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On the Number of 1-Factors of Locally Finite Graphs*

FRANÇOIS BRY

Université Pierre et Marie Curie, U.E.R. 48,
Équipe de Recherche Combinatoire,
4, place Jussieu, 75005 Paris, France

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Every infinite locally finite graph with exactly one 1-factor is at most 2-connected is shown. More generally a lower bound for the number of 1-factors in locally finite n-connected graphs is given.

1. INTRODUCTION

Kotzig has shown in [8] that every factorizable 2-edge-connected finite graph has at least two 1-factors. This result does not extend to infinite graph: there are 2-edge-connected infinite locally finite graphs with exactly one 1-factor (see Example 3.2). However, the following theorem holds:

Every locally finite graph with exactly one 1*-factor is at most* 2*-connected.* (Theorem 3.3), and then at most 2-edge-connected since the n-connectivity is a strengthening of the n-edge-connectivity.

Kotzig's theorem is actually a first step in the study of the number $f(G)$ of 1-factors of a finite graph G. Other contributions are due to Beineke and Plummer [1]($f(G) \geqslant n$ if G is n-connected) and Zaks [14]($f(G) \geqslant n!!$[1] if G is n-connected). Lovász [9] improved Zaks' theorem in certain cases. Mader [11] has given an exact lower bound depending on the minimal degree. Previously M. Hall [7] has given such a bound in the special case of bipartite graphs ($f(G) \geqslant n!$ if G is a bipartite graph with minimal degree n).

Other results presented in this note estimate the number of 1-factors of locally finite infinite graphs:

For all n there are n-connected locally finite infinite graphs with a finite number of 1*-factors.*

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[1] For a positive integer n, $n!!$ denote $n \cdot (n-2) \cdots 4.2.1$ if n is even and $n \cdot (n-2) \cdots 5.3.1$ if n is odd.

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A factorizable locally finite n-connected graph has at least $n!!/2$ 1-factors if n is even, and at least $\frac{2}{3}n!!$ 1-factors if n is odd.

This last theorem is improved in certain cases.

A new proof of Zaks' theorem is given.

2. Notations and Terminology

Graphs considered in this article are undirected without loops or multiple edges.

Let $G=(V,E)$ be a graph. A *1-factor*, or *perfect matching*, of G is a set of pairwise disjoint edges of G containing all vertices [2]. We say that G is *factorizable* if it contains at least one 1-factor, and *uniquely factorizable* if it contains exactly one 1-factor.

A finite graph is said to be *1-factor critical* if by deleting any vertex one obtains a factorizable graph. A 1-factor critical graph has clearly an odd number of vertices.

We denote by $C_1(G)$ the number of connected components with odd cardinalities of G, and by $C_{cr}(G)$ the number of connected components of G which are 1-factor critical.

Given $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by S. If no confusion results we abbreviate $C_1(G[S])$ and $C_{cr}(G[S])$ to $C_1(S)$ and $C_{cr}(S)$, respectively.

A graph is *locally finite* if every vertex is incident to finitely many edges.

A locally finite graph is said to be *bicritical* if it is factorizable and if by deleting any two (distinct) vertices one obtains a factorizable graph. Clearly every edge of a bicritical graph G belongs to some 1-factor of G.

3. Locally Finite Graphs with Exactly One 1-Factor

PROPOSITION 3.1. *Let $G=(V,E)$ be a locally finite graph with exactly one 1-factor F. Then the following three properties are equivalent*:

(1) *There is a finite nonempty subset S of V such that*

$$C_1(V\setminus S)=|S|$$

(2) *There is an isthmus $\{x,y\}$ of G which belongs to F.*

(3) *There is an isthmus $\{x,y\}$ of G which belongs to F such that*

$$C_1(V\setminus\{x\})=1 \qquad \text{or} \qquad C_1(V\setminus\{y\})=1.$$

Proposition 3.1 is an extension to locally finite graphs of a theorem of Kotzig characterizing uniquely factorizable finite graphs [8]:

(A) *If a finite graph is uniquely factorizable, then it has an isthmus belonging to the unique* 1*-factor.*

Property (1) holds trivially in any factorizable finite graph. We note that our proof of Proposition 3.1 uses Kotzig'z theorem. A proof of this theorem is given (Remark 3.4). To prove Proposition 3.1 we also use Tutte's 1-factor theorem [12]:

(B) *A locally finite graph* $G=(V,E)$ *is factorizable if and only if* $C_1(V\backslash S)\leqslant |S|$ *for all finite subsets* S *of* V.

Proof. We have clearly $(3)\Rightarrow(1)$ and $(3)\Rightarrow(2)$. We next show that $(1)\Rightarrow(3)$.

Let $C_1,\dots,C_p$ ($|S|=p$) be the odd components of $G[V\backslash S]$. Set $\tilde{X}=S\cup(\bigcup_{i=1}^p C_i)$ and $G'=G[\tilde{X}]$. If $\{s,t\}\in F$ and $s\in S$, necessarily $t\in\bigcup_{i=1}^p C_i$. Therefore G' has exactly one 1-factor F' and $F'\subseteq F$. Because of the odd cardinalities of the C_i's, there is no edge of F joining two vertices of S. Therefore we can assume that two vertices of S are adjacent in G', without forming another 1-factor of G'.

Since G' is finite, by (A) there is an edge $\{x,y\}$ of F' which is an isthmus of G' and therefore an isthmus of G. If $x\notin S$ and $y\notin S$, x and y are in the same component, say C_i. Since $\{x,y\}$ is an isthmus of G', there is a partition $C_i=X+Y$ with $x\in X$ and $y\in Y$, and one and only one of the two sets X and Y is adjacent to S. If X is adjacent to S, we have

$$C_1(G'[\tilde{X}\backslash\{x\}])=1$$

and then

$$C_1(G[V\backslash\{x\}])=1.$$

If $x\in S$, then $y\in C_i$ and we have

$$C_1(G[V\backslash\{x\}])=1.$$

So property (3) holds.

We finally show that $(2)\Rightarrow(1)$. Let $e=\{x,y\}$ be an isthmus of G belonging to F. Let X and Y denote the connected components of $G-e=(V,E\backslash\{e\})$ such that $x\in X$ and $y\in Y$. If (1) does not hold, we have

$$C_1(G[V\backslash(S\cup\{x\})])\leqslant|S|$$

for all finite subsets S of $V\backslash\{x\}$. Therefore by the 1-factor theorem (B), $G[V\backslash\{x\}]$ has a 1-factor L_x. The connected component of $(V,\ L_x\cup F)$ containing x is necessarily an infinite alternating path P_x issued from x. Clearly P_x has no other vertex in X than x. Similarly $G[V\backslash\{y\}]$ has a 1-factor L_y and the connected component of $(V,L_y\cup F)$ containing y is an

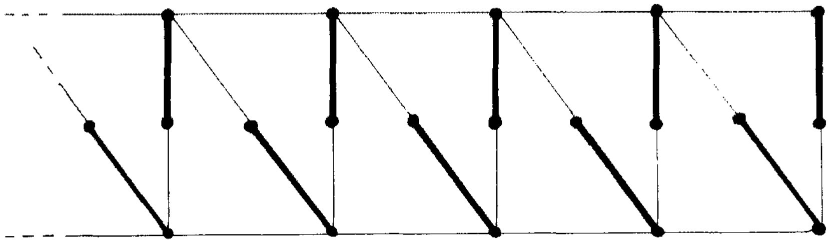

FIGURE 1

infinite alternating path P_y issued from y. The only vertex of P_y contained in Y is y. It follows that $P_x \cup P_y$ is an infinite elementary F-alternating path without end. This contradicts the uniqueness of the 1-factor F. Therefore (1) holds, achieving the proof of Proposition 3.1.

EXAMPLE 3.2. The locally finite graph depicted in Fig. 1 is 2-edge-connected and has exactly one 1-factor.

THEOREM 3.3. *Every locally finite graph with exactly one 1-factor is at most 2-connected.*

Our proof uses a strengthening of the 1-factor theorem proved in [3].

(C) *A locally finite graph $G = (V, E)$ has a 1-factor if and only if $C_{cr}(V \setminus S) \leqslant |S|$ for all finite subsets S of V.*

In the finite case, this result seems to be well known. However, we have been unable to find an explicit reference in the literature. Papers [4, 6] can be given as implicit references.

Proof. Let $G = (V, E)$ be a 3-connected infinite locally finite graph. Assume that G has exactly one 1-factor F. Let $e = \{x, y\}$ be an edge of F, and let G' denote the subgraph $(V, E \setminus \{e\})$. G' is not factorizable, and hence by (C) there exists a finite subset T of V such that

$$C_{cr}(G'[V \setminus T]) \geqslant |T| + 1.$$

Since G is 3-connected and uniquely factorizable, by Proposition 3.1 we have

$$C_{cr}(G[V \setminus T]) \leqslant C_1(G[V \setminus T]) \leqslant |T| - 1.$$

Hence e connects two 1-factor critical components A and B of $G'[V \setminus T]$, and we have

$$C_{cr}(G'[V \setminus T]) = |T| + 1$$

and

$$C_{cr}(G[V\setminus T]) = |T| - 1.$$

Note that T separates $A \cup B$, finite, from the (infinite) remaining of G; hence $|T| \geqslant 3$, since G is 3-connected. Let $C_1, \ldots, C_p(|T| = p+1, p \geqslant 2)$ be the 1-factor critical components of $G[V\setminus T]$. There is exactly one edge of F joining T and C_i for $i = 1, \ldots, p$. Otherwise, since all $|C_i|$ are odd, one C_i would be joined to T by at least 3 edges of F and there would be at least $p+2$ edges of F incidents to T, which is impossible.

Let t_i be the vertex of T incident to the edge of F touching C_i. Consider the bipartite graph H on vertex set $\{C_1, \ldots, C_p\} \cup \{t_1, \ldots, t_p\}$ with an edge $\{C_i, t_j\}$ if and only if C_i is adjacent to t_j in G. Since G is 3-connected, the degree of each C_i in H is at least 2, and hence by a theorem of Hall [7], H has at least two 1-factors. Now each of them can be enlarged into a 1-factor of $G[C_1 \cup \cdots \cup C_p \cup \{t_1, \ldots, t_p\}]$, since the C_i's are 1-factor critical, and hence into a 1-factor of G. It follows that G has more than one 1-factor, contradicting our assumption.

Remark 3.4. The proof of Theorem 3.3 contains a proof due to Mader [10] of Kotzig's theorem (A), which we give for completeness.

Let $G = (V, E)$ be a finite 2-connected graph. Assume that G has a unique 1-factor F. Let $e \in F$ and let G' denote the subgraph $(V, E\setminus\{e\})$. G' is not factorizable, therefore by Theorem (C) there is a subset T of V such that

$$C_{cr}(G'[V\setminus T]) \geqslant |T| + 1.$$

Since G is factorizable, we have $C_{cr}(G[V\setminus T]) \leqslant |T|$ and hence e connects two 1-factor critical components of $G'[V\setminus T]$ and we have $C_{cr}(G'[V\setminus T]) = |T| + 1$. Since e is not an isthmus of G, we have $T \neq \varnothing$. Since G is 2-connected, we have $|T| \geqslant 2$ and every 1-factor critical component of $G[V\setminus T]$ is adjacent to at least two vertices of T. The proof is achieved as above.

Remark 3.5. *For all n, there are locally finite 2-connected graphs of minimal degree n with exacctly one 1-factor.*

Proof. We first construct a finite graph G_n as follows: The graph G_1 is composed of two vertices joined by an edge (i.e., $G_1 \equiv K_2$). Suppose G_i has been constructed. Let G_i' and G_i'' be two disjoint suspensions of G_i obtained by joining two vertices v_i' and v_i'' to all vertices of two disjoint copies of G_i. The graph G_{i+1} is obtained by joining G_i' and G_i'' by the edge $\{v_i', v_i''\}$.

As is easily seen G_n is a uniquely factorizable finite graph with minimal degree n.

Let $G = (V, E)$ be a locally finite 2-connected graph with exactly one 1-factor F (Example 3.2). If x is a vertex of G with degree $k \leqslant n-1$, let $y \in V$

such that $\{x, y\} \in E \backslash F$. By joining x to every vertices of G'_{n-k-1} and y to every vertices of G''_{n-k-1}, we obtain a locally finite graph in which x is of degree at least n. Since G and G_n are factorizable, the constructed graph is also factorizable. One can easily prove that this graph has no more than one 1-factor.

4. Number of 1-Factors of n-connected Locally Finite graphs

First we give examples of n-connected locally finite infinite factorizable graphs with a finite number of 1-factors.

EXAMPLE 4.1. For $n \geqslant 3$, we define a locally finite graph T_n as follows: Let $(X_m/m \in N)$ be a sequence of pairwise disjoint sets, each of them with cardinality n. Put $X_m = \{x_1^m, ..., x_n^m\}$. T_n is the graph on vertex set $\bigcup_{m \in N} X_m$ and edge set E defined by: for m odd or $m = 0$

$$\{x_i^m, x_j^{m+1}\} \in E, \qquad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n$$

for m even and $m \neq 0$

$$\{x_i^m, x_i^{m+1}\} \in E, \qquad 1 \leqslant i \leqslant n.$$

The graph T_n is clearly n-connected and factorizable. It can easily be proved that T_n has exactly $n!$ 1-factors. See Fig. 2.

Remark 4.2. The conjecture of Van der Waerden [13], recently proved by Falikman [5] yields the lower bound $(n/p)^p p!$ on the number of 1-factors of a finite n-regular bipartite graph on p vertices.

In particular the number of 1-factors is not bounded when p tends to infinity for given $n \geqslant 3$.

The example $T_{n,p}$ below shows that this result cannot be extended to bipartite graphs with degrees at least n. The graph $T_{n,p}$ is n-connected on $2n(p+1)$ vertices and has exactly $(n!)^2$ 1-factors.

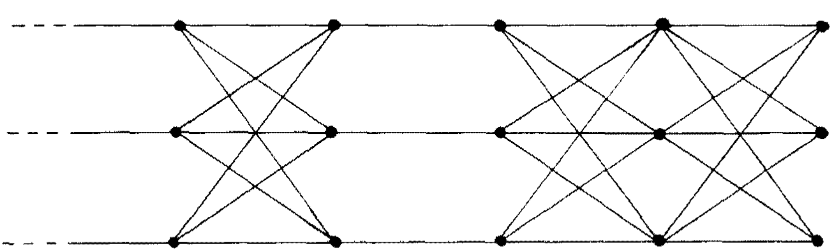

FIG. 2. The graph T_3.

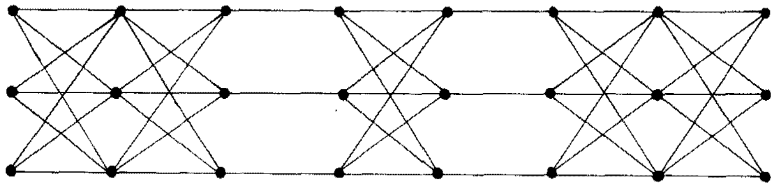

FIG. 3. The graph $T_{3,3}$.

We point out that by a slight modification of $T_{n,p}$ one can obtain a similar (not bipartite) graph with exactly $n!$ 1-factors.

Let $(X_m/0 \leqslant m \leqslant 2p+1)$ be a finite sequence of pairwise disjoint sets of cardinalities n. Put $X_m = \{x_1^m, ..., x_n^m\}$. $T_{n,p}$ is the graph on vertex-set $\bigcup_{m=0}^{2p+1} X_m$ with edge set E defined by: for $m=0$, $m=2p$ or for m odd

$$\{x_i^m, x_j^{m+1}\} \in E, \qquad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n$$

for m even and $m \neq 0$, $m \neq 2p$

$$\{x_i^m, x_i^{m+1}\} \in E, \qquad 1 \leqslant i \leqslant n.$$

See Fig. 3.

The following result extends Theorem 3.3, and is related to a theorem of Lovász.

THEOREM 4.3. *The number of* 1*-factors of a factorizable locally finite n-connected and not bicritical graph is at least* $(n-1)!$

Lovász has proved in [9] the following theorem: *The number of* 1*-factors of a factorizable finite n-connected and not bicritical graph is at least n*!

Our proof of Theorem 4.3 makes no use of this theorem of Lovász.

Proof. Let $G=(V, E)$ be a factorizable locally finite n-connected and not bicritical graph. Since G is factorizable we have $C_{\mathrm{cr}}(V \setminus S) \leqslant |S|$ for all finite subsets S of V.

Since G is not bicritical there is, by Theorem (C), a finite subset S of V such that $C_{\mathrm{cr}}(V \setminus S) \geqslant |S|-1$.

Case 1. There is a finite nonempty subset S of V such that

$$C_{\mathrm{cr}}(V \setminus S) = |S|.$$

Since S separates G and since G is n-connected, we have $|S| \geqslant n$. Let $C_1, ..., C_p$ $(p=|S|)$ be the 1-factor critical connected components of $G[V \setminus S]$. Consider the bipartite graph H on vertex set $\{C_1, ..., C_p\} \cup S$ with an edge $\{C_i, s\}$ $(s \in S)$ if and only if C_i is adjacent to s in G. Since G is n-connected, the degree of each C_i in H is at least n. By a theorem of M. Hall [7], H has at least $n!$ 1-factors. Since the C_i's are 1-factor critical every 1-factor of H can be enlarged into a 1-factor of $G[C_1 \cup \cdots \cup C_p \cup S]$, and hence into a 1-factor of G. Therefore G has at least $n!$ 1-factors.

Case 2. There is a finite nonempty subset S of V such that $C_{\mathrm{cr}}(V \backslash S) = |S| - 1$. As above $|S| \geqslant n$. Let F be a 1-factor of G. There is exactly one vertex s of S which is not joined by F to a 1-factor critical component of $G[V \backslash S]$. Let $C_1, \ldots, C_p$ $(p = |S| - 1)$ denote the 1-factor critical components of $G[V \backslash S]$. One can prove as above that $G[C_1 \cup \cdots \cup C_p \cup S \backslash \{S\}]$ has at least $(n-1)!$ 1-factors. It follows that G has at least $(n-1)!$ 1-factors.

THEOREM 4.4. *The number of* 1-*factors of a factorizable locally finite* n-*connected bicritical graph is at least* $n!!/2$ *if* n *is even, and at least* $\frac{2}{3}n!!$ *if* n *is odd.*

Proof. Let $f(n)$ denote the minimum number of 1-factors of a factorizable locally finite bicritical n-connected graph. Trivially, $f(2) \geqslant 1$ and by Theorem 3.3, $f(3) \geqslant 2$. By induction on n we prove that

$$f(n) \geqslant nf(n-2)$$

for each $n \geqslant 4$.

Let $G = (V, E)$ be a locally finite bicritical n-connected graph. If v is a vertex of G there are at least n pairwise distinct vertices $v_1, \ldots, v_n$ adjacent to v, since G is n-connected. Since G is bicritical, all the edge $\{v, v_i\}$ belong to some 1-factor of G. Let F_i be a 1-factor of G containing the edge $\{v, v_i\}$. $F_i' = F_i \backslash \{v, v_i\}$ is clearly a 1-factor of the subgraph $G_i = G[V \backslash \{v, v_i\}]$, and every 1-factor of G_i can be enlarged to a 1-factor of G containing the edge $\{v, v_i\}$. Since two 1-factors obtained from 1-factors of two distinct G_i's are clearly different, and since the G_i's are $(n-2)$-connected, it follows that G has at least $nf(n-2)$ 1-factors.

Therefore we have $f(n) \geqslant n \cdot (n-2) \cdots 4.1$, if n is even, and we have $f(n) \geqslant n \cdot (n-2) \cdots 5.2$, if n is odd.

Remark 4.5. Our proof of Theorem 4.4 is an extension to infinite graphs of a lemma due to Zaks [14].

Remark 4.6. In order to prove that every n-connected factorizable finite graph has at least $n!!$ 1-factors, Zaks needed to prove that in every n-connected factorizable finite graph there is a vertex v such that at least n edges incident to v belong to some 1-factor. This proof is long. One can easily prove Zaks' theorem by some slight modificatons of the proof of Theorem 4.3.

Let $G = (V, E)$ be a finite n-connected factorizable graph.

Case 1. There is a nonempty subset T of V such that. $C_1(V\backslash T)=|T|$. By [3] there is a nonempty subset S of V such that $C_{cr}(V\backslash S)-|S|\geqslant C_1(V\backslash T)-|T|=0$. Hence we have $V_{cr}(V\backslash S)=|S|$.

If $|S|=1$, put $S=\{s\}$. Therefore $G[V\backslash\{s\}]$ is 1-factor critical, and then every edge incident to s belongs to some 1-factor of G. Since G is n-connected there are at least n edges incident to s. Hence, by induction, since $G[V\backslash e]$ is $(n-2)$-connected for each edge e, G has at least $n\cdot(n-2)!!\geqslant n!!$ 1-factors.

If $|S|\geqslant 2$, then S separates G. Since G is n-connected, we have $|S|\geqslant n$. One can prove by the argument used in the proofs of Theorems 3.3 and 4.3 that the subgraph induced by S and the 1-factor critical components of $G[V\backslash S]$ has at least $n!$ 1-factors, and therefore G has at least $n!\geqslant n!!$ 1-factors.

Case 2. For parity reason, since G is finite, there is no subset S of V such that $C_1(V\backslash S)=|S|-1$.

Case 3. For every nonempty subset S of V we have $C_1(V\backslash S)\leqslant|S|-2$. The graph G is bicritical: see the proof of Theorem 4.4.

5. Questions

(1) It follows from Example 4.1 and Theorem 4.3 that $n!!/2\leqslant f(n)\leqslant n!$ if n is even, and $\frac{2}{3}n!!\leqslant f(n)\leqslant n!$ if n is odd. What is the exact value of $f(n)$?

(2) Using Theorem (A), one can easily construct every finite graph with exactly one 1-factor. Is there any construction of every locally finite 2-connected graph with exactly one 1-factor?

(3) An infinite locally finite bicrital graph seems to have an infinite number of 1-factors. It would be useful to prove this property.

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