} \chi_{\{\lambda=-k\}} = \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\tilde{\nu}_0=0\}} = \int_S \sigma(\mathrm{d}y) f(y).$$

Taking also (A.6) into account, we obtain (A.9).

We show now that

$$\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \nu_1 = 1. \tag{A.11}$$

Indeed, by the same method as above one can obtain that

$$\begin{aligned}
\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \nu_1 &= \sum_{k=1}^{\infty} k\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0, \tilde{v}_1 \le 0, \ldots, \tilde{v}_{k-1} \le 0, \tilde{v}_k > 0) \\
&= \sum_{k=1}^{\infty} k\tilde{\mathbf{P}}(\sup_{j<0} \tilde{v}_j < 0, \tilde{v}_1 \le 0, \ldots, \tilde{v}_{k-1} \le 0, \tilde{v}_k > 0) \\
&= \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \tilde{\mathbf{P}}(\sup_{j<-r} \tilde{v}_j < \tilde{v}_{-r}, \tilde{v}_{1-r} \le \tilde{v}_{-r}, \ldots, \tilde{v}_{k-r-1} \le 0, \tilde{v}_{k-r} > 0) \\
&= \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \tilde{\mathbf{P}}(\tilde{\nu}_0 = -r\,,\, \tilde{\nu}_1 = k-r)\,,
\end{aligned}$$

where $\tilde{\nu}_1 = \inf\{l > \tilde{\nu}_0 : \tilde{v}_l > \tilde{v}_{\tilde{\nu}_0}\}$ ($= \infty$ if no such n exists). Notice that the definition of $\tilde{\nu}_1$ implies that $\tilde{\nu}_1 \ge 1$ a.s.. Indeed, if $\tilde{\nu}_0 = 0$, then $\tilde{\nu}_1 \ge 1$ by definition. Now assume that $\tilde{\nu}_0 = l < 0$. Then

$$\sup_{j<l} \tilde{v}_j < \tilde{v}_l\,, \quad \tilde{v}_{l+1} \le \tilde{v}_l\,, \ldots\,, \tilde{v}_{-1} \le \tilde{v}_l$$

and $\tilde{v}_l \ge 0$. This follows from the fact that $\tilde{v}_l < 0$ implies $\sup_{j<0} \tilde{v}_j < 0 = \tilde{v}_0$. But this means that $\tilde{\nu}_0 = 0$ which contradicts $\tilde{\nu}_0 = l < 0$. Hence, $\tilde{\nu}_1 \ge 1$ a.s.

Condition $\mathbf{C}_2$ implies that $\tilde{\mathbf{P}}(\tilde{\nu}_1 < \infty) = 1$. Therefore,

$$\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \nu_1 = \sum_{r=0}^{\infty} \sum_{j=1}^{\infty} \tilde{\mathbf{P}}(\tilde{\nu}_0 = -r\,,\, \tilde{\nu}_1 = j) = 1.$$

Now let $Q(\cdot)$ be the probability measure on $(\Omega,\, \mathcal{F})$ defined by

$$Q(A) = \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{P}_y(A)\,,$$

where $\tilde{\sigma}$ is probability measure un S, i.e.

$$\tilde{\sigma}(\Gamma) = \frac{\sigma(\Gamma)}{\sigma(S)} = \frac{\sigma(\Gamma)}{\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)}\,, \quad \Gamma \in \mathcal{G}\,.$$

Define the stochastic process

$$Y_j = (\nu_j - \nu_{j-1},\, v_{\nu_j} - v_{\nu_{j-1}},\, x_{\nu_j})\,,\; j \in \mathbb{N}\,. \tag{A.12}$$

By (A.6) this process is a homogeneous Markov chain in $\mathbb{R}^3$ with respect to the measure Q, which is strictly stationary by Lemma A.2 below. Hence, by the Birkhoff-Khinchin theorem (see Doob [4]) we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\nu_n}{n} &= \lim_{n\to\infty}\frac{\sum_{j=1}^n \nu_j - \nu_{j-1}}{n} = \nu_\infty = \mathbf{E}^Q(\nu_1|\mathcal{I}) \quad Q-\text{a.s.},\\
\lim_{n\to\infty}\frac{v_{\nu_n}}{n} &= \lim_{n\to\infty}\frac{\sum_{j=1}^n v_{\nu_j} - v_{\nu_{j-1}}}{n} = v_\infty = \mathbf{E}^Q(v_{\nu_1}|\mathcal{I}) \quad Q-\text{a.s.},
\end{aligned}$$

where $\mathcal{I}$ is the Borel field of invariant sets, $\mathbf{E}^Q(\cdot)$ denotes the expectation with respect to the measure Q.

Condition $\mathbf{C}_2$ implies that $v_\infty = \beta\nu_\infty$. Therefore

$$\begin{aligned}
\int_S \sigma(\mathrm{d}y)\mathbf{E}_y v_{\nu_1} &= \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\,\mathbf{E}^Q v_{\nu_1} = \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\,\mathbf{E}^Q v_\infty\\
= \beta\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\,\mathbf{E}^Q \nu_\infty &= \beta\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\,\mathbf{E}^Q \nu_1 = \beta\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \nu_1\,.
\end{aligned}$$

Taking (A.11) into account we obtain (A.10). □

Lemma A.2. *The process $(Y_j)_{j\in\mathbb{N}}$ is a strictly stationary process; i.e. for all $m,k \in \mathbb{N}$ and $i_1 < \ldots < i_k$,*

$$\mathbf{E}^Q f_1(Y_{i_1+m})\cdots f_k(Y_{i_k+m}) = \mathbf{E}^Q f_1(Y_{i_1})\cdots f_k(Y_{i_k})\,, \tag{A.13}$$

for all measurable bounded functions $(f_i)_{1\le i\le k}$.

Proof. First notice that (A.6) implies that for all $n \in \mathbb{N}$

$$\mathbf{E}(f(Y_{j+n})|Y_0,\ldots,Y_j) = T^n(f)(Y_j)\,,$$

where $T^n(f)(\cdot)$ is nth power of the operator $T(f)$ which is defined for every bounded measurable function $f\,:\,\mathbb{R}^3 \to \mathbb{R}$ as

$$T(f)(Y) = \pi(f)(y_3)\,, \quad Y = (y_1,y_2,y_3) \in \mathbb{R}^3\,, \quad \pi(f)(y) = \mathbf{E}_y f(Y_1)\,.$$

We prove (A.13) by induction. For $k = 2$ we have

$$\begin{aligned}
\mathbf{E}^Q f_1(Y_{i_1+m})f_2(Y_{i_2+m}) &= \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y f_1(Y_{i_1+m})f_2(Y_{i_2+m})\\
&= \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y f_1(Y_{i_1+m})\mathbf{E}(f_2(Y_{i_2+m})|Y_0,\ldots,Y_{i_1+m})\\
&= \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y \tilde{f}_1(Y_{i_1+m})\,,
\end{aligned}$$

where $\tilde{f}_1(Y) = f_1(Y)T^{i_2-i_1}(f_2)(Y)$. Thus by (A.6) and (A.9) we obtain

$$\begin{aligned}\mathbf{E}^Q f_1(Y_{i_1+m})f_2(Y_{i_2+m}) &= \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y\pi(\tilde{f}_1)(x_{\nu_{i_1+m-1}}) \\ = \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y\pi(\tilde{f}_1)(x_{\nu_{i_1-1}}) &= \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y\tilde{f}_1(Y_{i_1}) \;=\; \mathbf{E}^Q f_1(Y_{i_1})f_2(Y_{i_2})\,.\end{aligned}$$

We assume now that (A.13) is true for some fixed $k \in \mathbb{N}$ and prove that it is true for $k+1$. By the same method as above we obtain

$$\mathbf{E}^Q f_1(Y_{i_1+m})\cdots f_k(Y_{i_k+m})f_{k+1}(Y_{i_{k+1}+m}) = \mathbf{E}^Q f_1(Y_{i_1+m})\cdots \tilde{f}_k(Y_{i_k+m})\,,$$

where $\tilde{f}_k(Y) = f_k(Y)T^{i_{k+1}-i_k}(f_{k+1})(Y)$. By assumption we obtain that

$$\begin{aligned}\mathbf{E}^Q f_1(Y_{i_1+m})\cdots f_k(Y_{i_k+m})f_{k+1}(Y_{i_{k+1}+m}) &= \mathbf{E}^Q f_1(Y_{i_1})\cdots \tilde{f}_k(Y_{i_k}) \\ = \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y f_1(Y_{i_1})\cdots f_k(Y_{i_k})T^{i_{k+1}-i_k}(f_{k+1})(Y_{i_k}) \\ = \int_S \tilde{\sigma}(\mathrm{d}y)\mathbf{E}_y f_1(Y_{i_1})\cdots f_k(Y_{i_k})f_{k+1}(Y_{i_{k+1}})\end{aligned}$$

which gives (A.13) for $k+1$. $\square$

A.2) Invariant measure for the process in continuous time (3.2).

In this section we show that the measure μ as defined in (4.3) (with σ defined in (A.4)) is an invariant measure for the process $(Z(t)\,, W(t))_{t\ge 0}$.

Recall the definition of ν_i in (A.5) and consider the function

$$\Theta(z,\Gamma,t) = \sum_{i=0}^{\infty}\mathbf{P}_Z(x_{\nu_i}\in\Gamma\,, v_{\nu_i}\le t)\,,\quad z\in S\,,\ \Gamma\in\mathcal{G}\,,\ t>0\,,$$

and $\Theta(z,\Gamma,t)=0$ for $t\le 0$. Notice that the inequality (5.4) implies that

$$\Theta(z,\Gamma,t) \le \sum_{i=0}^{\infty}\mathbf{P}_Z(v_{\nu_i}\le t) \le \sum_{i=0}^{\infty}\mathbf{P}_Z(v_i\le t) < \infty\,,\quad z\in S\,,\ \Gamma\in\mathcal{G}\,,\ t\ge 0\,.$$

For every measurable bounded function $f\,:\,S\times\mathbb{R}\to\mathbb{R}$ we define the operator

$$\Theta(f)(z,t) = \int_0^t f(x,t-\tau)\Theta(z,\mathrm{d}x,\mathrm{d}\tau) = \sum_{i=0}^{\infty}\mathbf{E}_z f^+(x_{\nu_i},t-v_{\nu_i}),$$

where $f^+(x,t) = f(x,t)\chi_{\{t\ge 0\}}$.

We show first that the function $\Theta(f)(z,t)$ satisfies

$$\Theta(f)(z,t) = f^+(z,t) + \mathbf{E}_z\Theta(f)(x_{\nu_1},t-v_{\nu_1})\,,\quad z\in S\,,\ t\ge 0\,. \tag{A.14}$$

Notice that (A.6) implies that for every bounded $f : S \times \mathbb{R} \to \mathbb{R}$-measurable function and for every $i \in \mathbb{N}$,

$$\mathbf{E}(f(x_{\nu_{i+1}}, v_{\nu_{i+1}})|\mathcal{F}_{\nu_i}) = \pi_0(f)(x_{\nu_i}, v_{\nu_i}),$$

where $\pi_0(f)(x,v) = \mathbf{E}_x f(x_{\nu_1},\, v_{\nu_1} + v)$. Therefore, for every $n \in \mathbb{N}$,

$$\mathbf{E}(f(x_{\nu_{i+n}}, v_{\nu_{i+n}})|\mathcal{F}_{\nu_i}) = \pi_0^n(f)(x_{\nu_i}, v_{\nu_i}),$$

where $\pi_0^n(f)(x\,,v) = \mathbf{E}_x f(x_{\nu_n}\,, v + v_{\nu_n})$. Moreover, we have for $t \geq 0$

$$\Theta(f)(z,t) = f(x,t) + \mathbf{E}_z \sum_{i=0}^{\infty} \mathbf{E}(f^+(x_{\nu_{i+1}}, t - v_{\nu_{i+1}})|\mathcal{F}_{\nu_1}) = f(x,t) + \mathbf{E}_z \sum_{i=0}^{\infty} \psi_i(x_{\nu_1}, t - v_{\nu_1}),$$

where $\psi_i(x,t) = \mathbf{E}_x f^+(x_{\nu_i}, t - v_{\nu_i})$. From this (A.14) follows.

Since σ is an invariant measure for $(x_{\nu_i})_{i\in\mathbb{N}}$ we obtain

$$\begin{aligned}
&\int_{S\times R} \Theta(f)(y,t-w)\mu(\mathrm{d}y,\mathrm{d}w) \\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \Theta(f)(x_{\nu_1}, t-w)\mathrm{d}w \\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\int_0^t \Theta(f)(y,t-w)\mathrm{d}w - \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_{v_{\nu_1}\wedge t}^t \Theta(f)(x_{\nu_1}, t-w)\mathrm{d}w \\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\int_0^t f(y,w)\mathrm{d}w + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\leq t\}} \int_0^{t-v_{\nu_1}} \Theta(f)(x_{\nu_1}, t-w-v_{\nu_1})\mathrm{d}w \\
&\quad -\frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\leq t\}} \int_0^{t-v_{\nu_1}} \Theta(f)(x_{\nu_1}, w)\mathrm{d}w\,.
\end{aligned}$$

By change of variables, setting $u = t - w - v_{\nu_1}$ in the second integral in the last equality, we obtain that

$$\int_{S\times\mathbb{R}} \Theta(f)(y,t-w)\mu(\mathrm{d}y,\mathrm{d}w) = \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\int_0^t f(y,w)\mathrm{d}w. \tag{A.15}$$

We show next that the measure μ is invariant for $(Z(t), W(t))_{z\geq 0}$; i.e. for every bounded measurable function $f : S \times \mathbb{R} \to \mathbb{R}$

$$\int_{S\times R} \mathbf{E}_{z,w} f(Z(t)\,, W(t))\mu(\mathrm{d}z,\mathrm{d}w) = \int_{S\times R} f(z\,,w)\mu(\mathrm{d}z,\mathrm{d}w)\,, \quad t \geq 0\,, \tag{A.16}$$

where $\mathbf{E}_{z,w}$ denotes the expectation by the distribution of the process $(Z(t)\,, W(t)\,, t \geq 0)$ under the initial condition $Z(0) = z\,, W(0) = w$. Notice that condition (2.1) and definitions (3.2) imply that for every measurable bounded function f and $t \geq 0$,

$$\mathbf{E}_{z,w} f(Z(t)\,, W(t)) = \begin{cases} f(z\,, w-t)\,, & \text{if } \ 0 \leq t < w; \\ \mathbf{E}_z f(Z(t-w)\,, W(t-w)), & \text{if } \ t \geq w; \end{cases} \tag{A.17}$$

Indeed, taking into account that on the set $\{N(0)=n\}$ for $t \geq w$ we have $v_n = v_{N(0)} = W(0) = w \leq t$ we conclude

$$\begin{aligned}
&\mathbf{E}_{z,w} f(Z(t)\,,W(t)) = \mathbf{E}_{z,w}\mathbf{E}(f(Z(t)\,,W(t))|\mathcal{F}_{N(0)}) \\
&= \mathbf{E}_{z,w}\sum_{n=1}^{+\infty}\chi_{\{N(0)=n\}}\mathbf{E}(f(Z(t)\,,W(t))|\mathcal{F}_n) \\
&= \mathbf{E}_{z,w}\sum_{n=1}^{+\infty}\chi_{\{N(0)=n\}}\mathbf{E}(\sum_{k=n+1}^{+\infty} f(x_k\,,v_k-t)\chi_{\{N(t)=k\}}|\mathcal{F}_n) \\
&= \mathbf{E}_{z,w}\sum_{n=1}^{+\infty}\chi_{\{v_1\leq 0,\ldots,v_{n-1}\leq 0,v_n>0\}}\mathbf{E}(\sum_{k=n+1}^{+\infty} f(x_k\,,v_k-t)\chi_{\{v_n\leq t,\ldots,v_{k-1}\leq t,v_k>t\}}|\mathcal{F}_n)\,.
\end{aligned}$$

By condition (2.1) we have also that

$$\mathbf{E}_{z,w} f(Z(t)\,,W(t)) \;=\; \mathbf{E}_{z,w}\sum_{n=1}^{+\infty}\chi_{\{N(0)=n\}}\Phi(x_n,v_n) \;=\; \mathbf{E}_{z,w}\Phi(x_{N(0)},v_{N(0)}) \;=\; \Phi(z,w)\,,$$

where for $t \geq v$

$$\begin{aligned}
\Phi(x,v) &= \mathbf{E}_x\sum_{k=1}^{+\infty} f(x_k\,,v_k+v-t)\chi_{\{v_1\leq t-v,\ldots,v_{k-1}\leq t-v,v_k>t-v\}} \\
&= \mathbf{E}_x\sum_{k=1}^{+\infty} f(x_k\,,v_k+v-t)\chi_{\{N(t-v)=k\}} \\
&= \mathbf{E}_x f(x_{N(t-v)}\,,v_{N(t-v)}+v-t) \\
&= \mathbf{E}_x f(Z(t-v)\,,W(t-v))\,.
\end{aligned}$$

From this we obtain (A.17).

It remains to show (A.16). Indeed, by definition $N(t)=\nu_i$ for some $i\in\mathbb{N}$. Therefore, for $t \geq 0$,

$$\begin{aligned}
\mathbf{E}_x f(Z(t)\,,W(t)) &= \sum_{i=1}^{\infty}\mathbf{E}_x f(x_{\nu_i}\,,v_{\nu_i}-t)\chi_{\{N(t)=\nu_i\}} \\
&= \sum_{i=0}^{\infty}\mathbf{E}_x\chi_{\{v_{\nu_i}\leq t\}} f^+(x_{\nu_{i+1}}\,,v_{\nu_{i+1}}-t) \\
&= \mathbf{E}_x\sum_{i=0}^{\infty}\chi_{\{v_{\nu_i}\leq t\}} g(x_{\nu_i}\,,t-v_{\nu_i}) = \Theta(g)(x,t)\,,
\end{aligned}$$

where $g(x,u)=\mathbf{E}_x f^+(x_{\nu_1}\,,v_{\nu_1}-u)$. Therefore, taking (A.15) into account we obtain that

$$
\begin{aligned}
&\int_{S\times R} \mathbf{E}_{z,w} f(Z(t)\,, W(t))\mu(\mathrm{d}z, \mathrm{d}w)\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^{v_{\nu_1}} \mathbf{E}_{x_{\nu_1},w} f(Z(t)\,, W(t))\mathrm{d}w\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \mathbf{E}_{x_{\nu_1},w} f(Z(t)\,, W(t))\mathrm{d}w + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_{v_{\nu_1}\wedge t}^{v_{\nu_1}} \mathbf{E}_{x_{\nu_1},w} f(Z(t)\,, W(t))\mathrm{d}w\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \mathbf{E}_{x_{\nu_1}} f(Z(t-w)\,, W(t-w))\mathrm{d}w\\
&\quad + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_t^{v_{\nu_1}} f(x_{\nu_1}\,, w-t)\mathrm{d}w\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \Theta(g)(x_{\nu_1}, t-w)\mathrm{d}w + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f(x_{\nu_1}\,, w)\mathrm{d}w\\
&= \int_{S\times\mathbb{R}} \Theta(g)(y, t-w)\mu(\mathrm{d}y\,, \mathrm{d}w) + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f(x_{\nu_1}\,, w)\mathrm{d}w\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y) \int_0^t g(y, w)\mathrm{d}w + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f(x_{\nu_1}\,, w)\mathrm{d}w\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^t f^+(x_{\nu_1}\,, v_{\nu_1}-w)\mathrm{d}w + \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f(x_{\nu_1}\,, w)\mathrm{d}w\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \left(\int_{v_{\nu_1}-v_{\nu_1}\wedge t}^{v_{\nu_1}} f(x_{\nu_1}\,, w)\mathrm{d}w + \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f(x_{\nu_1}\,, w)\mathrm{d}w\right)\\
&= \frac{1}{\beta}\int_S \sigma(\mathrm{d}y)\mathbf{E}_y \int_0^{v_{\nu_1}} f(x_{\nu_1}\,, w)\mathrm{d}w = \int_{S\times\mathbb{R}} f(y\,, w)\mu(\mathrm{d}y, \mathrm{d}w)\,.
\end{aligned}
$$

From this equality (A.16) follows.

A.3) Proof of (4.3).

For the function (4.2) in the definition of H for $t > \eta$ we have

$$
\begin{aligned}
&\int_{S\times\mathbb{R}} H(z, t-w))\mu(\mathrm{d}z, \mathrm{d}w)\\
&= \int_{-\infty}^{\infty} \theta(s)\left(\int_{S\times\mathbb{R}} \mathbf{E}_z f(Z(t+s-w), W(t+s-w))\chi_{\{t+s\ge w\}}\mu(\mathrm{d}z, \mathrm{d}w)\right)\mathrm{d}s\\
&= \int_{-\infty}^{\infty} \theta(s)\left(\int_{S\times\mathbb{R}} \mathbf{E}_{z,w} f(Z(t+s), W(t+s))\mu(\mathrm{d}z, \mathrm{d}w)\right)\mathrm{d}s - \delta(t)\,,
\end{aligned}
$$

where

$$
\delta(t) = \int_{-\infty}^{\infty} \theta(s)\int_{S\times\mathbb{R}} f(z, w-t-s)\chi_{\{t+s\le w\}}\mu(\mathrm{d}z, \mathrm{d}w)\mathrm{d}s.
$$

Taking (A.16) and the definitoin of the function $\theta(\cdot)$ in (4.2) into account we obtain that the first integral in the last equality is equal to the right hand side of (4.3), and by the dominated convergence theorem we get $\delta(t) \to 0$ as $t \to \infty$. Hence the relationship (4.3) holds.

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