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## Logic and Computation

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Wilfried Sieg, Editor

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## A NOTE ON POLYNOMIAL TIME COMPUTABLE ARITHMETIC

Wilfried Buchholz and Wilfried Sieg

**ABSTRACT.** In Ferreira's contribution to these Proceedings the class  $\mathcal{P}$  of polynomial time computable functions is characterized as the class of provably recursive functions of some weak formal theories. The first such characterization of  $\mathcal{P}$  was given, of course, by Buss. A form of Herbrand's theorem for partially normalized derivations is used in this note to obtain Ferreira's results. Such "Herbrand-analyses" have been applied in a variety of contexts (see [F/S], [L], [S1]): they are most appropriate if one wants to extract computational information from derivations; they are conceptually clear and technically strong.

**A. INTRODUCTION.** The class  $\mathcal{P}$  of polynomial time computable functions is characterized in [F] as the class of provably recursive functions of three restricted theories for binary trees or 0-1-words. The basic theory, PTCA, allows induction for polynomial time decidable predicates;  $PTCA^+$  is obtained from it by expanding the induction schema to NP-predicates. The third theory,  $(\Sigma_1^b\text{-PIND})$ , is like  $PTCA^+$ , but its language contains only symbols for some basic functions, not for all elements of  $\mathcal{P}$ . That the latter theory has exactly the elements of  $\mathcal{P}$  as its provably recursive functions is the analogue of the main theorem in [B] for  $n=1$ . Ferreira obtains this result by a mixture of model- and proof- theoretic techniques. We give a canonical, purely proof-theoretic Herbrand-analysis that yields Ferreira's result for  $PTCA^+$  and brings out most sharply the central problem; namely, the analysis of weak induction schemata by recursive functions of low complexity.

The main ideas for this paper emerged in the summer of 1988, when we gave a joint seminar at the Ludwigs-Maximilians-Universität in München. Buchholz presented [F] in the seminar; Sieg was working on his [S2] in which Herbrand-analyses for systems of (bounded) arithmetic are given. So it was natural to explore whether they can be given for Ferreira's theories of binary trees. Our note is thus complementing [F].

**B. BOUNDED LOGICAL COMPLEXITY.** We use the same formal framework as [F]; in particular,  $L$  is the first order language with constant symbols  $\emptyset, 0, 1$ , function symbols  $\wedge$  and  $\times$ , and two binary relation symbols  $\subseteq$  and  $=$ . The language  $L(\mathcal{P})$ , i.e.  $L_{\mathcal{P}}$  in [F], is obtained from  $L$  by adding function symbols for each element of  $\mathcal{P}$ . The

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latter class can be defined inductively as the smallest class  $X$  of functions, such that  $X$  contains certain initial functions ( $Z$ ;  $P_i^n$ ,  $1 \leq i \leq n$ ,  $n \in \mathbb{N}$ ;  $C_0, C_1$ , and  $Q$ ), and is closed under composition and bounded iteration. Let  $st$  abbreviate  $s \hat{t}$  for L-terms  $s$  and  $t$ :  $x|_z = y$  stands for  $(1 \times z \subseteq 1 \times x \ \& \ y \subseteq x \ \& \ 1 \times z = 1 \times y) \vee (1 \times x \subseteq 1 \times z \ \& \ y = x)$ . Using these abbreviations we formulate the schema of bounded iteration :

$f$  is defined by iteration from  $g, h_0, h_1$  with bound  $t[\mathbf{x}, y]$  if

$$f(\mathbf{x}, \emptyset) = g(\mathbf{x})$$

$$f(\mathbf{x}, yi) = h_i(\mathbf{x}, y, f(\mathbf{x}, y))|_{t[\mathbf{x}, y]} \quad (i = 0, 1)$$

where  $t$  is an L-term and  $\mathbf{x}$  indicates a possibly empty sequence of variables. As we are going to work in a Tait-style sequent calculus, it is convenient to build up formulas from literals (atomic or negations of atomic formulas) by using  $\&$ ,  $\vee$ ,  $\forall$ ,  $\exists$ . Negations of complex formulas, conditionals and biconditionals are defined as usual.

## 1. DEFINITIONS

- 1.1.  $QF(\mathcal{P})$  denotes the set of quantifier-free formulas of  $L(\mathcal{P})$ .
- 1.2.  $(\forall x \subseteq y) \varphi$  [ $(\exists x \subseteq y) \varphi$ ] abbreviates  $(\forall x)(x \subseteq y \rightarrow \varphi)$  [ $(\exists x)(x \subseteq y \ \& \ \varphi)$ , resp.].
- 1.3.  $(\exists x \leq t) \varphi$  abbreviates  $(\exists x)(x \leq t \ \& \ \varphi)$ , where  $s \leq t$  is  $1 \times s \subseteq 1 \times t$ .
- 1.4. A formula  $\varphi$  is in  $\Delta_0^b(\mathcal{P})$  [ $\Sigma_1^b(\mathcal{P})$ , resp.] if it has been obtained from literals in  $L(\mathcal{P})$  by  $\&$ ,  $\vee$ ,  $\forall \subseteq$ ,  $\exists \subseteq$ , [and  $\exists \leq$ , resp.].
- 1.5. An L( $\mathcal{P}$ )-formula  $\varphi$  is in  $s\text{-}\Sigma_1^b(\mathcal{P})$  if it is of the form  $(\exists y \leq t) \psi$  with  $\psi \in QF(\mathcal{P})$ .

The formulas in  $\Delta_0^b(\mathcal{P})$  are exactly the polynomial time computable matrices of [F]. – The theories for binary trees to be investigated contain the basic axioms for the non-logical symbols of  $L$  (see [F]), the defining equations for the elements of  $\mathcal{P}$  in case the theory is formulated in  $L(\mathcal{P})$ , and the induction principle on notations for classes of formulas  $\mathcal{F}$ :  $\varphi \emptyset \ \& \ (\forall x)(\varphi x \rightarrow \varphi x 0 \ \& \ \varphi x 1) \rightarrow (\forall x) \varphi x$  ( $\varphi \in \mathcal{F}$ ). The latter schema is denoted by  $\mathcal{F}$ -NIA; the resulting theory – always with classical logic – is called ( $\mathcal{F}$ -NIA).

We formulate a few properties of  $\mathcal{P}$  that are provable in ( $QF(\mathcal{P})$ -NIA).

## 2. LEMMA.

- (i) For every term  $s$  of  $L(\mathcal{P})$  there is a term  $t$  of  $L$ , such that ( $QF(\mathcal{P})$ -NIA) proves  $s \leq t$ .
- (ii) For any  $\varphi_1(\mathbf{x}), \dots, \varphi_n(\mathbf{x}) \in QF(\mathcal{P})$  and  $f_1, \dots, f_{n+1} \in \mathcal{P}$  there is an  $f \in \mathcal{P}$  such that ( $QF(\mathcal{P})$ -NIA) proves
 
$$\begin{aligned} (\varphi_1(\mathbf{x}) \ \& \ f(\mathbf{x}) = f_1(\mathbf{x})) \vee (\neg \varphi_1(\mathbf{x}) \ \& \ \varphi_2(\mathbf{x}) \ \& \ f(\mathbf{x}) = f_2(\mathbf{x})) \\ \vee (\neg \varphi_1(\mathbf{x}) \ \& \ \neg \varphi_2(\mathbf{x}) \ \& \ \varphi_3(\mathbf{x}) \ \& \ f(\mathbf{x}) = f_3(\mathbf{x})) \\ \vdots \\ \vee (\neg \varphi_1(\mathbf{x}) \ \& \ \dots \ \& \ \neg \varphi_n(\mathbf{x}) \ \& \ f(\mathbf{x}) = f_{n+1}(\mathbf{x})). \end{aligned}$$
- (iii) For any  $\varphi(\mathbf{x}, y) \in QF(\mathcal{P})$  there is an  $h \in \mathcal{P}$ , such that ( $QF(\mathcal{P})$ -NIA) proves
 
$$((\exists y \subseteq x) \varphi(\mathbf{x}, y) \leftrightarrow \varphi(\mathbf{x}, h(\mathbf{x}, x))).$$

The last part of the lemma allows us to prove that in ( $QF(\mathcal{P})$ -NIA) every  $\Delta_0^b(\mathcal{P})$ -formula is equivalent to a quantifier-free formula; proposition 6 of [F] establishes in turn that in  $(s\text{-}\Sigma_1^b(\mathcal{P})$ -NIA) every  $\Sigma_1^b(\mathcal{P})$ -formula is equivalent to one in  $s\text{-}\Sigma_1^b(\mathcal{P})$ .

Thus we have:

3. PROPOSITION.

- (i)  $(\Delta_0^b(\mathcal{P})\text{-NIA})$  is equivalent to  $(\text{QF}(\mathcal{P})\text{-NIA})$ .
- (ii)  $(s\text{-}\Sigma_1^b(\mathcal{P})\text{-NIA})$  is equivalent to  $(\Sigma_1^b(\mathcal{P})\text{-NIA})$ .

Notice that  $(\Delta_0^b(\mathcal{P})\text{-NIA})$  is Ferreira's PTCA, and  $(s\text{-}\Sigma_1^b(\mathcal{P})\text{-NIA})$  is his PTCA<sup>+</sup>. Now we turn our attention to bounding the complexity of formulas in derivations. The latter are now presented in a Tait-style calculus as in [Sch]; the induction principle is given equivalently by a rule  $\mathcal{F}\text{-NIR}^*$  of the form

$$\frac{\Delta, \neg\varphi x, \varphi x 0 \quad \Delta, \neg\varphi x, \varphi x 1}{\Delta, \neg\varphi \emptyset, \varphi s} \quad (\varphi \in \mathcal{F})$$

where  $s$  is a term, and  $x$  must not occur in the lower sequent. This new formulation of  $(\mathcal{F}\text{-NIA})$  has two virtues – it is equivalent to the earlier one and allows us to prove partial normalization theorems.

5. DEFINITION .

A derivation in  $(\mathcal{F}\text{-NIA})$  is called I-normal if and only if all its cuts are either I-cuts or have atomic cut-formulas; where a cut with cut-formula  $\varphi$  is called an I-cut if one of its premises is the conclusion of the induction rule with principal formula  $\varphi$  or  $\neg\varphi$ .

The standard proof of the normalization theorem for predicate logic can readily be adapted to show that any derivation in  $(\mathcal{F}\text{-NIA})$  can be I-normalized.

6. THEOREM. (I-normalization) If  $D$  is a derivation of  $\Gamma$  in  $(\mathcal{F}\text{-NIA})$ , then there is an I-normal derivation  $D^\circ$  of the same endsequent in  $(\mathcal{F}\text{-NIA})$ .

The length  $|D^\circ|$  of  $D^\circ$  can be bounded by  $2_{m}^{|D|}$ ,  $m = \rho(D) - 1$ ; the cut-rank function  $\rho$  takes into account only the complexity of cuts that are not I-cuts. I-normal derivations do not have the subformula property, but the complexity of formulas occurring in them can nevertheless be bounded significantly.

7. COROLLARY. Let  $\mathcal{F}$  and  $\mathcal{G}$  be classes of formulas that are closed under substitution. If  $D$  is an I-normal derivation of  $\Gamma$  in  $(\mathcal{F}\text{-NIA})$  with  $\Gamma \subseteq \mathcal{G}$ , then any formula in  $D$  is either atomic or a subformula of an element of  $\mathcal{F} \cup \{\neg\varphi : \varphi \in \mathcal{F}\} \cup \mathcal{G}$ .

C. EXTRACTING TERMS.

The I-normalization theorem will be used to establish a (generalized) Herbrand-theorem.

8. THEOREM. ( $\exists$ -inversion) Let  $\Gamma$  contain only existential formulas, and let  $\psi$  be quantifier-free; if  $\Gamma, (\exists y)\psi y$  is provable in  $(\text{QF}(\mathcal{P})\text{-NIA})$  then there is a term  $t^*$  such that  $\Gamma, \psi t^*$  is also provable in  $(\text{QF}(\mathcal{P})\text{-NIA})$ . ( $\psi$  may contain additional variables.)

PROOF. The proof proceeds by induction on I-normal  $(\text{QF}(\mathcal{P})\text{-NIA})$ -derivations  $D$ . We focus on the central step, when  $\text{NIR}^*$  is the last rule applied in  $D$ . (The other non-trivial cases, e.g.  $\exists$ -introduction, require definition by cases.)

Then  $D$  is of the form

$$D \left\{ \begin{array}{c} D_0 \left\{ \begin{array}{c} | \\ \Delta, \neg \varphi x, \varphi x 0, (\exists y) \psi y \end{array} \right. \quad \left. \begin{array}{c} | \\ \Delta, \neg \varphi x, \varphi x 1, (\exists y) \psi y \end{array} \right. \right\} D_1 \\ \hline \underbrace{\Delta, \neg \varphi \emptyset, \varphi s, (\exists y) \psi y}_{= \Gamma} \end{array} \right\}$$

The induction hypothesis applied to the  $D_i$  yields terms  $t_i[x]$  and derivations  $D_i^*$  of

$$(1) \quad \Delta, \neg \varphi x, \varphi x i, \psi t_i[x] \quad (i = 0, 1).$$

Obviously (QF( $\mathcal{P}$ )-NIA) proves

$$(2) \quad \neg \varphi \emptyset, \varphi s, \exists x \subseteq s[\varphi x \ \& \ \neg(\varphi x 0 \ \& \ \varphi x 1)].$$

By Lemma 2 (iii) there is an  $h \in \mathcal{P}$ , such that (QF( $\mathcal{P}$ )-NIA) proves

$$(3) \quad \neg \exists x \subseteq s[\varphi x \ \& \ \neg(\varphi x 0 \ \& \ \varphi x 1)], \varphi h(s) \ \& \ \neg(\varphi h(s) 0 \ \& \ \varphi h(s) 1).$$

From (2) and (3) we obtain:

$$(4) \quad \neg \varphi \emptyset, \varphi s, \varphi h(s).$$

$$(5) \quad \neg \varphi \emptyset, \varphi s, \neg \varphi h(s) 0, \neg \varphi h(s) 1.$$

From (1) (with  $h(s)$  substituted for  $x$ ) and (5) we obtain

$$\neg \varphi \emptyset, \varphi s, \Delta, \neg \varphi h(s), \psi t_0[h(s)], \psi t_1[h(s)]$$

$$\Delta, \neg \varphi \emptyset, \varphi s, \psi t_0[h(s)], \psi t_1[h(s)].$$

This together with Lemma 2(ii) gives us an  $f \in \mathcal{P}$  such that (QF( $\mathcal{P}$ )-NIA) proves

$$\Delta, \neg \varphi \emptyset, \varphi s, \psi f(s). \quad \text{Q.E.D.}$$

The  $\exists$ -inversion is crucial for establishing the main conservation result.

**9. THEOREM.** ( $s$ - $\Sigma_1^b(\mathcal{P})$ -NIA) is conservative over (QF( $\mathcal{P}$ )-NIA) with respect to  $\Pi_2^0$ -sentences  $\varphi$  of the form  $(\forall x)(\exists y)\varphi^*(x, y)$  with  $\varphi^* \in \text{QF}(\mathcal{P})$ .

**PROOF.** As (QF( $\mathcal{P}$ )-NIA) is contained in ( $s$ - $\Sigma_1^b(\mathcal{P})$ -NIA) we have to show only that every  $\Pi_2^0$ -sentence provable in the latter theory is provable in the former. This is achieved by transforming any  $l$ -normal derivation  $D$  in ( $s$ - $\Sigma_1^b(\mathcal{P})$ -NIA) of a sequent  $\Delta$ , where  $\Delta$  contains only existential formulas, into a derivation  $D'$  of  $\Delta$  in (QF( $\mathcal{P}$ )-NIA). We proceed by induction on the number  $\#$  of applications of  $\text{NIR}^*$  in  $D$ , not counting for sure  $\text{NIR}^*$ -instances with formulas in QF( $\mathcal{P}$ ). - The case  $\# = 0$  is trivial. So let  $\#$  be  $m+1$  and consider an uppermost instance of  $\text{NIR}^*$  with  $\psi$  of the form  $(\exists y)(y \leq t[x] \ \& \ \psi^*yx)$ , where  $\psi^*$  is in QF( $\mathcal{P}$ ); both  $\psi^*$  and  $t$  may contain additional variables. The subderivation  $E$  of  $D$  determined by that inference is of the form

$$E_0 \left\{ \begin{array}{c} | \\ \Gamma, \neg \psi x, \psi x 0 \end{array} \right. \quad \left. \begin{array}{c} | \\ \Gamma, \neg \psi x, \psi x 1 \end{array} \right\} E_1 \\ \hline \Gamma, \neg \psi \emptyset, \psi s \quad \left. \right\} E$$

Taking into account the form of  $\psi$  and the fact that  $D$  is an  $l$ -normal derivation in ( $s$ - $\Sigma_1^b(\mathcal{P})$ -NIA), we can obtain (recalling corollary 7) by repeated  $\forall$ -inversion from the  $E_i$  derivations of  $\Gamma^*, \neg(y \leq t[x] \ \& \ \psi^*yx), \psi x i$ , where  $\Gamma^*$  contains only existential formulas.  $\exists$ -inversion yields terms  $t_i[y, x]$  and derivations in (QF( $\mathcal{P}$ )-NIA) of

$$(\square_i) \quad \Gamma^*, \neg(y \leq t[x] \ \& \ \psi^*yx), t_i[y, x] \leq t[xi] \ \& \ \psi^*t_i[y, x]xi.$$

Now we define a function  $f$  by iteration with bound  $t[x_0]t[x_1]$  (using Lemma 2(i) and (ii) to bring the definition into the required form):  $f(y, \emptyset) = y$ ,  $f(y, xi) = t_i[f(y, x), x]$ . From the derivations leading to the  $\square_i$  and this definition we get derivations of

$$\Gamma^*, \neg(f(y, x) \leq t[x] \& \psi^* f(y, x) x), f(y, xi) \leq t[xi] \& \psi^* f(y, xi) xi$$

and by QF( $\mathcal{P}$ )-NIR\* of

$$\Gamma^*, \neg(y \leq t[\emptyset] \& \psi^* y \emptyset), f(y, s) \leq t[s] \& \psi^* f(y, s) s.$$

With a little bit of logic we finally obtain a derivation  $E'$  in (QF( $\mathcal{P}$ )-NIA) of  $\Gamma, \neg\psi\emptyset, \psi s$ . Replace  $E$  in  $D$  by  $E'$ . The resulting derivation has only  $m$  applications of  $s\text{-}\Sigma_1^b(\mathcal{P})\text{-NIR}^*$  and the induction hypothesis yields the above claim. Q.E.D.

#### D. CHARACTERIZING $\mathcal{P}$ .

Since for every  $L(\mathcal{P})$ -term  $t[x]$  the function  $\lambda x.t[x]$  is in  $\mathcal{P}$ , theorem 8 implies that the provably recursive functions of (QF( $\mathcal{P}$ )-NIA) are exactly the polynomial time computable ones. Using also theorem 9 and proposition 4 we obtain the sought after characterization result.

#### 10. THEOREM.

$\mathcal{P}$  is exactly the class of provably recursive functions of  $(\Sigma_1^b(\mathcal{P})\text{-NIA})$ .

#### REMARKS.

- (i) [F] establishes that  $(\Sigma_1^b(\mathcal{P})\text{-NIA})$  is a conservative extension of  $(\Sigma_1^b\text{-NIA})$ ; thus the theorem holds also for the latter theory.
- (ii) The Herbrand-analysis given in C is insensitive to extensions of the various theories by  $\Pi_1^0$ -sentences. Thus, the main results hold also for  $\Pi_1^0$ -extensions of the theories involved.

#### REFERENCES

- [B] **Samuel Buss**, Bounded Arithmetic, Bibliopolis, Napoli, 1986.
- [F/S] **Solomon Feferman and Wilfried Sieg**, "Proof theoretic equivalences between classical and constructive theories for analysis", Springer Lecture Notes in Mathematics, vol. 897, (1981), pp. 78-142.
- [F] **Fernando Ferreira**, "Polynomial time computable arithmetic", this volume.
- [L] **Horst Luckhardt**, "Herbrand-Analysen zweier Beweise des Satzes von Roth: polynomiale Anzahlschranken", to appear in the Journal of Symbolic Logic.
- [Sch] **Helmut Schwichtenberg**, "Proof theory: some applications of cut-elimination", Handbook of Math.Logic, J.Barwise (ed.), NHPC, Amsterdam 1977, pp. 867-895.
- [S1] **Wilfried Sieg**, "Fragments of arithmetic", Annals of Pure and Applied Logic, 28 (1985), pp. 33-71.
- [S2] **Wilfried Sieg**, "Derivations as computations", manuscript.

Wilfried Buchholz  
Mathematisches Institut  
Ludwig-Maximilians-Universität  
8000 MÜNCHEN 2, West-Germany

Wilfried Sieg  
Department of Philosophy  
Carnegie Mellon University  
PITTSBURGH, PA 15213, USA