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Homotopy Inverses for Nerve

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Section 1. Introduction

In [19] simple conditions on "categorical simplices" are given so that, up to homotopy, the only reasonable assignment of a (semi-) simplicial set to a small category is the standard nerve embedding. In this paper, more complicated conditions on "categorical simplices" are given so that certain cocontinuous functors from simplicial sets to small categories are homotopy inverses for the standard nerve embedding, and induce the "surprising" (adjoint, in certain cases) homotopy equivalence between the categories of simplicial sets and small categories. The results in this paper are only part of the larger long-term program to develop "useful" algebraic topology for small categories.

At least, since Gabriel-Zisman's seminar [10], it is known that the category $\mathscr{C}at$ of small categories is a full reflective subcategory of the category $\mathscr{K}$ of simplicial sets via the standard nerve embedding

$$N\colon \mathscr{C}at \to \mathscr{K}$$

and thus it inherits the elementary notions of homotopy theory, like weak homotopy equivalence (WHE), from $\mathscr{K}$. The "real" reason for this paper is that the reflector for the nerve embedding, categorical realization

$$c\colon \mathscr{K} \to \mathscr{C}at$$

is "wildly" wrong with respect to weak homotopy. For instance, it maps certain simplicial spheres to contractible small categories:

$$Nc(\Delta[k]/\dot{\Delta}[k]) \cong \Delta[0] \quad \text{for } k \geqq 2 \tag{1.1}$$

$$Nc(Sd(\Delta[k]/\dot{\Delta}[k])) \cong \Delta[1] \quad \text{for } k \geqq 2 \tag{1.2}$$

where $\Delta[k]$ is the standard k-dimensional (representable) simplicial set, $\dot{\Delta}[k]$ represents its boundary, and

$$\mathrm{Sd}\colon \mathscr{K} \to \mathscr{K}$$

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is the (first) barycentric subdivision functor [13; 7]. (For $k=1$:

$$c(\Delta[1]/\dot{\Delta}[1]) \cong \mathbb{N} \equiv \text{“natural numbers”}$$
$$Nc(Sd\Delta[1]/\dot{\Delta}[1]) = Sd(\Delta[1]/\dot{\Delta}[1]).$$

$N\mathbb{N}$ is not a sphere, but it is well known (e.g. [3; X. Cor. 4.2] or [23]) that the classifying space of the natural numbers still has the homotopy type of a 1-sphere). Moreover, Gabriel and Zisman show [10; II.4] that the categorical realization cX of any simplicial set X depends only on X^2, the 2-dimensional skeleton of X. Therefore, many workers believed for a long time that small categories were nearly homotopically trivial. But then, L. Illusie [11], M.J. Lee [20], and Latch [17] showed that the small categories ΛX and ΓX have the homotopy type of X, for *any* simplicial set X (i.e. $N\Lambda X$ and $N\Gamma X$ are both naturally weak homotopy equivalent to X). From the geometric point of view, both functors

$$\Lambda\colon \mathscr{K} \to \mathscr{C}at \quad \text{and} \quad \Gamma\colon \mathscr{K} \to \mathscr{C}at$$

are not quite satisfactory: $N\Lambda X$ and $N\Gamma X$ are both infinite dimensional simplicial sets for *every* simplicial set X, even the trivial simplicial point $\Delta[0]$.

Then R. Thomason conjectured that the “finite dimensional” construction

$$cSd^2\colon \mathscr{K} \to \mathscr{C}at,$$

where $Sd^2\colon \mathscr{K} \to \mathscr{K}$ is the twofold barycentric subdivision [13; 3], would preserve homotopy type. This appears astonishing at first; but results of J. Segal [29] and Fritsch [7] show, that after a suitable subdivision, the new 1-skeleton contains a lot of information about X. The conjecture was proved by R. Thomason [30] and independently by Fritsch using different methods. The first proof used an earlier version of Theorem 6.3 in this paper, while the second, depends directly on Theorem 4.11 here and is presented in Example 4.12(iv) and (v). Moreover, this second proof shows that every simplicial set after geometric realization is homeomorphic to the classifying space of a small category in the sense of G. Segal [28] (see Remark 4.13).

The starting point for this paper is the question, why do Lee's, Latch's and Thomason's constructions work and why other cocontinuous functors (particularly categorical realization $c\colon \mathscr{K} \to \mathscr{C}at$) from $\mathscr{K}$ to $\mathscr{C}at$ do not? The functors

$$\Gamma_\theta\colon \mathscr{K} \to \mathscr{C}at$$

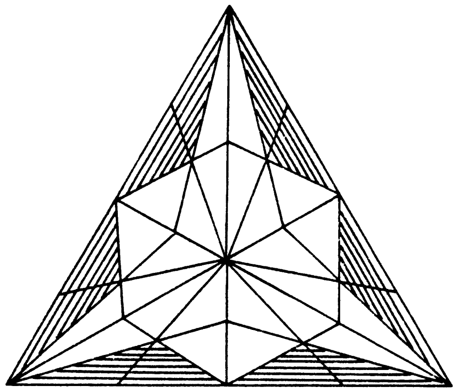
from $\mathscr{K}$ to $\mathscr{C}at$ considered are cocontinuous (i.e. preserve colimits) and are completely determined by their behavior on the contractible standard simplicial sets $\Delta[k]$ (for $k \geqq 0$), yielding the weakly contractible “categorical simplices”

$$\theta[k] \equiv \Gamma_\theta \Delta[k].$$

The main result (see Theorem 4.5) shows that such a cocontinuous functor Γ_θ: $\mathscr{K} \to \mathscr{C}at$ preserves homotopy type, whenever the canonical natural transformation

$$\rho\colon \Gamma_{N\theta} \dot{\to} N\Gamma_\theta\colon \mathscr{K} \to \mathscr{K}$$

is a natural WHE. But this global condition sometimes can be replaced by a local condition on the categorical simplices. A sufficient condition for this purpose in given Theorem 6.3. Roughly speaking it says that the objects in the boundary of a categorical simplex causing trouble under degeneracies can be collared in a suitable way. The geometric intuition for these collars is illustrated in the following picture of $cSd^2\Delta[2]$:



This last problem can be better understood by referring to the skeletal decomposition of simplicial sets and the process of attaching simplices [10; II.3]. Categorical realization $c\colon \mathscr{K} \to \mathscr{C}at$ is left adjoint to nerve $N\colon \mathscr{C}at \to \mathscr{K}$, and thus it preserves attachings, but in a strange way, due to the "ugly" form of pushouts in $\mathscr{C}at$. The functor associating to a small category **A** its morphism set Mor **A** does not preserve pushouts (compare the difference between the coproduct of two groups and the disjoint union of the underlying sets). In contrast, there are cocontinuous functors from $\mathscr{K}$ to the category $\mathscr{E}ns$ of sets which preserve pushouts and hence attachings, and which yield *all* information about the original simplicial set. Thus, in order to get a small category $\Gamma_\theta X$ having the homotopy type of a given simplicial set X (i.e., so that $N\Gamma_\theta X$ and X are naturally weak homotopy equivalent), it is necessary that to the k-dimensional simplices of X, there be associated categorical simplicies $\theta[k]$ for which the attaching process is preserved, at least up to weak homotopy equivalence. We study the behavior of attachings under nerve in Sect. 5 as far as it is necessary for the theorems in Sect. 6.

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Section 2. Preliminaries

This section contains notations, definitions and theorems which are well-known or have appeared in detail in other papers, and which will be used here and in subsequent sections.

2.1. Notation. The following large categories are denoted by:

(i) $\mathscr{E}ns \equiv$ the category of sets and functions;

(ii) $\mathscr{K} \equiv$ the category of (semi-)simplicial sets [10; II.2];

(iii) $\mathscr{W} \equiv$ the category of spaces of homotopy type of a CW-complex [10; VII.1];

(iv) $\mathscr{C}at \equiv$ the category of small categories [22; I.3].

2.2. Notation. Small categories will be represented by $\mathbf{A}$, $\mathbf{B}$, etc., while $f\colon \mathbf{A} \to \mathbf{B}$ denotes a functor, a morphism in $\mathscr{C}at$. For each small category $\mathbf{A}$,

$$|\mathbf{A}| \equiv \text{set of objects of } \mathbf{A};$$

and if S is a subset of $|\mathbf{A}|$, then

$$\langle S \rangle \equiv \text{full subcategory of } \mathbf{A} \text{ generated by } S.$$

If $p, q \in |\mathbf{A}|$, then

$$\mathbf{A}(p,q) \equiv \text{set of all morphisms } a \text{ in } \mathbf{A} \text{ with dom } a = p \text{ and cod } a = q.$$

If f, $g \in \mathscr{C}at(\mathbf{A}, \mathbf{B})$, i.e. f and g are functors with dom$f = \mathbf{A}$ and cod$f = \mathbf{B}$, then a natural transformation η from f to g is denoted by

$$\eta\colon f \overset{\cdot}{\to} g\colon \mathbf{A} \to \mathbf{B}.$$

(Similar notation is used even when one or both of the categories are not small.)

2.3. Proposition. *Let $\theta\colon \mathbf{D} \to \mathscr{C}$ be a functor with $\mathbf{D} \in |\mathscr{C}at|$ and $\mathscr{C}$ cocomplete, and let $D\colon \mathbf{D} \to [\mathbf{D}^{\mathrm{op}}, \mathscr{E}ns]$ be the Yondeda embedding* [22; X.6].

(i) *There is one and only one cocontinuous functor*

$$\Gamma_\theta\colon [\mathbf{D}^{\mathrm{op}}, \mathscr{E}ns] \to \mathscr{C}$$

such that the triangle

$$\begin{tikzcd} {[\mathbf{D}^{\mathrm{op}}, \mathscr{E}ns]} \arrow[dr, "\Gamma_\theta"] & \\ \mathbf{D} \arrow[u, "D"] \arrow[r] & \mathscr{C} \end{tikzcd}$$

commutes [22; X.3].

(ii) Γ_θ *has a right adjoint, the θ-singular functor,*

$$S_\theta\colon \mathscr{C} \to [\mathbf{D}^{\mathrm{op}}, \mathscr{E}ns],$$

explicitly defined by

$$(S_\theta C)\, d \equiv \mathscr{C}(\theta d, C)$$

where $C \in |\mathscr{C}|$ and $d \in |\mathbf{D}|$ [22; X.5, *dual* Cor. 4].

(iii) *The counit*

$$\varepsilon\colon \Gamma_\theta S_\theta \overset{\cdot}{\to} \mathrm{Id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$$

is an equivalence (a) *iff* S_θ *is full and faithful* [22; IV.3, *Prop.* 1], *or alternately* (b) *iff* θ *is dense;* i.e. *every object in* $\mathscr{C}$ *is a colimit of a diagram with vertices and edges in the image of* θ [22; X.6, *dual of Prop.* 2]. □

2.4. *Remarks.* Γ_θ is called the left Kan extension of θ along **D.** It is given on objects $X \in [\mathbf{D}^{\mathrm{op}}, \mathscr{E}ns]$ by

$$\Gamma_\theta X \equiv \operatorname*{colim}_{x \in X d} \theta d \qquad [22;\ \mathrm{X.3(10)}]$$

$$\equiv \int^{d} X d \cdot \theta d \qquad [22;\ \mathrm{X.4(1)}].$$

In the case $\mathscr{C} = \mathscr{E}ns$, the elements of $\Gamma_\theta X$ can be described explicitly as equivalence classes $[x, a]$ of pairs $(x, a) \in X d \times \theta d$ where the equivalence relation is generated by

$$(x \cdot X\delta, a) \sim (x, \theta(\delta) a)$$

for $x \in X d$, $a \in \theta d'$ and $\delta \in \mathbf{D}(d', d)$.

We now specialize the above abstract category theory.

2.5. *Notation.* The following small categories are used:

(i) $\mathbf{\Delta} \equiv$ category of finite ordinals

$$[k] \equiv \{0 < 1 < \ldots < k\}$$

(in [22; VIII.5] denoted by $\mathbf{\Delta}^+$), whose morphisms decompose into surjective degeneracy operators and injective face operators; it is considered to be a full and dense subcategory of $\mathscr{C}at$ ([10; II.4.1] or [19; 2.5]).

(ii) $\mathbf{M} \equiv$ subcategory of monomorphisms μ: $[p] \rightarrowtail [k]$ of $\mathbf{\Delta}$ with canonical inclusion

$$U\colon \mathbf{M} \hookrightarrow \mathbf{\Delta}.$$

Each small category yields a canonical functor category:

(iii) $\mathscr{K} \equiv [\mathbf{\Delta}^{\mathrm{op}}, \mathscr{E}ns] \equiv$ category of simplicial sets [10; II.2].

(iv) $\mathscr{L} \equiv [\mathbf{M}^{\mathrm{op}}, \mathscr{E}ns] \equiv$ category of nondegenerate simplicial sets [26].

The inclusion ι: $\mathbf{\Delta} \hookrightarrow \mathscr{C}at$ induces:

(v) $c \equiv \Gamma_\iota$: $\mathscr{K} \to \mathscr{C}at \equiv$ categorical realization (in analogy to the classical notion of geometric realization) and

(vi) $N \equiv S_\iota$: $\mathscr{C}at \to \mathscr{K} \equiv$ nerve. (*Name derivation:* Classically, the nerve of a covering of a topological space is an ordered simplicial complex. From the given covering, one forms a poset which can be considered as a small category **C** ([22; I.2] or [10; II.4.1]). The nerve of the *category* **C**, N**C**, is nothing but the classical nerve of the covering).

The inclusion U: $\mathbf{M} \hookrightarrow \mathbf{\Delta}$ composed with the Yoneda embedding Δ: $\mathbf{\Delta} \to \mathscr{K}$ yields the pair of adjoint functors:

(vii) $J \equiv \Gamma_{\Delta U}$: $\mathscr{L} \to \mathscr{K}$

(viii) $P \equiv S_{\Delta U}$: $\mathscr{K} \to \mathscr{L}$.

Proposition 2.3 implies:

2.6. Corollary. (i) *Nerve is a full and faithful embedding, given by:*

$$(N\mathbf{C})_k \equiv (N\mathbf{C})[k] \equiv \{p_k \xleftarrow{a_k} p_{k-1} \leftarrow \dots \leftarrow p_2 \xleftarrow{a_2} p_1 \xleftarrow{a_1} p_0\}$$

for all $\mathbf{C} \in |\mathscr{C}at|$ *and* $k \geq 0$.

(ii) *The counit*

$$\varepsilon \colon cN \stackrel{\cdot}{\to} \mathrm{Id}_{\mathscr{C}at} \colon \mathscr{C}at \to \mathscr{C}at$$

is an equivalence [10; II.4]. □

Moreover, we have:

2.7. Lemma. *P forgets degeneracies;* i.e.,

$$PX = X/\mathbf{M}^{\mathrm{op}}$$

for all $X \in |\mathscr{K}|$. □

Thus P is not only continuous (as a right adjoint), but in addition:

2.8. Corollary. *P is cocontinuous* [8; Satz 1 b]. □

Each of the categories $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{C}at$ inherits homotopy theory from the category $\mathscr{W}$ via the classical geometric realization functor $|-| \colon \mathscr{K} \to \mathscr{W}$ ([24, 28]):

2.9. Definitions. Let $\mathscr{C}$ represent either $\mathscr{K}$ or $\mathscr{L}$ or $\mathscr{C}at$ (in this order).

(i) A morphism $f \in \mathscr{C}$ is a *weak homotopy equivalence* (*WHE*) in $\mathscr{C}$ iff $|f|$, resp. $|Jf|$, resp. $|Nf|$ is a homotopy equivalence in $\mathscr{W}$.

(ii) An object C in $\mathscr{C}$ is *weakly contractible* (*WC*) in $\mathscr{C}$, whenever the terminal morphism $C \to \cdot$ is a WHE in $\mathscr{C}$.

(iii) Let $\mathscr{C}'$ be an arbitrary category. A natural transformation

$$\phi \colon F \stackrel{\cdot}{\to} G \colon \mathscr{C}' \to \mathscr{C}$$

is a *natural WHE* in $\mathscr{C}$ if, for every object C' in $\mathscr{C}'$, the corresponding morphism

$$\phi C' \colon FC' \to GC'$$

is a WHE in $\mathscr{C}$.

A tool used to attach simplices, and hence to do skeletal inductions, is the:

2.10. Glueing Lemma. *If*

$$\begin{tikzcd}
X \arrow[r, hook, "u"] \arrow[d, "\rho_1"] & Y \arrow[r, "f'"] \arrow[d, "\rho_2"] & P \arrow[d, "\rho"] & Z \arrow[l, hook', "u'"'] \arrow[d, "\rho_3"] & X \arrow[l, "f"'] \arrow[d, "\rho_1"] \\
\bar{X} \arrow[r, hook, "\bar{u}"] & \bar{Y} \arrow[r, "\bar{f}'"] & \bar{P} & \bar{Z} \arrow[l, hook', "\bar{u}'"'] & \bar{X} \arrow[l, "\bar{f}"']
\end{tikzcd}$$

is a commutative ladder in $\mathscr{K}$ *or* $\mathscr{L}$ *with the rows pushouts, with* u, $\bar{u}$ *inclusions, and with* ρ_1, ρ_2, ρ_3 *WHE's, then*

$$\rho \colon P \to \bar{P}$$

is also a WHE [2; 7.5.7]. □

One application of this lemma yields the following theorem which in embryo form, was used in connection with subdivision problems (e.g. see [8; 4.1. Satz 2]).

2.11. Comparison Theorem. *Let $\mathscr{C}'$ and $\mathscr{C}$ each either be $\mathscr{K}$ or $\mathscr{L}$. Suppose F, G: $\mathscr{C}' \to \mathscr{C}$ are cocontinuous functors which also preserve injections. In addition, assume*

$$\varphi\colon F \xrightarrow{\cdot} G\colon \mathscr{C}' \to \mathscr{C}$$

is a natural transformation such that its composition with the corresponding Yoneda embedding is a natural WHE. Then $\phi\colon F \xrightarrow{\cdot} G$ is itself a natural WHE. □

In particular, one verifies that the counit

$$JP\Delta \xrightarrow{\cdot} \Delta\colon \Delta \to \mathscr{K}$$

is a natural WHE.

2.12. Proposition. *The counit and the unit of the adjunction $J \dashv P$*

$$\tilde{\varepsilon}\colon JP \xrightarrow{\cdot} \mathrm{Id}_{\mathscr{K}}\colon \mathscr{K} \to \mathscr{K}$$
$$\tilde{\eta}\colon \mathrm{Id}_{\mathscr{L}} \xrightarrow{\cdot} PJ\colon \mathscr{L} \to \mathscr{L}$$

are both natural WHE's [5; 3.4]. □

2.13. Corollary. *P and J both preserve and reflect WHE's* (i.e., *Pf is a WHE in $\mathscr{L}$* iff *f is a WHE in $\mathscr{K}$, and similarly for J*). □

2.14. Remark. N and J both are inclusions. Thus $\mathscr{C}at$ and $\mathscr{L}$ both, can be considered as subcategories of $\mathscr{K}$. The essential difference lies in the fact that J is a left adjoint, while N is a right adjoint. Therefore, the Glueing Lemma and its consequences are only available for $\mathscr{L}$, but *not* for $\mathscr{C}at$. This is the reason for many of the difficulties in this paper.

The adjoint pair $J \dashv P$ is an example for the notion of "homotopy inverses" appearing in the title of this paper. More precisely:

2.15. Definitions. Let $\mathscr{C}'$ and $\mathscr{C}$ be each either $\mathscr{K}$, $\mathscr{L}$ or $\mathscr{C}at$.
(i) Two functors F, G: $\mathscr{C}' \to \mathscr{C}$ are called *WHE*, denoted $F \simeq G$, if there exists a zig-zag of functors F_i: $\mathscr{C} \to \mathscr{C}'$, $0 \leqq i \leqq n$ and natural WHE's

$$F = F_0 \xrightarrow{\cdot} F_1 \xleftarrow{\cdot} F_2 \ldots \xrightarrow{\cdot} F_{n-1} \xleftarrow{\cdot} F_n = G$$

connecting F and G.
(ii) Two functors H: $\mathscr{C}' \to \mathscr{C}$ and G: $\mathscr{C} \to \mathscr{C}'$ are *homotopy inverses for each other* or H is *a homotopy inverse for* G, if:

(a) at least one functor, both preserves and reflects WHE's;
(b) $GH \simeq \mathrm{Id}_{\mathscr{C}'}$ and $HG \simeq \mathrm{Id}_{\mathscr{C}}$.

2.16. Remark. It is easy to see that both functors in a pair of homotopy inverses must preserve and relfect WHE's. Thus $\simeq$ is an equivalence relation on the class of functors.

For later use, we fix the notation for some special functors $\theta\colon \Delta \to \mathscr{C}at$ and list some relations among them.

2.17. Notations.
(i) $\xi \equiv c\Delta': \mathbf{\Delta} \to \mathscr{C}at$
where $\Delta': \mathbf{\Delta} \to \mathscr{K}$ denotes the barycentric subdivision in the sense of Kan [13]. Note that we interchange the meanings of ξ and ξ^{op} in [19; 5.1].
For any $k \geqq 0$

$$\xi[k] = \mathbf{M} \downarrow [k]$$

in the language of comma categories [22; II.6]. Adopting Kan's notation [13; 7]

$$Sd \equiv \Gamma_{\Delta'}: \mathscr{K} \to \mathscr{K},$$

we introduce more generally, for any $n \geqq 1$

$$\xi^n \equiv cSd^{n-1}\Delta' = cSd^n\Delta: \mathbf{\Delta} \to \mathscr{C}at.$$

(ii) $$\gamma \equiv \mathbf{\Delta} \downarrow \_: \mathbf{\Delta} \to \mathscr{C}at;$$
that is

$$\gamma[k] \equiv \mathbf{\Delta} \downarrow [k]$$

for all $k \geqq 0$. Γ_γ: $\mathscr{K} \to \mathscr{C}at$ is the "category of simplices functor" which is a homotopy inverse to nerve ([11], [17]).

(iii) $$\omega \equiv U \downarrow \_: \mathbf{\Delta} \to \mathscr{C}at$$

extends to

$$\Lambda \equiv \Gamma_\omega: \mathscr{K} \to \mathscr{C}at$$

which is also a homotopy inverse to nerve [20].

2.18. Relations. (i) For all $n \geqq 1$,

$$\Gamma_{\xi^n} = cSd^n: \mathscr{K} \to \mathscr{C}at.$$

(ii) $$N\xi^n = Sd^{n-1}\Delta': \mathbf{\Delta} \to \mathscr{K},$$

and consequently, for all $n \geqq 1$,

$$\Gamma_{N\xi^n} = Sd^n: \mathscr{K} \to \mathscr{K}.$$

(iii) $$N\Lambda = Sd \cdot J \cdot P: \mathscr{K} \to \mathscr{K}$$

[17; correction].

(iv) $$\Lambda = \Gamma_{\xi U} P: \mathscr{K} \to \mathscr{C}at$$

(v) $$\Gamma_{\xi U} = cSdJ: \mathscr{L} \to \mathscr{C}at. \quad \square$$

Section 3. Homotopy Inverses for $\tilde{\mathbf{N}} \equiv \mathbf{PN}: \mathscr{C}at \to \mathscr{L}$

The category $\mathscr{L}$ lies "between" the category of ordered simplicial complexes and the category $\mathscr{K}$ of simplicial sets. From the geometric point of view, $\mathscr{K}$ is the more interesting category, but $\mathscr{L}$ has some better formal properties. Thus some classical results for $\mathscr{K}$ are proved via $\mathscr{L}$ (see e.g. [5; 3.8]). Similarly, in our context, to find homotopy inverses for nerve, it is useful to first study nerve *without* degeneracies, i.e., the composed functor

$$\tilde{N} \equiv PN \colon \mathscr{C}at \to \mathscr{L}.$$

Although $\tilde{N}$ is a right adjoint, its left adjoint

$$cJ \colon \mathscr{L} \to \mathscr{C}at$$

fails to be a homotopy inverse for nerve. For example

$$\tilde{N}cJP(\Delta[k]/\dot{\Delta}[k]) \cong PJP\Delta[0]$$

for $k \geqq 2$; but Proposition 2.12 guarantees that $P(\Delta[k]/\dot{\Delta}[k])$ has the homotopy type of a sphere, while $PJP\Delta[0]$ is WC.

In order that the possible homotopy inverse category ΓX reflect the original simplicial structure of X, we restrict our search for homotopy inverses for $\tilde{N}$ to cocontinuous functors $\Gamma \colon \mathscr{L} \to \mathscr{C}at$, which automatically preserve attachings. From Proposition 2.3 (i), such a functor $\Gamma \colon \mathscr{L} \to \mathscr{C}at$ is completely determined by its restriction to $\mathbf{M}$ (via the Yoneda embedding M)

$$\theta \equiv \Gamma\mathsf{M} \colon \mathbf{M} \to \mathscr{C}at.$$

Obviously, whenever Γ is a homotopy inverse for $\tilde{N}$, it is necessary that $\theta \equiv \Gamma\,\mathsf{M} \colon \mathbf{M} \to \mathscr{C}at$ be WC; i.e., for each $k \geqq 0$, $\theta[k]$ is WC. In particular, $\theta[k]$ would be WC whenever *all* the canonical simplicial maps arising in the string

$$N\theta[k] \leftarrow J\tilde{N}\theta[k] \equiv J\tilde{N}\Gamma_\theta\mathsf{M}[k] \leftarrow \ldots \to J\mathsf{M}[k] = \Delta[k] \to \Delta[0]$$

are WHE's.

To go further, we recall some familiar terminology form geometric simplices:

3.1. Definitions. Let $\mathscr{C}$ be either $\mathscr{K}$, $\mathscr{L}$, $\mathscr{C}at$, or $\mathscr{E}ns$ and let $\theta \colon \mathbf{M} \to \mathscr{C}$ be a functor.

(i) A simplex (object, morphism, element) $a \in \theta[k]$ is called *interior* if it is *not* in the image of any $\theta\mu$ with $\mu \in \mathbf{M} \setminus |\mathbf{M}|$ (i.e. μ a nonidentity injection in $\mathbf{M}$). The collection of interiors is denoted by $\mathring{\theta}[k]$.

(ii) θ *has normal forms* if the (always existing) representation of $a \in \theta[k]$ as

$$a = \theta\mu(b)$$

with $\mu \in \mathbf{M}$ and $b \in \mathring{\theta}[\dim \mu]$, is unique.

3.2. Remark. The condition for a functor $\theta \colon \mathbf{M} \to \mathscr{C}$ to have normal forms can be rephrased more categorically: First, augment the small category $\mathbf{M}$ to a small category $\mathbf{M}_\phi$ by adjoining an initial object ϕ; then $\mathbf{M}_\phi$ has pullbacks.

Next, extend θ to θ_ϕ

$$\begin{tikzcd}
\mathbf{M} \arrow[d, hook] \arrow[dr, "\phi"] & \\
\mathbf{M}_\phi \arrow[r, dashed, "\theta_\phi"'] & \mathscr{C}
\end{tikzcd}$$

by mapping the initial object of $\mathbf{M}_\phi$ onto the initial object of $\mathscr{C}$. Then θ has normal forms iff θ_ϕ preserves pullbacks. Thus a composition $Q\theta$ of θ with a functor Q preserving finite limits and having suitable codomain, has normal forms whenever θ has normal forms.

The essential feature of normal forms is given in:

3.3. Lemma. *Let $\mathscr{C}$ be either $\mathscr{K}$, $\mathscr{L}$, or $\mathscr{E}ns$ and let θ: $\mathbf{M}\to\mathscr{C}$ have normal forms. Then Γ_θ: $\mathscr{L}\to\mathscr{C}$* preserves injections. □

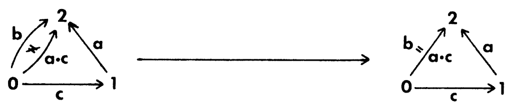
3.4. Remark. The statement of this lemma is not true for $\mathscr{C}=\mathscr{C}at$. To see this, note that the inclusion

$$\iota U\colon \mathbf{M}\hookrightarrow\mathscr{C}at$$

(notation introduced in 2.5) has normal forms and consider the injection

$$\dot{\mathsf{M}}[2]\hookrightarrow\mathsf{M}[2],$$

where $\dot{\mathsf{M}}[2]$ denotes the boundary of $\mathsf{M}[2]$ which is defined in the obvious way. The image of $\dot{\mathsf{M}}[2]\hookrightarrow\mathsf{M}[2]$ in $\mathscr{C}at$



$\Gamma_{\xi U}(\dot{\mathsf{M}}[2])$ $\Gamma_{\xi U}(M[2])$

is clearly not an inclusion. But there is an extra condition insuring that Γ_θ: $\mathscr{L}\to\mathscr{C}at$ preserves injections (see Corollary 3.11).

Proof of Lemma 3.3. Let X be a nondegenerate simplicial set. From Remarks 2.4, every element of $\Gamma_\theta X$ can be represented by a pair (x,a) with $x\in X$ and $a\in\theta[\dim x]$. If θ has normal forms, then this representation is unique, whenever a is required to be interior. Suppose f: $X\hookrightarrow Y$ is injective in $\mathscr{K}$, then

$$\Gamma_\theta f(x,a)=\Gamma_\theta f(\tilde{x},\tilde{a})$$

implies

$$fx=f\tilde{x}\quad\text{and}\quad a=\tilde{a}$$

if a, $\tilde{a}\in\mathring{\theta}[\dim x]$; and the injectivity of f implies

$$x=\tilde{x}.$$

Thus

$$(x,a)=(\tilde{x},\tilde{a})$$

insuring the injectivity of $\Gamma_\theta f$. □

We now state the main result of this section, which in the generality given, is due to the referee.

3.5. Theorem. *Let θ: $\mathbf{M}\to\mathscr{C}at$ be a functor which is WC and has normal forms. Then Γ_θ: $\mathscr{L}\to\mathscr{C}at$ is a homotopy inverse for $\tilde{N}$: $\mathscr{C}at\to\mathscr{L}$ whenever the canonical* *natural transformation*

$$\tilde{\rho}\colon \Gamma_{\tilde{N}\theta} \stackrel{\bullet}{\to} \tilde{N}\Gamma_\theta\colon \mathscr{L}\to\mathscr{L}$$

is a natural WHE. □

The key to this theorem lies in the following:

3.6. Lemma. *Let $\theta\colon \mathbf{M}\to\mathscr{K}$ be a functor which is WC and has normal forms. Then*

$$\Gamma_\theta \simeq J\colon \mathscr{L}\to\mathscr{K}$$

(*in the sense of Definition* 2.15).

Proof. Let $Ex^\infty\colon \mathscr{K}\to\mathscr{K}$ be Kan's extension functor [13; 4] turning every simplicial set into a WHE Kan set under the natural WHE

$$e^\infty\colon \mathrm{Id}_{\mathscr{K}} \stackrel{\bullet}{\to} \mathrm{Ex}^\infty\colon \mathscr{K}\to\mathscr{K}.$$

Then any simplicial map $\dot{\Delta}[k]\to \mathrm{Ex}^\infty\theta[k]$ has an extension

$$\begin{tikzcd} \dot{\Delta}[k] \arrow[r] \arrow[d] & \mathrm{Ex}^\infty\theta[k] \\ \Delta[k] \arrow[ur, dashed] & \end{tikzcd}$$

Using these extensions, one constructs easily (by induction on k) a natural WHE

$$\Delta U \stackrel{\bullet}{\to} (\mathrm{Ex}^\infty\theta \equiv \theta')\colon \mathbf{M}\to\mathscr{K}.$$

On the other hand, the restriction of e^∞ to the image of θ yields the natural WHE

$$\theta \stackrel{\bullet}{\to} \theta'\colon \mathbf{M}\to\mathscr{K}.$$

Cocontinuous extension produces the zig-zag

$$\Gamma_\theta \stackrel{\bullet}{\to} \Gamma_{\theta'} \stackrel{\bullet}{\leftarrow} \Gamma_{\Delta U} \equiv J$$

of natural transformations.

Now note that $\mathrm{Ex}^\infty\colon \mathscr{K}\to\mathscr{K}$ is a colimit of continuous functors but is not continuous itself. Nevertheless, it preserves finite limits [10; IV.3.2]; thus, in view of Remark 3.2, θ' also has normal forms. From Lemma 3.3, both Γ_θ and $\Gamma_{\theta'}$ to preserve injections. Therefore, the Comparison Theorem, Theorem 2.11, applies and gives the desired result. □

From this, we get the:

Proof of Theorem 3.5, Part 1. We show

$$\tilde{N}\Gamma_\theta \simeq \mathrm{Id}_{\mathscr{L}}.$$

By assumption,

$$\tilde{N}\Gamma_\theta \simeq \Gamma_{\tilde{N}\theta}.$$

Next, note that since θ: $\mathbf{M}\to\mathscr{C}at$ has normal forms, the same holds for $N\theta$: $\mathbf{M}\to\mathscr{K}$ (see Remark 3.2); hence the preceding lemma implies

$$\Gamma_{N\theta}\simeq J\colon \mathscr{L}\to\mathscr{K},$$

and thus

$$\Gamma_{\tilde{N}\theta}=P\Gamma_{N\theta}\underset{(2.13)}{\simeq}PJ\underset{(2.12)}{\simeq}\mathrm{Id}_{\mathscr{L}}.\quad\square$$

3.7. Remark. A proof of the second part of Theorem 3.4 would consist of showing

$$\Gamma_\theta\tilde{N}\simeq\mathrm{Id}_{\mathscr{C}at}.$$

This is very easy, provided that there exists at least one homotopy inverse $\tilde{\Gamma}$: $\mathscr{L}\to\mathscr{C}at$ for $\tilde{N}$: $\mathscr{C}at\to\mathscr{L}$.

Then one simply forms

$$\Gamma_\theta\tilde{N}\simeq\tilde{\Gamma}\tilde{N}\Gamma_\theta\tilde{N}\simeq\tilde{\Gamma}\tilde{N}\simeq\mathrm{Id}_{\mathscr{C}at},$$

using $\tilde{N}\Gamma_\theta\simeq\mathrm{Id}_{\mathscr{L}}$ and the fact that $\tilde{\Gamma}$ preserves WHE's (see Remark 2.16). Any homotopy inverse Γ: $\mathscr{K}\to\mathscr{C}at$ for N: $\mathscr{C}at\to\mathscr{K}$ (see Notation 2.17) gives rise to a homotopy inverse $\tilde{\Gamma}$ for $\tilde{N}$; namely

$$\tilde{\Gamma}\equiv\Gamma J\colon\mathscr{L}\to\mathscr{C}at;$$

and such Γ's can be found in the literature!

But our intention is to give a self-contained theory for homotopy inverses for N, $\tilde{N}$ respectively. Thus we will also prove explicitly the second part of Theorem 3.4 for a special functor θ at the end of this section (see Lemma 3.14).

Before doing so, we describe a condition on functors θ: $\mathbf{M}\to\mathscr{C}at$ which assures that $\tilde{\rho}$: $\Gamma_{\tilde{N}\theta}\dot{\to}\tilde{N}\Gamma_\theta$ is a WHE, the required hypotheses in Theorem 3.5. This condition is fulfilled in all known examples.

3.8. Definition. A functor θ: $\mathbf{M}\to\mathscr{C}at$ is called *divided*, if

(i) it has normal forms, and

(ii) the functors $\theta\mu$ are inclusions of right (left) ideals for all face operators $\mu\in\mathbf{M}$. (Recall that a subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is a *right ideal* in $\mathbf{B}$, iff $a\in\mathbf{A}$, $b\in\mathbf{B}$ implies $ab\in\mathbf{A}$, whenver the composition is defined.)

3.9. Note that in the presence of (ii), condition (i) could be weakened to requiring normal forms for the composed functor

$$|\theta|\colon\mathbf{M}\to\mathscr{E}ns.$$

3.10. Lemma. *If θ: $\mathbf{M}\to\mathscr{C}at$ is divided, then for any nondegenerate simplicial set $X\in\mathscr{L}$, each morphism of $\Gamma_\theta X$ has a unique representation by pairs (x,a) with $x\in X$ and $a\in\mathring{\theta}[\dim x]$ (see Remarks 2.4).*

Proof. We assume $\theta\mu$ to be an inclusion as a right ideal for each $\mu\in\mathbf{M}$. Then we have, for each interior morphism $a\in\theta[k]$,

(i) $$\operatorname{cod} a \in \mathring{\theta}[k]$$

(ii) $\operatorname{dom} a$ has a unique representation

$$\operatorname{dom} a = \theta\mu(p)$$

with $p \in \mathring{\theta}[\dim \mu]$

(iii) $a \circ \theta\mu(b) \in \theta[k]$, for all $\mu \in \mathbf{M}$ and $b \in \theta[\dim \mu]$ for which the composition is defined in $\theta[k]$.

Now consider the set

$$\underline{X} = \{(x, a) \mid x \in X,\ a \text{ an interior morphism in } \theta[\dim x]\}$$

and provide it with a category structure by defining:

$$\operatorname{cod}(x, a) \equiv (x, \operatorname{cod} a)$$
$$\operatorname{dom}(x, a) \equiv (x\mu, p)$$

where $\operatorname{dom} a = \theta\mu(p)$ as defined in (i), and composition by

$$(x, a) \circ (x\mu, b) \equiv (x, a \circ \theta\mu(b)).$$

The resulting category is easily seen to be isomorphic to $\Gamma_\theta X$. □

From this, it follows directly that

3.11. Corollary. *If θ: $\mathbf{M} \to \mathscr{C}at$ is divided, then Γ_θ: $\mathscr{L} \to \mathscr{C}at$ transforms injections into inclusions of right (left) ideals and $\tilde{\rho}$: $\Gamma_{\tilde{N}\theta} \xrightarrow{\bullet} \tilde{N}\Gamma_\theta$ is a natural equivalence (not just a natural WHE). If, in addition, θ is WC, then Γ_θ is a homotopy inverse for $\tilde{N}$.* □

In order to justify the terminology "divided", we show that there is a strong connection between divided functors and geometric subdivision processes. Recall the functor ξ: $\mathbf{\Delta} \to \mathscr{C}at$, derived from the barycentric subdivision of standard simplices (see Notation 2.17(i)). Its restriction (see Lemma 2.7)

$$\xi/\mathbf{M} = \xi U\colon\ \mathbf{M} \to \mathscr{C}at$$

is not only divided, but it also has a universal property among all divided functors θ: $\mathbf{M} \to \mathscr{C}at$.

3.12. Proposition. *A functor θ: $\mathbf{M} \to \mathscr{C}at$ is divided iff it has normal forms and there exists a (necessarily unique) natural transformation*

$$\phi\colon\ \theta \xrightarrow{\bullet} \xi U\colon\ \mathbf{M} \to \mathscr{C}at$$

(or ϕ: $\theta \to \xi^{\mathrm{op}} U$) which preserves interior objects, i.e. $\phi_k(p) \in |\mathring{\widehat{\xi U}}[k]|$ *for all* $p \in |\mathring{\theta}[k]|$. □

Besides giving authenticity to the term "divided", the functor ξU also has another useful property. The natural transformation ("last evaluation")

(3.13) $$\delta\colon\ \Delta' \xrightarrow{\bullet} \Delta\colon\ \mathbf{\Delta} \to \mathscr{K}$$

considered by Kan [13; 2.2] induces a natural transformation

$$\phi\colon \xi U \stackrel{\cdot}{\rightarrow} \iota U\colon \mathbf{M} \rightarrow \mathscr{C}\!at$$

which extends to a natural transformation

$$\Gamma_\phi\colon \Gamma_{\xi U} \stackrel{\cdot}{\rightarrow} (\Gamma_{\iota U} = cJ)\colon \mathscr{L} \rightarrow \mathscr{C}\!at.$$

This allows us finally to prove:

3.14. Lemma.

$$\Gamma_{\xi U}\tilde{N} \simeq \mathrm{Id}_{\mathscr{C}\!at}.$$

Proof. We connect $\Gamma_{\xi U}N$ and $\mathrm{Id}_{\mathscr{C}\!at}$ by the following composition of natural transformations:

$$\Gamma_{\xi U}\tilde{N} \xrightarrow[\Gamma_\phi\tilde{N}]{\cdot} (cJ = cJPN) \xrightarrow[c\tilde{\varepsilon}N]{\cdot} cN \xrightarrow[\varepsilon]{\cdot} \mathrm{Id}_{\mathscr{C}\!at}.$$

Next, note that a natural transformation Φ between endofunctors in $\mathscr{C}\!at$ is a natural WHE in $\mathscr{C}\!at$ (see Definition 2.9(iii)) iff $N\Phi$ is a natural WHE in $\mathscr{K}$; and that Corollary 2.13 insures this is the case, iff $PN\Phi = \tilde{N}\Phi$ is a natural WHE in $\mathscr{L}$. Thus it is sufficient to show that

$$\tilde{N}(\varepsilon \circ c\tilde{\varepsilon}N \circ \Gamma_\phi\tilde{N})\colon \Gamma_{\xi U} \circ \tilde{N} \stackrel{\cdot}{\rightarrow} \tilde{N}\colon \mathscr{C}\!at \rightarrow \mathscr{L}$$

is a natural WHE in $\mathscr{L}$.

To this end, consider the diagram

$$\begin{tikzcd}
\tilde{N}\Gamma_{\xi U}\tilde{N} \arrow[r, "\cdot", "\tilde{N}\Gamma_\phi\tilde{N}"'] & \tilde{N}\Gamma_{\iota U}\tilde{N} \arrow[r, phantom, "="] & PNcJPN \arrow[r, "\cdot", "PNC\tilde{\varepsilon}N"'] & PNcN \arrow[r, "\cdot"] & PN \equiv \tilde{N} \\
\Gamma_{\tilde{N}\xi U}\tilde{N} \arrow[u, "\tilde{\rho}\tilde{N}", "\cdot"'] \arrow[r, "\cdot", "\Gamma_{\tilde{N}\varphi}\tilde{N}"'] & \Gamma_{\tilde{N}\iota U}\tilde{N} \arrow[u, "\tilde{\rho}\tilde{N}", "\cdot"'] \arrow[r, phantom, "="] \arrow[ur, phantom, "="] & PJPN \arrow[u, "P\eta JPN", "\cdot"'] \arrow[r, "\cdot", "P\tilde{\varepsilon}N"'] & PN \equiv \tilde{N} \arrow[u, "P\eta N", "\cdot"'] \arrow[ur, "\equiv"'] &
\end{tikzcd}$$

The universal property of Kan extensions [22; X.3] insures that the two lefthand squares commute. The third square commutes by the naturality of the unit

$$\eta\colon \mathrm{Id}_{\mathscr{K}} \stackrel{\cdot}{\rightarrow} Nc\colon \mathscr{K} \rightarrow \mathscr{K}.$$

Lastly, the triangle on the right is (up to P) just one of the adjunction equalities [22; IV.1]. Corollary 3.11 insures the left vertical arrow is an equivalence.

Both $J = \Gamma_{N\iota U}$ and $SdJ = \Gamma_{N\xi U}$ preserve injections; thus $\Gamma_{\tilde{N}\xi U} = P\Gamma_{N\xi U}$ and $SdJ = \Gamma_{N\xi U}$ also do. Moreover,

$$(P\delta = \Gamma_{\tilde{N}\phi}\mathsf{M})\colon (\Gamma_{\tilde{N}\xi U}\mathsf{M} = P\Delta') \stackrel{\cdot}{\rightarrow} (P\Delta = \Gamma_{\tilde{N}\iota U}\mathsf{M})\colon \mathbf{M} \rightarrow \mathscr{L}$$

is a natural WHE in $\mathscr{L}$; hence, the Comparison Theorem, Theorem 2.11, applies and $\Gamma_{\tilde{N}\phi}\tilde{N}$ is a natural WHE. Proposition 2.12 and Corollary 2.13 guarantee that $P\tilde{\varepsilon}N$ is also a natural WHE. Hence, the proof follows (again) from Corollary 3.11 and from Proposition 3.12. □

3.15. Remark. Thus we have shown that

$$cSdJ = \Gamma_{\xi U}\colon \mathscr{L} \to \mathscr{C}at$$

is a homotopy inverse for $\tilde{N}$. Moreover, for all $n \geqq 1$,

$$cSd^n J/\mathbf{M}\colon \mathbf{M} \to \mathscr{C}at$$

is divided. Hence, all the functors

$$cSd^n J\colon \mathscr{L} \to \mathscr{C}at$$

are also homotopy inverses for $\tilde{N}$.

Section 4. Homotopy Inverses for $N\colon \mathscr{K} \to \mathscr{C}at$

In this section, we study homotopy inverses for nerve, up to crucial conditions which will be analyzed in more detail in Sect. 6. Note first that any homotopy inverse to $\tilde{N}$ gives rise to a homotopy inverse for N.

4.1. Proposition. *Let $\tilde{\Gamma}\colon \mathscr{L} \to \mathscr{C}at$ be any homotopy inverse for $\tilde{N}\colon \mathscr{C}at \to \mathscr{L}$. Then $\tilde{\Gamma}P\colon \mathscr{K} \to \mathscr{C}at$ is a homotopy inverse for nerve, $N\colon \mathscr{C}at \to \mathscr{K}$.*

Proof. (i) $\tilde{\Gamma}PN = \tilde{\Gamma}\tilde{N} \simeq \mathrm{Id}_{\mathscr{C}at}$

(ii) $N\tilde{\Gamma}P \underset{(2.12)}{\simeq} JPN\tilde{\Gamma}P = J\tilde{N}\tilde{\Gamma}P \underset{(2.13)}{\simeq} JP \underset{(2.12)}{\simeq} \mathrm{Id}_{\mathscr{K}}$. □

4.2. Remarks. (i) The homotopy inverses for nerve generated by the preceding proposition are quite unsatisfactory from the geometric point of view, since the functor P associates with every nonempty simplicial set, even the point $\Delta[0]$, an object of infinite geometric dimension. This motivates the search for other homotopy inverses for nerve which "preserve" geometric dimension.
(ii) If $\tilde{\Gamma}\colon \mathscr{L} \to \mathscr{C}at$ is a homotopy inverse for nerve of the form $\tilde{\Gamma} = \Gamma J\colon \mathscr{L} \to \mathscr{C}at$ for some functor $\Gamma\colon \mathscr{K} \to \mathscr{C}at$, it does not necessarily follow that Γ is a homotopy inverse for nerve. In particular, the functor

$$\tilde{\Gamma} = \Gamma_{\xi U} \underset{(2.18(\mathrm{v}))}{=} cSdJ\colon \mathscr{L} \to \mathscr{C}at$$

was shown to be a homotopy inverse for $\tilde{N}$ (see Remark 3.15); however,

$$\Gamma_\xi = cSd\colon \mathscr{K} \to \mathscr{C}at$$

is *not* a homotopy inverse for N (see Eq. (1.2)).
(iii) If $\tilde{\Gamma}\colon \mathscr{L} \to \mathscr{C}at$ is cocontinuous, Corollary 2.8 insures that $\Gamma = \tilde{\Gamma}P$ is also cocontinuous. By Proposition 2.3(i), Γ may be written in the form $\Gamma = \Gamma_\theta$ with

$$\theta = \tilde{\Gamma}P\Delta\colon \mathbf{\Delta} \to \mathscr{C}at.$$

In particular, for $\tilde{\Gamma} = \Gamma_{\xi U}$, we get from 2.17(iii) and 2.18(iv) that

$$\theta = \Gamma_{\xi U}P\Delta = \omega\colon \mathbf{\Delta} \to \mathscr{C}at.$$

Thus, from the notation introduced in 2.17(iii),

$$\Gamma = \Lambda\colon \mathscr{K} \to \mathscr{C}at$$

is the homotopy inverse for nerve defined by Lee [20].

From another point of view, we start from a given functor

$$\theta\colon \mathbf{\Delta} \to \mathscr{C}at$$

and determine conditions under which

$$\Gamma_\theta\colon \mathscr{K} \to \mathscr{C}at$$

is a homotopy inverse for nerve.

4.3. Definition. The functor $\theta\colon \mathbf{\Delta} \to \mathscr{C}at$ *has normal forms* (*is divided*), if its restriction $\theta/\mathbf{M} = \theta U\colon \mathbf{M} \to \mathscr{C}at$ has normal forms (is divided). □

That θ has normal forms, is a rather mild restriction, because of the following result.

4.4. Proposition: *For a functor $\theta\colon \mathbf{\Delta} \to \mathscr{C}at$ to have normal forms it is sufficient that*

$$(\theta\delta^0)a = (\theta\delta^1)a$$

for every morphism $a \in \theta[0]$. (Here δ^0 and δ^1 denote the two elements of $\mathbf{M}([0],[1])$.) □

The straightforward proof is left for the reader. The statement is essentially due to D. Puppe (unpublished) and may be also found in the paper [27] of C. and R. Ruiz Salguero, which deals, in great detail, with the question of normal forms for functors $\theta\colon \mathbf{\Delta} \to \mathscr{E}ns$.

There is an analogue of Theorem 3.5 in this case:

4.5. Theorem. *Let $\theta\colon \mathbf{\Delta} \to \mathscr{C}at$ be a functor which is WC and has normal forms. Then $\Gamma_\theta\colon \mathscr{K} \to \mathscr{C}at$ is a homotopy inverse for nerve whenever the canonical natural transformation*

$$\rho\colon \Gamma_{N\theta} \dot{\to} N\Gamma_\theta\colon \mathscr{K} \to \mathscr{K}$$

is a natural WHE. □

4.6. Remarks. (i) Although this appears to be literally the same as Theorem 3.5, it is quite different in content. The condition that ρ be a natural *WHE* is much stronger than the condition on $\tilde{\rho}$ in Theorem 3.5. If the functor $\theta U\colon \mathbf{M} \to \mathscr{C}at$ is divided, then $\tilde{\rho}$ is always, not only a natural WHE, but even an honest natural equivalence (see Corollary 3.11); nevertheless, in this case ρ may fail to a natural WHE. In particular, from Remark 4.2(ii), it follows that, for $\xi = cSd\Delta\colon \mathbf{\Delta} \to \mathscr{C}at$ and $k \geqq 2$,

$$\rho(\Delta[k]/\dot{\Delta}[k])\colon (\Gamma_{N\xi}(\Delta[k]/\dot{\Delta}[k]) \underset{(2.18(\mathrm{ii}))}{=} Sd(\Delta[k]/\dot{\Delta}[k]))$$
$$\to (N\Gamma_\xi(\Delta[k]/\dot{\Delta}[k]) = NcSd(\Delta[k]/\dot{\Delta}[k]) \underset{(1.2)}{=} \Delta[1]);$$

but $\Delta[1]$ is WC and $Sd(\Delta[k]/\dot{\Delta}[k])$ is WHE to the k-dimensional simplicial sphere [13; 7.5].
(ii) If one strengthens the assumption WC to the existence of a natural transformation

$$\phi\colon \theta \dot{\to} \iota\colon \mathbf{\Delta} \to \mathscr{C}at$$

such that $\phi[k]\colon \Delta[k] \to \iota[k]$ is a strong homotopy equivalence, in the sense of [19; 3], then by the Main Theorem of [19], the right adjoint $S_\theta\colon \mathscr{C}at \to \mathscr{K}$ of $\Gamma_\theta\colon \mathscr{K} \to \mathscr{C}at$ is connected to nerve by the natural WHE

$$S\phi\colon (N \equiv S_\iota) \dot{\to} S_\theta\colon \mathscr{C}at \to \mathscr{K}.$$

Thus S_θ becomes a homotopy inverse for Γ_θ. Moreover, the proof of Corollary 4.7 in [19] applies to such a θ, instead of to just γ appearing there, and guarantees that both the unit and counit of the adjunction $\Gamma_\theta \dashv S_\theta$ are natural WHE's. Thus we have *an adjoint homotopy equivalence* between the categories $\mathscr{C}at$ and $\mathscr{K}$. Note that such a natural transformation ϕ exists for $\theta \in \{\gamma, \omega, \xi^n, n \geqq 1\}$ (see [19: 5.13, 5.16, and 5.4].)

The formal analogy between Theorem 3.5 and Theorem 4.5 extends to their proofs. The "key" to Theorem 4.5 is a suitable reformulation of Lemma 3.6:

4.7. Lemma. *Let $\theta\colon \mathbf{\Delta} \to \mathscr{K}$ be a functor which is WC and has normal forms. Then $\Gamma_\theta \simeq \mathrm{Id}_{\mathscr{K}}$. In particular, Γ_θ preserves and reflects WHE's.*

Proof. From Lemma 3.6,

$$\Gamma_\theta J = \Gamma_{\theta U} \simeq J.$$

The functor $\Gamma_{\theta U}$ preserves injections, since θU has normal forms (see Lemma 3.3); and also the composition $\Gamma_\theta JP$ does, since P is a singular functor (right adjoint). Moreover, for each $k \geqq 0$

$$\Gamma_\theta JP\Delta[k] \simeq JP\Delta[k] \underset{(2.12)}{\simeq} \Delta[k];$$

thus both

$$\Gamma_\theta JP\Delta\colon \mathbf{\Delta} \to \mathscr{K} \quad \text{and} \quad \Gamma_\theta \Delta = \theta\colon \mathbf{\Delta} \to \mathscr{K}$$

are WC. Therefore

$$\Gamma_\theta \tilde{\varepsilon} \Delta\colon \Gamma_\theta JP\Delta \dot{\to} \Gamma_\theta \Delta\colon \mathbf{\Delta} \to \mathscr{K}$$

is forced to be a natural WHE. Hence, the hypotheses of the Comparison Theorem, Theorem 2.11, are satisfied for the natural transformation

$$\Gamma_\theta \tilde{\varepsilon}\colon \Gamma_\theta JP \dot{\to} \Gamma_\theta\colon \mathscr{K} \to \mathscr{K};$$

and it is a natural WHE. Thus

$$\Gamma_\theta \simeq \Gamma_\theta JP \underset{(3.6)}{\simeq} JP \underset{(2.12)}{\simeq} \mathrm{Id}_{\mathscr{K}}. \quad \square$$

Proof of Theorem 4.5. The preceding lemma implies $\Gamma_{N\theta} \simeq \mathrm{Id}_{\mathscr{K}}$. Thus the hypothesis on ρ gives

$$N\Gamma_\theta \simeq \Gamma_{N\theta} \simeq \mathrm{Id}_{\mathscr{K}}.$$

There are homotopy inverses for nerve (see, e.g., Proposition 4.1). Thus, in the same way as in Remark 3.7., we get

$$\Gamma_\theta N \simeq \mathrm{Id}_{\mathscr{C}at}. \quad \square$$

In contrast to the situation in Section 3, the assumption "divided" does not imply the hypothesis on ρ in Theorem 4.5; its role is clarified in the following statement which is due to the referee.

4.8. Theorem. *For a divided functor $\theta:\Delta\to\mathscr{C}at$ the following conditions are equivalent:*

(i) *Γ_θ is a homotopy inverse for nerve;*

(ii) *θ is WC and Γ_θ preserves WHE's;*

(iii) *θ is WC and*

$$\rho: \Gamma_{N\theta} \xrightarrow{\cdot} N\Gamma_\theta : \mathscr{K}\to\mathscr{K}$$

is a natural WHE.

Proof. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i) was actually proved as part of the proof of Theorem 4.5, because to show (i), we needed only (a) that Γ_θ preserves WHE's and (b) that $\tilde{N}$ and $\Gamma_{\theta U}$ are homotopy inverses of each other which is guaranteed by Corollary 3.11.

(iii)$\Rightarrow$(ii) was also shown in the proof of Theorem 4.5

(ii)$\Rightarrow$(iii): Consider the commutative square

$$\begin{tikzcd}
(\Gamma_{\tilde{N}\theta U}P = \Gamma_{\tilde{N}\theta}JP) \arrow[r, "\cdot", "\Gamma_{N\theta\tilde{\varepsilon}}"'] \arrow[d, "\tilde{\rho}P"', "\cdot"] & (\Gamma_{\tilde{N}\theta} = P\Gamma_{N\theta}) \arrow[d, "P\rho"'] \\
(\tilde{N}\Gamma_{\theta U}P = \tilde{N}\Gamma_\theta JP) \arrow[r, "\cdot", "\tilde{N}\Gamma_{\theta\tilde{\varepsilon}}"'] & (\tilde{N}\Gamma_\theta = PN\Gamma_\theta).
\end{tikzcd}$$

The lefthand vertical arrow is a natural equivalence by Corollary 3.11. Lemma 4.7 guarantees that $\Gamma_{\tilde{N}\theta} = P\Gamma_{N\theta}$ preserves WHE's; thus the top arrow is a natural WHE. The functor Γ_θ preserves WHE's by hypothesis; so does $\tilde{N}\Gamma_\theta$. Therefore, the bottom arrow is also a natural WHE. Hence, $P\rho$ is a natural WHE; and since P reflects WHE's, the assertion follows. $\square$

There is another interesting variant of Theorem 4.5 and Theorem 4.8. In order to state it, we first recall:

4.9. Definition. A simplicial set X is said to be *regulated* (see [21; III.8]), if for any nondegenerate simplex x and any pair of face operators $\mu \neq \bar{\mu} \in \Delta[\dim x]$ with $0 \in \mathrm{Im}\,\mu$,

$$x\mu \neq x\bar{\mu}. \quad \square$$

4.10. Remark. If X is any simplicial set, then $Sd^{\mathrm{op}}X$ is regulated. Here,

$$Sd^{\mathrm{op}}: \mathscr{K}\to\mathscr{K}$$

denotes the cocontinuous functor which is obtained from Sd by reversing the ordering of barycenters.

4.11. Theorem. *Let $\theta\colon \Delta\to\mathscr{C}at$ be a functor which is WC and divided. Assume the canonical natural transformation*

$$\rho\colon \Gamma_{N\theta}\dot{\to} N\Gamma_\theta\colon \mathscr{K}\to\mathscr{K}$$

gives a WHE for every regulated simplicial set. Then the composition

$$\Gamma_\theta Sd^{\mathrm{op}}\colon \mathscr{K}\to\mathscr{C}at$$

is a homotopy inverse for nerve.

Proof. Define

$$\theta'\equiv\Gamma_\theta Sd^{\mathrm{op}}\Delta\colon \Delta\to\mathscr{C}at.$$

Then Proposition 2.3 implies $\Gamma_\theta Sd^{\mathrm{op}}=\Gamma_{\theta'}$. Furthermore, note $Sd^{\mathrm{op}}\Delta[k]$ is in the image of $J\colon \mathscr{L}\to\mathscr{K}$ for all $k\geqq 0$ [21; III.7]; thus by Corollary 3.11, $(N\theta'=N\Gamma_\theta Sd^{\mathrm{op}}\Delta=\Gamma_{N\theta}Sd^{\mathrm{op}}\Delta)\colon \Delta\to\mathscr{C}at$. But

$$\rho'=\rho Sd^{\mathrm{op}}\colon (\Gamma_{N\theta'}=\Gamma_{N\theta}Sd^{\mathrm{op}})\dot{\to}(N\Gamma_\theta Sd^{\mathrm{op}}=N\Gamma_{\theta'})\colon \mathscr{K}\to\mathscr{K}$$

is a natural WHE; and the result follows from Theorem 4.5. □

This ends, for the moment, the abstract theory. Let us now consider:

4.12. Examples. (i) The functors

$$\xi^n, \gamma, \omega\colon \Delta\to\mathscr{C}at$$

introduced in Notations 2.17, all are divided.

(ii) We already proved that

$$\Gamma_\omega\equiv\Lambda\colon \mathscr{K}\to\mathscr{C}at$$

is a homotopy inverse for nerve (see Remark 4.2(iii)). Theorem 4.8 implies that the canonical natural transformation

$$\rho\colon \Gamma_{N\omega}\dot{\to}(N\Gamma_\omega=N\Lambda)\colon \mathscr{K}\to\mathscr{K}$$

is a natural WHE. It is not hard to see that it is even a natural equivalence, which will follow from the later development (see Remark 6.7).

(iii) The natural transformation

$$\rho\colon \Gamma_{N\gamma}\dot{\to}N\Gamma_\gamma\colon \mathscr{K}\to\mathscr{K}$$

is a natural equivalence. To see this, we refer, for the moment, to [17; Lemma B], from which this statement is an immediate consequence. In the next two sections, we improve this result (see Remark 6.7). Consequently, $\Gamma_\gamma\colon \mathscr{K}\to\mathscr{C}at$ is a homotopy inverse for nerve, as mentioned before.

(iv) As already mentioned (see Remark 4.2(ii)), the functor

$$\Gamma_\xi = cSd\colon \mathcal{K} \to \mathcal{C}at$$

is *not* a homotopy inverse for nerve; thus, by Theorem 4.8, the corresponding

$$\rho\colon (\Gamma_{N\xi} = Sd) \dot{\to} (NcSd = N\Gamma_\xi)\colon \mathcal{K} \to \mathcal{K}$$

cannot be a natural WHE. But it fulfills the weaker condition in Theorem 4.11. To see this, we prove the following proposition. We assume that X is a regulated simplicial set.

Proposition. *The small category $\Gamma_\xi X$ is a poset.*

Proof. Using the explicit description for categorical realization

$$c\colon \mathcal{K} \to \mathcal{C}at$$

given in [10; II.4.2] and the existence of normal forms for

$$Sd\colon \mathcal{K} \to \mathcal{K}$$

(see Proposition 3.12), it is clear that objects of $cSdX$ can be identified with the nondegenerate simplices of X. By similar reasoning, each morphism from y to x in $cSdX$ is an equivalence class of face operators

$$\mu\colon [\dim y] \rightarrowtail [\mathit{dim}\, x]$$

such that $x\mu = y$. To show that $cSdX$ is a poset, it is sufficient to show that if the equivalence classes $[\mu]$ and $[\nu]$ are both in $cSdX(y, x)$, then $[\mu] = [\nu]$. If such $[\mu]$ and $[\nu]$ are given, then let

$$\mu \cup \nu\colon [m] \rightarrowtail [\dim x]$$

denote the unique face operator in $\mathbf{\Delta}$ satisfying

$$(\operatorname{Im} \mu) \cup (\operatorname{Im} \nu) = \operatorname{Im}(\mu \cup \nu).$$

Note that y is also the unique nondegenerate part of $x(\mu \cup \nu)$. The result follows from the description of $c\colon \mathcal{K} \to \mathcal{C}at$ in [10; II.4.2] and the consideration of the 2-simplices in SdX represented in normal form by

$$(x; (\mu, \mu \cup \nu, 1d_{[\dim x]})),$$

and

$$(x; (\nu, \mu \cup \nu, 1d_{[\dim x]})),$$

where $(\mu, \mu \cup \nu, 1d_{[\dim x]})$ and $(\nu, \mu \cup \nu, 1d_{[\dim x]})$ are in $\Delta'[\dim x]_2$. (For the explicit form of $\Delta'[\dim x]$ used here see [13; I.2].) $\square$

Now, the proof of this proposition implies, in addition,

$$N\Gamma_\xi X = {}^*X,$$

where $*: \mathscr{K} \to \mathscr{K}$ is the old "star" functor originially defined by Barratt [1] (or see also [21; III.9]) in order to prove that geometric realizations of simiplicial sets are triangulable [6]! The natural map

$$\Gamma_\delta X: \Gamma_{N\xi} X \to X$$

(for the definition of δ see (3.13)) is a WHE by the Comparison Theorem, Theorem 2.11, and decomposes into

$$\Gamma_\delta X = \psi \circ \rho X$$

where $|\psi|$ is homotopic to a (non-natural) homeomorphism (see [1] or [21] in connection with the corrections given in [6]) and thus ψ is a WHE. Hence ρX is a WHE.

(v) From (iv), we know

$$\Gamma_\xi Sd^{\mathrm{op}} = cSdSd^{\mathrm{op}} \cong cSd^2 = \Gamma_{\xi^2}: \mathscr{K} \to \mathscr{C}at$$

to be a homotopy inverse for nerve. According to Theorem 4.8, the canonical natural transformation

$$\rho^2: \Gamma_{N\xi^2} \dot{\to} N\Gamma_{\xi^2}: \mathscr{K} \to \mathscr{K}$$

is also a natural WHE. An inductive argument, using Theorem 4.11, insures that all the canonical natural transformations

$$\rho^n: \Gamma_{N\xi^n} \dot{\to} N\Gamma_{\xi^n}: \mathscr{K} \to \mathscr{K}$$

for all $n \geqq 2$, are natural WHE's.

4.13. Essential Remark. The considerations, in Example 4.12(iv), lead to another interesting interpretation. According to [5; II], we have $|X|$ homeomorphic to $|Sd^{\mathrm{op}}X|$, for every simplicial set X.

Thus

$$|X| \approx |Sd^{\mathrm{op}}X| \approx |{*Sd^{\mathrm{op}}X}| = |N\Gamma_\xi Sd^{\mathrm{op}}X| = B(\Gamma_\xi Sd^{\mathrm{op}}X)$$

where $B = |N\_|: \mathscr{C}at \to \mathscr{W}$ denotes the classifying space functor as introduced by G. Segal [28]. We can interpret this as follows: an arbitrary simplicial set has not only the homotopy type of a small category, but its geometric realization is also homeomorphic to the classifying space of a small category. Roughly speaking: $\mathscr{C}at$ is (via N) not a "thin" subcategory of $\mathscr{K}$, but it exhausts $\mathscr{K}$ up to homeomorphism!

Section 5. Intermezzo: Nerve and Pushouts

This section contains technical definitions, lemmas, and hypotheses under which nerve does preserve certain pushouts up to isomorphism or WHE. These conditions will be summed up in Sect. 6 to a condition on a divided functor $\theta: \Delta \to \mathscr{C}at$ assuring that the canonical natural transformation

$$\rho: \Gamma_{N\theta} \dot{\to} N\Gamma_\theta: \mathscr{K} \to \mathscr{K}$$

is a natural WHE. Many of the tedious details are left for the reader. Furthermore, each of the hypotheses, and most definitions have dual formulations; these are also left for the reader to complete.

Throughout this section, let the square

(5.1)
$$\begin{array}{ccc} \mathbf{A} & \xhookrightarrow{i} & \mathbf{B} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathbf{C} & \xrightarrow{i'} & \mathbf{D} \end{array}$$

be a pushout in $\mathscr{C}at$ with i an inclusion.

5.2. Proposition. *If i is full, then i' is also a full inclusion.*

Proof. $\mathbf{D}$ can be viewed to be the following category. As object set, we take

$$|\mathbf{D}| \equiv |\mathbf{C}| \sqcup (|\mathbf{B}| \setminus |\mathbf{A}|)$$

(where $\sqcup$ represents the disjoint union of the sets in question). In order to get the morphism sets of $\mathbf{D}$, we first set

$$\mathbf{D}(p, q) \equiv \mathbf{C}(p, q)$$

for $p, q \in \mathbf{C}$. In the case where $p \in |\mathbf{C}|$, $q \in |\mathbf{B}| \setminus |\mathbf{A}|$, we consider pairs

$$(b, c) \in \mathbf{B} \times \mathbf{C}$$

such that

$$\operatorname{dom} c = p, \operatorname{dom} b \in |\mathbf{A}|, f \operatorname{dom} b = \operatorname{cod} c, \operatorname{cod} b = q.$$

On the set of pairs, generate an equivalence relation by the relation

$$(b \circ a, c) \sim (b, fa \circ c)$$

for $a \in \mathbf{A}$ such that the compositions involved are defined. Now, let $\mathbf{D}(p, q)$ be the set of the corresponding equivalence classes. The case $p \in \mathbf{B} \setminus \mathbf{A}$, $q \in \mathbf{C}$ is dual. Finally for $p, q \in \mathbf{B} \setminus \mathbf{A}$, define $\mathbf{D}(p, q)$ to be equivalence classes of the set

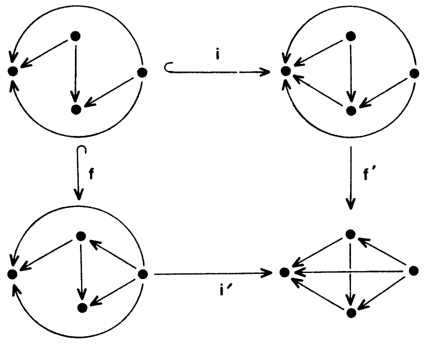
$$\begin{aligned} &\mathbf{B}(p, q) \sqcup \{(b_1, c, b_2) \in \mathbf{B} \times \mathbf{C} \times \mathbf{B} \mid \\ &\operatorname{cod} b_1 = q, \operatorname{dom} b_1 \in \mathbf{A}, f(\operatorname{dom} b_1) = \operatorname{cod} c, \\ &\operatorname{dom} b_2 = p, \operatorname{cod} b_2 \in \mathbf{A}, f(\operatorname{cod} b_2) = \operatorname{dom} c \}. \end{aligned}$$

Here, the equivalence relation is generated by the following three relations:

$$\begin{aligned} &b_1 \circ a \circ b_2 \sim (b_1, fa, b_2) \\ &(b_1 \circ a, c, b_2) \sim (b_1, fa \circ c, b_2) \\ &(b_1, c, a \circ b_2) \sim (b_1, c \circ fa, b_2) \end{aligned}$$

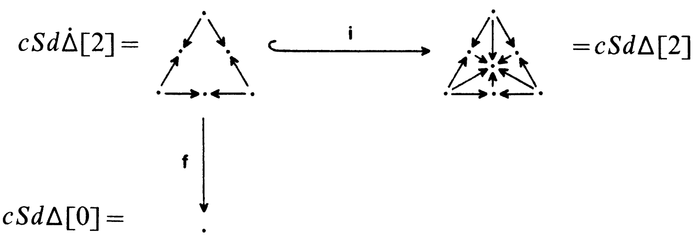
for $a \in \mathbf{A}$ such that the compositions involved are defined. Now the assertion is clear. $\square$

5.3. Remarks. (i) The question arises: Is the condition that i is an inclusion sufficient for i' to be an inclusion? The following picture, inspired from an example, due to Kimura [15], concerning semigroups, shows that this is not true. (All triangles involved commute.):



But there are situations with i' inclusion without i being full. One trivial case occurs when **C** is a discrete category. More interesting is the classical result if **A**, **B**, **C** are groups and f is an inclusion, then **D** is the "amalgamated product" of **B** with **C**. It seems that weakening the condition on the fullness of i requires a condition on f in order to force i' to be an inclusion. The example given above also demonstrates that, in contrast to the group case, the assumption "f inclusion" is not enough.

(ii) Already, the example in (i) above shows that nerve does not preserve pushouts. Requiring fullness for i does not help. Consider the following situation:



The pushout in $\mathscr{C}at$ yields a 1-simplex (after embedding in $\mathscr{K}$ via nerve); the pushout in $\mathscr{K}$ leads to a 2-sphere (see Remark 4.2(ii)).

(iii) Finally, note that since $cN \simeq \mathrm{Id}_{\mathscr{C}at}$, $N: \mathscr{C}at \to \mathscr{K}$ reflects pushouts.

A careful analysis of the proof of Proposition 5.2 yields the following variation on this result:

5.4. Proposition. *If* **A** *is a right (left) ideal in* **B** (see Definition 3.8(ii)), *then* **C** *is a right (left) ideal in* **D**. □

The example in Remark 5.3(ii) also shows that not only fullness, but also ideal preservation properties, do not insure that a certain pushout in $\mathscr{C}at$ is at least a homotopy pushout in $\mathscr{K}$. For this reason, the condition of being divided for a functor $\theta: \Delta \to \mathscr{C}at$ is not enough to insure that $\Gamma_\theta: \mathscr{K} \to \mathscr{C}at$ is a homotopy inverse for nerve, as explained in Sect. 4. Some of the necessary stronger conditions are described in:

5.5. Definition. An object $p \in \mathbf{A}$ is said to be *f-regular*, if f induces an isomorphism $p\mathbf{A} \xrightarrow{\cong} fp\mathbf{C}$ between the principal right ideals $p\mathbf{A}$, $fp\mathbf{C}$ generated by p, fp resp. in $\mathbf{A}$, $\mathbf{C}$ resp. Dually p is *f-coregular*, if f induces an isomorphism $\mathbf{A}p \xrightarrow{\cong} \mathbf{C}fp$ between left ideals.

The first result, in the desired direction, is now contained in:

5.6. Proposition. *Assume there are subcategories* $\mathbf{A}'$, $\mathbf{A}''$ *of* $\mathbf{A}$ *such that:*

(i) $|\mathbf{A}'| \cup |\mathbf{A}''| = |\mathbf{A}|$;
(ii) $f|\mathbf{A}'| \cap f|\mathbf{A}''| = \phi$;
(iii) $\mathbf{A}'(\mathbf{A}'')$ *is a right (left) ideal in* $\mathbf{B}$;
(iv) *Each object in* $\mathbf{A}'(\mathbf{A}'')$ *is f-regular (f-coregular).*

Then the nerve of square (5.1) *is a pushout in* $\mathscr{K}$.

Proof. Pushouts in $\mathscr{K}$ are computed pointwise. Therefore, it is sufficient to note that under the given assumptions, a string of morphisms in $\mathbf{D}$ is of one of the following forms:

(i) $$p_0 \xrightarrow{c_1} p_2 \to \dots \to p_{n-1} \xrightarrow{c_n} p_n, \quad c_i \in \mathbf{C};$$

(ii) $$p_0 \xrightarrow{fa_1} p_1 \to \dots \to p_{k-1} \xrightarrow{fa_k} p_k \xrightarrow{b_{k+1}} p_{k+1} \to \dots \to p_{m-1} \xrightarrow{b_m}$$

$$P_m \xrightarrow{fa_{m+1}} P_{m+1} \to \dots \to P_{n-1} \xrightarrow{fa_n} p_n, \quad a_i \in \mathbf{A}, \quad b_i \in \mathbf{B};$$

where the c_i are uniquely determined if all the p_i belong to $\mathbf{C}$, and otherwise, the a_i and b_i are unique. □

5.7. Remark. That the functors Γ_γ and $\Gamma_\omega = \Lambda$ may serve as homotopy inverses for nerve depends on the fact that in the pushouts involved, the corresponding regularity conditions are fulfilled. On the other hand, in the case of $cSd^n (n \geqq 2)$, one can neglect regularity and use further properties of the inclusion i. The following considerations shall show how one can compensate "f-singularities" by stronger assumptions on i.

5.8. Definitions. (i) An object $p \in \mathbf{A}$ is said to be *f-singular*, if it is not f-regular. By S_f we denote the set of f-singular objects, which is a subset of $|\mathbf{A}|$, and also a subset of $|\mathbf{B}|$ via i.
(ii) The *singular collar* $\mathbf{W}_f$ *of* f *in* $\mathbf{B}$ is the smallest full subcategory of $\mathbf{B}$ containing $\mathbf{A}$ and containing $\mathbf{B}S_f$, the left ideal of $\mathbf{B}$ generated by S_f.

Now we are able to state the main result of this section:

5.9. Theorem. *Suppose*

(i) $\mathbf{A}$ *is a right ideal in* $\mathbf{B}$;
(ii) $\mathbf{A}$ *is a coreflective subcategory of the singular collar* $\mathbf{W}_f$.

Then the canonical simplicial map

$$\rho\colon N\mathbf{C}\coprod_{N\mathbf{A}} N\mathbf{B} \to N\mathbf{D}$$

is a WHE in $\mathscr{K}$. □

For the proof, we need three lemmas.

5.10. Lemma. *The nerve of the pushout*

$$\begin{tikzcd} \mathbf{A} \arrow[r, hook] \arrow[d, "f"'] & \mathbf{W}_f \arrow[d] \\ \mathbf{C} \arrow[r, hook] & \mathbf{E} \end{tikzcd}$$

is a pushout, up to homotopy, in $\mathscr{K}$; i.e., *the canonical simplicial map*

$$\rho\colon N\mathbf{C}\coprod_{N\mathbf{A}} N\mathbf{W}_f \to N\mathbf{E}$$

is a WHE.

Proof. Since $\mathbf{A}$ is coreflective in $\mathbf{W}_f$, so is $\mathbf{C}$ in $\mathbf{E}$. A natural transformation $\phi\colon F \dot{\to} G$ in $\mathscr{C}at$ can be viewed as a functor $\operatorname{dom} F \times \imath[1] \to \operatorname{cod} F$; thus the classifying space $|N\mathbf{A}|$ of $\mathbf{A}$ becomes a strong deformation retract of $|N\mathbf{W}_f|$. For the same reason, $|N\mathbf{C}|$ is a strong deformation retract of $|N\mathbf{E}|$. Since pushouts in $\mathscr{W}$ preserve strong deformation retracts, $|N\mathbf{C}|$ is also a strong deformation retract of $|N\mathbf{C}\coprod_{N\mathbf{A}} N\mathbf{W}_f|$. Since ρ composed with the embedding $N\mathbf{C} \hookrightarrow N\mathbf{C}\coprod_{N\mathbf{A}} N\mathbf{W}_f$ yields the embedding $N\mathbf{C} \hookrightarrow N\mathbf{E}$, ρ itself must be a WHE. □

5.11. Lemma. *The commutative diagram*

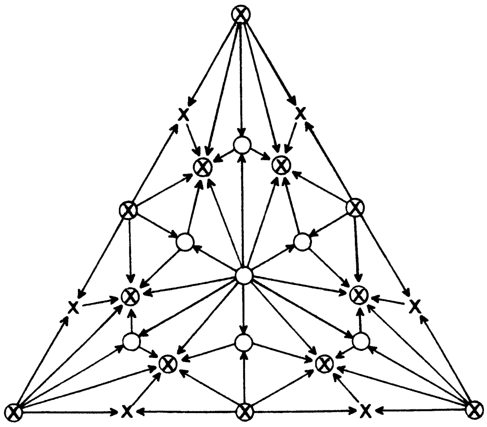
$$\begin{tikzcd} N\mathbf{W} \arrow[r, hook] \arrow[d, hook] & N\mathbf{R}_f \arrow[d, hook] \\ N\mathbf{W}_f \arrow[r, hook] & N\mathbf{B} \end{tikzcd}$$

with $\mathbf{R}_f = \langle |\mathbf{B}| \setminus S_f \rangle$ *(see Notation* 2.2*) and* $\mathbf{W} = \mathbf{W}_f \cap \mathbf{R}_f$, *is bicartesian in* $\mathscr{K}$; i.e., *it is both a pullback and a pushout.* □

Before giving the proof, we indicate by an illustration what is happening. Let

$$\mathbf{A} = cSd^2\,\dot{\Delta}[2], \qquad \mathbf{B} = cSd\,\Delta[2], \qquad \mathbf{C} = cSd^2\,\Delta[0] = \imath[0].$$

We draw $\mathbf{B}$ as follows: the objects of $\mathbf{R}_f$ are marked by small circles ○; the objects of $\mathbf{W}_f$ are represented by ×; and thus the objects of $\mathbf{W}$ bear tensor ⊗.



Proof of Lemma 5.11. The pullback property follows from the definition of **W** and the adjointness of nerve. Since pushouts in $\mathscr{K}$ are computed pointwise, it is sufficient to show that any string of morphisms

$$p_n \xleftarrow{b_n} p_{n-1} \leftarrow \cdots \leftarrow p_1 \xleftarrow{b_1} p_0$$

in **B** has a unique preimage either in $\mathbf{R}_f$ or in $\mathbf{W}_f$. Since all functors involved are inclusions, the uniqueness is clear. If all the objects p_i belong to $\mathbf{R}_f$, then there is a preimage for the string in $\mathbf{R}_f$, by the fullness of $\mathbf{R}_f$ in **B**. If one p_j is not in $\mathbf{R}_f$, then it is in **A** and f-singular; thus p_j is in $\mathbf{W}_f$. Since **A** is a right ideal in **B**, all the objects with $i<j$ belong to **A**, and thus also to $\mathbf{W}_f$. Since $\mathbf{W}_f$ contains the left ideal of **B** generated by p_j, all the objects p_i with $i>j$, also belong to $\mathbf{W}_f$. Now, from the fullness of $\mathbf{W}_f$ in **B**, the whole string lies in $\mathbf{W}_f$. $\square$

Since nerve reflects pushouts (see Remark 5.3 (iii)), the diagram

$$\begin{tikzcd} \mathbf{W} \arrow[r, hook] \arrow[d, hook] & \mathbf{R}_f \arrow[d, hook] \\ \mathbf{W}_f \arrow[r, hook] & \mathbf{B} \end{tikzcd}$$

is bicartesian in $\mathscr{C}at$. We compose this diagram with the induced pushout

$$\begin{tikzcd} \mathbf{W}_f \arrow[r, hook] \arrow[d] & \mathbf{B} \arrow[d] \\ \mathbf{E} \arrow[r, hook] & \mathbf{D} \end{tikzcd}$$

to show:

5.12. Lemma. *The composed diagram*

$$\begin{tikzcd} \mathbf{W} \arrow[r, hook] \arrow[d] & \mathbf{R}_f \arrow[d] \\ \mathbf{E} \arrow[r, hook] & \mathbf{D} \end{tikzcd}$$

is not only a pushout in $\mathscr{C}at$, but, after embedding via nerve, also a pushout in $\mathscr{K}$.

Proof. We check the hypotheses of Proposition 5.6 by taking

$$\mathbf{A}' \equiv \mathbf{W} \cap \mathbf{A}, \qquad \mathbf{A}'' \equiv \langle |\mathbf{W}| \setminus |\mathbf{A}'| \rangle.$$

(Confused? Check the illustration after the statement of Lemma 5.11.) The only problem is to verify the regularity conditions, hypothesis (iv).

First, consider an object $p \in \mathbf{A}'$. It is f-regular, and thus

$$p\mathbf{W} = p\mathbf{A} = fp\mathbf{C} = fp\mathbf{E}$$

via canonical isomorphisms; the last follows from Proposition 5.4 and the fact that $\mathbf{C}$ is a right ideal in $\mathbf{E}$.

For an object $p \in \mathbf{A}''$,

$$\mathbf{W}p = \mathbf{W}_f p \quad \text{and} \quad \mathbf{W}_f p \cap \mathbf{A} = \phi.$$

The constructive proof of Proposition 5.2 insures that

$$\mathbf{W}_f p = \mathbf{E}p. \quad \square$$

Finally, the

Proof of Theorem 5.9. Consider the diagram

$$\begin{tikzcd}
 & N\mathbf{W} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "③"] & N\mathbf{R}_f \arrow[d] \\
N\mathbf{A} \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "①"] & N\mathbf{W}_f \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "②"] & N\mathbf{B} \arrow[d, "f'"] \\
N\mathbf{C} \arrow[r, hook] & N\mathbf{E} \arrow[r, hook] & N\mathbf{D}.
\end{tikzcd}$$

By Lemma 5.12 the vertical composed rectangle ②∘③ is a pushout, and from Lemma 5.11, square ③ is a pushout; thus, square ② is also a pushout. Lemma 5.10 implies square ① is a pushout up to homotopy. Finally, from the Glueing Lemma, Lemma 2.10, the rectangle ②∘①, composed of a pushout up to homotopy and a real pushout, is a pushout up to homotopy. $\square$

5.13. Remarks. We developed our collaring condition from ideas of R. Thomason. He proved Theorem 5.9 under the stronger hypothesis that $\mathbf{A}$ is a coreflective subcategory, not only of the singular collar $\mathbf{W}_f$, but also of the complete left ideal in $\mathbf{B}$ generated by $\mathbf{A}$ (see [30; Prop. 4.3]). Our proof is modeled on his; but we need the more subtle condition 5.9 (ii), since his condition is not fulfilled for the pushouts occurring in dealing with Γ_γ and $\Gamma_\omega = \Lambda$ (see Remark 5.7).

There are also other pushouts in $\mathscr{C}at$, which are, via nerve, pushouts up to homotopy in $\mathscr{K}$. We mention the classical result of J.H.C. Whitehead [32; Theorem 5] for the case $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ groups and f injective. Moreover, J. Hardy and D. Puppe found the following assumptions to be sufficient for this purpose: $\mathbf{A} = |\mathbf{B}|$, $\mathbf{C} = |\mathbf{D}|$ and $\mathbf{B}$ a groupoid (still unpublished). The methods in these cases are quite different from ours; nevertheless, we hope, by combining all these methods, to find a general theorem as a common cover for the above results.

Section 6. Divided Functors θ: $\Delta \to \mathscr{C}at$

Throughout this section let θ: $\Delta \to \mathscr{C}at$ be a divided functor. Without loss of generality we assume that all functors

$$\theta\mu\colon \theta[p] \hookrightarrow \theta[k]$$

to be inclusions of right ideals (see Definition 3.8(ii)), for every $\mu\colon [p] \to [k]$ in **M**.
First note (In contrast, see Lemma 3.3.):

6.1. Proposition. *The cocontinuous extension*

$$\Gamma_\theta\colon \mathscr{K} \to \mathscr{C}at$$

of θ: $\Delta \to \mathscr{C}at$ *transforms injections in* $\mathscr{K}$ *into inclusions of right ideals in* $\mathscr{C}at$.

Proof. Let $i'\colon X \hookrightarrow Y$ be an inclusion in $\mathscr{K}$. Using the standard skeletal induction, it is sufficient to assume i' can be embedded in a pushout in $\mathscr{K}$ of the form

$$\begin{tikzcd} \dot\Delta[k] \arrow[r, hook, "i"] \arrow[d, "f"'] & \Delta[k] \arrow[d, "f'"] \\ X \arrow[r, hook, "i'"] & Y. \end{tikzcd}$$

Next, $i = J\tilde{i}$ for the inclusion $\tilde{i}$: $\dot{\mathsf{M}}[k] \hookrightarrow \mathsf{M}[k]$ in $\mathscr{L}$; thus it follows from Corollary 3.11 that

$$\Gamma_\theta i = \Gamma_\theta J\tilde{i} = \Gamma_{\theta U}\tilde{i}\colon (\dot\theta[k] \equiv \Gamma_\theta\dot\Delta[k]) \hookrightarrow (\Gamma_\theta\Delta[k] \equiv \theta[k])$$

is an inclusion of a right ideal.
Apply the functor Γ_θ to the above pushout. This yields a pushout in $\mathscr{C}at$ with

$$\Gamma_\theta i'\colon \Gamma_\theta X \hookrightarrow \Gamma_\theta Y$$

an inclusion of a right ideal from Proposition 5.4. $\square$

In order to describe the extra condition on θ: $\Delta \to \mathscr{C}at$ which we need, some further:

6.2. Notations. (i) For a fixed $k > 0$. *Let*

$$\tau\colon \dot\Delta[k] \to \Delta[0]$$

denote the terminal simplicial map in $\mathscr{K}$, and set

$$t \equiv \Gamma_\theta\tau\colon \Gamma_\theta\dot\Delta[k] \to \Gamma_\theta\Delta[0].$$

(ii) The functor t can be embedded in the pushout (in $\mathscr{C}at$)

$$\begin{tikzcd} \dot\theta[k] = \Gamma_\theta\dot\Delta[k] \arrow[r, hook] \arrow[d, "t"'] & \Gamma_\theta\Delta[k] = \theta[k] \arrow[d] \\ \theta[0] = \Gamma_\theta\Delta[0] \arrow[r, hook] & \Gamma_\theta(\Delta[k]/\dot\Delta[k]), \end{tikzcd}$$

and thus it gives rise to a singular collar $\mathbf{W}_t$, in the sense of Definition 5.8(ii). (Note, Proposition 6.1 insures that the top arrow in this pushout is an inclusion of a right ideal.)

Next, we state the main result of this section:

6.3. Theorem. *The canonical natural transformation*

$$\rho\colon \Gamma_{N\theta} \dot{\to} N\Gamma_\theta\colon \mathscr{K} \to \mathscr{K}$$

is a natural WHE, whenever $\dot{\theta}[k]$ is a coreflective subcategory of $\mathbf{W}_t$, for every $k>0$. $\square$

6.4. Remark. In the cases of $\theta = \gamma, \omega$,

$$\mathbf{W}_t = \dot{\theta}[k];$$

so the condition in Theorem 6.3 is trivially satisfied. It is also easy, but tedious, to verify this condition for $\theta = \xi^n$, $n \geqq 2$; we therefore refer the reader to the "picture" after the statement of Lemma 5.11. For the case $\theta = \xi$, we find that $\mathbf{W}_t = \theta[k]$; thus the condition is not satisfied.

For the proof of Theorem 6.3, note that the composed functor $N\Gamma_\theta\colon \mathscr{K} \to \mathscr{K}$, although not cocontinuous, preserves the simplicial decomposition of a simplicial set; more precisely:

6.5. Lemma. *Let X be a simplicial set. Then $N\Gamma_\theta$ of the following pushout in $\mathscr{K}$*

$$\begin{tikzcd} \coprod_{x \in X_k} \langle \dot{x} \rangle \arrow[r, hook] \arrow[d] & \coprod_{x \in X_k} \langle x \rangle \arrow[d] \\ X^{k-1} \arrow[r, hook] & X^k \end{tikzcd}$$

is again a pushout in $\mathscr{K}$, where $\langle x \rangle$ represents the subsimplicial set of X generated by the simplex x and $\langle \dot{x} \rangle$ the subsimplicial set of X generated by the boundaries of x.

Proof. Since $\Gamma_\theta\colon \mathscr{K} \to \mathscr{C}at$ is cocontinuous, Γ_θ of the above square is a pushout in $\mathscr{C}at$. From Proposition 6.1, the top arrow is an inclusion of a right ideal. Proposition 6.1, also insures that $\Gamma_\theta \langle \dot{x} \rangle \hookrightarrow \Gamma_\theta X^{k-1}$ are inclusions of right ideals. Thus every object in

$$\Gamma_\theta \coprod_{x \in X_k} \langle \dot{x} \rangle = \coprod_{x \in X_k} \Gamma_\theta \langle \dot{x} \rangle$$

is regular, and the conclusion follows from Proposition 5.6. $\square$

Proof of Theorem 6.3. Since any simplicial set is the sequential colimit of its skeletons and a sequential colimit of WHE's is a WHE ([10; II.3 and VII.1]), it suffices to consider

$$\rho X\colon \Gamma_{N\theta} X \to N\Gamma_\theta X$$

for finite dimensional simplicial sets X. Thus, we use an induction on $\dim X$. The induction begins with

$$\Gamma_{N\theta} \phi = \phi = N\Gamma_\theta \phi.$$

Next, assume $\dim X = k$, and consider the following commutative ladder

$$\begin{tikzcd}
\coprod\limits_{x\in X_k} \Gamma_{N\theta}\langle\dot{x}\rangle \arrow[r, hook] \arrow[d, "\rho_1"] & \coprod\limits_{x\in X_k} \Gamma_{N\theta}\langle x\rangle \arrow[r] \arrow[d, "\rho_2"] & \Gamma_{N\theta}X \arrow[d, "\rho"] & \Gamma_{N\theta}X^{k-1} \arrow[l, hook'] \arrow[d, "\rho_3"] & \coprod\limits_{x\in X_k} \Gamma_{N\theta}\langle\dot{x}\rangle \arrow[l] \arrow[d, "\rho_1"] \\
\coprod\limits_{x\in X_k} N\Gamma_\theta\langle\dot{x}\rangle \arrow[r, hook] & \coprod\limits_{x\in X_k} N\Gamma_\theta\langle x\rangle \arrow[r] & N\Gamma_\theta X & N\Gamma_\theta X^{k-1} \arrow[l, hook'] & \coprod\limits_{x\in X_k} N\Gamma_\theta\langle\dot{x}\rangle \arrow[l]
\end{tikzcd}$$

By the cocontinuity of $\Gamma_{N\theta}$, the top row is a pushout (in $\mathscr{K}$). Lemma 6.5 and the fact that nerve preserves coproducts together imply that the bottom row is also a pushout. By induction hypothesis, ρ_1, and ρ_3 are WHE's. Thus, in view of the Glueing Lemma, Lemma 2.10, it suffices to show the map

$$\rho\langle x\rangle: \Gamma_{N\theta}\langle x\rangle \to N\Gamma_\theta\langle x\rangle$$

is a WHE, for all $x\in X_k$.

Fix $x\in X_k$ and apply Γ_θ to the pushout

$$\begin{tikzcd}
\dot{\Delta}[k] \arrow[r, hook, "\hat{\imath}"] \arrow[d, "\hat{f}"'] & \Delta[k] \arrow[d] \\
\langle\dot{x}\rangle \arrow[r, hook] & \langle x\rangle
\end{tikzcd} \tag{6.6}$$

which yields a pushout square in $\mathscr{C}at$. Let $i\equiv\Gamma_\theta\hat{\imath}$ and $f\equiv\Gamma_\theta\hat{f}$. Proposition 6.1 insures that the functor i is an inclusion of a right ideal. Hence $\mathbf{W}_f$ is a subcategory of $\mathbf{W}_i$; and therefore, it contains $\dot{\theta}[k]\equiv\Gamma_\theta\dot{\Delta}[k]$ as a coreflective subcategory. Thus Theorem 5.9 applies and gives the induced WHE

$$\psi: (Y\equiv N\theta[k]) \coprod_{N\dot{\theta}[k]} (N\Gamma_\theta\langle\dot{x}\rangle) \to N\Gamma_\theta\langle x\rangle.$$

Next, note that Nf factors into

$$Nf = \rho\langle\dot{x}\rangle \circ \Gamma_{N\theta}f;$$

and thus we get the following composition of pushouts

$$\begin{tikzcd}
N\dot{\theta}[k] \arrow[r, hook] \arrow[d, "\Gamma_{N\theta}f"'] & N\theta[k] \arrow[d] \\
\Gamma_{N\theta}\langle\dot{x}\rangle \arrow[r, hook] \arrow[d, "\rho\langle\dot{x}\rangle"'] & N\theta\langle x\rangle \arrow[d, "\dot{\psi}"] \\
N\Gamma_\theta\langle\dot{x}\rangle \arrow[r, hook] & Y.
\end{tikzcd}$$

By induction hypothesis, $\rho\langle\dot{x}\rangle$ is a WHE; thus, so is $\dot{\psi}$ (using a special form of the Glueing Lemma, Lemma 2.10). Clearly, since WHE is closed under composition,

$$(\rho\langle x\rangle = \psi\circ\dot{\psi}): \Gamma_{N\theta}\langle x\rangle \to N\Gamma_\theta\langle x\rangle$$

is a WHE. □

6.7. Remark. If the pushout in 6.2(ii) does not contain singular objects, then one is able to use Proposition 5.6 instead of Theorem 5.9; and the natural transformation ρ of Theorem 6.3 becomes a natural isomorphism, not only a WHE. This, for instance, applies in the cases $\theta=\gamma$, ω.

In view of Theorem 4.11, it seems to be interesting how the condition in Theorem 6.3 can be weakened to assure only that ρX is a WHE for a regulated simplicial set X [21; III.8]. To this end, we introduce the following:

6.8. Notations. (i) For a fixed $k>0$, *let*

$$\delta^0\colon [k-1]\rightarrowtail[k]$$

denote the unique injective map with $0\notin \operatorname{Im}\delta^0$.

(ii) If $\tau\colon \Delta[k-1]\twoheadrightarrow\Delta[0]$ denotes the terminal simplicial map, let

$$\hat{t}=\Gamma_\theta\tau\colon (\theta[k-1]=\Gamma_\theta\Delta[k-1])\to(\Gamma_\theta\Delta[0]=\theta[0])$$

denote its image in $\mathscr{C}at$.

(iii) The functor $\hat{t}$ can be embedded in the pushout

$$\begin{tikzcd}
\theta[k-1] \arrow[r, hook, "\theta\delta^0"] \arrow[d, "\hat{t}"] & \theta[k] \arrow[d] \\
\theta[0] \arrow[r, hook] & \theta[k] \coprod\limits_{\theta[k-1]} \theta[0]
\end{tikzcd}$$

and it generates a singular collar $\mathbf{W}_{\hat{t}}$.

Theorem 6.3 can be replaced by:

6.9. Theorem. *The canonical natural transformation*

$$\rho\colon \Gamma_{N\theta}\dot{\to}N\Gamma_\theta\colon \mathscr{K}\to\mathscr{K}$$

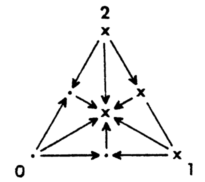
is a natural WHE, at least for regulated simplicial sets, whenever $\theta[k-1]$ is a coreflective subcategory of $\mathbf{W}_{\hat{t}}$, for every $k>0$.

Proof. The proof of this theorem is a slight modification of the proof of Theorem 6.3. In this case, consider, instead of pushout (6.6), the square

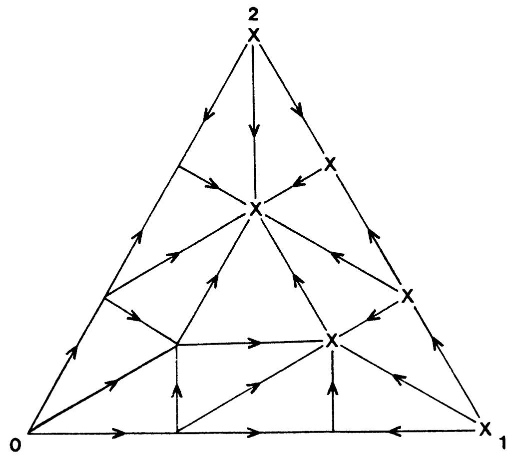
$$\begin{tikzcd}
\Delta[k-1] \arrow[r, hook, "\Delta\delta^0"] \arrow[d] & \Delta[k] \arrow[d] \\
\langle x\delta^0\rangle \arrow[r, hook] & \langle x\rangle
\end{tikzcd}$$

which is a pushout whenever x is a nondegenerate simplex of a regulated simplicial set. □

6.10. Picture. For $\theta=\xi\colon \Delta\to\mathscr{C}at$ and $k=2$, the objects of $\mathbf{W}_{\hat{t}}$ are marked by crosses:



6.11. One More Example. In [7], the functor $\mathscr{G}$: $\Delta \to \mathscr{K}$ is defined; it "subdivides" a simplex by plugging into its interior, a standard simplex of the same dimension. We picture $\mathscr{G}[2]$:



The functor

$$\lambda \equiv c\mathscr{G}: \Delta \to \mathscr{C}at$$

is explicitly described in [19; 5.10]. As in the case of $\theta = \xi$, we find $\mathbf{W}_t = \lambda[k]$; so the condition of Theorem 6.3 is not fulfilled and Γ_λ: $\mathscr{K} \to \mathscr{C}at$ is not a homotopy inverse for nerve (in particular, $N\Gamma_\lambda(\Delta[2]/\dot{\Delta}[2]) = \Delta[3]$). But again, Im $\lambda\delta^0$ is a coreflective subcategory of $\mathbf{W}_t$ (the vertices corresponding to objects of $\mathbf{W}_t$ are marked by crosses, $\times$, in the picture above).

Note that there exists a commutative diagram

$$\begin{tikzcd} \xi^2 \arrow[rr, "\cdot"] \arrow[dr, "\cdot"'] & & \lambda \arrow[dl, "\cdot"] \\ & \xi & \end{tikzcd}$$

of natural WHE's. Thus λ can be viewed as "sitting" between ξ^2 and ξ. Hence ξ^2 "seems" to be the "smallest" functor having its cocontinuous extension $\Gamma_{\xi^2} = cSd^2$: $\mathscr{K} \to \mathscr{C}at$, a homotopy inverse for nerve.

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Added in Proof. We note that D. McDuff developed another type of homotopy inverse for nerve in: On the classifying space of discrete monoids. [Topology **18**, 313–320 (1979)].