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# Computational complexity in additive hedonic games\*

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## Abstract

We investigate the computational complexity of several decision problems in hedonic coalition formation games and demonstrate that attaining stability in such games remains NP-hard even when they are additive. Precisely, we prove that when either core stability or strict core stability is under consideration, the existence problem of a stable coalition structure is NP-hard in the strong sense. Furthermore, the corresponding decision problems with respect to the existence of a Nash stable coalition structure and of an individually stable coalition structure turn out to be NP-complete in the strong sense.

*JEL classification:* C63; C70; C71; D02; D70; D71

*Keywords:* additive preferences; coalition formation; computational complexity; hedonic games; NP-hard; NP-complete

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# 1 Introduction

The study of computational complexity in *hedonic coalition formation* games has a short history, although these issues in cooperative and non-cooperative game theory are being gradually recognized.<sup>1</sup> The formal model of a hedonic game was only recently introduced (cf. Banerjee et al. (2001) and Bogomolnaia and Jackson (2002)). This model consists of a finite set of players and a preference relation for each player defined over the set of all coalitions containing that player. The outcome of a hedonic game is a coalition structure, i.e., a partition of the set of players into coalitions. Thus, in a hedonic game one explicitly takes into account the dependence of an agent's utility on the identity of the members of his or her coalition as recognized in the seminal paper of Drèze and Greenberg (1980), and as it applies to many social and economic situations like the formation of social clubs, groups, societies, etc. The focus in the above mentioned works is on different stability concepts like the (*strict*) *core*, *Nash stability* and *individual stability*, and on conditions guaranteeing the existence of stable coalition structures.

Computational complexity issues related to hedonic games in a general setting are studied by Ballester (2004). As shown by this author, the problems to decide whether for a hedonic game there exists at least one core stable, Nash stable, or individually stable partition are *NP-complete*. In a less general setting (i.e., with some preference restrictions), Cechlarová and Hajduková (2002, 2004) study hedonic games, where the ranking over coalitions for each player is guided either by his most preferred member of the

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<sup>1</sup> For complexity considerations in cooperative games the reader is referred, among many others, to Faigle et al. (1997) and Faigle et al. (1998). With respect to non-cooperative games, see Baron et al. (2008), Ben-Porath (1990), Gilboa (1988), Gilboa and Zemel (1989), Koller and Megiddo (1992), and Koller et al. (1996).

group or by his least preferred member of the group, and consider computational complexity issues with respect to core related concepts. Recently, Dimitrov et al. (2006) also study the computational complexity for finding a core element in hedonic games; in particular, preference profiles based on aversion to enemies that constitute a small subdomain of the domain of *additive preferences* (players' preferences are represented by an additive utility function) are considered by these authors and it is shown that finding a core member for such games is NP-hard. In addition, Sung and Dimitrov (2007a) study the problem of core membership testing for hedonic games which is to decide whether a coalition structure belongs to the core of the game and show that this problem is *co-NP complete* when players' preferences are additive; indeed, the co-NP completeness is shown by a reduction to hedonic games in which players' preferences are based on aversion to enemies.

Notice that the existence of an additive utility function defined on the player set allows each player to easily calculate his or her utility from joining a certain coalition. Moreover, each such game can be described by  $n^2$  numbers, where  $n$  is the number of the players in the game. Hence, the computational task seems to become less demanding since the input size of the problem is polynomial of  $n$  when additivity is imposed. Despite these facts, the above mentioned works show that it is hard to find a coalition structure which is core stable. This let us conjecture a possible referential role the domain of additive hedonic games could play in the study of computational complexity issues. The aim of this paper is to present a detailed analysis of this conjecture by studying the computational complexity of several decision problems with respect to the existence of stable coalition structures in additive hedonic games. Precisely, we prove that when either core stability or strict core stability is under consideration, the existence problem of a stable coalition

structure is NP-hard in the strong sense. Furthermore, the corresponding decision problems with respect to the existence of a Nash stable coalition structure and of an individually stable coalition structure turn out to be NP-complete in the strong sense.

The rest of the paper is structured as follows. Section 2 contains basic definitions and concepts from both the theory of hedonic games and the theory of computational complexity. Sections 3 and 4 are devoted to the computational complexity of the existence problems with respect to the core and the strict core (Section 3) and with respect to Nash stable and individually stable coalition structures (Section 4). We introduce the corresponding stability notions, formulate the decision problems, and prove our results by explicitly providing the intuition when constructing the additive hedonic games in the proofs. Section 5 concludes then with some final remarks.

## 2 Preliminaries

Throughout this paper we will need the following basic notions and concepts.

### 2.1 Hedonic games

We denote by  $N = \{1, 2, \dots, n\}$  a finite set of players. A *coalition* is a nonempty subset of  $N$ . For each player  $i \in N$ , we denote by  $\mathcal{A}^i = \{X \subseteq N \mid i \in X\}$  the collection of all coalitions containing  $i$ . A collection  $\Pi$  of coalitions is called a *coalition structure* of  $N$  if it is a partition of  $N$ , i.e., all coalitions in  $\Pi$  are pairwise disjoint and their union is  $N$ . For each coalition structure  $\Pi$  of  $N$  and for each player  $i \in N$ , we denote by  $\Pi(i)$  the coalition in  $\Pi$  containing  $i$ .

A *hedonic game* is a pair  $\langle N, \succeq \rangle$  of a finite set  $N$  of players and a preference profile  $\succeq = (\succeq_1, \succeq_2, \dots, \succeq_n)$ . That is, in a hedonic game  $\langle N, \succeq \rangle$ , each

player  $i \in N$  is endowed with a complete and transitive binary relation  $\succeq_i$  over the coalitions in  $\mathcal{A}^i$ . Moreover, the preference of each player  $i \in N$  over all coalition structures is assumed to be *purely hedonic*, i.e., it is completely characterized by  $\succeq_i$  in such a way that, for every two coalition structures  $\Pi$  and  $\Pi'$ , each player  $i$  weakly prefers  $\Pi$  to  $\Pi'$  if and only if  $\Pi(i) \succeq_i \Pi'(i)$ .

A preference profile  $\succeq = (\succeq_1, \succeq_2, \dots, \succeq_n)$  is called *additive* if, for each  $i \in N$ , there exists a real-valued function  $v_i : N \rightarrow \mathbb{R}$  such that for all  $X, Y \in \mathcal{A}^i$ ,

$$X \succeq_i Y \quad \Leftrightarrow \quad \sum_{j \in X} v_i(j) \geq \sum_{j \in Y} v_i(j).$$

A hedonic game with an additive preference profile is called an *additive hedonic game*. Observe that the preference of each player  $i \in N$  can be represented by the vector  $(v_i(1), v_i(2), \dots, v_i(n))$ , and thus, an additive hedonic game  $\langle N, \succeq \rangle$  can be described by  $n^2$  numbers.

In the study of hedonic games one is usually interested in the existence of coalition structures that are *stable* in the sense that they are immune against either group or individual deviations (cf. Sung and Dimitrov (2007b)). We introduce these notions in Section 3 and 4, respectively.

## 2.2 Computational complexity

In this paper we formulate the problem of the existence of a stable coalition structure in (additive) hedonic games as a specific *decision problem* which can be described by *instances*, which are the inputs, and a *question*, which has either “YES” or “NO” as an answer. Thus, an instance for each of the decision problems we consider will be an additive hedonic game and the question will be whether there exists a stable coalition structure for the game.

Once a decision problem has been formulated, we may next ask how *fast* the problem can be correctly solved by an *algorithm*. Generally, the running

time of an algorithm increases as the size of problem instances (the games) increases. An algorithm is said to be *efficient* if its running time is bound above by a polynomial function of the size of the instance (game).

The class of problems that admit at least one efficient algorithm is denoted by P. NP is the class of all decision problems such that if the answer to a problem instance is “YES”, then there exists a certificate (string of symbols) of polynomial length so that, in polynomial time, an algorithm accepts the certificate as proof for a “YES” answer. The class NP contains P and it is generally accepted that NP and P are different classes of problems.

While many problems have been proved to be in P (generally by explicitly giving an efficient algorithm solving the problem), it is difficult to prove that a problem is *not* in P (and hence, it is hard to be solved). Instead of this, one usually shows that if the problem under consideration can be solved efficiently, then so can every member of a certain class C of problems. Such a problem is said then to be *C-hard* (and *C-complete* if, additionally, the problem has also been shown to lie in C). Furthermore, if a C-hard (C-complete) problem remains C-hard (C-complete) even if each of its instance parameters is bounded by a constant, then the problem is said to be C-hard (C-complete) in the *strong sense*. For a list of problems known to be NP-hard the reader is referred to Garey and Johnson (1979) and Ausiello et al. (1999).

Once one problem P1 has been shown to be NP-hard, the task of proving that another problem P2 is also NP-hard becomes much easier: one can do so by *reducing* P1 to P2. Informally, a reduction maps every instance of problem P1 to a corresponding instance of problem P2, in such a way that the answer to the former instance can be easily inferred from the answer to the latter instance. Moreover, the reduction itself should be efficiently

computable. If such an efficient reduction exists, then problem P1 can be seen as computationally *at most* as hard to solve as problem P2. If P1 is NP-hard, then the existence of an efficient reduction tells us that we cannot hope to find an efficient algorithm for P2 without (implicitly) finding such an efficient algorithm for the NP-hard problem P1. Notice finally, that, in order P2 to be NP-hard in the strong sense, the following two requirements have to be satisfied: (1) each of the P2's instance parameters created by the reduction should be bounded by a constant, and (2) P1 has to be NP-hard in the strong sense.

The NP-hardness in the strong sense of all decision problems we consider in the next sections are shown by efficient reductions from a problem known to be NP-complete in the strong sense. This problem is called *Exact Cover by 3 Sets* and it is defined as follows.

**Exact Cover by 3 Sets (E3C):**

**Instance:** A pair  $(R, \mathcal{S})$ , where  $R$  is a set and  $\mathcal{S}$  is a collection of subsets of  $R$  such that  $|R| = 3m$  for some positive integer  $m$  and  $|s| = 3$  for each  $s \in \mathcal{S}$ .

**Question:** Is there a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ ?

As an example, let us consider an instance  $(R, \mathcal{S})$  with  $R = \{a, b, c, d, e, f\}$  and  $\mathcal{S} = \{\{a, b, c\}, \{c, d, e\}, \{d, e, f\}\}$ . Then the answer to **E3C** is “YES” since the sub-collection  $\mathcal{S}' = \{\{a, b, c\}, \{d, e, f\}\}$  is a partition of  $R$ .

It is known that **E3C** remains NP-complete even if each  $r \in R$  occurs in *at most three* members of  $\mathcal{S}$ . Moreover, in order to exclude some trivial cases, we assume that each  $r \in R$  occurs in at least one member of  $\mathcal{S}$  (otherwise the answer to **E3C** is “NO”).

### 3 Group deviations and stability

Let us start by introducing two stability notions for hedonic games that are based on group deviations and consider then the corresponding decision problems.

Let  $\langle N, \succeq \rangle$  be a hedonic game,  $\Pi$  be a coalition structure of  $N$ , and  $X$  be a coalition. We say that

- $X$  is a *strong deviation* from  $\Pi$  in  $\langle N, \succeq \rangle$  if  $X \succ_i \Pi(i)$  for each  $i \in X$ ;
- $X$  is a *weak deviation* from  $\Pi$  in  $\langle N, \succeq \rangle$  if  $X \succeq_i \Pi(i)$  for each  $i \in X$ , and  $X \succ_i \Pi(i)$  for some  $i \in X$ .

Moreover, we say that

- $\Pi$  is *core stable* if there is no strong deviation from  $\Pi$ ;
- $\Pi$  is *strictly core stable* if there is no weak deviation from  $\Pi$ .

Thus,  $\Pi$  is core stable if there is no coalition such that each of its members is strictly better off in comparison to his or her corresponding coalition according to  $\Pi$ . For strict core stability, one requires that there is no deviation with at least one member being strictly better off and no one being worse off in comparison to the corresponding coalitions in  $\Pi$ .

The first decision problem we will be concerned with is the following:

#### **Existence of a core stable coalition structure (HC):**

**Instance:** A hedonic game  $\langle N, \succeq \rangle$ , where  $N$  is a set of players and  $\succeq$  is an additive preference profile.

**Question:** Is there a coalition structure which is core stable in  $\langle N, \succeq \rangle$ ?

Analogously, the decision problem of the existence of a strictly core stable partition is as follows.

**Existence of a strict core stable coalition structure (HS):**

**Instance:** A hedonic game  $\langle N, \succeq \rangle$ , where  $N$  is a set of players and  $\succeq$  is an additive preference profile.

**Question:** Is there a coalition structure which is strictly core stable in  $\langle N, \succeq \rangle$ ?

The existence problem **HC** in the general setting is considered in Ballester (2004), and is shown to be NP-complete. Observe that, in the general setting, each player  $i$ 's preference is given as a binary relation over  $\mathcal{A}^i$ , where the cardinality of  $\mathcal{A}^i$  is  $2^{n-1}$ . Hence, the input size of the problems becomes exponential of  $n$ . However, it is not known whether **HC** and **HS** belong to NP when additivity is imposed.

Notice that the NP-hardness of either of the above problems does not imply that the other problem is also NP-hard. The reason is that, since strict core stability implies core stability, a “YES” answer to **HS** implies a “YES” answer to **HC**. However, a “NO” answer to **HS** does not necessary imply a “NO” answer to **HC** since there are core stable coalition structures which are not strictly core stable.

We show in what follows that, when an additive hedonic game is under consideration, *both* **HC** and **HS** are NP-hard in the strong sense. The proofs are based on polynomial time reductions from **E3C**. That is, for a given instance  $(R, \mathcal{S})$  of **E3C**, we construct in polynomial time of  $|R|$  and  $|\mathcal{S}|$  an additive hedonic game  $\langle N, \succeq \rangle$ , in which all parameters are bounded by a constant.

The constructions of the corresponding games for **HC** and **HS** are slightly different and they are presented in Section 3.1 and 3.2, respectively. In these constructions, for each  $i, j \in N$  with  $i \neq j$ , we define  $v_i(j)$  only when it has a positive value. For all other  $v_i(j)$ s (that are *not explicitly defined* in the corresponding proofs) we assume that

$$v_i(j) = \begin{cases} 0 & \text{if } j = i, \\ -(M_i + 1) & \text{otherwise,} \end{cases}$$

where  $M_i = \sum_{k \in N} \max \{v_i(k), 0\}$ . As a consequence, for each  $i \in N$  and for each  $X \in \mathcal{A}^i$ ,  $\{i\} \succ_i X$  when  $v_i(j) < 0$  for some  $j \in X$ . It follows then that a coalition structure  $\Pi$  cannot be stable in any sense when  $v_i(j) < 0$  for some  $i \in N$  and  $j \in \Pi(i)$ . Conversely, if  $v_i(j) \geq 0$  for each  $i \in N$  and for each  $j \in \Pi(i)$ , then each  $X \subseteq N$  such that  $v_i(j) < 0$  for some  $i, j \in X$  can be neither a strong nor a weak deviation from  $\Pi$ .

### 3.1 Core stability

We start by first explaining and exemplifying how we construct an additive hedonic game from an instance  $(R, \mathcal{S})$  of **E3C**, and then continue with the formal proof of the NP-hardness of **HC** as to show that the constructed hedonic game has a core stable coalition structure if and only if there is a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ .

For simplicity, let us consider again the instance  $(R, \mathcal{S})$  of **E3C** with  $R = \{a, b, c, d, e, f\}$  and  $\mathcal{S} = \{\{a, b, c\}, \{c, d, e\}, \{d, e, f\}\}$ . To each element of  $R$  we first attach a 5-player hedonic game adapted from one in Bogomolnaia and Jackson (2002, Example 5). If we take  $a \in R$ , the hedonic game attached to  $a$  has the following structure: the player set is

$$\{\alpha_a, \beta_a, \gamma_a, \delta_a, \varepsilon_a\},$$

and players' preferences are

$$v_{\alpha_a}(\beta_a) = v_{\beta_a}(\gamma_a) = v_{\gamma_a}(\delta_a) = v_{\delta_a}(\varepsilon_a) = v_{\varepsilon_a}(\alpha_a) = 17$$

and

$$v_{\alpha_a}(\varepsilon_a) = v_{\varepsilon_a}(\delta_a) = v_{\delta_a}(\gamma_a) = v_{\gamma_a}(\beta_a) = v_{\beta_a}(\alpha_a) = 18.$$

In this game the players form a cycle such that each player likes the following player more than the previous one, and hates each of the other players. Notice that if a coalition structure contains a coalition with three or more players, then it cannot be core stable. Thus, the only possible coalitions are of size one or two, where coalitions of size two must contain consecutive players - for instance  $\{\{\alpha_a\}, \{\beta_a, \gamma_a\}, \{\delta_a, \varepsilon_a\}\}$ . This coalition structure is not core stable since the coalition  $\{\alpha_a, \beta_a\}$  blocks it. In a similar way one can show that there is no core stable coalition structure for this game. Observe that the values 17 and 18 are not essential for the above argument, but these values play an important role in the reduction used in our proof.

Notice additionally, that the players in the coalitions  $\{\beta_a, \gamma_a\}$  and  $\{\delta_a, \varepsilon_a\}$  would stick together if one provides an incentive for  $\alpha_a$  not to be interested in  $\beta_a$ . In order to do this, it is maybe helpful first to interpret  $s = \{a, b, c\} \in \mathcal{S}$  as a committee with  $a$  being represented by  $\alpha_a$ ,  $b$  being represented by  $\alpha_b$ , and  $c$  being represented by  $\alpha_c$ , where  $\alpha_b$  and  $\alpha_c$  are the corresponding players (one should provide a similar incentive to as the one for  $\alpha_a$ ) in the 5-player games attached to  $b \in R$  and  $c \in R$ , respectively.

One possibility to construct these incentives is to first make  $\alpha_a$ ,  $\alpha_b$ , and  $\alpha_c$  like each other and then let them stick together (as being representatives of the members in committee  $s$ ). If we define  $v_{\alpha_r}(\alpha_{r'}) = 2$  for  $r, r' \in \{a, b, c\}$  with  $r \neq r'$ , then we accomplish only partially the task: player  $\alpha_a$  still has an incentive to form a strong deviation with  $\beta_a$  since the value of the coalition

$\{\alpha_a, \alpha_b, \alpha_c\}$  for  $\alpha_a$  is only 4. Thus, we need an additional player who helps the representatives for the members of  $s$  stick together. We denote this player by  $\zeta_s$  and define  $v_{\alpha_r}(\zeta_s) = 14$  and  $v_{\zeta_s}(\alpha_r) = 2$  for  $r \in \{a, b, c\}$  (see Fig. 1). Notice then that, in such a case, player  $\alpha_a$  does not have an incentive to form a strong deviation with  $\beta_a$  since his utility from being a member of  $\{\alpha_a, \alpha_b, \alpha_c, \zeta_s\}$  equals to 18, while his utility from being together with  $\beta_a$  equals to 17.

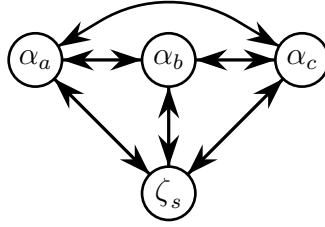


Figure 1:

Observe finally, that  $\alpha_c$  is also a representative for  $c$  in the committee  $s' = \{c, d, e\}$ ; thus, one has also to guarantee that there is no incentive for  $\alpha_c$  to be part of a strong deviation by the coalition  $\{\alpha_c, \alpha_d, \alpha_e, \zeta_{s'}\}$ . For this, we make  $\alpha_c$  being indifferent between representing  $c$  in  $s$  and representing  $c$  in  $s'$  by defining  $v_{\alpha_r}(\zeta_s) = 14$  and  $v_{\zeta_s}(\alpha_r) = 2$  for each  $s \in \mathcal{S}$  and for each  $r \in s$ . The result of this construction is that any representative of an element in a committee belonging to  $\mathcal{S}' = \{\{a, b, c\}, \{d, e, f\}\}$  ( $\mathcal{S}'$  is a partition of  $R$ ) has no incentive to strongly deviate together with representatives of some elements in committees belonging to  $\mathcal{S} \setminus \mathcal{S}'$ .

We are ready now to present the formal proof of our first result.

**Theorem 1** *HC is NP-hard in the strong sense.*

**Proof.** Let  $(R, \mathcal{S})$  be an instance of **E3C** such that each  $r \in R$  occurs in at

most three members of  $\mathcal{S}$ . From  $(R, \mathcal{S})$ , an instance of **HC**, i.e., an additive hedonic game  $\langle N, \succeq \rangle$ , is constructed in polynomial time of  $|R|$  and  $|\mathcal{S}|$ .

Let

$$N = \{\alpha_r, \beta_r, \gamma_r, \delta_r, \varepsilon_r \mid r \in R\} \\ \cup \{\zeta_s \mid s \in \mathcal{S}\}.$$

Observe that  $|N| = 5|R| + |\mathcal{S}|$ . Players' preferences are defined as follows.

- For each  $s \in \mathcal{S}$  and for each  $r \in s$ ,  $v_{\alpha_r}(\zeta_s) = 14$  and  $v_{\zeta_s}(\alpha_r) = 2$ .
- For each  $s \in \mathcal{S}$  and for each  $r, r' \in s$  with  $r \neq r'$ ,  $v_{\alpha_r}(\alpha_{r'}) = v_{\alpha_{r'}}(\alpha_r) = 2$ .
- For each  $r \in R$ ,  $v_{\alpha_r}(\beta_r) = v_{\beta_r}(\gamma_r) = v_{\gamma_r}(\delta_r) = v_{\delta_r}(\varepsilon_r) = v_{\varepsilon_r}(\alpha_r) = 17$  and  $v_{\alpha_r}(\varepsilon_r) = v_{\varepsilon_r}(\delta_r) = v_{\delta_r}(\gamma_r) = v_{\gamma_r}(\beta_r) = v_{\beta_r}(\alpha_r) = 18$ .

In the following, we show that there exists a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$  if and only if there exists a core stable coalition structure  $\Pi$  in the constructed additive hedonic game.

( $\Rightarrow$ ) Suppose there exists a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ . Then, consider the following partition of  $N$ .

$$\Pi = \{\{\zeta_s\} \cup \{\alpha_r \mid r \in s\} \mid s \in \mathcal{S}'\} \\ \cup \{\{\zeta_s\} \mid s \in \mathcal{S} \setminus \mathcal{S}'\} \\ \cup \{\{\beta_r, \gamma_r\}, \{\delta_r, \varepsilon_r\} \mid r \in R\}.$$

Observe that  $v_i(j) \geq 0$  for each  $i \in N$  and for each  $j \in \Pi(i)$ . By assumption,  $X \subseteq N$  such that  $v_i(j) < 0$  for some  $i, j \in X$  cannot be a strong (or weak) deviation from  $\Pi$ .

For each  $s \in \mathcal{S}'$ ,

$$\{i \in N \mid v_{\zeta_s}(i) \geq 0\} = \{\zeta_s\} \cup \{\alpha_r \mid r \in s\}.$$

Thus, for each  $s \in \mathcal{S}'$ , there is no strong (or weak) deviation from  $\Pi$  containing  $\zeta_s$ .

For each  $s \in \mathcal{S} \setminus \mathcal{S}'$ , the existence of a strong deviation  $X$  from  $\Pi$  containing  $\zeta_s$  requires the inequality

$$\sum_{i \in X} v_{\zeta_s}(i) > \sum_{i \in \Pi(\zeta_s)} v_{\zeta_s}(i)$$

to be satisfied. Thus, there should exist  $r \in s$  with  $\alpha_r \in X$ . However, we have  $\sum_{i \in \Pi(\alpha_r)} v_{\alpha_r}(i) = 18$  for each  $r \in R$ , and  $\sum_{i \in X} v_{\alpha_r}(i) \leq 18$  for each  $X \subseteq \{\zeta_s\} \cup \{\alpha_r \mid r \in s\}$ . Thus, for each  $s \in \mathcal{S} \setminus \mathcal{S}'$ , there is no strong deviation from  $\Pi$  containing  $\zeta_s$ .

We show next that there is no strong deviation from  $\Pi$  containing any of the players  $\beta_r$ ,  $\gamma_r$ ,  $\delta_r$ , and  $\varepsilon_r$ . By definition, we have

$$\sum_{i \in \Pi(\gamma_r)} v_{\gamma_r}(i) = \sum_{i \in \Pi(\varepsilon_r)} v_{\varepsilon_r}(i) = 18$$

and for all  $X \subseteq N$ , the values  $\sum_{i \in X} v_{\gamma_r}(i)$  and  $\sum_{i \in X} v_{\varepsilon_r}(i)$  are at most 18. Moreover, the existence of  $X \subseteq N$  with

$$\sum_{i \in X} v_{\beta_r}(i) > \sum_{i \in \Pi(\beta_r)} v_{\beta_r}(i)$$

would imply  $X = \{\beta_r, \alpha_r\}$ , and the existence of  $X' \subseteq N$  with

$$\sum_{i \in X'} v_{\delta_r}(i) > \sum_{i \in \Pi(\delta_r)} v_{\delta_r}(i)$$

would imply  $X' = \{\delta_r, \gamma_r\}$ . However,

$$v_{\alpha_r}(\alpha_r) + v_{\alpha_r}(\beta_r) = 17 < 18 = \sum_{i \in \Pi(\alpha_r)} v_{\alpha_r}(i)$$

and

$$v_{\gamma_r}(\gamma_r) + v_{\gamma_r}(\delta_r) = 17 < 18 = \sum_{i \in \Pi(\gamma_r)} v_{\gamma_r}(i).$$

Thus, there is no strong (or weak) deviation from  $\Pi$  containing anyone of  $\beta_r$ ,  $\gamma_r$ ,  $\delta_r$ , and  $\varepsilon_r$ .

Recall that each  $r \in R$  occurs in at most three members of  $\mathcal{S}$ . Then,

$$|\{\alpha_{r'} \mid r' \in R, v_{\alpha_r}(\alpha_{r'}) > 0\}| \leq 6$$

and thus, for each  $X \subseteq \{\alpha_{r'} \mid r' \in R\}$ ,

$$\sum_{i \in X} v_{\alpha_r}(i) \leq 12 < 18 = \sum_{i \in \Pi(\alpha_r)} v_{\alpha_r}(i).$$

We conclude that  $X \subseteq \{\alpha_r \mid r \in R\}$  cannot be a strong (or weak) deviation from  $\Pi$ . Therefore,  $\Pi$  is core stable.

( $\Leftarrow$ ) Suppose there exists a core stable coalition structure  $\Pi$  for the constructed additive hedonic game. First, observe that we have  $\Pi(\alpha_r) \neq \{\alpha_r, \beta_r\}$  for each  $r \in R$ ; otherwise,  $\Pi(\alpha_r) = \{\alpha_r, \beta_r\}$  implies either

- $\Pi(\varepsilon_r) = \{\varepsilon_r, \delta_r\}$  and  $\Pi(\gamma_r) = \{\gamma_r\}$  in which case  $\{\delta_r, \gamma_r\}$  becomes a strong (and weak) deviation from  $\Pi$ , or
- $\Pi(\varepsilon_r) = \{\varepsilon_r\}$  in which case  $\{\alpha_r, \varepsilon_r\}$  is a strong (and weak) deviation from  $\Pi$ .

Similarly, we have  $\Pi(\alpha_r) \neq \{\alpha_r, \varepsilon_r\}$ . Moreover,  $\sum_{i \in \Pi(\alpha_r)} v_{\alpha_r}(i) \geq 17$  for each  $r \in R$ ; otherwise  $\{\alpha_r, \beta_r\}$  becomes a strong (and weak) deviation from  $\Pi$ . It follows that, for each  $r \in R$ , there exists  $s \in \mathcal{S}$  such that  $\Pi(\alpha_r) = \{\zeta_s\} \cup \{\alpha_r \mid r \in s\}$ . Therefore

$$\{s \in \mathcal{S} \mid \{\alpha_r \mid r \in s\} \subseteq \Pi(\zeta_s)\}$$

is a partition of  $R$ . ■

### 3.2 Strict core stability

Consider again the instance  $(R, \mathcal{S})$  of **E3C** with  $R = \{a, b, c, d, e, f\}$  and  $\mathcal{S} = \{\{a, b, c\}, \{c, d, e\}, \{d, e, f\}\}$ , and the game used in the proof of Theorem 1. Let  $s^* = \{c, d, e\}$ . Since  $\mathcal{S}' = \{\{a, b, c\}, \{d, e, f\}\}$  is a partition of  $R$ , we already know that the coalition structure

$$\begin{aligned} \Pi = & \{ \{ \zeta_s \} \cup \{ \alpha_r \mid r \in s \} \mid s \in \mathcal{S}' \} \\ & \cup \{ \zeta_{s^*} \} \cup \{ \{ \beta_r, \gamma_r \}, \{ \delta_r, \varepsilon_r \} \mid r \in R \} \end{aligned}$$

is core stable. However,  $\Pi$  is not strictly core stable since the coalition  $\{ \alpha_c, \alpha_d, \alpha_e, \zeta_{s^*} \}$  is a weak deviation from  $\Pi$ :  $\alpha_c, \alpha_d$ , and  $\alpha_e$  are all indifferent between  $\{ \alpha_c, \alpha_d, \alpha_e, \zeta_{s^*} \}$  and their corresponding coalitions according to  $\Pi$ ; and  $\zeta_{s^*}$  strictly prefers  $\{ \alpha_c, \alpha_d, \alpha_e, \zeta_{s^*} \}$  over  $\Pi(\zeta_{s^*}) = \{ \zeta_{s^*} \}$ .

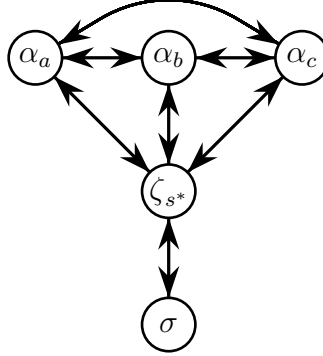


Figure 2:

Hence, when constructing the game in the proof of our next result, we have to eliminate all players' incentive for becoming members of a weak deviation. We do this by introducing  $|\mathcal{S} \setminus \mathcal{S}'|$  new players in the game constructed in the previous section. Each of these new players likes and is liked only by the  $\zeta_s$ -players, where the value of the corresponding utility function is 6.

The reason for selecting this value is that we are going to bring together in a coalition a player as  $\zeta_{s^*}$  above with a newly introduced player  $\sigma$ ; for  $(R, \mathcal{S})$  as above,  $\sigma$  is the only new player added to the game (see Fig. 2).

Then, both coalitions  $\{\zeta_{s^*}, \sigma\}$  and  $\{\alpha_c, \alpha_d, \alpha_e, \zeta_{s^*}\}$  would have value 6 for  $\zeta_{s^*}$  (recall that for each  $s \in \mathcal{S}$  and for each  $r \in s$ ,  $v_{\zeta_s}(\alpha_r) = 2$ ). That is, the strong incentive for player  $\zeta_{s^*}$  has been eliminated.

**Theorem 2** *HS is NP-hard in the strong sense.*

**Proof.** In addition to the additive hedonic game constructed in Theorem 1,  $\ell = |\mathcal{S}| - |R|/3 = |\mathcal{S}| - m$  new players  $\sigma_1, \sigma_2, \dots, \sigma_\ell$  are included in the game. The preferences related to these newly added players are as follows.

- For each  $s \in \mathcal{S}$  and for each  $1 \leq k \leq \ell$ ,  $v_{\zeta_s}(\sigma_k) = v_{\sigma_k}(\zeta_s) = 6$ .

Observe that the number of players remains polynomial of  $|R|$  and  $|\mathcal{S}|$ , and all parameters are bounded above by a constant. Again, we show that there exists a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$  if and only if there exists a core stable coalition structure  $\Pi$ .

( $\Rightarrow$ ) Suppose there exists a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ . Let  $\{s_1, s_2, \dots, s_\ell\} = \mathcal{S} \setminus \mathcal{S}'$ . Then, consider the following partition of  $N$ .

$$\begin{aligned} \Pi &= \{ \{\zeta_s\} \cup \{\alpha_r \mid r \in s\} \mid s \in \mathcal{S}' \} \\ &\quad \cup \{ \{\zeta_{s_1}, \sigma_1\}, \{\zeta_{s_2}, \sigma_2\}, \dots, \{\zeta_{s_\ell}, \sigma_\ell\} \} \\ &\quad \cup \{ \{\beta_r, \gamma_r\}, \{\delta_r, \varepsilon_r\} \mid r \in R \}. \end{aligned}$$

As shown in the proof of Theorem 1, there is no weak deviation from  $\Pi$  containing anyone of  $\alpha_r, \beta_r, \gamma_r, \delta_r$ , and  $\varepsilon_r$  for all  $r \in R$ . Moreover, for each  $s \in \mathcal{S}$  and for each  $X \subseteq N$ ,

$$\sum_{i \in X} v_{\zeta_s}(i) > 6 \text{ only if } v_i(j) < 0 \text{ for some } i, j \in X,$$

and  $\sum_{i \in X} v_{\zeta_s}(i) = 6$  only if either

- $X = \{\zeta_s\} \cup \{\alpha_r \mid r \in s\}$ , or
- $X = \{\zeta_s, \sigma_k\}$  for some  $1 \leq k \leq \ell$ .

Notice that, for each  $X \subseteq N$ ,

$$\sum_{i \in \Pi(\alpha_r)} v_{\alpha_r}(i) \geq \sum_{i \in X} v_{\alpha_r}(i)$$

and

$$\sum_{i \in \Pi(\sigma_k)} v_{\sigma_k}(i) \geq \sum_{i \in X} v_{\sigma_k}(i).$$

Thus, for each  $s \in S$ , there is no weak deviation from  $\Pi$  containing  $\zeta_s$ . Therefore  $\Pi$  is strictly core stable.

( $\Leftarrow$ ) Suppose there exists a core stable coalition structure  $\Pi$  for the constructed additive hedonic game. By the same argument as in the proof of Theorem 1,

$$\{s \in \mathcal{S} \mid \{\alpha_r \mid r \in s\} \subseteq \Pi(\zeta_s)\}$$

is a partition of  $R$ . ■

## 4 Individual deviations and stability

Now let us introduce the stability concepts for hedonic games, which are based on individual deviations. Let  $\langle N, \succeq \rangle$  be a hedonic game, and  $\Pi$  be a coalition structure of  $N$ . We say that,

- $\Pi$  is a *Nash stable* in  $\langle N, \succeq \rangle$  if, for each  $i \in N$  and for each  $X \in \Pi \cup \{\emptyset\}$ ,

$$- \Pi(i) \succeq_i X \cup \{i\}$$

- $\Pi$  is a *individually stable* in  $\langle N, \succeq \rangle$  if, for each  $i \in N$  and for each  $X \in \Pi \cup \{\emptyset\}$ ,
  - $\Pi(i) \succeq_i X \cup \{i\}$ , or
  - there exists  $j \in X$  such that  $X \cup \{i\} \prec_j X$ .

In other words,  $\Pi$  is Nash stable if no player is strictly better off by either staying alone or by moving to another coalition in  $\Pi$ . Individual stability additionally requires that the coalitions in  $\Pi$  do not accept entering members who make some player worse off.

The first decision problem we consider in this section is the following:

**Existence of a Nash stable coalition structure (HN):**

**Instance:** A hedonic game  $\langle N, \succeq \rangle$ , where  $N$  is a set of players and  $\succeq$  is an additive preference profile.

**Question:** Is there a coalition structure which is Nash stable in  $\langle N, \succeq \rangle$ ?

Analogously, the decision problem of the existence of an individually stable coalition structure is as follows.

**Existence of an individually stable coalition structure (HI):**

**Instance:** A hedonic game  $\langle N, \succeq \rangle$ , where  $N$  is a set of players and  $\succeq$  is an additive preference profile.

**Question:** Is there a coalition structure which is individually stable in  $\langle N, \succeq \rangle$ ?

The existence problem **HN** and **HI** in the general setting is considered in Ballester (2004), and are shown to be NP-complete. Notice gain that the

NP-hardness of either of the above problems does not imply that the other problem is also NP-hard. The reason is similar to the one about **HC** and **HS** from the previous section and it is based on the fact that Nash stability implies individual stability but the reverse implication does not hold.

We show in what follows that, when an additive hedonic game is under consideration, *both* **HN** and **HI** are NP-complete in the strong sense<sup>2</sup>. For this, we first show that these problems lie in NP.

**Lemma 1** ***HN** and **HI** belong to NP.*

**Proof.** In order to show that **HN** belongs to NP, it suffices to provide a polynomial time algorithm for the following test.

- For a given hedonic game  $\langle N, \succeq \rangle$  and a given coalition structure  $\Pi$ , test whether  $\Pi$  is Nash stable in  $\langle N, \succeq \rangle$ .

This test can be done in an obvious way, i.e., test whether  $\Pi(i) \succeq_i X \cup \{i\}$  for each  $i \in N$  and for each  $X \in \Pi \cup \{\emptyset\}$ . Observe that the test whether  $\Pi(i) \succeq_i X \cup \{i\}$  can be in  $O(n)$  time<sup>3</sup>, because players' preferences are additive. From  $|\Pi| \leq n$ , the test, whether  $\Pi$  is Nash stable in  $\langle N, \succeq \rangle$ , can be done in  $O(n^3)$  time.

Similarly, **HI** belongs to NP, because for a given hedonic game  $\langle N, \succeq \rangle$  and a given coalition structure  $\Pi$ , the test, whether  $\Pi$  is individually stable in  $\langle N, \succeq \rangle$ , can be done in  $O(n^4)$  time. ■

In order to show the NP-hardness of these two existence problems we use polynomial time reductions again from **E3C**.

<sup>2</sup> Thus, our result is stronger than the corresponding result in Olsen (2007). In that work, the NP-completeness of **HN** is shown by reduction from a problem (PARTITION) which is known to be NP-hard but not in the strong sense.

<sup>3</sup> That is, the running time of the test is bounded from above by  $n$ , up to a constant factor.

## 4.1 Nash stability

More precisely, for the problem **HN**, we use in our reduction a similar trick to the one used in the construction of the games for **HC** and **HS**. That is, given an instance  $(R, \mathcal{S})$  of **E3C**, we attach a game with no Nash stable coalition structure to each element of  $R$  and add an additional player for each  $s \in \mathcal{S}$ .

The game we attach to each  $r \in R$  is very simple: the player set is  $\{\alpha_r, \beta_r\}$  with  $v_{\alpha_r}(\beta_r) < 0$  and  $v_{\beta_r}(\alpha_r) > 0$ . Observe that there are only two coalition structures  $\{\{\alpha_r\}, \{\beta_r\}\}$  and  $\{\{\alpha_r, \beta_r\}\}$ , and neither of them is Nash stable, because player  $\alpha_r$  prefers to be alone (i.e.,  $\{\alpha_r\} \succ_{\alpha_r} \{\alpha_r, \beta_r\}$ ) but player  $\beta_r$  prefers to be with player  $\alpha_r$  (i.e.,  $\{\alpha_r, \beta_r\} \succ_{\beta_r} \{\alpha_r\}$ ). Thus, when defining the corresponding utilities with respect to the additional player  $\zeta_s$  with  $r \in s$ , the task will be to provide a good incentive for  $\beta_r$  to stick together with  $\zeta_s$ . Notice that such an incentive should be given to each  $\beta_{r'}$  with  $r' \in s$  and hence, any two players  $\beta_r$  and  $\beta_{r'}$  with  $r, r' \in s$  and  $r \neq r'$  should also like each other; otherwise, one of these players would be strictly better off by staying single no matter how much he likes  $\zeta_s$ .

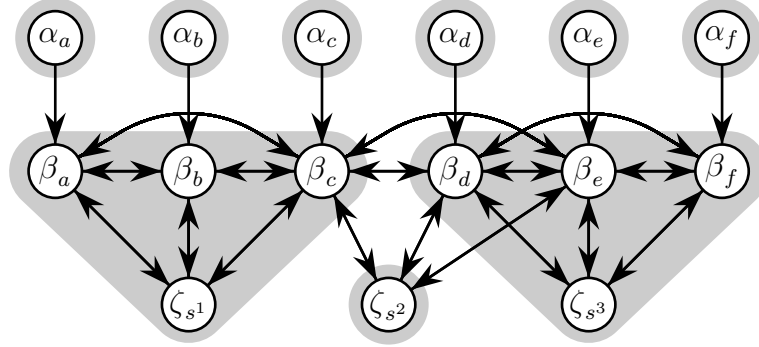


Figure 3:

Having done this, we can consider the coalition structure shown in Fig. 3. It consists of

- all  $\alpha_r$  players being single,
- the coalitions, each of which consisting of all  $\beta_r$  players together with  $\zeta_s$ , where  $r \in s$  and  $s \in \mathcal{S}'$ , and
- all  $\zeta_{s'}$  players for  $s' \in \mathcal{S} \setminus \mathcal{S}'$  being single.

As we show next, we can use the above construction as to prove the NP-hardness of **HN**.

**Theorem 3** *HN is NP-Complete.*

**Proof.** Let  $(R, \mathcal{S})$  be an instance of **E3C**. From  $(R, \mathcal{S})$ , an instance of **HN**, i.e., an additive hedonic game  $\langle N, \succeq \rangle$ , is constructed in polynomial time of  $|R|$  and  $|\mathcal{S}|$ .

Let  $N = \{\alpha_r, \beta_r \mid r \in R\} \cup \{\zeta_s \mid s \in \mathcal{S}\}$ . Players' preference are defined as follows.

- $v_{\beta_r}(\beta_{r'}) = 2$  for all  $r, r' \in R$  with  $r \neq r'$ ,
- $v_{\beta_r}(\zeta_s) = 2|R|$  and  $v_{\zeta_s}(\beta_r) = 1$  if  $r \in s$ ,
- $v_{\beta_r}(\alpha_r) = 2|R| + 3$  for all  $r \in R$ .

The remaining  $v_i(j)$ s are negative, and each of which is defined by

$$v_i(j) = -(6|R| + 2).$$

Observe that  $\sum_{j \in N} \max\{v_i(j), 0\} \leq 6|R| + 1$  for each  $i \in N$ . Thus,  $\{i\} \succ_i X$  if  $v_i(j) < 0$  for some  $j \in X \subseteq N$ , i.e.,  $X$  is not individually rational for  $i$ . In other words,

- a coalition  $X$  is individually rational if and only if  $v_i(j) > 0$  for all  $i, j \in X$ .

Notice that a coalition structure  $\Pi$  is Nash stable only if every coalition in  $\Pi$  is individually rational. Hence, only individually rational coalitions are considered in the following.

Now, we have the following observations.

- for each  $X \in \mathcal{A}^{\alpha_r}$  with  $X \neq \{\alpha_r\}$ ,  $X$  is not individually rational since  $\{\alpha_r\} \succ_{\alpha_r} X$ .

For each  $s \in \mathcal{S}$ , let  $P_s = \{\beta_r \mid r \in s\} \cup \{\zeta_s\}$ . Then,

- for each  $s \in \mathcal{S}$ ,  $X \in \mathcal{A}^{\zeta_s}$  is individually rational only if  $X \subseteq P_s$ , since  $\{i \in N \mid v_{\zeta_s}(i) > 0\} = P_s$ ; and
- for each  $r \in s$  and for each  $X \in \mathcal{A}^{\beta_r}$ ,  $X$  is individually rational and  $X \succeq_{\beta_r} P_s$  if and only if  $X = P_s$ .

Suppose now that there exists a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ . Then, consider the following coalition structure.

$$\{\{\alpha_r\} \mid r \in R\} \cup \{P_s \mid s \in \mathcal{S}'\} \cup \{\{\zeta_s\} \mid s \in \mathcal{S} \setminus \mathcal{S}'\}.$$

Observe that all  $\alpha_r$ s and all  $\beta_r$ s have no incentive to deviate. Moreover, for each  $s \in \mathcal{S}$ , each  $X \in \Pi \setminus \{\Pi(\zeta_s)\}$  contains a member  $i$  such that  $v_{\zeta_s}(i) < 0$ , and thus, each  $\zeta_s$  has no incentive to deviate. Hence,  $\Pi$  is Nash stable.

Suppose next that there exists a coalition structure  $\Pi$  which is Nash stable in the above additive hedonic game. From the above observations, we have  $\Pi(\alpha_r) = \{\alpha_r\}$  for each  $r \in R$ . It follows that, for each  $r \in R$ , we have  $\Pi(\beta_r) \subseteq \{\beta_{r'} \mid r' \in R\}$  or  $\Pi(\beta_r) \subseteq P_s$  for some  $s \in \mathcal{S}$ ; otherwise  $\Pi(\beta_r)$  is not individually rational. Moreover, if  $\Pi(\beta_r) \neq P_s$  for some  $s \in \mathcal{S}$ , we have

$$\sum_{i \in \Pi(\beta_r)} v_{\beta_r}(i) \leq 2|R| + 2 < v_{\beta_r}(\alpha_r)$$

and thus,  $\Pi(\alpha_r) \cup \{\beta_r\} \succ_{\beta_r} \Pi(\beta_r)$ , which implies that  $\Pi$  is not Nash stable. Therefore, for each  $r \in R$ , there exists  $s \in \mathcal{S}$  such that  $\Pi(\beta_r) = P_s$ . In other words, we have

$$\bigcup_{s \in \mathcal{S}'} s = R,$$

where

$$\mathcal{S}' = \{s \in \mathcal{S} \mid P_s \in \Pi\}.$$

Since  $\Pi$  is a partition of  $N$ ,  $\mathcal{S}'$  is a partition of  $R$ . Hence, from Lemma 1, we can conclude that **HN** is NP-complete. ■

Notice that in the proof of the above theorem we did not use the fact that **E3C** remains NP-complete even when restricted to instances in which each  $r \in R$  occurs in at most three members of  $\mathcal{S}$ . By considering those instances satisfying such a restriction, we are able to show a stronger result.

**Theorem 4** *HN is NP-complete in the strong sense.*

**Proof.** Let  $(R, \mathcal{S})$  be an instance of **E3C** such that each  $r \in R$  occurs in at most three members of  $\mathcal{S}$ . From  $(R, \mathcal{S})$ , an instance of **HN** is constructed as follows.

Let  $N = \{\alpha_r, \beta_r \mid r \in R\} \cup \{\zeta_s \mid s \in \mathcal{S}\}$ . Players' preferences are defined as follows.

- $v_{\beta_r}(\beta_{r'}) = 2$  if  $r \neq r'$  and  $r, r' \in s$  for some  $s \in \mathcal{S}$ .
- $v_{\beta_r}(\zeta_s) = 13$  and  $v_{\zeta_s}(\beta_r) = 1$  if  $r \in s$ ,
- $v_{\beta_r}(\alpha_r) = 16$  for all  $r \in R$ .

All the remaining  $v_i(j)$ s are negative, and each of them is defined by

$$v_i(j) = -42.$$

By a similar argument to the one in the proof of Theorem 3, it can be verified that such an additive hedonic game has a Nash stable coalition structure if and only if there exists a sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ . Observe that each  $v_i(j)$  is bounded above and below by constants. Therefore, **HN** is NP-complete in the strong sense. ■

## 4.2 Individual stability

In the reduction of **HI** from **E3C**, we use the following trick. Let  $w$  and  $z$  be real numbers satisfying  $w > z > 0$  and let us consider the following additive hedonic game with 5 players.

- $N = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ ,
- $v_{\alpha_1}(\alpha_2) = v_{\alpha_2}(\alpha_3) = v_{\alpha_3}(\alpha_4) = v_{\alpha_4}(\alpha_5) = v_{\alpha_5}(\alpha_1) = w$ ,
- $v_{\alpha_1}(\alpha_5) = v_{\alpha_2}(\alpha_1) = v_{\alpha_3}(\alpha_2) = v_{\alpha_4}(\alpha_3) = v_{\alpha_5}(\alpha_4) = z$ , and
- all the remaining  $v_{\alpha_i}(\alpha_j)$ s are defined by  $v_{\alpha_i}(\alpha_j) = -(w + z + 1)$ .

It is shown that this game does not have an individually stable coalition structure (cf. Example 5 in Bogomolnaia and Jackson (2002)). Now suppose a player  $\beta$  is introduced such that

- $v_\beta(\alpha_1) = v_{\alpha_1}(\beta) = z$ ,
- $v_{\alpha_j}(\beta) = v_\beta(\alpha_j) = -(w + z + 1)$  for  $j \in \{2, 3, 4, 5\}$ .

Then, this hedonic game has an individually stable coalition structure, namely  $\{\{\alpha_1, \beta\}, \{\alpha_2, \alpha_3\}, \{\alpha_4, \alpha_5\}\}$ .

The structure of the above game can be generalized as follows. Let  $\ell$  be a positive integer with  $\ell > 2$ . Define the player set to be  $N = A \cup B$  with

- $A = \{\alpha_j^k \mid j \in \{1, 2, 3, 4, 5\}, k \in \{1, \dots, \ell - 1\}\}$  and  $B = \{\beta_k \mid k \in \{1, \dots, \ell\}\}$ ,

and players' preferences be as follows. For each  $k \in \{1, 2, \dots, \ell - 1\}$ ,

- $v_{\alpha_1^k}(\alpha_2^k) = v_{\alpha_2^k}(\alpha_3^k) = v_{\alpha_3^k}(\alpha_4^k) = v_{\alpha_4^k}(\alpha_5^k) = v_{\alpha_5^k}(\alpha_1^k) = w$ ,
- $v_{\alpha_1^k}(\alpha_5^k) = v_{\alpha_2^k}(\alpha_1^k) = v_{\alpha_3^k}(\alpha_2^k) = v_{\alpha_4^k}(\alpha_3^k) = v_{\alpha_5^k}(\alpha_4^k) = z$ , and

For each  $k \in \{1, 2, \dots, \ell - 1\}$  and  $k' \in \{1, 2, \dots, \ell\}$ ,

- $v_{\beta_{k'}}(\alpha_1^k) = v_{\alpha_1^k}(\beta_{k'}) = z$ ,
- $v_{\alpha_j^k}(\beta_{k'}) = v_{\beta_{k'}}(\alpha_j^k) = -(w + z + 1)$  for  $j \in \{2, 3, 4, 5\}$ .

Moreover, each of the remaining  $v_i(j)$ s is defined as  $v_i(j) = -(w + z + 1)$ . For this hedonic game, a coalition structure  $\Pi$  is individually stable only if, for each  $k \in \{1, 2, \dots, \ell - 1\}$ , there exists  $k' \in \{1, 2, \dots, \ell\}$  such that  $\Pi(\alpha_1^k) = \{\alpha_1^k, \beta_{k'}\}$ .

Let us now explain how we use the above facts in the game defined in the proof of our last result. We start by attaching a  $\beta$ -player  $\beta_{rs}$  to each  $r \in R$  and  $s \in \mathcal{S}$  with  $r \in s$ , and a  $\zeta_s$ -player to each  $s \in \mathcal{S}$ . The corresponding preferences are defined in such a way that there is a unique individually stable partition and each of its elements is of the form  $P_s = \{\beta_{rs} \mid r \in s\} \cup \{\zeta_s\}$ . Notice however, that this specification of the game and the corresponding individually stable coalition structure do not always imply that there is a subcollection  $\mathcal{S}' \subseteq \mathcal{S}$  which is a partition of  $R$ ; the reason is that there might exist two (different) players  $\beta_{rs}$  and  $\beta_{r's'}$  with  $r = r'$  and  $s \neq s'$  since an element of  $R$  may belong to more than one element of  $\mathcal{S}$ . Thus, we have to redefine the game such that for each  $r \in R$ , there exists *at most one*  $s \in \mathcal{S}$  with  $r \in s$  and  $P_s$  being an element of an individually stable partition. Roughly speaking,

we have to get rid of all but one of the players from  $\{\beta_{rs} \mid s \in \mathcal{S} \text{ s.t. } r \in s\}$  as being members of a coalition in an individually stable partition. For this, we use the 5-player game defined above and illustrate the procedure in Fig. 4 for  $\{\beta_{r's'} \mid s \in \mathcal{S} \text{ s.t. } r \in s\} = \{\beta_{rs^1}, \beta_{rs^2}, \beta_{rs^3}\}$ .

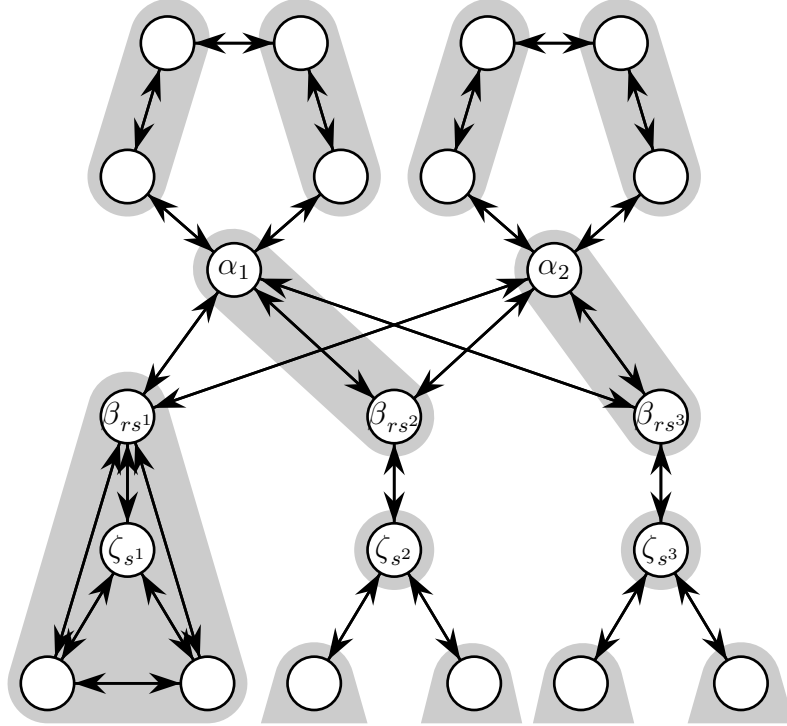


Figure 4:

Let the players we would like to get rid off be  $\beta_{rs^2}$  and  $\beta_{rs^3}$ . Let us then add  $\beta_{rs^2}$  to the above 5-player game and do the same operation for  $\beta_{rs^3}$  with respect to an analogous 5-player game. Then, in an individually stable partition,  $\beta_{rs^2}$  and  $\beta_{rs^3}$  will be attracted by the corresponding  $\alpha$ -players. The final result of this construction, together with the specification of players' preferences, is that it singles out, for each  $r \in R$ , only one  $s \in \mathcal{S}$  ( $s = s^1$ ) with  $r \in s$  that guarantees the existence of a sub-collection of  $\mathcal{S}$

which partitions  $R$ .

**Theorem 5** *HI is NP-complete in the strong sense.*

**Proof.** Let  $(R, \mathcal{S})$  be an instance of **E3C**. In order to avoid trivial cases, we assume that each  $r \in R$  is included in at least one member of  $\mathcal{S}$ . From  $(R, \mathcal{S})$ , an instance of **HI** is constructed as follows.

Let  $M = \{\beta_{rs} \mid s \in \mathcal{S}, r \in s\} \cup \{\zeta_s \mid s \in \mathcal{S}\}$  be a set of  $4|\mathcal{S}|$  players. Players' preferences are defined as follows.

- $v_{\beta_{rs}}(\zeta_s) = 1$  and  $v_{\zeta_s}(\beta_{rs}) = 1$ .
- $v_{\beta_{rs}}(\beta_{r's}) = 1$  if  $r \neq r'$ , and
- $v_{\beta_{rs}}(\beta_{rs'}) = 0$  if  $s \neq s'$ .

All the remaining  $v_i(j)$ s with  $i, j \in M$  is defined by  $v_i(j) = -4$ . Let  $P_s = \{\beta_{rs} \mid r \in s\} \cup \{\zeta_s\}$  for each  $s \in \mathcal{S}$ . Observe that among players belonging to  $M$ , individually stable coalition structure exists and is unique, namely,

$$\Pi = \{P_s \mid s \in \mathcal{S}\}.$$

Now we introduce more players in order to have an additive hedonic game such that a coalition structure  $\Pi$  is individually stable if and only if  $\{s \in \mathcal{S} \mid P_s \in \Pi\}$  is a partition of  $R$ . In other words, an additive hedonic game is constructed in such a way that a coalition structure  $\Pi$  is not individually stable if

- there exist  $s, s' \in \mathcal{S}$  such that  $s \neq s'$ ,  $s \cap s' \neq \emptyset$ , and  $P_s, P_{s'} \in \Pi$ , or
- there exists  $r \in R$  such that  $\Pi(\beta_{rs}) \neq P_s$  for each  $s \in \mathcal{S}$ .

Suppose  $r \in R$  is included in  $\ell$  members of  $\mathcal{S}$ , where  $\ell > 1$ . Then, by using the trick mentioned above, and introducing  $5(\ell - 1)$  new players with  $w = 2$  and  $z = 1$ , a coalition structure  $\Pi'$  is individually stable only if

- for each  $r \in R$ , there exists at most one  $s \in \mathcal{S}$  such that  $r \in s$  and  $P_s \in \Pi'$ .

Moreover, we have  $\{\beta_{rs}, \zeta_s\} \succ_{\beta_{rs}} \{\beta_{rs}\}$ , and for each  $X \in \mathcal{A}^{\zeta_s}$ ,  $X \succeq_{\zeta_s} \{\zeta_s\}$  if and only if  $X \subseteq P_s$ . Hence, a coalition structure  $\Pi'$  is individually stable only if  $\Pi'(\beta_{rs}) = \Pi'(\zeta_s)$  for some  $s \in \mathcal{S}$  with  $r \in s$ . Moreover, if  $\Pi'(\beta_{rs}) = \Pi'(\zeta_s) \subseteq P_s$  but  $\Pi'(\zeta_s) \neq P_s$ , then  $\Pi'(\zeta_s) \cup \{\beta_{r's}\} \succ_{\beta_{r's}} \Pi'(\beta_{r's})$  for some  $r' \in s$ . It follows that a coalition structure  $\Pi'$  is individually stable only if

- for each  $r \in R$ , there exists  $s \in \mathcal{S}$  such that  $r \in s$  and  $P_s \in \Pi'$ .

Conversely, it can be verified that an individually stable coalition structure  $\Pi'$  exists among all coalition structures satisfying the above conditions.

Finally, the hedonic game we have constructed has  $9|\mathcal{S}| - |R|$  players and each  $v_i(j)$  is bounded below and above by constants. Therefore, **HI** is NP-complete in the strong sense. ■

## 5 Concluding remarks

We provided reductions from the NP-complete problem Exact Cover by 3 Sets that demonstrate that in additive hedonic games:

- it is NP-hard in the strong sense to determine (1) whether a core partition exists, and (2) whether a strict core stable partition exists. Moreover,

- the problem of deciding (1) whether a Nash stable partition exists, and (2) whether an individually stable partition exists are NP-complete in the strong sense.

In all reductions we used procedures with some common properties we would like to stress now. Given an instance  $(R, \mathcal{S})$  of **E3C**, all additive hedonic games were constructed by respecting the following pattern. A set of players was first attached to each element of  $R$ ; these players were involved in a basic additive game with no stable coalition structure. Each of the two types of basic games (the 5-player game and the 2-player game) were selected in such a way as to have a boundary property in the sense that when adding or removing a player we were able to construct games for which a stable coalition structure do exist. Then, the operations of adding or removing a player from the basic games were done by attaching a player to each element of  $\mathcal{S}$ . This common pattern allowed us to derive a sub-collection of  $\mathcal{S}$  which is a partition of  $R$  in order to complete the corresponding reductions.

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