COMMISSION OF THE EUROPEAN COMMUNITIES

radiation protection

Eighth Symposium on Microdosimetry

Kernforschungsanlage Jülich GmbH
Jülich, Federal Republic of Germany
27 September - 1 October 1982

Proceedings edited by
J. BOOZ and H. G. EBERT

Directorate-General
Science, Research and Development
Radiation Protection

1983 EUR 8395 EN

Opening Lecture

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THE KINEMATIC FUNDAMENTAL FORMULA

- A Theorem with Implications to Dosimetry and Microdosimetry -

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ABSTRACT: The random superposition of geometric objects - charged particle tracks and cellular structures - is a central problem of microdosimetry. Similar problems of random intersection occur in conventional dosimetry.

The fundamental formula of integral geometry relates to this problem and contains, as special cases, a variety of notable results.

The familiar Cauchy theorem and its generalizations are chosen to illustrate the fundamental formula.

Work supported by Euratom Contract BIO-286-81 D (B)

INTRODUCTION

The development of the computational approach to microdosimetry requires the exploration of underlying mathematical principles that belong largely to the field of geometrical probability. A contribution to the preceding symposium (1) has outlined - without mathematical rigor and without detailed derivation - some relevant concepts and results. The present report deals with the deeper root of certain results that may appear puzzling due to their general but seemingly isolated character. The well known Cauchy theorem together with its less familiar generalizations will be utilized as example of a relation that can be traced back to the *Fundamental Kinematic Formula of Blaschke* (see (2) or (3)). A recent result on the mean number of clumps resulting from figures randomly placed on a plane (4) would be an equally good example, and there are numerous other remarkable results of geometric probability that are rooted in the fundamental theorem.

The present note indicates merely the general nature of one of the deep relations of integral geometry. The reader is referred to an adequate treatment, such as the monograph of Santalô (3).

The mathematical essence of microdosimetry is the problem of the random overlap of two types of geometrical structures the *receptors*, i.e. certain cellular structures, and the charged particle *tracks*, i.e. the configurations of energy deposits produced by ionizing particles. The notion of the *product of two geometric objects* introduced by Minkowski determines the mean overlap in the uniform, isotropic random intersection of two geometrical objects. It encompasses the more narrow concept of the associated volume that has been introduced by Lea into radiation biology, and that permits the derivation of the frequency-mean event sizes. The variance of the overlap of two geometrical objects is determined by their *proximity functions*, that are in essence the point pair distance distributions. Applied to microdosimetry the relation permits the derivation of the dose-average event sizes (5).

In the following only one specific aspect in this general context - the mean overlap, not the variance - will be considered. This will serve to exemplify the role of integral geometry in a wider range of microdosimetric problems.

The Cauchy Theorem and its Generalizations

The Cauchy theorem (6,7) specifies the mean chord length that results when a uniform isotropic field of infinite straight lines in three-dimensional space, R_3, intercepts a convex body of volume V and surface S:

$$\lambda = 4V/S \qquad \text{in } R_3 \tag{1}$$

The relation for a convex figure of perimeter s and area a in two-dimensional space is:

$$\lambda = \pi a/s \qquad \text{in } R_2 \tag{2}$$

The result (see Fig.1) is notable for its generality. It is even more striking that the relation applies equally for non-convex volumes or

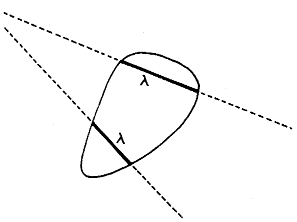

Fig. 1

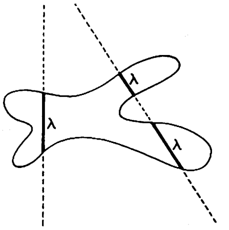

Fig. 2

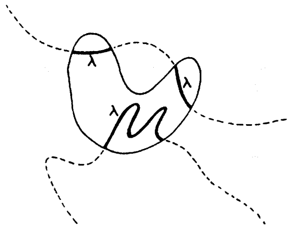

Fig. 3

figures, provided one considers each connected intercept separately, as indicated in Fig.2. However, one can go further, and finds that the same relations remain valid with curved random lines as in Fig.3. A uniform and isotropic distribution of the random lines (or the domains) is the only precondition.

A further step is the consideration of random lines that are of finite length; the Cauchy theorem retains its validity in a slightly modified form:

$$\overline{\lambda} = (1/\,\overline{l} + 1/\,\overline{t}\,)^{-1} \qquad (3)$$

$\overline{\lambda}$ is the mean length of intercepts, with individual segments taken separately as indicated in Fig.4. $\overline{t}$ is the mean length of the random lines, while $\overline{l}$ equals 4V/S in R_3 and πa/s in R_2.

Even this is not the ultimate generalization. The line structures may

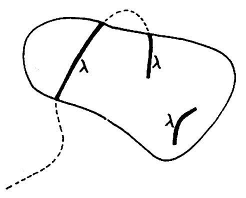

Fig. 4

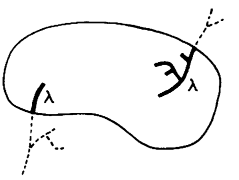

Fig. 5

also be branched as represented in Fig.5, and Eq(3) will remain valid. However, the structures must not contain loops, a somewhat puzzling restriction at this point but one that will be understood on the basis of the subsequent considerations.

The proof of the generalized Cauchy theorem (8) is remarkably simple, but evokes the feeling that there must be a deeper root to this seemingly isolated result. Consideration of the formula of Blaschke will indeed show that it contains Eqs(1) to (3) as mere special cases.

The Fundamental Formula of Blaschke

Towards the end of the nineteenth century important work had been done in the field of geometric probability; the findings of Crofton seemed to have exhausted the essential results. Turning to the article on 'Probability' in the familiar 11th Edition of the Encyclopaedia Britannica one encounters a survey of geometrical probability, and Crofton's article in the 10th Edition had been even more specialized. Subsequently the field appears to have lost its attraction. It was resurrected as *Integral Geometry* in the nineteen thirties by the Hamburg school around Wilhelm Blaschke. The field has attained considerable recent interest in the context of automatic texture analysis and computerized image analysis.

One of the most striking results is the fundamental kinematic formula obtained by Blaschke (2). The following is a summary indication of its essential content.

For closed convex geometric objects [+)] certain measures, m_k, can be

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+) *The term closed implies that the body contains its border; the term convex implies that it contains the straight connection between any pair of its points.*

defined. The familiar measure, m_o, is the volume in R_3, the area in R_2, or the length of an interval in R_1. The additional parameters m_k can be understood in terms of the projections of the body into sub-spaces averaged over all orientations:

m_o = n-dimensional measure
m_k = average measure of the (n-k)-dimensional projections
m_n = Euler characteristic, χ (=1 for convex bodies).

In R_3 the parameter m_1 is the average projection area, and m_2 is the average projection on a line, i.e. the mean width of the body.

The fundamental measures can be generalized to apply also to non-convex geometric objects that are constructed out of a finite number of convex bodies. The definition is made unique by the requirement of additivity:

$$m_k(A_1 \cup A_2) = m_k(A_1) + m_k(A_2) - m_k(A_1 \cap A_2) \tag{4}$$

For non-convex bodies the parameters can not be visualized in terms of projections, but they can still be interpreted in terms of more or less familiar concepts. In R_3 one has:

$$\begin{aligned} m_o &= \text{volume} = V \\ m_1 &= \text{surface}/4 = S/4 \\ m_2 &= \text{integral of the mean curvature} = M \\ m_3 &= \text{Euler characteristic} = \chi \end{aligned} \tag{5}$$

In R_2:

$$\begin{aligned} m_o &= \text{area} = a \\ m_1 &= \text{perimeter}/\pi = s/\pi \\ m_2 &= \text{Euler characteristic} = \chi \end{aligned} \tag{6}$$

The Euler characteristic, a topological parameter, equals i if the body consists of i simply connected parts. When holes occur the value of χ is reduced. In R_2 the Euler characteristic is equal to the number of outer contours minus inner contours. For example $\chi = 0$ for a ring, and $\chi = -1$ for a domain with two holes.

Assume that two bodies A_1 and A_2 intersect randomly. Then a random region of intersection results. Its measures, m_k, are, of course, random variables. The fundamental formula expresses the mean values of these parameters in terms of the measures, $\underset{1}{m}_k$ and $\underset{2}{m}_k$, of the two intersecting regions.

A specification of the process of random intersection is required. Assume a fixed position of A_1 and let the position of a reference point of A_2 be uniformly distributed throughout a sufficiently large+) neighbourhood, G, of A_1. The random orientation of A_2 is assumed to be isotropic. All positions of A_2 that do not result in an intercept are disregarded. Let γ be the integral over G of the probability of intercept. Then γ is the measure of the *Minkowski product,* $A_1 \oplus A_2$, averaged over all orientations. The Minkowski product is also called the *dilatation* (9). It is, as stated earlier, the generalization of the *associated volume*; $A_1 \oplus A_2$ is called the associated volume of A_1 if A_2 is a sphere.

There are, as is well known, other types of randomness that must be

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+) *The requirement is $A_1 \cap A_2 = o$ for all positions with the reference point outside the region G.*

distnguished from uniform isotropic randomness and that are not all relaed to it (see (10,11) for random chords, and (1) for the general case .

With the stated conventions one obtains the formula for the mean values of tie fundamental parameters of the region of intersection:+)

$$\boxed{\overline{m}_k = \frac{1}{\gamma} \sum_{i+j=k} \binom{k}{j} \frac{b_k}{b_i\, b_j}\, m_{1i}\, m_{2j}} \tag{7}$$

b_k is the measure of the k-dimensional unit ball (with b_o=1).

This is the fundamental formula, obtained by Blaschke for R_2 and exteided to R_n by Santaló.

The explicit equations for 2-dimensional space are:

$$\begin{aligned} \overline{a} &= \frac{1}{\gamma}\, a_1\, a_2 \\ \overline{s} &= \frac{1}{\gamma}\, (a_1\, s_2 + a_2\, s_1) \\ \overline{\chi} &= \frac{1}{\gamma}\, (a_1\, \chi_2 + s_1\, s_2\, /\, 2\pi + a_2\, \chi_1) \end{aligned} \tag{8}$$

The indices identify the parameters of the intersecting bodies, A_1 and A_2, and $\overline{a}$, $\overline{s}$ and $\overline{\chi}$ are the mean area, perimeter and Euler characteristic of the intersection.

+) *This relation combines results that are conventionally stated in separate form (e.g. Eqs(15.20), (15.72) and (15.36) in Santalô's monograph).- I am indebted to H.-G.Kellerer for the elegant condensation of Blaschke's theorem in terms of the parameters m_k.*

With the analogous notation one obtains for 3-dimensional space:

$$\overline{V} = \frac{1}{\gamma} V_1 V_2$$

$$\overline{S} = \frac{1}{\gamma} (V_1 S_2 + V_2 S_1) \tag{9}$$

$$\overline{M} = \frac{1}{\gamma} (V_1 M_2 + \pi^2 S_1 S_2 / 16 + V_2 M_1)$$

$$\overline{\chi} = \frac{1}{\gamma} (V_1 \chi_2 + (S_1 M_2 + S_2 M_1)/4\pi + V_2 \chi_1)$$

If both A_1 and A_2 are convex one has $\chi_1 = 1 = \chi_2$ and the intersection is also simply connected, $\overline{\chi} = 1$. Therefore:

$$\gamma = a_1 + s_1 s_2 / 2\pi + a_2 \qquad \text{in } R_2 \tag{10}$$

$$\gamma = V_1 + (S_1 M_2 + S_2 M_1) / 4\pi + V_2 \qquad \text{in } R_3 \tag{11}$$

The average intersections of convex bodies are therefore:

$$\overline{a} = \frac{a_1 a_2}{a_1 + s_1 s_2/2\pi + a_2} \qquad \text{in } R_2 \tag{12}$$

$$\overline{V} = \frac{V_1 V_2}{V_1 + (S_1 M_2 + S_2 M_1)/4\pi + V_2} \qquad \text{in } R_3 \tag{13}$$

and analogous formulae apply for $\overline{s}$, $\overline{S}$ or $\overline{M}$. Streit (12) has given the corresponding relations for the random intersection of more than two convex bodies.

The formulae have here been listed for isotropic randomness. However, they can also be given for other conditions. Blaschke has obtained his original result (2) for unidirectional randomness.

One may, furthermore, note that the results can also be formulated for the intersection of a body, A_1, from a random set with a body, A_2, from another random set. The formulae contain then the averages of the measures of A_1 and the averages of the measures of A_2.

Illustration of the applicability of Eq(13):

Hadwiger (13) lists the measures for elementary bodies in R_3.
For the sphere of radius R:

$$V_s = 4\pi R^3/3 \qquad S_s = 4\pi R^2 \qquad M_s = 4\pi R \qquad |1|$$

For the right circular cylinder of radius r and height h:

$$V_c = \pi h r^2 \qquad S_c = 2\pi r^2 + 2\pi h r \qquad M_c = \pi^2 r + \pi h \qquad |2|$$

Example 1:

The mean intersection volume of a cylinder and a convex body A (with V,S,M) is (see Eqs(13) and |2|):

$$\overline{V} = \left(\frac{1}{V} + \frac{S\ (\pi^2 r+\pi h)\ +\ (2\pi r^2+2\pi h r)M}{4\pi^2 h r^2 V} + \frac{1}{\pi h r^2}\right)^{-1}$$

$$= \left(\frac{1}{V} + \frac{(\pi r+h)S}{4\pi h r^2 V} + \frac{(r+h)M}{2\pi h r V} + \frac{1}{\pi h r^2}\right)^{-1} \qquad |3|$$

Example 2:

The mean intersection of the cylinder and the sphere is (see Eqs |3| and |1|):

$$\overline{V} = \left(\frac{3}{4\pi R^3} + \frac{3(\pi r+h)}{4\pi h r^2 R} + \frac{3(r+h)}{2\pi h r R^2} + \frac{1}{\pi h r^2}\right)^{-1} \qquad |4|$$

To obtain this or related results by explicit integration would cause evident difficulties.

Example 3:

Taking the limit $r \to \infty$ and $h \to o$ in Eq|3| one obtains the mean intersection area , $\overline{a} = \overline{V}/h$, of a random plane with a convex body:

$$\overline{a} = 2\pi V/M \qquad |5|$$

Example 4:

Taking the limit $r \to o$ in Eq|3| one obtains the mean intersection length, $\overline{\lambda} = \overline{V}/\pi r^2$, with a straight line of length h, i.e. the Cauchy theorem for straight line segments:

$$\overline{\lambda} = (S/4V + 1/h)^{-1} \qquad |6|$$

4V/S equals $\overline{l}$, and h corresponds to $\overline{t}$ in Eq(3).

Other examples are left to the reader.

Relation to the Generalized Cauchy Theorem

For non-convex bodies the fundamental formulae contain the factor γ, (the average size of the Minkowski product that is proportional to the frequency of overlap of A_1 and A_2 in a Poisson process). The magnitude of γ is not always readily obtained, because it is, unlike the m_k, not a simple integral over local properties of the geometric objects. γ depends instead on the overall configuration of A_1 and A_2.

However, one can consider the ratios $\overline{a}/\overline{\chi}$ or $\overline{V}/\overline{\chi}$. Then the factor γ cancels and one obtains relations that are analogous to those for convex bodies (see Eqs(12) and (13)):

$$\langle\alpha\rangle = \frac{\bar{a}}{\bar{\chi}} = \frac{a_1 a_2}{a_1 \chi_2 + s_1 s_2/2\pi + a_2 \chi_1} \qquad (14)$$

$$\langle V\rangle = \frac{\bar{V}}{\bar{\chi}} = \frac{V_1\ V_2}{V_1\ \chi_2 + (S_1\ M_2 + S_2\ M_1)/4\pi + V_2\ \chi_1} \tag{15}$$

Te disappearance of the factor γ , even for non-convex bodies, is the ky to the generalized Cauchy theorem. It is obtained from Eqs(14) and (5) in the limit where A_2 consists of thin ribbons or fibers, i.e. c linear structures. However the derivation requires that the linear sructures contain no loops (χ_2=1). The intersecting parts of the linear sructures will then also have no loops, so that $\bar{\chi}$ is simply the average nmber of separate segments per intersection. $\bar{a}\,/\bar{\chi}$ and $\bar{V}\,/\bar{\chi}$ are therefre the average sizes of the *separate* segments of the intersection. Tis is the essence of the generalized Cauchy theorem.

Oe can go one final step further and observe that, in the plane, the itersection of simply connected figures will never contain holes. $\langle\alpha\rangle$ i therefore always the average area of separate parts of the intersecton of two figures. This is indicated in Fig.6 as a final generalizaton of the Cauchy theorem. The mean size of separate domains of intersction of two simply connected figures A_1 and A_2 is:

$$\langle\alpha\rangle = (1/a_1 + s_1\ s_2\ /2\pi\ a_1\ a_2 + 1/a_2)^{-1} \tag{16}$$

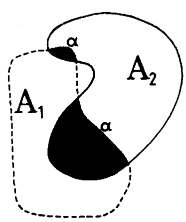

Fig. 6

The solution for linear structures is derived from Eq(14) or Eq(16) as follows. Let A_2 be a ribbon like structure (possibly with branches but without loops) of narrow width ε and of total length t. Then $a_2=\varepsilon t$, $s_2=2t$ and $\chi_2=1$. Omitting the indices in a_1, s_1, and χ_1 one obtains the mean segment length $\bar{\lambda}$:

$$\bar{\lambda} = \lim_{\varepsilon \to 0} \left(\frac{\langle a \rangle}{\varepsilon} = \frac{1}{\varepsilon} \left(\frac{1}{a\chi} + \frac{2\,s\,t}{2\pi a t \varepsilon} + \frac{1}{\varepsilon t} \right)^{-1} \right) = (s/\pi a + 1/t)^{-1} \qquad (17)$$

This is the generalized Cauchy theorem (see Eq(3)) for linear structures, in R_2. For R_3 one can apply analogous considerations to Eq(13) utilizing the fact that the integral mean curvature of a thin fiber of length t equals πt; although the derivation is slightly more complex it, too, is elementary.

While the Cauchy theorem exemplifies well the implications of the fundamental formula, it is only one example among numerous others. The results of geometric probability are comparatively little known to the non-specialist, but they can be remarkably powerful tools in the analysis of random structures and in all problems of random superposition.

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