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Neutral Free Logic: Motivation, Proof Theory and Models

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Abstract

Free logics are a family of first-order logics which came about as a result of examining the existence assumptions of classical logic (Hintikka *The Journal of Philosophy*, *56*, 125–137 1959; Lambert *Notre Dame Journal of Formal Logic*, *8*, 133–144 1967, 1997, 2001). What those assumptions are varies, but the central ones are that (i) the domain of interpretation is not empty, (ii) every name denotes exactly one object in the domain and (iii) the quantifiers have existential import. Free logics reject the claim that names need to denote in (ii). *Positive* free logic concedes that some atomic formulas containing non-denoting names (including self-identity) are true, *negative* free logic treats them as uniformly false, and *neutral* free logic as taking a third value. There has been a renewed interest in analyzing proof theory of free logic in recent years, based on intuitionistic logic in Maffezioli and Orlandelli (*Bulletin of the Section of Logic*, *48*(2), 137–158 2019) as well as classical logic in Pavlović and Gratzl (*Journal of Philosophical Logic*, *50*, 117–148 2021), there for the positive and negative variants. While the latter streamlines the presentation of free logics and offers a more unified approach to the variants under consideration, it does not cover neutral free logic, since there is some lack of both clear formal intuitions on the semantic status of formulas with empty names, as well as a satisfying account of the conditional in this context. We discuss extending the results to this third major variant of free logics. We present a series of G3 sequent calculi adapted from Fjellstad (*Studia Logica*, *105*(1), 93–119 2017, *Journal of Applied Non-Classical Logics*, *30*(3), 272–289 2020), which possess all the desired structural properties of a good proof system, including admissibility of contraction and all versions of the cut rule. At the same time, we maintain the unified approach to free logics and moreover argue that greater clarity of intuitions is achieved once neutral free logic is conceptualized as consisting of two sub-varieties.

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1 Introduction

"To be, or not to be, that is the Question..."
W. Shakespeare: *Hamlet*, Act III, Scene 1

This paper marks the second entry in the more general endeavor of unifying the diverse, as well as extremely rich and useful, family of free logics (FL) [5, 33]. This family of first-order logics came about as a result of examining the existence assumptions of classical logics [19, 24–26], with the name due to Karel Lambert. The definitional hallmarks of a free logic are: we take a logic to be free iff (1) it is free of existential presuppositions with respect to its singular terms, (2) it is free of existential presuppositions with respect to its general terms and finally (3) its quantifiers have existential import, or are more broadly limited to the predicate $E!$ (most commonly read as *existence* [3, 30, 31, 38] and *definedness* [2, 12]). Its main variants are *positive* FL, which allows for some atoms with terms outside of $E!$ to remain true (at the very least self-identities), *negative* FL, which treats all such atoms as uniformly false, and neutral FL, which assigns them a third value. This paper is concerned with the last version.

In modern analytic or stronger mathematical philosophy the origins of neutral free logic(s) date back at least to Gottlob Frege [15]. Frege advocated the idea that in a language that serves a scientific purpose every sentence should have determinate truth-value. Having said this, Frege also discusses sentences containing (what we call today) non-denoting or empty singular terms. Being in the spirit of free logic these sentences takes a value other than true or false (if at all). More recently Scott Lehmann in a series of papers [27–29] discusses approaches to neutral free logic(s):

> The underlying semantic rationale for neutral free semantics is Frege's functional view of reference: predicates and '=' name functions from individuals to truth-values. If functions are operations, as Frege seems to have thought, then the semantic rules governing subject-predicate and identity constructions are [such that] where there is no input to an operation, there is no output either. The truth-functional connectives name truth-functions, so the same line of thought dictates the weak tables for them [29, p.234].

Naturally, different approaches are possible, but since one has to start somewhere, in keeping with this idea we will open the discussion of the proof theory of neutral free logic with a weak Kleene system [7, 32] for it.[1] However, different choices of underlying logic are possible. We here investigate weak and strong Kleene [22, 23] logics, as suggested in [42].

[1] One way to acknowledge the neutral phenomena, but avoid them by reducing them to positive free logic, are *supervaluations* [49, 53]. As the title of the paper might suggest, it is not a goal of this paper to avoid neutral free logic.

Proof theory, and especially sequent calculi, of various systems of free logic are being extensively investigated at the moment, and recently. In [31] the authors develop a sequent calculus for an intuitionistic free logic with existence, [40] offered a unified approach to free logics on a classical base, while [20] further extended them with functions, and also added a treatment of quasi-free logics.

Of these, the present paper follows [40] most directly. The (somewhat) unified approach there comes with a caveat, namely the treatment focused only on positive and negative free logics and omitted the third major variety, neutral free logics, since there is some lack of both clear formal intuitions on the semantic status of formulas with empty names, as well as a satisfying account of the conditional in this context.

In this paper we suggest that the clash of intuitions might arise from various choices of the propositional base since somewhat unexpectedly (for the obvious bases we discuss here) the treatment of quantifiers, which is what free logics concern themselves with, does not vary.

Ultimately, on the philosophical side of things we propose that the previous lack of clarity potentially stems from the strain between the facts that, on the one hand, neutral free logic requires a unified account of quantification, while on the other the choice of the base logic for it influences the choice of the interpretation of the conditional, but not the quantifiers. Once this is made apparent, two separate systems emerge, distinguished by their propositional base and unified by their use of free version of quantification. Of course, this should be seen as a beginning of the debate, not its end. It is a suggestion to consider a more fine-grained look at the relationship between various intuitions regarding neutral free logic which was previously obfuscated by a uniform approach to quantification.

On the technical side we present two sequent calculi, one based on weak Kleene, and another on strong Kleene logic, and demonstrate that they posses all the desirable properties of a good proof system, including admissibility of contraction and cut. We moreover present a simple and unified system of generalized semantics, like that in [40], and show that these systems are each sound and complete for their respective version. Finally, we outline the many strands that still need to be researched.

1.1 Preliminaries

The language of free logic $\mathcal{L}$ utilized in this paper is a standard one, as in [40] (that one adapted from [17]), which is to say the typical language of first-order logic enriched with the predicate $E!$. As in [40], for simplicity the language does not contain function symbols, but [20] shows how those can be added. Likewise, identity is omitted from the current presentation, and the list of connectives and quantifiers is kept to a minimum. All these restrictions make the systems more manageable.

Definition 1.1 (Alphabet $\mathcal{L}$) The alphabet of the language $\mathcal{L}$ consists of:

1. Terms: $t, s, \ldots$, consisting of:
   (a) Denumerable list of free individual variables (names): $a, b, c, \ldots$
   (b) Denumerable list of bound individual variables: $x, y, z, \ldots$

2. Denumerable list of n-ary predicates, including a unary predicate $E!$
3. $\neg, \rightarrow, \forall, (,)$

A formula of the language $\mathcal{L}$ is then defined as

Definition 1.2 (Formula of $\mathcal{L}$)

$$A ::= P^n(t_1 ... t_n) \mid \neg A \mid A \rightarrow A \mid \forall x A$$

As a starting point of our proof-theoretic presentation we take the sequent calculus for weak Kleene logics from [14] (we prefer the G3 approach it takes to the LK ones of [10, 39, 52]). The system there is a five-sided calculus, with the fifth side introduced to account for crispness of formulas (formulas are crisp when they are either true or false), specifically in order to deal with the falsity conditions of the universal quantifier – these, for weak Kleene logics, require all instances to be crisp.

However, in neutral free logic, quantification is limited to the extension of the predicate $E!$, for which every atom containing it is crisp (thus, when it is understood as existence, the answer to Prince Hamlet's Question is "yes"), and which determines the crispness of every other atom (namely, $P(t_1 ... t_n)$ is crisp iff $E!t_i$ for $1 \leq i \leq n$). Consequently, if for some t_i such that $E!t_i$ the instantiated formula $A[t_i/x]$ is not crisp, then for any such t_i it is not crisp (intuitively, it is due to some term other than t_i in A that A is not crisp). Therefore, it follows from $A[t_i/x]$ being false that it is crisp for every t_i s.t. $E!t_i$, and therefore the additional crispness condition is not required. This enables us to drop the fifth side in adopting the rules of [14], making it an interesting (and unusual) case where the free version of a logic is a simplification of the base logic it departs from. It is technically very convenient, and we believe philosophically revealing (as to what might be fueling the clash of intuitions), that this means that the treatment of quantification can remain uniform between the systems under considerations.

We moreover move some of the multiset variables around, since we prefer that a single sequent have only a single sequent arrow. So the basic building block of the system is a (dual-)sequent of the form

$$\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$$

where Γ, Γ', Δ and Δ' are multisets. We call Γ the *antecedent* and Γ' the *antecedent prime*, and similarly for Δ and succedent. We find this notation additionally appropriate since here the vertical bar '|' is a structural comma (i.e. a structural conjunction between antecedents and a structural disjunction between succedents). Ours is essentially a version of the presentation from [16], but we use the vertical bar notation to facilitate later extension into modal logics via a labelled calculus. In effect, this system is a slight notational variation of the generalized propositional sequent calculi for weak and strong Kleene in [21] (see also [6, 11]), extended to quantification (which is then modified to represent free logic quantification). Several advantages of this mode of presentation are discussed there, and these carry over to the present approach. Notably, it facilitates easy transitions between two intuitive readings of the sequents.

One intuitive reading of a sequent is "if everything in Γ is true and everything in Γ' non-false, then either something in Δ is non-false or something in Δ' is true" in an implication format, and "it is not the case that everything in Γ is true, everything in Δ is false, everything in Γ' is non-false and everything in Δ' is non-true," in the negative-conjunction one.

In the sequent calculi below, all the formulas except Γ, Γ', Δ and Δ' (i.e. all the displayed formulas) are called *active* formulas of the rule if they occur only in the upper sequents (or premises) and *principal* if they occur in the lower sequent (or conclusion) of the rule. Left (left prime) rules have the principal formula in the antecedent (antecedent prime), and the right (right prime) rules in the succedent (succedent prime).

A *derivation* of a sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is a finite tree whose root is the said sequent (called the *endsequent*) and ends of all the branches are initial sequents. A *branch* is a series of sequents, starting with the endsequent, in which every element is a conclusion of a rule of which the following element is a premise. The *height* of a derivation is the length (the number of sequents) of its longest branch.

2 Weak Neutral Free Logic

As explained in the introduction, we open the discussion of proof theory of neutral free logics with the treatment of weak Kleene propositional base enriched with free quantification (see e.g. [48, 50, 54]). Consequently, the approach here is akin to the 'nonsense' reading of the third value in the vein of [7], or 'off-topic' of the approach of [4] (one would then take $E!$ to be a topic marker), or any other interpretation that leads to the truth value being the third value whenever it is assigned to some subformula. In the vein of [40], we strive for maximal generality and do not commit to any one understanding of the third value, nor claim that one interpretation would cover all cases.

As in [14], the system used here is G3-style following [35, 36]. This will make the demonstration of the structural properties, which will follow immediately, significantly easier (although this is not to say it's easy). The most substantial piece of work here is the demonstration of admissibility of cut rules, of which there are six. Fortunately, this work only needs to be done once and can then be reused for subsequent versions.

Sequent Calculus for Weak Neutral Free Logic G3$_{wnf}$

Initial sequents (is):

(is$_1$) $p, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', p$ (is$_2$) $\Gamma \mid \Gamma', p \Rightarrow p, \Delta \mid \Delta'$

(is$_3$) $p, \Gamma \mid \Gamma' \Rightarrow p, \Delta \mid \Delta'$

(is$_4$) $\{E!t_i\}_{1 \leq i \leq n}, \Gamma \mid \Gamma', P(t_1...t_n) \Rightarrow \Delta \mid \Delta', P(t_1...t_n)$

Propositional rules:

$$\frac{\Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'}{\neg A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}\neg \qquad \frac{A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \neg A, \Delta \mid \Delta'}\ \mathrm{R}\neg$$

$$\frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A}{\Gamma \mid \Gamma', \neg A \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}'\neg \qquad \frac{\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \neg A}\ \mathrm{R}'\neg$$

$$\frac{A, B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta' \qquad \Gamma \mid \Gamma' \Rightarrow A, B, \Delta \mid \Delta' \qquad B, \Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'}{A \to B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'} \text{L}\!\to$$

$$\frac{A, \Gamma \mid \Gamma' \Rightarrow B, \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow A \to B, \Delta \mid \Delta'} \text{R}\!\to$$

$$\frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \qquad \Gamma \mid B, \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid A \to B, \Gamma' \Rightarrow \Delta \mid \Delta'} \text{L}'\!\to$$

$$\frac{\Gamma \mid \Gamma', A \Rightarrow B, \Delta \mid \Delta' \qquad \Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta', A \qquad \Gamma \mid \Gamma', B \Rightarrow \Delta \mid \Delta', B}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \to B} \text{R}'\!\to$$

Quantifier rules:

$$\frac{E!t, \forall x A, A[t/x], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \forall x A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'} \text{L}\forall \qquad \frac{E!t, \Gamma \mid \Gamma' \Rightarrow A[t/x], \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \forall x A, \Delta \mid \Delta'} \text{R}\forall^*$$

$$\frac{E!t, \Gamma \mid \forall x A, A[t/x], \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \Gamma \mid \forall x A, \Gamma' \Rightarrow \Delta \mid \Delta'} \text{L}'\forall \qquad \frac{E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A} \text{R}'\forall^*$$

$E!$ rules:

$$\frac{E!t, P[t], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{P[t], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'} \text{L}E! \qquad \frac{E!t, \Gamma \mid \Gamma' \Rightarrow P[t], \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow P[t], \Delta \mid \Delta'} \text{R}E!$$

$$\frac{\{E!t_i\}_{1 \le i \le n}, P(t_1 ... t_n), \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{\{E!t_i\}_{1 \le i \le n}, \Gamma \mid \Gamma', P(t_1 ... t_n) \Rightarrow \Delta \mid \Delta'} \text{L}'E!$$

$$\frac{\{E!t_i\}_{1 \le i \le n}, \Gamma \mid \Gamma' \Rightarrow P(t_1 ... t_n), \Delta \mid \Delta'}{\{E!t_i\}_{1 \le i \le n}, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', P(t_1 ... t_n)} \text{R}'E!$$

$$\frac{E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid \Gamma', E!t \Rightarrow \Delta \mid \Delta'} \text{L}Tr_{E!} \qquad \frac{\Gamma \mid \Gamma' \Rightarrow E!t, \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', E!t} \text{R}Tr_{E!}$$

Where p is atomic, $P[t]$ and $P(t_1 ... t_n)$ are atoms other than $E!t$ and t is fresh in rules marked with $*$.

2.1 Structural Properties

Here the language is the same as in [40], the weight of the formula is defined in a standard way, and likewise the height of the derivation, except that the height of an initial sequent is 1 (in line with the explanation in Section 1.1).

Lemma 2.1 (Substitution) *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ *is derivable in* $G3_{wnf}$ *(where* $\vdash_n$ *denotes derivability with height bounded by* n*), then the sequent* $\vdash_n \Gamma[t/s] \mid \Gamma[t/s] \Rightarrow \Delta[t/s] \mid \Delta'[t/s]$ *is likewise derivable.*

Proof Routine by induction on the height of the derivation. Since substitution is height-preserving, in case of a clash of terms in rules with a freshness condition, we first apply the inductive hypothesis to replace the offending term and then again to produce the desired sequent. □

Lemma 2.2 (Axiom generalization) *For any formula A, the following are derivable:*

1. $A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A,$
2. $\Gamma \mid \Gamma', A \Rightarrow A, \Delta \mid \Delta',$
3. $A, \Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'$
4. $\{E!t_i\}_{1\leq i\leq n}, \Gamma \mid \Gamma', A[t_1...t_n] \Rightarrow \Delta \mid \Delta', A[t_1...t_n].$

Proof By simultaneous induction on the weight of A. Straightforward (if tedious) for all cases, so we just illustrate with the case of $\forall$:

1.

$$\dfrac{\dfrac{\dfrac{\text{i.h.}}{E!t, \forall x A, A[t/x], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]}}{E!t, \forall x A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]}\,\mathrm{L}\forall}{\forall x A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A}\,\mathrm{R'}\forall$$

2.

$$\dfrac{\dfrac{\dfrac{\text{i.h.}}{E!t, \Gamma \mid \Gamma', \forall x A, A[t/x] \Rightarrow A[t/x], \Delta \mid \Delta'}}{E!t, \Gamma \mid \Gamma', \forall x A \Rightarrow A[t/x], \Delta \mid \Delta'}\,\mathrm{L'}\forall}{\Gamma \mid \Gamma', \forall x A \Rightarrow \forall x A, \Delta \mid \Delta'}\,\mathrm{R}\forall$$

3.

$$\dfrac{\dfrac{\dfrac{\text{i.h.}}{E!t, \forall x A, A[t/x], \Gamma \mid \Gamma' \Rightarrow A[t/x], \Delta \mid \Delta'}}{E!t, \forall x A, \Gamma \mid \Gamma' \Rightarrow A[t/x], \Delta \mid \Delta'}\,\mathrm{L}\forall}{\forall x A, \Gamma \mid \Gamma' \Rightarrow \forall x A, \Delta \mid \Delta'}\,\mathrm{R}\forall$$

4.

$$\dfrac{\dfrac{\dfrac{\text{i.h.}}{\{E!t_i\}_{1\leq i\leq n+1}, \Gamma \mid \Gamma', \forall x A[t_1...t_n], A[t_1...t_n][t_{n+1}/x] \Rightarrow \Delta \mid \Delta', A[t_1...t_n][t_{n+1}/x]}}{\{E!t_i\}_{1\leq i\leq n+1}, \Gamma \mid \Gamma', \forall x A[t_1...t_n] \Rightarrow \Delta \mid \Delta', A[t_1...t_n][t_{n+1}/x]}\,\mathrm{L'}\forall}{\{E!t_i\}_{1\leq i\leq n}, \Gamma \mid \Gamma', \forall x A[t_1...t_n] \Rightarrow \Delta \mid \Delta', \forall x A[t_1...t_n]}\,\mathrm{R'}\forall$$

□

Notice that simultaneous induction is necessary to deal with the cases for negation and implication.

Lemma 2.3 (Weakening) *Weakening is height-preserving admissible in* $G3_{wnf}$*:*

1. *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ *then* $\vdash_n C, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta',$
2. *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ *then* $\vdash_n \Gamma \mid \Gamma', C \Rightarrow \Delta \mid \Delta',$
3. *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ *then* $\vdash_n \Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta',$
4. *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ *then* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C.$

Proof By induction on the height of a derivation. □

Lemma 2.4 (Invertibility) *All the rules of $G3_{wnf}$ are height-preserving invertible.*

Proof Routine by induction on the height of the derivation, using Lemma 2.1 when necessary. We illustrate for the case of R′∀ (the cases for all quantifier rules are very similar to the corresponding proofs for $G3_{pf}$/$G3_{nf}$ in [40]):

If $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A$ then $\vdash_n E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]$.

Basic step. If $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A$ is an initial sequent, then so is $E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]$.

Inductive step. If $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A$ is derived by some rule R other than R′∀ with $\forall x A$ principal, we then apply Lemma 2.1 if the rule R has a freshness condition, then the inductive hypothesis to the premise(s) of the rule, and finally reapply the rule R to obtain the required sequent (note that both substitution and the inductive hypothesis are height-preserving).

Otherwise, if the last rule used is R′∀ with $\forall x A$ principal, then either the premise of that rule is already the required sequent, or we apply Lemma 2.1 to the premise to obtain it. □

Lemma 2.5 (Contraction) *Contraction is height-preserving admissible in $G3_{wnf}$:*

1. *If* $\vdash_n C, C, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ *then* $\vdash_n C, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$,
2. *If* $\vdash_n \Gamma \mid \Gamma', C, C \Rightarrow \Delta \mid \Delta'$ *then* $\vdash_n \Gamma \mid \Gamma', C \Rightarrow \Delta \mid \Delta'$,
3. *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow C, C, \Delta \mid \Delta'$ *then* $\vdash_n \Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta'$,
4. *If* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C, C$ *then* $\vdash_n \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C$.

Proof Routine by simultaneous induction on the height of the derivation. □

Lemma 2.6 (Transfer) *The following rules are height-preserving admissible in $G3_{wnf}$:*

$$\frac{\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta'}{A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ LTr \qquad\qquad \frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A}{\Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'}\ RTr$$

Proof By simultaneous induction on the height of the derivation. Since this is relatively novel, we present the proof almost in its entirety.

(LTr)

Basic case. If $\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta'$ is an initial sequent, then either

1. $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is an initial sequent, or
2. A is atomic and occurs in Δ (is_2), or
3. A is of the form $P(t_1 ... t_n)$ and occurs in Δ', while $E!t_1 ... E!t_n$ occur in Γ (is_4).

In any case $A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is also an initial sequent (in case 1 it is an initial sequent of the same form, in 2 it is (is_3) and in 3 it is (is_1)).

Inductive case. If the formula A is not principal in the last step, then it has the following form (square brackets indicate a possible second premise, and double square brackets a possible third one):

$$\dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1' \qquad [\Gamma_2 \mid \Gamma_2', A \Rightarrow \Delta_2 \mid \Delta_2'] \qquad [[\Gamma_3 \mid \Gamma_3', A \Rightarrow \Delta_3 \mid \Delta_3']]}{\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta'}\ \mathrm{R}$$

We apply the inductive hypothesis to the premise(s), and then the rule R, to obtain:

$$\dfrac{A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad [A, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'] \qquad [[A, \Gamma_3 \mid \Gamma_3' \Rightarrow \Delta_3 \mid \Delta_3']]}{A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{R}$$

If the formula A is principal in the last step, there are several cases to consider.
(a) If A is of the form $\neg C$, then the last step has the following form:

$$\dfrac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C}{\Gamma \mid \Gamma', \neg C \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}'\neg$$

We apply the inductive hypothesis [RTr] to the premise to obtain $\Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta'$, and then the rule L$\neg$ to obtain $\neg C, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$.
(b) If A is of the form $C \rightarrow D$, then the last step has the following form:

$$\dfrac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C \qquad \Gamma \mid D, \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid C \rightarrow D, \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}'\rightarrow$$

We can then obtain the derivation (here, and in the remainder of the paper, D1-D5 signify fragments of a derivation to be reconstituted immediately below):
(D1)

$$\dfrac{\dfrac{\Gamma \mid D, \Gamma' \Rightarrow \Delta \mid \Delta'}{D, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \text{i.h.[L}Tr\text{]}}{C, D, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \text{Lemma 2.3}$$

(D2)

$$\dfrac{\dfrac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C}{\Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta'}\ \text{i.h.[R}Tr\text{]}}{\Gamma \mid \Gamma' \Rightarrow C, D, \Delta \mid \Delta'}\ \text{Lemma 2.3}$$

(D3)

$$\dfrac{\dfrac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C}{\Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta'}\ \text{i.h.[R}Tr\text{]}}{D, \Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta'}\ \text{Lemma 2.3}$$

$$\dfrac{\text{(D1)} \qquad \text{(D2)} \qquad \text{(D3)}}{C \rightarrow D, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}\rightarrow$$

(c) If A is of the form $E!t$ then it is principal in $\mathrm{L}Tr_{E!}$ and the required sequent is the premise of that rule. If it is some other atomic formula then it is principal in $\mathrm{L}'E!$ and the required sequent is likewise the premise.

(d) If A is of the form $\forall x C$ then the last step has the following form:

$$\frac{E!t, \Gamma \mid \forall x C, C[t/x], \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \Gamma \mid \forall x C, \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}'\forall$$

We apply the inductive hypothesis to the premise twice, and then the rule, to obtain the derivation:

$$\cfrac{\cfrac{E!t, \Gamma \mid \forall x C, C[t/x], \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \forall x C, C[t/x], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \text{i.h.}[\mathrm{L}Tr]}{E!t, \forall x C, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}\forall$$

($\mathrm{R}Tr$)
Basic case. If $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A$ is an initial sequent, then either

1. $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is an initial sequent, or
2. A is atomic and occurs in Γ (is_1), or
3. A is of the form $P(t_1...t_n)$ and occurs in Γ', while $E!t_1...E!t_n$ occur in Γ (is_4).

In any case $\Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'$ is also an initial sequent (in case 1 it is an initial sequent of the same form, in 2 it is (is_3), and in case 3 it is (is_2)).

Inductive case. Very similar to that for $\mathrm{L}Tr$, with the important case to check one where the formula A is principal and of the form $C \to D$. Then the last step has the form:

$$\frac{\Gamma \mid \Gamma', C \Rightarrow D, \Delta \mid \Delta' \qquad \Gamma \mid \Gamma', C \Rightarrow \Delta \mid \Delta', C \qquad \Gamma \mid \Gamma', D \Rightarrow \Delta \mid \Delta', D}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C \to D}\ \mathrm{R}'\!\to$$

We can then obtain the derivation:

$$\cfrac{\cfrac{\Gamma \mid \Gamma', C \Rightarrow D, \Delta \mid \Delta'}{C, \Gamma \mid \Gamma' \Rightarrow D, \Delta \mid \Delta'}\ \text{i.h.}[\mathrm{L}Tr]}{\Gamma \mid \Gamma' \Rightarrow C \to D, \Delta \mid \Delta'}\ \mathrm{R}\!\to$$

□

In the standard order of derivations of the structural properties, we now arrive to the demonstration of the admissibility of cut. Using the method of exhausting the possible valuations of a formula from [44, 46, 47], in [14], five different cut rules are listed. Since $\mathrm{G3}_{wnf}$ allows us to omit the crisp side of the sequents, we are then left with four rules. Then, we also need to add the rules for $E!$, one stating $E!$ is crisp and another that any atom where all free variables are in $E!$ is crisp (these take the structure of the more familiar cut rules). Thus we wind up with the list of cut rules below. Since this is central to the paper and relatively novel, with a multitude of cuts checked for a plethora of cases, we will go through a lot of detail. The structure of the proofs mostly follows the standard presentation as [36].

We break the six cuts into two groups. The first two in Theorem 2.7 are those concerning $E!$, and those are established independently of each other.

The remaining four, in Theorem 2.8, will require *simultaneous* demonstration, whereby the inductive hypothesis for each states that the property holds up to that point for all. This is because in some cases a reduction of one cut rule requires the assumption that it *and others* are admissible for previous cases.

As in [36], some cases can be bypassed if we first consider those where at least one on the premises of cut is initial, and we will likewise do so here.

Theorem 2.7 *The following are admissible in $G3_{wnf}$:*

1. *$E!$ cut:*

$$\frac{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1' \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ E!\text{-}Cut$$

2. *Left cut:*

$$\frac{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1' \qquad \{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ L\text{-}Cut$$

Proof By induction on the height of the cut (the sum of heights of premises of cut).

$E!$ **cut**.

$$\frac{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1' \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ E!\text{-Cut}$$

If $\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1'$ is initial, then either

1. $\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$ is initial. In this case we obtain the endsequent from it by Lemma 2.3.
2. Γ_1 contains $E!t$ (is_3). In this case we obtain the endsequent by Lemma 2.3 from $E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$.
3. Γ_1' contains $E!t$ (is_2). In this case we obtain the endsequent by $LTr_{E!}$ and Lemma 2.3 from $E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$.

Otherwise $E!$ is not principal in $\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1'$ and the last step is of the form (single premise rule for simplicity):

$$\frac{\dfrac{\Gamma_1^* \mid \Gamma_1'^* \Rightarrow E!t, \Delta_1^* \mid \Delta_1'^*}{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1'}\ \text{R} \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ E!\text{-Cut}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma_1^* \mid \Gamma_1'^* \Rightarrow E!t, \Delta_1^* \mid \Delta_1'^* \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1^*, \Gamma_2 \mid \Gamma_1'^*, \Gamma_2' \Rightarrow \Delta_1^*, \Delta_2 \mid \Delta_1'^*, \Delta_2'}\ E!\text{-Cut}}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{R}$$

where the $E!$ cut is of lesser height. Similar for two- or three-premise R.
Left cut.

$$\frac{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1' \qquad \{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \text{ L-Cut}$$

If $\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1'$ is initial, then either

1. $\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$ is initial. In this case we obtain the endsequent from it by Lemma 2.3.
2. Γ_1 contains $P(t_1...t_n)$ (is$_3$). In this case we obtain the endsequent by Lemma 2.3 from $\{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$.
3. Γ_1' contains $P(t_1...t_n)$ (is$_2$). In this case we obtain the endsequent by L$'E!$ and Lemma 2.3 from $\{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$.
   The case where $\{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$ is initial is similar.

The next important step to check is when $P(t_1...t_n)$ is principal in the left premise. Then we have the following derivation:

$$\frac{\dfrac{E!t_i, \{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1'} \text{R}E! \qquad \{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \text{ L-Cut}$$

This is transformed into:

$$\dfrac{\dfrac{E!t_i, \{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1' \qquad \{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{E!t_i, \{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \text{ L-Cut}}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \text{ Lemma 2.5}$$

where the instance of L-Cut is of lower height.
Similar if $P(t_1...t_n)$ is principal in the right premise and otherwise routine. □

With these established we move to the more involved proof for the remaining four rules.

Theorem 2.8 (Cut) *The following are admissible in G3$_{wnf}$:*

1. *Outside cut:*
$$\frac{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A \qquad A, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \textit{ O-Cut}$$
2. *Inside cut:*
$$\frac{\Gamma_1 \mid \Gamma_1' \Rightarrow A, \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2', A \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \textit{ I-Cut}$$
3. *Right cut:*
$$\frac{\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'} \textit{ R-Cut}$$
4. *Triple cut:*

$$\frac{A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2' \qquad \Gamma_3 \mid \Gamma_3', A \Rightarrow \Delta_3 \mid \Delta_3', A}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\ \textit{3-Cut}$$

Proof By *simultaneous* induction on the weight of the cut formula with a subinduction on the height of the cut.

Outside cut.

$$\frac{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A \qquad A, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{O-Cut}$$

If $\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A$ is initial, then either

1. $\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$ is initial. In this case from it we obtain the endsequent by Lemma 2.3.
2. A is atomic and occurs in Γ_1 (is_1). In this case we obtain the endsequent by Lemma 2.3 from $A, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$.
3. A is of the form $P(t_1 ... t_n)$ and occurs in Γ_1', and $E!t_1 ... E!t_n$ occur in Γ_1 (is_4). Then we have the following derivation:

$$\cfrac{\cfrac{\cfrac{P(t_1 ... t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1 \le i \le n}, P(t_1 ... t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}\ \text{Lemma 2.3}}{\{E!t_i\}_{1 \le i \le n}, \Gamma_2 \mid \Gamma_2', P(t_1 ... t_n) \Rightarrow \Delta_2 \mid \Delta_2'}\ \text{L}'E!}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{Lemma 2.3}$$

If $A, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$ is initial, then either

1. $\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$ is initial. In this case from it we obtain the endsequent by Lemma 2.3.
2. A is atomic and occurs in Δ_2' (is_1). In this case we obtain the endsequent by Lemma 2.3 from $\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A$.
3. A is atomic and occurs in Δ_2 (is_3). In this case we have the following derivation:

$$\cfrac{\cfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A}{\Gamma_1 \mid \Gamma_1' \Rightarrow A, \Delta_1 \mid \Delta_1'}\ \text{R}Tr}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{Lemma 2.3}$$

4. If A is $E!t_i$, we have several cases depending on the left premise of the O-Cut. If it is initial, we proceed according to the steps for that case above. If not and $E!t_i$ is principal, then the last step used is $\text{R}Tr_{E!}$ and this transforms to $E!$-cut. Otherwise the last step is an application of some rule R and this is transformed into one to three applications of O-Cut (possibly first applying Lemma 2.1) of lower height, followed by an application of the rule R.

Otherwise the first important case we need to check here is when the cut formula is $E!t$ and principal in the left premise. Then the derivation has the following form:

$$\dfrac{\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1'}{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', E!t}\, \mathrm{R}Tr_{E!} \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\, \text{O-Cut}$$

This is transformed into:

$$\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1' \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\, E!\text{-Cut}$$

The next case is when the cut formula is $P(t_1...t_n)$ and principal in the left premise. Then the derivation is of the form:

$$\dfrac{\dfrac{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', P(t_1...t_n)}\, \mathrm{R}'E! \qquad P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\, \text{O-Cut}$$

This is transformed into:

(D1)

$$\dfrac{P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}\, \text{Lemma 2.3}$$

$$\dfrac{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1' \qquad \text{(D1)}}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\, \text{L-Cut}$$

In the case when the cut formula is of the form $A \rightarrow B$ and principal in both premises, the derivation is of the form:
(D1)

$$\dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow B, \Delta_1 \mid \Delta_1' \qquad \Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1', A \qquad \Gamma_1 \mid \Gamma_1', B \Rightarrow \Delta_1 \mid \Delta_1', B}{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A \rightarrow B}\, \mathrm{R}'\!\rightarrow$$

(D2)

$$\dfrac{A, B, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A, B, \Delta_2 \mid \Delta_2' \qquad B, \Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2'}{A \rightarrow B, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}\, \mathrm{L}\!\rightarrow$$

$$\dfrac{\text{(D1)} \qquad \text{(D2)}}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\, \text{O-Cut}$$

This is transformed into:
(D1)

$$\dfrac{\dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1', A \qquad A, B, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2' \qquad B, \Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2'}{B, B, \Gamma_1, \Gamma_2, \Gamma_2 \mid \Gamma_1', \Gamma_2', \Gamma_2' \Rightarrow \Delta_1, \Delta_2, \Delta_2 \mid \Delta_1', \Delta_2', \Delta_2'}\, \text{3-Cut}}{B, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\, \text{Lemma 2.5}$$

(D2)

$$\dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1', A \qquad \dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow B, \Delta_1 \mid \Delta_1'}{A, \Gamma_1 \mid \Gamma_1' \Rightarrow B, \Delta_1 \mid \Delta_1'}\,\mathrm{L}Tr \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A, B, \Delta_2 \mid \Delta_2'}{\dfrac{\Gamma_1, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_1', \Gamma_2' \Rightarrow B, B, \Delta_1, \Delta_1, \Delta_2 \mid \Delta_1', \Delta_1', \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow B, \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{Lemma 2.5}}\,\text{3-Cut}$$

$$\dfrac{(\mathrm{D1}) \qquad (\mathrm{D2}) \qquad \Gamma_1 \mid \Gamma_1', B \Rightarrow \Delta_1 \mid \Delta_1', B}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{3-Cut, Lemma 2.5}$$

where each instance of 3-Cut is of lower weight.

The last important case is when the cut formula is of the form $\forall x A$ and principal in both premises. Then the derivation is of the form:

$$\dfrac{\dfrac{E!s, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A[s/x]}{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', \forall x A}\,\mathrm{R}'\forall \qquad \dfrac{E!t, \forall x A, A[t/x], \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{E!t, \forall x A, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}\,\mathrm{L}\forall}{\Gamma_1, E!t, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{O-Cut}$$

This is transformed as follows:

(D1)

$$\dfrac{\dfrac{E!s, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A[s/x]}{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', \forall x A}\,\mathrm{R}'\forall \qquad E!t, \forall x A, A[t/x], \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, E!t, A[t/x], \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{O-Cut}$$

$$\dfrac{\dfrac{\dfrac{E!s, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A[s/x]}{E!t, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A[t/x]}\,\text{Lemma 2.1} \qquad (\mathrm{D1})}{\Gamma_1, \Gamma_1, E!t, E!t, \Gamma_2 \mid \Gamma_1', \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_1, \Delta_2 \mid \Delta_1', \Delta_1', \Delta_2'}\,\text{O-Cut}}{\Gamma_1, E!t, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{Lemma 2.5}$$

where the upper O-Cut is of lesser height and the lower of lesser weight.

The remaining cases are familiar.

Inside cut.

$$\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow A, \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2', A \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{I-Cut}$$

The case where either of the premises of cut is initial is very similar to the case for Outside cut.

Otherwise the first important case we need to check is when the cut formula is $E!t$ and principal in the right premise. In that case the last step has the following form:

$$\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1' \qquad \dfrac{E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_2 \mid \Gamma_2', E!t \Rightarrow \Delta_2 \mid \Delta_2'}\,\mathrm{L}Tr_{E!}}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,\text{I-Cut}$$

This is transformed into:

$$\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow E!t, \Delta_1 \mid \Delta_1' \qquad E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\,E!\text{-Cut}$$

The next case is when the cut formula is $P(t_1...t_n)$ and principal in the right premise. Then the derivation is of the form:

$$\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1' \qquad \dfrac{\{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_2 \mid \Gamma_2', P(t_1...t_n) \Rightarrow \Delta_2 \mid \Delta_2'}\text{L}'E!}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{I-Cut}$$

This is transformed into:
(D1)

$$\dfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1 \mid \Gamma_1' \Rightarrow P(t_1...t_n), \Delta_1 \mid \Delta_1'}\text{Lemma 2.3}$$

$$\dfrac{(\text{D1}) \qquad \{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{L-Cut}$$

In the case when the cut formula is of the form $A \rightarrow B$ and principal in both premises the derivation is of the form:

$$\dfrac{\dfrac{A, \Gamma_1 \mid \Gamma_1' \Rightarrow B, \Delta_1 \mid \Delta_1'}{\Gamma_1 \mid \Gamma_1' \Rightarrow A \rightarrow B, \Delta_1 \mid \Delta_1'}\text{R}\rightarrow \qquad \dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A \qquad \Gamma_2 \mid \Gamma_2', B \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_2 \mid \Gamma_2', A \rightarrow B \Rightarrow \Delta_2 \mid \Delta_2'}\text{L}'\rightarrow}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{I-Cut}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A \qquad \dfrac{A, \Gamma_1 \mid \Gamma_1' \Rightarrow B, \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2', B \Rightarrow \Delta_2 \mid \Delta_2'}{A, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{I-Cut}}{\Gamma_1, \Gamma_2, \Gamma_2 \mid \Gamma_1', \Gamma_2', \Gamma_2' \Rightarrow \Delta_1, \Delta_2, \Delta_2 \mid \Delta_1', \Delta_2', \Delta_2'}\text{O-Cut}}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{Lemma 2.5}$$

The remaining cases are similar as in the case of O-cut.

Right cut.

$$\dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{R-Cut}$$

If $\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1'$ is initial, then either

1. $\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$ is initial. In this case from it we obtain the endsequent by Lemma 2.3.
2. A is of the form $P(t_1...t_n)$ and occurs in Δ_1', while $E!t_1...E!t_n$ occur in Γ_1 (is$_4$). In this case we obtain the endsequent by Lemma 2.3 from $\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A$.
3. A is atomic and occurs in Δ_1 (is$_2$). In this case we have the following derivation:

$$\dfrac{\dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A}{\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2'}\text{R}Tr}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{Lemma 2.3}$$

The case where $\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A$ is initial is similar.

We now check the case when the cut formula is of the form $E!t$ and principal in the left premise. Then the derivation is of the form:

$$\cfrac{\cfrac{E!t, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}{\Gamma_1 \mid \Gamma_1', E!t \Rightarrow \Delta_1 \mid \Delta_1'}\ \mathrm{L}Tr_{E!} \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', E!t}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{R-Cut}$$

This is transformed into:

$$\cfrac{E!t, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', E!t}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{O-Cut}$$

where the instance of O-cut is again of lower height.

Similar if the cut formula is of the form $E!t$ and principal in the right premise (reducing to I-cut), and similar for the respective cases when the cut formula is of the form $P(t_1...t_n)$.

The next important case is when the cut formula is of the form $A \rightarrow B$ and principal in both premises. Then the derivation is of the form:

(D1)

$$\cfrac{\Gamma_2 \mid \Gamma_2', A \Rightarrow B, \Delta_2 \mid \Delta_2' \qquad \Gamma_2 \mid \Gamma_2', A \Rightarrow \Delta_2 \mid \Delta_2', A \qquad \Gamma_2 \mid \Gamma_2', B \Rightarrow \Delta_2 \mid \Delta_2', B}{\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A \rightarrow B}\ \mathrm{R}'\!\!\rightarrow$$

$$\cfrac{\cfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A \qquad \Gamma_1 \mid \Gamma_1', B \Rightarrow \Delta_1 \mid \Delta_1'}{\Gamma_1 \mid \Gamma_1', A \rightarrow B \Rightarrow \Delta_1 \mid \Delta_1'}\ \mathrm{L}'\!\!\rightarrow \qquad (\mathrm{D1})}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{R-Cut}$$

This is transformed into:

$$\cfrac{\cfrac{\Gamma_1 \mid \Gamma_1', B \Rightarrow \Delta_1 \mid \Delta_1' \qquad \cfrac{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A \qquad \Gamma_2 \mid \Gamma_2', A \Rightarrow B, \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, B, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{R-Cut}}{\Gamma_1, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_1, \Delta_2 \mid \Delta_1', \Delta_1', \Delta_2'}\ \text{I-Cut}}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{Lemma 2.5}$$

where both instances of cut are of lower weight.

Finally, we check the case where the cut formula is of the form $\forall x A$ and principal in both premises. Then the derivation is of the form:

$$\cfrac{\cfrac{E!t, \Gamma_1 \mid \forall x A, A[t/x], \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}{E!t, \Gamma_1 \mid \forall x A, \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}\ \mathrm{L}'\forall \qquad \cfrac{E!s, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A[s/x]}{\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', \forall x A}\ \mathrm{R}'\forall}{E!t, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{R-Cut}$$

This is transformed into:

(D1)

$$\cfrac{E!t, \Gamma_1 \mid \forall x A, A[t/x], \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', \forall x A}{E!t, \Gamma_1, \Gamma_2 \mid A[t/x], \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\ \text{R-Cut}$$

$$\dfrac{\text{(D1)} \quad \dfrac{E!s, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A[s/x]}{E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2', A[t/x]}\text{ Lemma 2.1}}{\dfrac{E!t, \Gamma_1, \Gamma_2, \Gamma_2 \mid \Gamma_1', \Gamma_2', \Gamma_2' \Rightarrow \Delta_1, \Delta_2, \Delta_2 \mid \Delta_1', \Delta_2', \Delta_2'}{E!t, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{ Lemma 2.5}}\text{ R-Cut}$$

where the upper R-Cut is of lesser height and the lower of lesser weight.

The remaining cases are familiar.

Triple cut.

$$\dfrac{A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2' \qquad \Gamma_3 \mid \Gamma_3', A \Rightarrow \Delta_3 \mid \Delta_3', A}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ 3-Cut}$$

If $A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$ is initial, then either

1. $\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$ is initial. In this case we obtain the endsequent by Lemma 2.3.
2. A is atomic and occurs in Δ_1 (is$_3$). In this case we obtain the endsequent by Lemma 2.3 from $\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2'$.
3. A is atomic and occurs in Δ_1' (is$_1$). In this case we have the following derivation:

$$\dfrac{\dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2' \qquad \Gamma_3 \mid \Gamma_3', A \Rightarrow \Delta_3 \mid \Delta_3', A}{\Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3' \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', A}\text{ I-Cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ Lemma 2.3}$$

where the I-Cut is of lesser height.

If $\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2'$ is initial, then either

1. $\Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'$ is initial. In this case we obtain the endsequent by Lemma 2.3.
2. A is atomic and occurs in Γ_2 (is$_3$). In this case we obtain the endsequent by Lemma 2.3 from $A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$.
3. A is atomic and occurs in Γ_2' (is$_2$). In this case we have the following derivation:

$$\dfrac{\dfrac{A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_3 \mid \Gamma_3', A \Rightarrow \Delta_3 \mid \Delta_3', A}{\Gamma_1, \Gamma_3 \mid \Gamma_1', \Gamma_3', A \Rightarrow \Delta_1, \Delta_3 \mid \Delta_1', \Delta_3'}\text{ O-Cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ Lemma 2.3}$$

   where the O-Cut is of lesser height.

If $\Gamma_3 \mid \Gamma_3', A \Rightarrow \Delta_3 \mid \Delta_3', A$ is initial, then either

1. $\Gamma_3 \mid \Gamma_3' \Rightarrow \Delta_3 \mid \Delta_3'$ is initial. In this case we obtain the endsequent by Lemma 2.3.
2. A is atomic and occurs in Δ_3 (is$_2$). In this case we obtain the endsequent by Lemma 2.3 from $\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2'$.
3. A is atomic and occurs in Γ_3 (is$_1$). In this case we obtain the endsequent by Lemma 2.3 from $A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'$.
4. A is of the form $P(t_1...t_n)$ and $E!t_1...E!t_n$ occur in Γ_3 (is$_4$). Then we have the following derivation:

   (D1)

$$
\cfrac{
  \cfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow P(t_1...t_n), \Delta_2 \mid \Delta_2'}{\{E!t_i\}_{1\leq i\leq n}, \Gamma_2 \mid \Gamma_2' \Rightarrow P(t_1...t_n), \Delta_2 \mid \Delta_2'}\ \text{Lemma 2.3}
  \qquad
  \cfrac{P(t_1...t_n), \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}{\{E!t_i\}_{1\leq i\leq n}, P(t_1...t_n), \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}\ \text{Lemma 2.3}
  \qquad (D1)
}{
  \cfrac{\{E!t_i\}_{1\leq i\leq n}, \Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\ \text{Lemma 2.3}
}\ \text{L-Cut}
$$

The next important case is when the cut formula is of the form $A \to B$ and principal in all premises. Then there are two possible derivations, depending on which formula is principal in the rightmost premise. One derivation is of the form (for clarity premises are enumerated and listed vertically in the three-premise rules):

(D1)

$$
\cfrac{
\begin{array}{l}
(1)\ A, B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \\
(2)\ \Gamma_1 \mid \Gamma_1' \Rightarrow A, B, \Delta_1 \mid \Delta_1' \\
(3)\ B, \Gamma_1 \mid \Gamma_1' \Rightarrow A, \Delta_1 \mid \Delta_1'
\end{array}
}{A \to B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}\ \text{L}\to
$$

(D2)

$$
\cfrac{
\begin{array}{l}
(5)\ \Gamma_3 \mid \Gamma_3', A \to B, A \Rightarrow B, \Delta_3 \mid \Delta_3' \\
(6)\ \Gamma_3 \mid \Gamma_3', A \to B, A \Rightarrow \Delta_3 \mid \Delta_3', A \\
(7)\ \Gamma_3 \mid \Gamma_3', A \to B, B \Rightarrow \Delta_3 \mid \Delta_3', B
\end{array}
}{\Gamma_3 \mid \Gamma_3', A \to B \Rightarrow \Delta_3 \mid \Delta_3', A \to B}\ \text{R}'\to
$$

$$
\cfrac{(D1) \qquad \cfrac{(4)\ A, \Gamma_2 \mid \Gamma_2' \Rightarrow B, \Delta_2 \mid \Delta_2'}{\Gamma_2 \mid \Gamma_2' \Rightarrow A \to B, \Delta_2 \mid \Delta_2'}\ \text{R}\to \qquad (D2)}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}
$$

This is transformed into:

(D1)

$$
\cfrac{\Gamma_3 \mid \Gamma_3', A \to B, A \Rightarrow \Delta_3 \mid \Delta_3', A \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A \to B, \Delta_2 \mid \Delta_2'}{\Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3', A \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', A}\ \text{I-Cut}
$$

(D2)

$$
\cfrac{\cfrac{A, \Gamma_2 \mid \Gamma_2' \Rightarrow B, \Delta_2 \mid \Delta_2' \qquad \Gamma_1 \mid \Gamma_1' \Rightarrow A, B, \Delta_1 \mid \Delta_1' \qquad (D1)}{\Gamma_1, \Gamma_2, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_2', \Gamma_3' \Rightarrow B, \Delta_1, \Delta_2, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_2', \Delta_3'}\ \text{3-Cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow B, \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\ \text{Lemma 2.5}
$$

(D3)

$$
\cfrac{\Gamma_3 \mid \Gamma_3', A \to B, A \Rightarrow \Delta_3 \mid \Delta_3', A \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A \to B, \Delta_2 \mid \Delta_2'}{\Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3', A \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', A}\ \text{I-Cut}
$$

(D4)

$$\dfrac{\dfrac{A, B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad B, \Gamma_1 \mid \Gamma_1' \Rightarrow A, \Delta_1 \mid \Delta_1' \qquad (D3)}{B, \Gamma_1, \Gamma_2, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_2', \Delta_3'}\text{ 3-Cut}}{B, \Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ Lemma 2.5}$$

(D5)

$$\dfrac{\Gamma_3 \mid \Gamma_3', A \to B, B \Rightarrow \Delta_3 \mid \Delta_3', B \qquad \Gamma_2 \mid \Gamma_2' \Rightarrow A \to B, \Delta_2 \mid \Delta_2'}{\Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3', B \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', B}\text{ I-Cut}$$

$$\dfrac{(D2) \qquad (D4) \qquad (D5)}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ 3-Cut, Lemma 2.5}$$

where every instance of I-Cut is of lesser height and every instance of 3-Cut of lesser weight.

The other derivation is of the form:

(D1)

$$\dfrac{\begin{array}{l}(1)\ A, B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \\ (2)\ \Gamma_1 \mid \Gamma_1' \Rightarrow A, B, \Delta_1 \mid \Delta_1' \\ (3)\ B, \Gamma_1 \mid \Gamma_1' \Rightarrow A, \Delta_1 \mid \Delta_1'\end{array}}{A \to B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}\text{ L}\to$$

$$\dfrac{(D1) \qquad \dfrac{(4)\ A, \Gamma_2 \mid \Gamma_2' \Rightarrow B, \Delta_2 \mid \Delta_2'}{\Gamma_2 \mid \Gamma_2' \Rightarrow A \to B, \Delta_2 \mid \Delta_2'}\text{ R}\to \qquad \dfrac{\begin{array}{l}(5)\ \Gamma_3 \mid \Gamma_3' \Rightarrow \Delta_3 \mid \Delta_3', A, A \to B \\ (6)\ \Gamma_3 \mid \Gamma_3', B \Rightarrow \Delta_3 \mid \Delta_3', A \to B\end{array}}{\Gamma_3 \mid \Gamma_3', A \to B \Rightarrow \Delta_3 \mid \Delta_3', A \to B}\text{ L}'\to}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}$$

This is transformed into:

(D1)

$$\dfrac{A \to B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_3 \mid \Gamma_3' \Rightarrow \Delta_3 \mid \Delta_3', A, A \to B}{\Gamma_1, \Gamma_3 \mid \Gamma_1', \Gamma_3' \Rightarrow \Delta_1, \Delta_3 \mid \Delta_1', \Delta_3', A}\text{ O-Cut}_1$$

(D2)

$$\dfrac{A, \Gamma_2 \mid \Gamma_2' \Rightarrow B, \Delta_2 \mid \Delta_2' \qquad (D1)}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow B, \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ O-Cut}_2$$

$$\dfrac{(D2) \qquad \dfrac{A \to B, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \qquad \Gamma_3 \mid \Gamma_3', B \Rightarrow \Delta_3 \mid \Delta_3', A \to B}{\Gamma_1, \Gamma_3 \mid \Gamma_1', \Gamma_3', B \Rightarrow \Delta_1, \Delta_3 \mid \Delta_1', \Delta_3'}\text{ O-Cut}_3}{\Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\text{ I-Cut, Lemma 2.5}$$

where the instances of O-Cut labeled 1 and 3 are of lesser height, while the one labeled 2, as well as I-Cut, are of lesser weight.

Finally, we check the case where the cut formula is of the form $\forall x A$ and principal in all premises. Then there are two possible derivations, depending on which formula is principal in the rightmost premise. One derivation is of the form:

(D1)

$$\dfrac{E!t, \forall x A, A[t/x], \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}{E!t, \forall x A, \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1'}\,\mathrm{L}\forall$$

(D2)

$$\dfrac{E!t', \Gamma_3 \mid \Gamma_3', \forall x A, A[t'/x] \Rightarrow \Delta_3 \mid \Delta_3', \forall x A}{E!t', \Gamma_3 \mid \Gamma_3', \forall x A \Rightarrow \Delta_3 \mid \Delta_3', \forall x A}\,\mathrm{L}'\forall$$

$$\dfrac{(\mathrm{D1}) \qquad \dfrac{E!s, \Gamma_2 \mid \Gamma_2' \Rightarrow A[s/x], \Delta_2 \mid \Delta_2'}{\Gamma_2 \mid \Gamma_2' \Rightarrow \forall x A, \Delta_2 \mid \Delta_2'}\,\mathrm{R}\forall \qquad (\mathrm{D2})}{E!t, E!t', \Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\,\text{3-Cut}$$

This is transformed into:

(D1)

$$\dfrac{\begin{array}{l}(1)\ E!t, \forall x A, A[t/x], \Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1' \\ (2)\ \Gamma_2 \mid \Gamma_2' \Rightarrow \forall x A, \Delta_2 \mid \Delta_2' \\ (3)\ E!t', \Gamma_3 \mid \Gamma_3', \forall x A \Rightarrow \Delta_3 \mid \Delta_3', \forall x A\end{array}}{E!t, E!t', A[t/x], \Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\,\text{3-Cut}_1$$

(D2)

$$\dfrac{\dfrac{\dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow \forall x A, \Delta_2 \mid \Delta_2' \qquad E!t', \Gamma_3 \mid \Gamma_3', \forall x A \Rightarrow \Delta_3 \mid \Delta_3', \forall x A}{E!t', \Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3' \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', \forall x A}\,\text{I-Cut}}{E!t, E!t', \Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3' \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', A[t/x]}\,\text{Lemma 2.4}}{E!t, E!t', \Gamma_2, \Gamma_3 \mid \Gamma_2', \Gamma_3', A[t/x] \Rightarrow \Delta_2, \Delta_3 \mid \Delta_2', \Delta_3', A[t/x]}\,\text{Lemma 2.3}$$

$$\dfrac{(\mathrm{D1}) \qquad \dfrac{E!s, \Gamma_2 \mid \Gamma_2' \Rightarrow A[s/x], \Delta_2 \mid \Delta_2'}{E!t, \Gamma_2 \mid \Gamma_2' \Rightarrow A[t/x], \Delta_2 \mid \Delta_2'}\,\text{Lemma 2.1} \qquad (\mathrm{D2})}{E!t, E!t, \Gamma_1, \Gamma_2, \Gamma_3 \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Delta_1', \Delta_2', \Delta_3'}\,\text{3-Cut}_2\text{,Lemma 2.5}$$

Here 3-Cut$_1$ and I-Cut are of lesser height, while 3-Cut$_2$ is of lesser weight.

The case of the other formula principal in the rightmost premise is very similar and slightly simpler.

Simple for the remaining cases. □

We thus arrive at the end of the proof of Theorem 2.8. While it is quite long, the good news is that, as we already noted, the cost of proving it is one we only incur once. In the following section, on strong neutral free logic, we will be able to re-use virtually all of it.

3 Strong Neutral Free Logic

We now move on to neutral free logic based on the strong Kleene logic K_3 (see e.g. [42]). Since the propositional basis is limited to just the negation and implication, ultimately the only difference from weak neutral free logic will be the treatment of the conditional, so the presentation here will be significantly more schematic than in the previous section to avoid unnecessary repetition. Given such a minute difference, we can see a possible source of confusion in intuitions about neutral free logic. And since the treatment of quantification remains unaltered, it makes sense to see these as different versions of a single logic, rather than separate logics.

While our investigation in this paper stops at these two, it is not our claim that all the versions are exhausted. Rather, as we already stated, one needs to start somewhere, and the two systems under consideration were already suggested in [42].

The system here, while taken over from the previous section, is essentially a modification of the one in [13] along the same lines as the one in the previous section was of [14]. Another sequent calculus for K_3 appeared recently in [9], utilizing a [16]-style calculus. As before, our formulation is between the two approaches, and in addition to the reasons for utilizing our own presentation discussed in the introduction, having it uniform in the paper allows us to re-use significant portions of the previous section.

Sequent Calculus for Strong Neutral Free Logic $G3_{snf}$

Initial sequents:

$(\text{is}_1)\ p, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', p$ $\qquad (\text{is}_2)\ \Gamma \mid \Gamma', p \Rightarrow p, \Delta \mid \Delta'$

$(\text{is}_3)\ p, \Gamma \mid \Gamma' \Rightarrow p, \Delta \mid \Delta'$

$(\text{is}_4)\ \{E!t_i\}_{1\leq i\leq n}, \Gamma \mid \Gamma', P(t_1...t_n) \Rightarrow \Delta \mid \Delta', P(t_1...t_n)$

Propositional rules:

$$\frac{\Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'}{\neg A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\,\mathrm{L}\neg \qquad \frac{A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \neg A, \Delta \mid \Delta'}\,\mathrm{R}\neg$$

$$\frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A}{\Gamma \mid \Gamma', \neg A \Rightarrow \Delta \mid \Delta'}\,\mathrm{L}'\neg \qquad \frac{\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \neg A}\,\mathrm{R}'\neg$$

$$\frac{\Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta' \qquad B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{A \to B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\,\mathrm{L}\to$$

$$\frac{A, \Gamma \mid \Gamma' \Rightarrow B, \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow A \to B, \Delta \mid \Delta'}\,\mathrm{R}\to$$

$$\frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \qquad \Gamma \mid B, \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid A \to B, \Gamma' \Rightarrow \Delta \mid \Delta'} \mathrm{L}'\!\to$$

$$\frac{\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta', B}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \to B} \mathrm{R}'\!\to$$

Quantifier rules:

$$\frac{E!t, \forall x A, A[t/x], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \forall x A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'} \mathrm{L}\forall \qquad \frac{E!t, \Gamma \mid \Gamma' \Rightarrow A[t/x], \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \forall x A, \Delta \mid \Delta'} \mathrm{R}\forall^*$$

$$\frac{E!t, \Gamma \mid \forall x A, A[t/x], \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \Gamma \mid \forall x A, \Gamma' \Rightarrow \Delta \mid \Delta'} \mathrm{L}'\forall \qquad \frac{E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A} \mathrm{R}'\forall^*$$

$E!$ rules:

$$\frac{E!t, P[t], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{P[t], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'} \mathrm{L}E! \qquad \frac{E!t, \Gamma \mid \Gamma' \Rightarrow P[t], \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow P[t], \Delta \mid \Delta'} \mathrm{R}E!$$

$$\frac{\{E!t_i\}_{1 \leq i \leq n}, P(t_1 ... t_n), \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{\{E!t_i\}_{1 \leq i \leq n}, \Gamma \mid \Gamma', P(t_1 ... t_n) \Rightarrow \Delta \mid \Delta'} \mathrm{L}'E!$$

$$\frac{\{E!t_i\}_{1 \leq i \leq n}, \Gamma \mid \Gamma' \Rightarrow P(t_1 ... t_n), \Delta \mid \Delta'}{\{E!t_i\}_{1 \leq i \leq n}, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', P(t_1 ... t_n)} \mathrm{R}'E!$$

$$\frac{E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid \Gamma', E!t \Rightarrow \Delta \mid \Delta'} \mathrm{L}Tr_{E!} \qquad \frac{\Gamma \mid \Gamma' \Rightarrow E!t, \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', E!t} \mathrm{R}Tr_{E!}$$

Where p is atomic, $P[t]$ and $P(t_1 \dots t_n)$ are not $E!t$ and t is fresh in rules marked with $*$.

Clearly, this system differs from the previous only in the rules L$\to$ and R$'\to$, i.e. only the conditions under which an implication is true or not have changed (cf. [21, p. 290]). It therefore follows straightforwardly that

Theorem 3.1 (Structural properties G3$_{snf}$) *Axiom generalization and height-preserving substitution and invertibility hold for G3$_{snf}$, weakening, contraction and transfer are height-preserving admissible, and Outside, Inside, Right, Triple, E! and Left cuts are admissible.*

Proof Routine with only minor modifications of the previous proofs, in the usual order as presented in the previous section. For the standard structural properties, the proof more closely resembles those in [36].

For transfer, a new case is when the transfer formula A is of the form $C \to D$ and principal in the last step. Then for LTr the last step has the following form:

$$\frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C \qquad \Gamma \mid D, \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid C \to D, \Gamma' \Rightarrow \Delta \mid \Delta'} \mathrm{L}'\!\to$$

We can then obtain the derivation:

$$\dfrac{\dfrac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C}{\Gamma \mid \Gamma' \Rightarrow C, \Delta \mid \Delta'}\text{ i.h.[R}Tr\text{]} \qquad \dfrac{\Gamma \mid D, \Gamma' \Rightarrow \Delta \mid \Delta'}{D, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\text{ i.h.[L}Tr\text{]}}{C \to D, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\text{ L}\to$$

Similarly, for RTr the last step then has the following form:

$$\dfrac{\Gamma \mid \Gamma', C \Rightarrow \Delta \mid \Delta', D}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', C \to D}\text{ R}'\to$$

We can then obtain the derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \mid \Gamma', C \Rightarrow \Delta \mid \Delta', D}{\Gamma \mid \Gamma', C \Rightarrow D, \Delta \mid \Delta'}\text{ i.h.[R}Tr\text{]}}{C, \Gamma \mid \Gamma' \Rightarrow D, \Delta \mid \Delta'}\text{ i.h.[L}Tr\text{]}}{\Gamma \mid \Gamma' \Rightarrow C \to D, \Delta \mid \Delta'}\text{ R}\to$$

Note that transfer is height-preserving.

For the cut rules, the only important new case is when the cut formula is of the form $A \to B$ and principal in all premises of any cut rule. We illustrate on the case of O-Cut, since there both new rules are used and the derivation is of the form:

$$\dfrac{\dfrac{\Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1', B}{\Gamma_1 \mid \Gamma_1' \Rightarrow \Delta_1 \mid \Delta_1', A \to B}\text{ R}'\to \qquad \dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2' \qquad B, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{A \to B, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}\text{ L}\to}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{ O-Cut}$$

This is transformed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma_2 \mid \Gamma_2' \Rightarrow A, \Delta_2 \mid \Delta_2' \qquad \Gamma_1 \mid \Gamma_1', A \Rightarrow \Delta_1 \mid \Delta_1', B}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', B, \Delta_2'}\text{ I-Cut} \qquad B, \Gamma_2 \mid \Gamma_2' \Rightarrow \Delta_2 \mid \Delta_2'}{\Gamma_1, \Gamma_2, \Gamma_2 \mid \Gamma_1', \Gamma_2', \Gamma_2' \Rightarrow \Delta_1, \Delta_2, \Delta_2 \mid \Delta_1', \Delta_2', \Delta_2'}\text{ O-Cut}}{\Gamma_1, \Gamma_2 \mid \Gamma_1', \Gamma_2' \Rightarrow \Delta_1, \Delta_2 \mid \Delta_1', \Delta_2'}\text{ Contraction}$$

Where both the I-Cut and the O-Cut are of lesser weight.

Similar for all the other cases. □

4 Generalized Semantics for Neutral Free Logic

A significant impediment to a unified exploration of free logics is that in its literature, as we have already mentioned, the crucial predicate $E!$ comes in two very distinct flavors (existence and definedness). In both cases terms outside of $E!$ fail to denote, but where the fault lies is different. In the case of definedness the problem is with the term - it is non-sensical, disallowed, etc. On the other hand, with existence the fault is with the world, as it fails to contain the objects for (even perfectly fine) terms to pick out. Consequently, these do not agree even on the category of things they are attributed to - definedness is a property of terms, while existence belongs to objects. This makes comparing the two, to say nothing of unifying, a difficult prospect.

To get around this difficulty, while also facilitating meta-theory, the approach of utilizing *generalized semantics* was introduced in [40], taking a cue from [18] in having the language be its own model. In a nutshell, the idea of this approach is that for the proofs of soundness and completeness, a *general description* of a model, rather than a full-blown model, will suffice. This allows for a very simplified presentation of semantics and, given that it has a greater level of generality, can be applied to different particular (potentially incompatible at a lower level) semantic approaches. Great use was made of this feature of generalized semantics to offer a uniform picture of a variety of positive and negative free logics.

Adding neutral free logics to the overall picture in a similar way is left for another time, as it would take this paper beyond a manageable length (for the same reason we forgo a discussion of some nuances of the relationship between free and bound variables). Nonetheless, we notice that, as can be expected, the structures needed for the two systems under consideration here are identical and vary only at the level of valuation, specifically for the implication.

Definition 4.1 (Neutral structure $\mathcal{S}_{nt}$) A neutral structure $\mathcal{S}_{nt}$ is a pair $\langle \mathcal{D}, \varphi \rangle$, where $\mathcal{D} = a_1, \ldots, b_1, \ldots$ is a countable list of free individual variables, and φ an *interpretation function* on $\mathcal{L}$:

- $\varphi(t) = t$, where $t \in \mathcal{D}$ (to emphasize its dual role we will abuse the notation slightly and write $\mathcal{D}$ as $\varphi(\mathcal{D})$)
- $\varphi(E!) \subseteq \varphi(\mathcal{D})$,
- $\varphi(P^n) \subseteq \varphi(E!)^n$.

Note that here the structure is defined in a "negative-like" style, in that the extension of predicates is limited to the extension of $E!$. This is simply a practical decision and not a fundamental feature of the semantics. Since the extensions of predicates beyond $E!$ are not relevant, this allows for a simpler formulation of the valuations. If the extensions of the predicates instead stretched to the entirety of $\mathcal{D}$ (a "positive-like" structure), a limitation to $E!$ would need to be incorporated into the valuation for atoms. This formulation saves us some space.

Definition 4.2 (Weak Valuation $\mathcal{V}^-$) The truth-value assignment $\mathcal{V}^-$ on the structure $\langle \mathcal{D}, \varphi \rangle$ is defined as

1.

$$\mathcal{V}^-(E!t) = \begin{cases} \top, & \textit{if } t \in \varphi(E!) \\ \bot, & \textit{otherwise} \end{cases}$$

2.

$$\mathcal{V}^-(P^n(t_1 \ldots t_n)) = \begin{cases} +, & \textit{if for some } 1 \leq i \leq n, t_i \notin \varphi(E!) \\ \top, & \textit{if } \langle t_1, \ldots, t_n \rangle \in \varphi(P^n) \\ \bot, & \textit{otherwise} \end{cases}$$

3.

$$\mathcal{V}^-(\neg A) = \begin{cases} \top, & \textit{if } \mathcal{V}^-(A) = \bot \\ \bot, & \textit{if } \mathcal{V}^-(A) = \top \\ +, & \textit{otherwise} \end{cases}$$

4.

$$\mathcal{V}^-(A \to B) = \begin{cases} +, & \textit{if } \mathcal{V}^-(A) = + \textit{ or } \mathcal{V}^-(B) = + \\ \bot, & \textit{if } \mathcal{V}^-(A) = \top \textit{ and } \mathcal{V}^-(B) = \bot \\ \top, & \textit{otherwise} \end{cases}$$

5.

$$\mathcal{V}^-(\forall x A) = \begin{cases} \top, & \textit{if for every } t \in \varphi(E!), \mathcal{V}^-(A[t/x]) = \top \\ \bot, & \textit{if for some } t \in \varphi(E!), \mathcal{V}^-(A[t/x]) = \bot \\ +, & \textit{otherwise} \end{cases}$$

Definition 4.3 (Strong Valuation $\mathcal{V}^3$) The truth-value assignment $\mathcal{V}^3$ on the structure $\langle \mathcal{D}, \varphi \rangle$ is defined as

1.

$$\mathcal{V}^3(E!t) = \begin{cases} \top, & \textit{if } t \in \varphi(E!) \\ \bot, & \textit{otherwise} \end{cases}$$

2.

$$\mathcal{V}^3(P^n(t_1 \dots t_n)) = \begin{cases} +, & \textit{if for some } 1 \leq i \leq n, t_i \notin \varphi(E!) \\ \top, & \textit{if } \langle t_1, \dots, t_n \rangle \in \varphi(P^n) \\ \bot, & \textit{otherwise} \end{cases}$$

3.

$$\mathcal{V}^3(\neg A) = \begin{cases} \top, & \textit{if } \mathcal{V}^3(A) = \bot \\ \bot, & \textit{if } \mathcal{V}^3(A) = \top \\ +, & \textit{otherwise} \end{cases}$$

4.

$$\mathcal{V}^3(A \to B) = \begin{cases} \top, & \textit{if } \mathcal{V}^3(A) = \bot \textit{ or } \mathcal{V}^3(B) = \top \\ \bot, & \textit{if } \mathcal{V}^3(A) = \top \textit{ and } \mathcal{V}^3(B) = \bot \\ +, & \textit{otherwise} \end{cases}$$

5.

$$\mathcal{V}^3(\forall x A) = \begin{cases} \top, & \textit{if for every } t \in \varphi(E!), \mathcal{V}^3(A[t/x]) = \top \\ \bot, & \textit{if for some } t \in \varphi(E!), \mathcal{V}^3(A[t/x]) = \bot \\ +, & \textit{otherwise} \end{cases}$$

We list full definitions separately to make them self-standing, but as with the sequent calculus rules, it should be clear these differ only in the rules for implication.

4.1 Meta-theoretic Properties

We open the discussion of meta-theoretic properties of the two systems with the notion of *validity*.

Definition 4.4 (Validity) A sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is *valid* when if every formula in Γ is $\top$ and no formula in Γ' is $\bot$, then either not everything in Δ is $\bot$ or something in Δ' is $\top$.

We now begin to show that soundness holds for weak neutral free logic.

Theorem 4.5 (Soundness $G3_{wnf}$) *If the sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is derivable in $G3_{wnf}$, then it is valid under any valuation $\mathcal{V}^-$.*

Proof By induction on the height of the derivation.

Basic step. Let the initial sequent be of the form $P, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', P$. Assume everything in the antecedent is $\top$ and nothing in antecedent prime is $\bot$. Then something in succedent prime, namely P, is $\top$.

Let the initial sequent be of the form $P, \Gamma \mid \Gamma' \Rightarrow P, \Delta \mid \Delta'$ with the same assumptions. Since everything in the antecedent is $\top$ then not everything in the succedent is $\bot$, namely P is not.

Let the initial sequent be of the form $\Gamma \mid \Gamma', P \Rightarrow P, \Delta \mid \Delta'$, with the same assumptions. Since nothing in antecedent prime is $\bot$, then not everything in the succedent is $\bot$, namely P is not.

Finally, let the initial sequent be of the form $\{E!t_i\}_{1 \leq i \leq n}, \Gamma \mid \Gamma', P(t_1...t_n) \Rightarrow \Delta \mid \Delta', P(t_1...t_n)$ with the same assumptions. Since $P(t_1...t_n)$ is not $\bot$ but for every $1 \leq i \leq n$, by assumption $t_i \in \varphi(E!)$, it follows that $P(t_1...t_n)$, and therefore something in succedent prime, is $\top$.

Inductive step.

If the last step of the derivation is obtained by L$\rightarrow$, then it has the form

$$\dfrac{A, B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta' \qquad \Gamma \mid \Gamma' \Rightarrow A, B, \Delta \mid \Delta' \qquad B, \Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta'}{A \rightarrow B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\,\mathrm{L}\!\rightarrow$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Then, since $A \rightarrow B$ is $\top$, either (i) A and B are both $\top$, (ii) A and B are both $\bot$, or (iii) A is $\bot$ and B is $\top$.

In case (i) everything in the antecedent of the first premise is $\top$ and nothing in its antecedent prime is $\bot$, so by the inductive hypothesis either not everything in Δ is $\bot$ or something in Δ' is $\top$.

In case (ii) by the inductive hypothesis on the second premise it follows that either not everything in A, B, Δ is $\bot$ or something in Δ' is $\top$. But since A and B are both $\bot$, then either not everything in Δ is $\bot$ or something in Δ' is $\top$.

In case (iii) everything in the antecedent of the third premise is $\top$ and nothing in its antecedent prime is $\bot$, so by the inductive hypothesis either not everything in A, Δ is $\bot$ or something in Δ' is $\top$. But since A is $\bot$, then either not everything in Δ is $\bot$ or something in Δ' is $\top$.

If the last step of the derivation is obtained by R→, then it has the form

$$\frac{A, \Gamma \mid \Gamma' \Rightarrow B, \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow A \to B, \Delta \mid \Delta'}\ \mathrm{R}{\to}$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Moreover, assume for *reductio* that everything in the succedent is $\bot$ and nothing in succedent prime is $\top$. Then, since $A \to B$ is $\bot$, A is $\top$ and B is $\bot$. Therefore, everything in the antecedent of the premise (including A) is $\top$ and nothing in its antecedent prime is $\bot$, and so by the inductive hypothesis either not everything in B, Δ is $\bot$ or something in Δ' is $\top$. But since B is $\bot$ and we have assumed for *reductio* that everything in Δ is $\bot$, the first disjunct leads to a contradiction. Likewise, since we have assumed for *reductio* that nothing in succedent prime is $\top$, the second disjunct leads to a contradiction. So we can reject the assumption and conclude that either not everything in $A \to B, \Delta$ is $\bot$ or something in Δ' is $\top$.

If the last step of the derivation is obtained by L′→, then it has the form

$$\frac{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \qquad \Gamma \mid B, \Gamma' \Rightarrow \Delta \mid \Delta'}{\Gamma \mid A \to B, \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}'{\to}$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Then everything in the antecedent of the left premise is $\top$ and nothing in its antecedent prime is $\bot$. Therefore, by the inductive hypothesis, either not everything in Δ is $\bot$ or something in Δ', A is $\top$. In the first case we are done, and likewise if something in Δ' is $\top$. Now assume A is $\top$. Since $A \to B$ is not $\bot$, it follows that B is not $\bot$. Therefore, everything in the antecedent of the right premise is $\top$ and nothing in its antecedent prime is $\bot$ and so by the inductive hypothesis either not everything in Δ is $\bot$ or something in Δ' is $\top$.

If the last step of the derivation is obtained by R′→, then it has the form

$$\frac{\Gamma \mid \Gamma', A \Rightarrow B, \Delta \mid \Delta' \qquad \Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta', A \qquad \Gamma \mid \Gamma', B \Rightarrow \Delta \mid \Delta', B}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \to B}\ \mathrm{R}'{\to}$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. If $A \to B$ is $\top$ we are done, so assume it is not. Then there are three options - either (i) A is $\top$ and B is $\bot$, (ii) A is $+$, or (iii) B is $+$.

In case (i) everything in the antecedent of the first premise is $\top$ and nothing in its antecedent prime is $\bot$. Therefore, by the inductive hypothesis, either not everything in B, Δ is $\bot$ or something in Δ' is $\top$, and since B is $\bot$ it follows that either not everything in Δ is $\bot$ or something in Δ' is $\top$.

In case (ii) everything in the antecedent of the second premise is $\top$ and nothing in its antecedent prime is $\bot$. Therefore, by the inductive hypothesis, either not everything in Δ is $\bot$ or something in Δ', A is $\top$, and since A is $+$ it follows that either not everything in Δ is $\bot$ or something in Δ' is $\top$.

Case (iii) mirrors case (ii) exactly, for B and third premise.

Similar, but simpler, for negation and straightforward for $E!$ rules. What remains to be examined are the rules for the universal.

If the last step of the derivation is obtained by L∀, then it has the form

$$\frac{E!t, \forall x A, A[t/x], \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \forall x A, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}\forall$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Then, since $E!t$ and $\forall x A$ are $\top$, so is $A[t/x]$. Therefore, everything in the antecedent of the premise is $\top$ and nothing in its antecedent prime is $\bot$ and so by the inductive hypothesis either not everything in Δ is $\bot$ or something in Δ' is $\top$.

If the last step of the derivation is obtained by R$\forall$, then it has the form

$$\frac{E!t, \Gamma \mid \Gamma' \Rightarrow A[t/x], \Delta \mid \Delta'}{\Gamma \mid \Gamma' \Rightarrow \forall x A, \Delta \mid \Delta'}\ \mathrm{R}\forall$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Moreover, assume for *reductio* that everything in the succedent is $\bot$ and nothing in succedent prime is $\top$. Then, since $\forall x A$ is $\bot$, there is some t such that $E!t$ is $\top$ and $A[t/x]$ is $\bot$. Let it just be t (otherwise use Lemma 2.1). Then everything in the antecedent of the premise is $\top$ and nothing in its antecedent prime is $\bot$, so by the inductive hypothesis either not everything in $A[t/x]$, Δ is $\bot$ or something in Δ' is $\top$. But $A[t/x]$ is $\bot$ and by the assumption for *reductio* everything in the succedent is likewise $\bot$ and nothing in succedent prime is $\top$, so contradiction either way, and we can reject the assumption.

If the last step of the derivation is obtained by L$'\forall$, then it has the form

$$\frac{E!t, \Gamma \mid \forall x A, A[t/x], \Gamma' \Rightarrow \Delta \mid \Delta'}{E!t, \Gamma \mid \forall x A, \Gamma' \Rightarrow \Delta \mid \Delta'}\ \mathrm{L}'\forall$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Moreover assume for *reductio* that $A[t/x]$ is $\bot$. Then $\forall x A$ is $\bot$, contradictory to our initial assumption. Therefore $A[t/x]$ is not $\bot$. Then everything in the antecedent of the premise is $\top$ and nothing in its antecedent prime is $\bot$, and so by the inductive hypothesis either not everything in Δ is $\bot$ or something in Δ' is $\top$.

If the last step of the derivation is obtained by R$'\forall$, then it has the form

$$\frac{E!t, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[t/x]}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', \forall x A}\ \mathrm{R}'\forall$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Now take every s such that $E!s$ is $\top$. For each such s, by Lemma 2.1 $E!s, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A[s/x]$ is derivable (we know by the freshness condition that t does not occur in Γ, Γ', Δ or Δ') with the same height. In each case, it follows by the inductive hypothesis that either (i) not everything in Δ is $\bot$ or (ii) something in $\Delta', A[s/x]$ is $\top$. In case (i) holds we are done and in case (ii) if something in Δ' is $\top$ we are done. Otherwise for every t such that $E!t$ holds, $A[t/x]$ is $\top$, and therefore $\forall x A$ is $\top$ and we are likewise done.

Finally, straightforward for $E!$ rules. $\square$

This concludes the proof of soundness for weak neutral free logic, and we move on to the proof of the same property for the strong version.

Theorem 4.6 (Soundness $G3_{snf}$) *If the sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is derivable in $G3_{snf}$, then it is valid under any valuation $\mathcal{V}^3$.*

Proof By induction on the height of the derivation. For most cases the same as the corresponding case for $G3_{wnf}$, so we just examine the rules for implication, and specifically for L→ and R′→, since for the other two the proof is identical to $G3_{wnf}$ (it relies only on the condition for falsity for implication, which is the same for $\mathcal{V}^-$ and $\mathcal{V}^3$).

If the last step of the derivation is obtained by L→, then it has the form

$$\frac{\Gamma \mid \Gamma' \Rightarrow A, \Delta \mid \Delta' \qquad B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}{A \to B, \Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'}\, \mathrm{L}\!\to$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. Therefore everything in the antecedent of the left premise is $\top$ and nothing in its antecedent prime is $\bot$, and so by the inductive hypothesis it follows that either not everything in A, Δ is $\bot$ or something in Δ' is $\top$. Since $A \to B$ is $\top$ it follows that either (i) A is $\bot$ or (ii) B is $\top$. In case (i) it follows that either not everything in Δ is $\bot$ (from the fact not everything in A, Δ is $\bot$) or something in Δ' is $\top$ and we are done. In case (ii) everything in the antecedent of the right premise is $\top$ and nothing in its antecedent prime is $\bot$, and so by the inductive hypothesis it follows that either not everything in Δ is $\bot$ or something in Δ' is $\top$.

If the last step of the derivation is obtained by R′→, then it has the form

$$\frac{\Gamma \mid \Gamma', A \Rightarrow \Delta \mid \Delta', B}{\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta', A \to B}\, \mathrm{R}'\!\to$$

Assume everything in the antecedent of the conclusion is $\top$ and nothing in its antecedent prime is $\bot$. If $A \to B$ is $\top$ we are done, so assume it is not. So then either (i) A is $\top$ and B is $\bot$ or (ii) both A and B are $+$. In either case A is not $\bot$, so everything in the antecedent of the premise is $\top$ and nothing in its antecedent prime is $\bot$. Therefore, by the inductive hypothesis either not everything in Δ is $\bot$ or something in Δ', B is $\top$. But in both case (i) and case (ii) B is not $\top$, so it follows that either not everything in Δ is $\bot$ or something in Δ' is $\top$. □

4.1.1 Completeness

The proof of completeness is significantly more involved and moves through several auxiliary lemmas, taken over with minor modifications from [40]. As above, we begin with the proof of the property for weak neutral free logic.

Definition 4.7 (Active sequent) A sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is called active if *none* of the below hold:

1. the same atomic formula occurs in Γ and Δ',

2. the same atomic formula occurs in Γ and Δ,
3. the same atomic formula occurs in Γ' and Δ,
4. the same atomic formula occurs in Γ' and Δ' and for every free variable t_i in it, $E!t_i$ occurs in Γ.

Definition 4.8 (Reduction tree) A reduction tree for a sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is built in steps. At step 0, the tree is just $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$.

Each subsequent step consists of stages. At each stage and for each sequent $\Gamma_i \mid \Gamma_i' \Rightarrow \Delta_i \mid \Delta_i'$ active at the beginning of it, we apply to any eligible (pair of) formulas in the sequent the rule of the stage once (thereby extending the height of the tree by n, for n such formulas in $\Gamma_i \mid \Gamma_i' \Rightarrow \Delta_i \mid \Delta_i'$, and creating at most 3^n branches, before proceeding to the next stage). We call an application of a rule to the formula(s) their *reduction*.

The order of stages is:

(1) L$\neg$ (2) R$\neg$ (3) L$'\neg$ (4) R$'\neg$ (5) L$\to$ (6) R$\to$ (7) L$'\to$ (8) R$'\to$

(9) L$\forall$, for every pair of formulas $\forall x A$ and $E!t$ in Γ_i.

(10) R$\forall$, taking for the reduction of each formula $\forall x A$ in Δ_i from the denumerable list of free individual variables the first such variable t not yet used in the reduction tree.

(11) L$'\forall$, for every pair of formulas $\forall x A$ in Γ_i' and $E!t$ in Γ_i.

(12) R$'\forall$, taking for the reduction of each formula $\forall x A$ in Δ_i' from the denumerable list of free individual variables the first such variable t not yet used in the reduction tree.

(13) L$E!$ (14) R$E!$ (15) L$'E!$ (16) R$'E!$ (17) L$Tr_{E!}$ (18) R$Tr_{E!}$

Each active sequent to which no rule can be applied we just copy.

We now show that

Lemma 4.9 (Reduction) *For any sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ its reduction tree either produces a proof or it produces a structure and a valuation that makes all the formulas in Γ true, no formula in Γ' false, all formulas in Δ false and no formula in Δ' true.*

Proof It is clear that a reduction tree with no active sequents at the top of any branch will produce, read top down (and thus beginning with initial sequents and ending with $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$), a finite derivation of that sequent. The second part is more involved and goes through several lemmas below. □

We now build an invalidating structure and a valuation from a reduction tree to prove the second part. The existence of an infinite branch is guaranteed by the Kőnig's lemma in the usual way [36, p. 82].

Definition 4.10 (Refutation structure $\mathcal{C}$) Take an infinite branch

$$\mathcal{B} \equiv \Gamma_0 \mid \Gamma_0' \Rightarrow \Delta_0 \mid \Delta_0', \ldots, \Gamma_i \mid \Gamma_i' \Rightarrow \Delta_i \mid \Delta_i', \ldots$$

of a reduction tree for a sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ (where $\Gamma_0 \mid \Gamma_0' \Rightarrow \Delta_0 \mid \Delta_0'$ is $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$) and consider sets $\Gamma^* \equiv \bigcup \Gamma_i$, $\Delta^* \equiv \bigcup \Delta_i$, $\Gamma'^* \equiv (\bigcup \Gamma_i') - \Gamma^*$ and $\Delta'^* \equiv (\bigcup \Delta_i') - \Delta^*$ for $0 \leq i$.

A refutation structure $\mathcal{C}$ for a sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is built by assigning $\top$ to all atomic formulas in Γ^*, $\bot$ to all atomic formulas in Δ^* and other $E!t$ and $+$ to all other atomic formulas $P(t_1 \dots t_n)$ (therefore including all atomic formulas in Γ'^* and Δ'^*), and otherwise according to $\mathcal{V}^-$.

Lemma 4.11 (Uniqueness of assignment) *Any formula A of the language is assigned precisely one of the values $\{\top, +, \bot\}$ in a refutation structure $\mathcal{C}$.*

Proof Straightforward by induction on the weight of the formula A from Definitions 4.7, 4.8 and 4.10. □

We can now show

Lemma 4.12 (Refutation) *For any sequent $\Gamma_i \mid \Gamma_i' \Rightarrow \Delta_i \mid \Delta_i'$ along the infinite branch $\mathcal{B}$, the refutation structure $\mathcal{C}$ assigns $\top$ to any formula A in Γ_i, doesn't assign $\bot$ to any formula A' in Γ_i', assigns $\bot$ to any formula B in Δ and doesn't assign $\top$ to any formula B' in Δ_i'.*

Proof By simultaneous induction on the weights of the formulas A, A', B and B'.

Basic step. Immediate by Definitions 4.7 and 4.10.

Namely, all atoms in any Γ_i are in Γ^* and therefore assigned $\top$.

Atoms in any Γ_i' are either in Γ^* or Γ'^*, and are therefore assigned either $\top$ or $+$.

Similarly, all atoms in any Δ_i are in Δ^* and therefore assigned $\bot$, atoms in any Δ_i' are either in Δ^* or Δ'^*, and are therefore assigned either $\bot$ or $+$.

Inductive step. Straightforward, if tedious, so we just illustrate for the case of the universal.

Let A be $\forall x C$. Then, given the step for L$\forall$ in Definition 4.8, for every t such that $E!t$ is in Γ^* (and by the inductive hypothesis $\top$), $C[t/x]$ is in Γ^* (and by the inductive hypothesis $\top$), and therefore $\forall x C$ is assigned $\top$.

Let A' be $\forall x C$. Then, given the step for L$'\forall$ in Definition 4.8, for every t such that $E!t$ is in Γ^* (and by the inductive hypothesis $\top$), $C[t/x]$ is either in Γ^* or Γ'^* (and by the inductive hypothesis either $\top$ or $+$), and therefore $\forall x C$ is assigned either $\top$ or $+$.

Let B be $\forall x C$. Then, given the step for R$\forall$ in Definition 4.8, for some t such that $E!t$ is in Γ^* (and by the inductive hypothesis $\top$), $C[t/x]$ is in Δ^* (and by the inductive hypothesis $\bot$), and therefore $\forall x C$ is assigned $\bot$.

Let B' be $\forall x C$. Then, given the step for R$'\forall$ in Definition 4.8, for some t such that $E!t$ is in Γ^* (and by the inductive hypothesis $\top$), $C[t/x]$ is either in Δ^* or Δ'^* (and by the inductive hypothesis either $\bot$ or $+$), and therefore $\forall x C$ is assigned either $\bot$ or $+$. □

Therefore, finally,

Theorem 4.13 (Completeness $G3_{wnf}$) *If the sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is valid under any valuation $\mathcal{V}^-$, then it is derivable in $G3_{wnf}$.*

Proof By contraposition. We prove that if a sequent is not derivable, then it is not valid. Immediate from Lemmas 4.9 and 4.12. □

As before, the proof for stong neutral free logic is a minor modification of the one for the weak version. We can therefore likewise prove

Theorem 4.14 (Completeness $G3_{snf}$) *If the sequent $\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta'$ is valid under any valuation $\mathcal{V}^3$, then it is derivable in $G3_{snf}$.*

Proof Same, *mutatis mutandis*, like the proof of Theorem 4.13. We illustrate on a central lemma corresponding to Lemma 4.12 for the case of implication (the only case that is different).

Let A be $C \to D$. Then, given the step for L$\to$ in Definition 4.8, either C is in Δ^*, and by the inductive hypothesis $\bot$, or D is in Γ^*, and by the inductive hypothesis $\top$. In either case, $C \to D$ is assigned $\top$.

Let A' be $C \to D$. Then, given the step for L$'\to$ in Definition 4.8, either C is in Δ'^*, and by the inductive hypothesis not $\top$, or D is in Γ'^*, and by the inductive hypothesis not $\bot$. In either case, $C \to D$ is not assigned $\bot$.

Let B be $C \to D$. Then, given the step for R$\to$ in Definition 4.8, C is in Γ^*, and by the inductive hypothesis $\top$, and D is in Δ^*, and by the inductive hypothesis $\bot$. Therefore, $C \to D$ is assigned $\bot$.

Let B' be $C \to D$. Then, given the step for R$'\to$ in Definition 4.8, C is in Γ'^*, and by the inductive hypothesis not $\bot$, and D is in Δ'^*, and by the inductive hypothesis not $\top$. Therefore, $C \to D$ is not assigned $\top$. □

5 Concluding Remarks

In this paper we presented two systems of neutral free logic, based on either the weak or the strong Kleene understanding of the conditional. We suggest that the previous clash of intuitions regarding the meaning of the conditional in the context of neutral free logic could arise from this ambiguity. Since at the same time the rules for quantification remain unaltered (and all of free logics are precisely and primarily theories of quantification), one can still see it as a single logic, but admitting variant bases. So the technical work here opens up new vistas of philosophical research, where terms which up to now have been conflated can be explored in separation.

Of course, the potential bases are not exhausted in this paper and it is left to future work to try to figure out further ones. Moreover, some approaches present a mix of those discussed here (e.g. in [54] the conditional is read in the weak style, but conjunction and disjunction in the strong, while [51] uses a conditional between the two). The challenge here will be to come up with a satisfactory motivation. One consideration we offer here is that potentially some interpretations of $E!$ mesh better with some propositional bases. For instance, failure of definedness, at least when understood as a syntactic error (e.g. division by zero in Python [43]), clearly propagates to superordinate formulas and thus matches the weak Kleene reading. On the other hand, absence of existence, especially in a more ordinary sense, might lends itself better to gaps, and thus suggest a strong Kleene base. These considerations obviously require further development, but having suggested that so far multiple systems were conflated, it would be interesting to see how far they can be differentiated and to what the results might correspond. We wrap this part up by offering that, while neutral free logic is obviously a part of the family of free logics, it might nonetheless be strong enough to form its own family within them.

Moreover, the generalized semantics utilized here, in the same vein as in [40], provide a simple and useful semantic counterpart to our sequent calculi, but are nonetheless highly artificial. Letting them out, so to speak, into the wild and correlating them to existing systems in the literature provides another open avenue of research. In particular, our work can be related to the 'four validities'of [8], as was suggested in [14] - by omitting the prime part of the antecedent (non-prime part of the succedent) we obtain the strict validity on the respective side of the sequent arrow, and by omitting the non-prime part of the antecedent (prime part of the succedent) we obtain the tolerant validity. Another alternative approach to validity would be to use one of Logic of Paradox (LP) [41], or the generalization of both it and K_3 in First Degree Entailment (FDE) [1, 45]. All these avenues of research are left for the (near) future work.

Finally, as was already announced, the format of the rules presented here was chosen in part because it is easy to extend it into a modal logic using labeled sequent calculi [34, 37] in the same manner as was done in [40], and this presents another potential future development.

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