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On the effective quasi-bosonic Hamiltonian of the electron gas: collective excitations and plasmon modes

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Abstract

We consider an effective quasi-bosonic Hamiltonian of the electron gas which emerges naturally from the random phase approximation and describes the collective excitations of the gas. By a rigorous argument, we explain how the plasmon modes can be interpreted as a special class of approximate eigenstates of this model.

Keywords Electron gas · Collective excitations · Plasmon modes

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1 Introduction

In a series of four seminal papers [8–10, 19] published in the early 1950s, Bohm and Pines proposed the random phase approximation (RPA) as an effective theory to describe the collective excitations of jellium, a homogeneous high-density electron gas moving in a background of uniform positive charge. In particular, they predicted that the electron gas will be decoupled into quasi-free electrons which emerge from the usual mean-field approximation for independent particles, and collective plasmon excitations which correspond to correlated particle motion.

Although the plasmons were quickly detected by experiments [13, 23] after the works of Bohm and Pines, their theoretical explanation remains an important open question in condensed matter and nuclear physics. In 1957, Gell-Mann and Brueckner [15] gave a microscopic derivation of the RPA using a formal summation of a diagrammatic expansion, in which the leading diagrams describe the interaction of pairs of fermions, one from inside and one from outside the Fermi ball. This approach was pushed further by Sawada [21] and Sawada–Brueckner–Fukuda–Brout [22] who interpreted these pairs of electrons as bosons, obtaining an effective Hamiltonian which is quadratic with respect to the bosonic particle pairs.

Recently, the bosonization argument in [21, 22] has been made rigorous in [2–4, 6, 11, 18] for bounded interaction potentials in the mean-field regime, in which the interaction potential is coupled with a small constant such that the interaction energy and the kinetic energy are comparable. In these works, the non-bosonizable terms of the interaction energy are negligible and the rest can be diagonalized by adapting Bogolubov's method [7] to the quasi-bosonic setting. On the mathematical side, the main challenge in this approach is to realize the bosonization structure, which only holds in a very weak sense, making even perturbative results highly nontrivial [18]. In the first non-perturbative results in [3, 4], the correlation energy was computed exactly to the leading order by using a patching technique (averaging fermionic pairs in patches of the Fermi sphere) to enhance the bosonization structure. This approach has been developed further in [6] to improve the analysis of the ground state energy and in [5] to address the dynamics. In [11] we proposed an alternative approach where the weak bosonization structure was used directly (without relying on the patching technique) to approximately diagonalize the fermionic Hamiltonian. One of the advantages of this approach is that it allows us to derive an effective quasi-bosonic Hamiltonian which describes both the correlation energy and the elementary excitations of the system. In the mean-field regime there are, however, no approximate eigenstates corresponding to collective plasmon modes.

The aim of the present paper is to give an explanation of the collective plasmon excitations by taking the quasi-bosonic Hamiltonian derived in [11], extrapolating for the Coulomb potential and going beyond the mean-field regime. We hope that our analysis here will provide useful insights towards the ultimate goal of deriving this effective Hamiltonian and understanding the plasmons from first principles.

2 Derivation of the effective Hamiltonian

In this section we give a heuristic derivation of the effective quasi-bosonic Hamiltonian from the microscopic theory, by summarizing the approach in [11].

We consider a system of N (spinless) fermions on the torus $\mathbb{T}^3 = [0, 2\pi]^3$ (with periodic boundary conditions), interacting via a repulsive potential $V : \mathbb{T}^3 \to \mathbb{R}$, which is to say

$$V(x) = \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k e^{ik\cdot x} \quad \text{with}$$
$$\hat{V}_k = \int_{\mathbb{T}^3} V(x)\, e^{-ik\cdot x}\, dx \geq 0, \quad \forall k \in \mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{0\}, \tag{2.1}$$

and which satisfies the square summability condition on the Fourier transform

$$\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 < \infty. \tag{2.2}$$

Here we ignore the contribution of the zero-momentum mode (equivalently we set $\hat{V}_0 = 0$) as it corresponds to a trivial energy shift of the system. (Physically, this is understood to be compensated for by the uniformly charged background.) The reader may keep in mind the typical situation of the Coulomb potential where $\hat{V}_k = g\,|k|^{-2}$ with $g > 0$, although our analysis applies to a larger class of potentials.

In the many-body Schrödinger theory, the system is described by the Hamiltonian

$$H_N = H_{\text{kin}} + H_{\text{int}} = \sum_{i=1}^{N} (-\Delta_i) + \sum_{1 \leq i < j \leq N} V\left(x_i - x_j\right) \tag{2.3}$$

which acts on the fermionic space

$$\mathcal{H}_N = \bigwedge^N \mathfrak{h}, \quad \mathfrak{h} = L^2\left(\mathbb{T}^3\right). \tag{2.4}$$

Under our assumption, H_N is bounded from below and it can be extended to be a self-adjoint operator on $\mathcal{H}_N$ with domain $D(H_N) = D(H_{\text{kin}}) = \bigwedge^N H^2\left(\mathbb{T}^3\right)$. Moreover, H_N has compact resolvent and we are interested in the low-lying spectrum of H_N when $N \to \infty$.

In general, if $V \not\equiv 0$ and N is large, computing the spectrum of H_N directly from the microscopic formulation (2.3) is impossible, both analytically and numerically.

Consequently one must turn to efficient approximations. One of the most famous approximations for fermions is Hartree–Fock theory, where one restricts the consideration to Slater determinants $g_1 \wedge g_2 \cdots \wedge g_N$ with $\{g_i\}_{i=1}^N$ orthonormal in $L^2\left(\mathbb{T}^3\right)$, which are the least correlated states among all fermionic wave functions. The precision of the Hartree–Fock energy for Coulomb systems can be estimated using general *correlation inequalities* of Bach [1] and Graf–Solovej [17]. Within Hartree–Fock theory, it turns out that the ground state energy can be well approximated by the Fermi state, which is the Slater determinant of the plane waves with momenta inside the Fermi ball B_F, namely

$$\psi_{\mathrm{FS}} = \bigwedge_{p \in B_F} u_p, \quad u_p(x) = (2\pi)^{-\frac{3}{2}} e^{ip\cdot x}, \tag{2.5}$$

with

$$B_F = \overline{B}(0, k_F) \cap \mathbb{Z}^3, \quad N = |B_F|, \tag{2.6}$$

for some $k_F > 0$ (the Fermi momentum); see [16] and [4, Appendix A]. Here for simplicity we assume that the Fermi ball B_F is completely filled by N integer points, which implies that the Fermi state ψ_{FS} is the unique, non-degenerate ground state of the kinetic operator H_{kin}. Without this simplification, the Fermi state is not uniquely defined and the degeneracy of the elementary excitation introduced in the next subsection has to be factored out properly, which complicate the notation but do not improve the physical insight that we want to discuss.

In order to focus on the correlation structure of the interacting system, we need to extract the energy of the Fermi state. For this purpose, it is convenient to write the second-quantized form of the Hamiltonian operator H_N in (2.3):

$$H_N = H_{\mathrm{kin}} + H_{\mathrm{int}} = \sum_{p \in \mathbb{Z}^3_*} |p|^2 c_p^* c_p + \frac{1}{2(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in \mathbb{Z}^3} \hat{V}_k c_{p+k}^* c_{q-k}^* c_q c_p \tag{2.7}$$

where

$$c_p^* = c^*(u_p), \quad c_p = c(u_p), \quad \forall p \in \mathbb{Z}^3, \tag{2.8}$$

are the usual Fermionic creation and annihilation operators associated with the plane wave states u_p. Note that although the second-quantized form in (2.7) can be defined on the fermionic Fock space, we will always consider its restriction to the N particle space which coincides with the original Hamiltonian in (2.3).

Using the canonical anticommutation relations (CAR)

$$\left\{c_p, c_q\right\} = \left\{c_p^*, c_q^*\right\} = 0, \quad \left\{c_p, c_q^*\right\} = \delta_{p,q}, \quad p, q \in \mathbb{Z}^3, \tag{2.9}$$

where $\{A, B\} = AB + BA$, it is straightforward to compute the energy of the Fermi state (see, e.g., [11, Eqs. (1.10) and (1.20)])

$$E_{\mathrm{FS}} = \langle \psi_{\mathrm{FS}}, H_N \psi_{\mathrm{FS}} \rangle = \langle \psi_{\mathrm{FS}}, H_{\mathrm{kin}} \psi_{\mathrm{FS}} \rangle + \langle \psi_{\mathrm{FS}}, H_{\mathrm{int}} \psi_{\mathrm{FS}} \rangle \\ = \sum_{p \in B_F} |p|^2 + \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \left(|L_k| - N \right) \tag{2.10}$$

where we define the *lune* of relative momentum $k \in \mathbb{Z}^3_*$ by

$$L_k = B_F^c \cap (B_F + k) = \left\{ p \in \mathbb{Z}^3 \mid |p - k| \le k_F < |p| \right\}. \tag{2.11}$$

Now we extract the contribution of the Fermi state on the operator level, namely we rewrite the operator in (2.7) as

$$H_N = E_{\mathrm{FS}} + H'_{\mathrm{kin}} + H'_{\mathrm{int}} \tag{2.12}$$

for suitable operators $H'_{\mathrm{kin}}, H'_{\mathrm{int}} : D(H_{\mathrm{kin}}) \subset \mathcal{H}_N \to \mathcal{H}_N$. To be precise, we define the *localized kinetic operator* as

$$H'_{\mathrm{kin}} = H_{\mathrm{kin}} - \langle \psi_{\mathrm{FS}}, H_{\mathrm{kin}} \psi_{\mathrm{FS}} \rangle \ge 0 \tag{2.13}$$

and define the *localized interaction operator* as

$$H'_{\mathrm{int}} = H_{\mathrm{int}} - \langle \psi_{\mathrm{FS}}, H_{\mathrm{int}} \psi_{\mathrm{FS}} \rangle = \sum_{k \in \mathbb{Z}^3_*} \left(H^k_{\mathrm{int}} - \frac{\hat{V}_k}{2\,(2\pi)^3} |L_k| \right) \\ + \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \left(\sum_{p \in L_k} b^*_{k,p} D_k + D^*_k \sum_{p \in L_k} b_{k,p} + \frac{1}{2} D^*_k D_k \right) \tag{2.14}$$

where

$$H^k_{\mathrm{int}} = \sum_{p,q \in L_k} \frac{\hat{V}_k}{2\,(2\pi)^3} \left(b^*_{k,p} b_{k,q} + b_{k,q} b^*_{k,p} \right) + \sum_{p \in L_k} \sum_{q \in L_{-k}} \frac{\hat{V}_k}{2\,(2\pi)^3} \\ \times \left(b^*_{k,p} b^*_{-k,q} + b_{-k,q} b_{k,p} \right) \tag{2.15}$$

for

$$b^*_{k,p} = c^*_p c_{p-k}, \quad D_k = \sum_{p \in B_F \cap (B_F + k)} c^*_{p-k} c_p + \sum_{p \in B_F^c \cap (B_F^c + k)} c^*_{p-k} c_p. \tag{2.16}$$

We interpret $b^*_{k,p}$ as an *excitation operator*, since it creates a state with momentum $p \in B_F^c$ and annihilates a state with momentum $p - k \in B_F$.

2.1 The effective quasi-bosonic Hamiltonian

So far, the decomposition of (2.12) is exact, but to proceed further we now make some simplifications. Roughly speaking, the RPA in the physics literature [15, 21, 22] suggests that the fermionic correlation structure can be described by a bosonic quadratic Hamiltonian. As explained in [11], this bosonic analogy can be summarized in three steps:

Step 1 The excitation operators $b_{k,p}^*$, $b_{k,p}$ in (2.16) should be treated as *bosonic* creation and annihilation operators, where the operators $b_{k,p}$ and $b_{l,q}$ with $k \neq l$ can be considered as acting on independent Fock spaces.

On the mathematical side, we expect the canonical commutation relations (CCR) to hold in an appropriate sense:

$$\left[b_{k,p}, b_{l,q}\right] = \left[b_{k,p}^*, b_{l,q}^*\right] = 0, \quad \left[b_{k,p}, b_{l,q}^*\right] \approx \delta_{k,l}\delta_{p,q}. \tag{2.17}$$

To motivate (2.17), let us consider the simple case $k = l$ where we have the exact relations

$$\left[b_{k,p}, b_{k,q}\right] = [b_{k,p}^*, b_{k,q}^*] = 0, \quad \left[b_{k,p}, b_{k,q}^*\right] = \delta_{p,q} - \delta_{p,q}\left(c_p^* c_p + c_{p-k} c_{p-k}^*\right) \tag{2.18}$$

for all $p, q \in L_k$. The last error terms in (2.18) are not small individually (as we only know $c_p^* c_p,\ c_p c_p^* \leq 1$ by Pauli's exclusion principle), but they are small on average. To make it transparent, let us introduce the *excitation number operator*

$$\mathcal{N}_E := \sum_{p \in B_F^c} c_p^* c_p = \sum_{p \in B_F} c_p c_p^* \quad \text{on } \mathcal{H}_N \tag{2.19}$$

where the last identity in (2.19) follows from the assumption $|B_F| = N$ via the *particle-hole symmetry*[1]. Then it is obvious that

$$\sum_{p,q \in L_k} \delta_{p,q}\left(c_p^* c_p + c_{p-k} c_{p-k}^*\right) \leq 2\mathcal{N}_E \tag{2.20}$$

while for the low-lying eigenfunctions of H_N the excitation number operator $\mathcal{N}_E$ is expected to be of lower order than

$$\sum_{p,q \in L_k} \delta_{p,q} = |L_k| \sim \min(|k|, k_F) k_F^2. \tag{2.21}$$

See, e.g., [11, Proposition A.1] for estimates related to (2.21).

[1] Namely, the *excitation number operator* (which counts the number of particles outside the Fermi state) coincides with the *hole number operator* (which counts the number of holes inside the Fermi state).

Step 2 The full operator in (2.12) is approximated by a quadratic Hamiltonian of $b_{k,p}^*$ and $b_{k,p}$. Concretely, the *non-bosonizable terms*, which are the last sum in (2.14), are ignored, so that

$$H'_{\text{int}} \approx \sum_{k \in \mathbb{Z}_*^3} \left(H_{\text{int}}^k - \frac{\hat{V}_k}{2\,(2\pi)^3}\,|L_k| \right) \tag{2.22}$$

with H_{int}^k given in (2.15), and the localized kinetic operator is thought of as

$$H'_{\text{kin}} \approx \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 2\lambda_{k,p} b_{k,p}^* b_{k,p}, \quad \lambda_{k,p} = \frac{1}{2}\left(|p|^2 - |p-k|^2\right). \tag{2.23}$$

The latter approximation (2.23) is motivated by the commutation relations

$$\left[H'_{\text{kin}}, b_{k,p}^*\right] = 2\lambda_{k,p} b_{k,p}^* \approx \left[\sum_{\ell \in \mathbb{Z}_*^3} \sum_{q \in L_\ell} 2\lambda_{\ell,q} b_{\ell,q}^* b_{\ell,q}, b_{k,p}^*\right] \tag{2.24}$$

where the first identity follows from the (exact) CAR (2.9) and the second relation follows from the (approximate) CCR (2.17).

Step 3 If the effective Hamiltonian

$$\sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 2\lambda_{k,p} b_{k,p}^* b_{k,p} + \sum_{k \in \mathbb{Z}_*^3} \left(H_{\text{int}}^k - \frac{\hat{V}_k}{2\,(2\pi)^3}\,|L_k| \right) \tag{2.25}$$

were an exact bosonic quadratic operator, then it could be diagonalized by a Bogolubov transformation (see, e.g., [11, Sect. 3.2]), resulting in the effective operator

$$E_{\text{corr}} + \sum_{k \in \mathbb{Z}_*^3} 2 \sum_{p,q \in L_k} \left\langle e_p, \tilde{E}_k e_q \right\rangle b_{k,p}^* b_{k,q}. \tag{2.26}$$

Here we introduced the correlation energy

$$\begin{aligned} E_{\text{corr}} &= \sum_{k \in \mathbb{Z}_*^3} \left(\operatorname{tr}\left(\widetilde{E}_k - h_k\right) - \frac{\hat{V}_k}{2\,(2\pi)^3}\,|L_k| \right) \\ &= \sum_{k \in \mathbb{Z}_*^3} \frac{1}{\pi} \int_0^\infty F\left(\frac{\hat{V}_k}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \right) dt \end{aligned} \tag{2.27}$$

with $F(x) = \log(1+x) - x$, and for every $k \in \mathbb{Z}_*^3$ we defined the following real, symmetric operators on $\ell^2(L_k)$:

$$\widetilde{E}_k = \left(h_k^{\frac{1}{2}}\left(h_k + 2P_{v_k}\right)h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}, \quad h_k e_p = \lambda_{k,p} e_p, \quad P_v = |v_k\rangle\langle v_k|,$$
$$v_k = \sqrt{\frac{\hat{V}_k}{2(2\pi)^3}} \sum_{p\in L_k} e_p, \tag{2.28}$$

with $\left(e_p\right)_{p\in L_k}$ the standard orthonormal basis of $\ell^2(L_k)$. However, the quadratic kinetic approximation of (2.23) only holds in the weak sense of (2.24), so the difference

$$H'_{\text{kin}} - \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k} 2\lambda_{k,p} b^*_{k,p} b_{k,p} \tag{2.29}$$

is only essentially invariant under the Bogolubov transformation, rather than close to 0 in a direct sense. Therefore, adding (2.29) to (2.26) we obtain the more realistic approximation, up to a unitary transformation, that

$$H_N \approx E_{\text{FS}} + E_{\text{corr}} + H_{\text{eff}}, \tag{2.30}$$

where we introduced the effective quasi-bosonic Hamiltonian

$$H_{\text{eff}} = H'_{\text{kin}} + 2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} \left\langle e_p, \left(\tilde{E}_k - h_k\right) e_q\right\rangle b^*_{k,p} b_{k,q}. \tag{2.31}$$

which is an operator on the fermionic space $\mathcal{H}_N = \bigwedge^N \mathfrak{h}$.

All in all the bosonization procedure of the random phase approximation thus suggests that (2.30) holds at least for states with few excitations (when $\mathcal{N}_E$ is not too large).

For regular potentials in the mean-field regime, i.e., when V is replaced by $k_F^{-1} W$ for a fixed potential W satisfying $\sum_{k\in\mathbb{Z}^3_*} |k||\hat{W}(k)| < \infty$, the operator approximation (2.30) has been justified rigorously in [11]. To be precise, we proved in [11, Theorem 1] that there exists a unitary operator $\mathcal{U} : \mathcal{H}_N \to \mathcal{H}_N$ such that

$$\mathcal{U} H_N \mathcal{U}^* = E_{\text{FS}} + E_{\text{corr}} + H_{\text{eff}} + \mathcal{E}_{\mathcal{U}} \tag{2.32}$$

where the error operator satisfies

$$\pm\mathcal{E}_{\mathcal{U}} \le C k_F^{-\frac{1}{94}+\epsilon}\left(k_F^{-1}\mathcal{N}_E H'_{\text{kin}} + H'_{\text{kin}} + k_F\right), \quad k_F \to \infty, \tag{2.33}$$

for any fixed $\epsilon > 0$. Moreover, thanks to [11, Theorem 1.2], the bound in (2.33) suffices to show that $\mathcal{E}_{\mathcal{U}}$ is negligible when applied to low-lying eigenstates Ψ of $\mathcal{U} H_N \mathcal{U}^*$ satisfying $\langle \Psi, \mathcal{U} H_N \mathcal{U}^* \Psi\rangle = E_{\text{FS}} + E_{\text{corr}} + O(k_F)$, namely

$$|\langle \Psi, \mathcal{E}_{\mathcal{U}} \Psi\rangle| \le C k_F^{1-\frac{1}{94}+\epsilon}, \quad k_F \to \infty, \tag{2.34}$$

while both E_{corr} and $\langle \Psi, H_{\mathrm{eff}} \Psi \rangle$ are of order k_F.

Note that even in the mean-field regime, the Coulomb potential is still excluded in [11]. In this case, when $\hat{V}_k$ is replaced by $g k_F^{-1} |k|^{-2}$, the correlation energy E_{corr} is of order $k_F \log(k_F)$ instead of k_F, and existing techniques seem insufficient to estimate the error terms for the energy lower bound. We refer to the recent work [12] for a rigorous upper bound for the correlation energy. The operator approximation (2.30) for the Coulomb gas in the mean-field regime remains completely open, let alone the corresponding result beyond the mean-field regime.

In the present paper, we will consider the effective operator H_{eff} in more detail, without imposing the mean-field and regularity restrictions on the interaction. In particular, we will focus on the most interesting case of the Coulomb potential $\hat{V}_k = g|k|^{-2}$ for which the plasmon modes can be interpreted as a special class of approximate eigenstates of H_{eff}.

2.2 Elementary excitations and the plasmon frequency

As explained in [11], since the effective Hamiltonian H_{eff} in (2.31) commutes with $\mathcal{N}_E$, we can without loss of generality restrict H_{eff} to the eigenspaces $\{\mathcal{N}_E = M\}$ with $M = 0, 1, 2, \ldots$

The case $M = 0$ is trivial since the eigenspace $\{\mathcal{N}_E = 0\}$ is the one-dimensional space spanned by the Fermi state. In the first non-trivial case, $M = 1$, the identity

$$\sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} 2 \lambda_{k,p} b^*_{k,p} b_{k,p} = \mathcal{N}_E H'_{\mathrm{kin}} \tag{2.35}$$

(see [11, Eq. (1.55)]) implies that the relation of (2.23) is in fact valid, whence

$$H_{\mathrm{eff}}|_{\mathcal{N}_E = 1} = 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \left\langle e_p, \widetilde{E}_k e_q \right\rangle b^*_{k,p} b_{k,q}. \tag{2.36}$$

This operator can be diagonalized explicitly on $\{\mathcal{N}_E = 1\}$. More precisely, it was proved in [11, Theorem 1.4] that by introducing the unitary transformation

$$\tilde{U} : \bigoplus_{k \in \mathbb{Z}^3_*} \ell^2 (L_k) \to \{\Psi \in \mathcal{H}_N \mid \mathcal{N}_E \Psi = \Psi\}, \tag{2.37}$$

$$\tilde{U} \bigoplus_{k \in \mathbb{Z}^3_*} \varphi_k = \sum_{k \in \mathbb{Z}^3_*} b^*_k (\varphi_k) \, \psi_{\mathrm{FS}}, \tag{2.38}$$

where for any $\varphi \in \ell^2(L_k)$ the *generalized excitation operator* $b^*_k(\varphi)$ is defined by

$$b^*_k(\varphi) = \sum_{p \in L_k} \left\langle e_p, \varphi \right\rangle b^*_{k,p}, \tag{2.39}$$

we have the identity

$$\tilde{U}^* \left(H_{\text{eff}}|_{\mathcal{N}_E=1} \right) \tilde{U} = \bigoplus_{k \in \mathbb{Z}^3_*} 2\widetilde{E}_k \quad \text{on} \bigoplus_{k \in \mathbb{Z}^3_*} \ell^2 (L_k) . \tag{2.40}$$

Consequently, the spectrum of $H_{\text{eff}}|_{\mathcal{N}_E=1}$ is fully determined by the eigenvalues of $2\widetilde{E}_k$. Note that every eigenvalue ϵ of $2\widetilde{E}_k = 2(h_k^{\frac{1}{2}} \left(h_k + 2P_{v_k}\right) h_k^{\frac{1}{2}})^{\frac{1}{2}}$ solves the equation

$$\left(\epsilon^2 - 4h_k^2\right) w = 4\widetilde{E}_k^2 w - 4h_k^2 w = 8 \left\langle h_k^{\frac{1}{2}} v_k, w \right\rangle h_k^{\frac{1}{2}} v_k \tag{2.41}$$

for a normalized eigenvector w. Therefore, if ϵ is not an eigenvalue of $2h_k$, we can take the inner product with $\langle h_k^{\frac{1}{2}} v_k, w\rangle^{-1} (\epsilon^2 - 4h_k^2)^{-1} h_k^{\frac{1}{2}} v_k$ and obtain

$$\begin{aligned} 1 = 8\langle v_k, h_k \left(\epsilon^2 - 4h_k^2\right)^{-1} v_k \rangle &= \frac{4\hat{V}_k}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\epsilon^2 - 4\lambda_{k,p}^2} \\ &= \frac{2\hat{V}_k}{(2\pi)^3} \sum_{p \in B_F} \frac{|k|^2}{(\epsilon - 2k \cdot p)^2 - |k|^4}, \end{aligned} \tag{2.42}$$

which coincides with [22, Eq. (6)]. Since the last equality in (2.42) is not obvious, let us add an explanation for the reader's convenience. Using the algebraic identity

$$\frac{\lambda_{k,p}}{\epsilon^2 - 4\lambda_{k,p}^2} = \frac{\lambda_{k,p}}{(\epsilon - 2\lambda_{k,p})(\epsilon + 2\lambda_{k,p})} = \frac{1}{4} \left(\frac{1}{\epsilon - 2\lambda_{k,p}} - \frac{1}{\epsilon + 2\lambda_{k,p}} \right) \tag{2.43}$$

and the definition $L_k = B_F^c \cap (B_F + k) = (B_F + k) \backslash B_F$ we can write

$$\begin{aligned} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\epsilon^2 - 4\lambda_{k,p}^2} &= \frac{1}{4} \sum_{p \in L_k} \left(\frac{1}{\epsilon - 2\lambda_{k,p}} - \frac{1}{\epsilon + 2\lambda_{k,p}} \right) \\ &= \frac{1}{4} \sum_{p \in (B_F + k)} \left(\frac{1}{\epsilon - 2\lambda_{k,p}} - \frac{1}{\epsilon + 2\lambda_{k,p}} \right) \\ &\quad - \frac{1}{4} \sum_{p \in (B_F + k) \cap B_F} \left(\frac{1}{\epsilon - 2\lambda_{k,p}} - \frac{1}{\epsilon + 2\lambda_{k,p}} \right). \end{aligned} \tag{2.44}$$

Using $2\lambda_{k,p} = 2k \cdot (p - \frac{k}{2})$ and substituting $p \mapsto p - \frac{1}{2}k$ in the last sum in (2.44) we get

$$\frac{1}{4}\sum_{p\in(B_F+k)\cap B_F}\left(\frac{1}{\epsilon-2\lambda_{k,p}}-\frac{1}{\epsilon+2\lambda_{k,p}}\right)$$
$$=\frac{1}{4}\sum_{p\in(B_F+\frac{k}{2})\cap(B_F-\frac{k}{2})}\left(\frac{1}{\epsilon-2k\cdot p}-\frac{1}{\epsilon+2k\cdot p}\right)=0 \tag{2.45}$$

where the cancelation comes from the symmetry $p\mapsto -p$. On the other hand, by substituting $p\mapsto p+k$ in the first sum in (2.44) we can write

$$\begin{aligned}
&\frac{1}{4}\sum_{p\in(B_F+k)}\left(\frac{1}{\epsilon-2\lambda_{k,p}}-\frac{1}{\epsilon+2\lambda_{k,p}}\right)\\
&\quad=\frac{1}{4}\sum_{p\in B_F}\left(\frac{1}{\epsilon-2k\cdot p-|k|^2}-\frac{1}{\epsilon+2k\cdot p+|k|^2}\right)\\
&\quad=\frac{1}{4}\sum_{p\in B_F}\left(\frac{1}{\epsilon-2k\cdot p-|k|^2}-\frac{1}{\epsilon-2k\cdot p+|k|^2}\right)\\
&\quad=\frac{1}{2}\sum_{p\in B_F}\frac{|k|^2}{(\epsilon-2k\cdot p)^2-|k|^4}
\end{aligned} \tag{2.46}$$

where we also transformed $p\mapsto -p$ on the second term. Thus (2.42) holds.

In summary, (2.42) characterizes all eigenvalues of $2\widetilde{E}_k$ outside the spectrum of $2h_k$. In the case of the Coulomb potential $\hat{V}_k=g\,|k|^{-2}$, with a constant $g>0$, the k-dependence in (2.42) is simplified and we obtain

$$1=\frac{2g}{(2\pi)^3}\sum_{p\in B_F}\frac{1}{(\epsilon-2k\cdot p)^2-|k|^4}. \tag{2.47}$$

In this case, among all eigenvalues described in (2.47), the largest one is special as it is proportional to $k_F^{3/2}$ while the other eigenvalues are bounded from above by

$$2\lambda_{k,\max}:=\sup_{p\in B_F}(2k\cdot p+|k|^2)\le 2|k|k_F+|k|^2\ll k_F^{3/2}\quad\text{if}\quad |k|\ll k_F^{1/2}. \tag{2.48}$$

Indeed, note that the function

$$f(\epsilon)=\frac{2g}{(2\pi)^3}\sum_{p\in B_F}\frac{1}{(\epsilon-2k\cdot p)^2-|k|^4} \tag{2.49}$$

is strictly decreasing on $(2\lambda_{k,\max},\infty)$ and

$$\lim_{\epsilon\to(2\lambda_{k,\max})^+}f(\epsilon)=\infty,\qquad \lim_{\epsilon\to\infty}f(\epsilon)=0. \tag{2.50}$$

Therefore, the equation $f(\epsilon) = 1$ has a unique solution on $(2\lambda_{k,\max}, \infty)$. Moreover, this solution satisfies

$$\epsilon^2 = \frac{2g}{(2\pi)^3} \sum_{p\in B_F} \frac{\epsilon^2}{(\epsilon - 2k\cdot p)^2 - |k|^4} \geq \frac{2g}{(2\pi)^3} \sum_{p\in B_F} 1 = \frac{2gN}{(2\pi)^3} \tag{2.51}$$

namely

$$\epsilon \geq \sqrt{\frac{2gN}{(2\pi)^3}} = \sqrt{\frac{g}{3\pi^2}} k_F^{3/2} (1 + o(1)_{k_F\to\infty}). \tag{2.52}$$

When $|k| \ll k_F^{1/2}$, the lower bound in (2.52) implies that

$$\frac{\epsilon^2}{(\epsilon - 2k\cdot p)^2 - |k|^4} \approx 1, \tag{2.53}$$

and hence (2.52) is asymptotically sharp, namely we have

$$\epsilon = \sqrt{\frac{g}{3\pi^2}} k_F^{3/2} (1 + o(1)_{k_F\to\infty}). \tag{2.54}$$

In summary, if $|k| \ll k_F^{1/2}$, then the largest eigenvalue ϵ of $2\widetilde{E}_k$ is proportional to $k_F^{3/2}$, while all other eigenvalues of $2\widetilde{E}_k$, either being characterized by (2.47) or belonging to the spectrum of $2h_k$, are always bounded by $2\lambda_{k,\max} \ll k_F^{3/2}$.

In the physics literature, the largest eigenvalue of $2\widetilde{E}_k$ is often computed in the thermodynamic limit, where we replace Riemann sums by integrals and obtain

$$\epsilon^2 \approx 2gn \tag{2.55}$$

where

$$n = \frac{N}{\mathcal{V}} = \frac{\mathrm{Vol}\,(B\,(0, k_F))}{(2\pi)^3} = \frac{1}{6\pi^2} k_F^3 \tag{2.56}$$

is the number density of the system[2]. By taking $g = 4\pi e^2$, and also inserting $\frac{\hbar^2}{2m} = 1$, we find that the largest eigenvalue of $2\widetilde{E}_k$ is

$$\epsilon \approx \sqrt{2gn} = \hbar\sqrt{\frac{4\pi n e^2}{m}} = \hbar\omega_0 \tag{2.57}$$

[2] Here we consider the spinless fermions for simplicity. If we include a factor of q for the electron spin states (e.g., $q = 2$ for electrons), the equality (2.55) is still correct provided that $n = \frac{q}{6\pi^2} k_F^3$.

where $\omega_0 = \sqrt{\frac{4\pi n e^2}{m}}$ is exactly the plasmon frequency written in [20, Eqs. (3–90)] and [14, Eqs. (15.16–15.18)].

In the present paper, we will study H_{eff} in (2.31) for a general $M \geq 1$. In this case, the spectrum of H_{eff} corresponds to not only the *elementary excitations* but also all of the *collective excitations* of the system. Unlike the simple case $M = 1$ discussed above, for $M \geq 2$ the operator $H_{\text{eff}}|_{\mathcal{N}_E=M}$ can not be diagonalized explicitly as in (2.40), and hence understanding the spectrum of H_{eff} is both interesting and difficult. We will focus on the part of the spectrum of $H_{\text{eff}}|_{\mathcal{N}_E=M}$ which can be interpreted as describing the collective plasmon modes.

3 Main results

Consider the effective Hamiltonian H_{eff} in (2.31), i.e.,

$$H_{\text{eff}} = H'_{\text{kin}} + 2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\left\langle e_p, (\tilde{E}_k - h_k)e_q\right\rangle b^*_{k,p}b_{k,q}, \tag{3.1}$$

which is an operator on the fermionic N-particle space $\mathcal{H}_N = \bigwedge^N L^2\left(\mathbb{T}^3\right)$ with domain $D(H_N) = \bigwedge^N H^2\left(\mathbb{T}^3\right)$.

As discussed above, for $M = 1$ the eigenfunctions of H_{eff} are precisely the states of the form $b^*_k(\varphi)\psi_{\text{FS}}$, where $\varphi \in \ell^2(L_k)$ is an eigenvector of $2\widetilde{E}_k$. In the exact bosonic case, the eigenfunctions of $H_{\text{eff}}|_{\mathcal{N}_E=M}$ would be the states of the form

$$b^*_{k_1}(\varphi_1)\cdots b^*_{k_M}(\varphi_M)\psi_{\text{FS}} \tag{3.2}$$

where each $\varphi_i \in \ell^2(L_k)$ is an eigenvector of $2\widetilde{E}_{k_i}$ for $1 \leq i \leq M$.

For the effective Hamiltonian this is generally no longer true when $M \geq 2$. However, we will show that in the case that $k_1 = \cdots = k_M =: k$, where $\varphi_1 = \cdots = \varphi_M =: \varphi_k$ is the eigenvector of the greatest eigenvalue of $2\widetilde{E}_k$ (and so describes the plasmon mode), this is nonetheless approximately correct.

For the specific case of the Coulomb potential, we prove the following:

Theorem 1 *Let $k_F > 0$ be a large parameter. Let $\hat{V}_k = g\,|k|^{-2}$ with a constant $g > 0$. Let $\delta \in \left(0, \frac{1}{2}\right)$, $\varepsilon \in (0,1)$, $|k| \leq k_F^\delta$ and $1 \leq M \leq k_F^\varepsilon$ be given. Let $\varphi_k \in \ell^2(L_k)$ denote the normalized eigenvector corresponding to the greatest eigenvalue, ϵ_k, of $2\widetilde{E}_k$, and define $\Psi_M \in \{\mathcal{N}_E = M\}$ by*

$$\Psi_M = b^*_k(\varphi_k)^M\psi_{\text{FS}}.$$

Then the normalized state $\hat{\Psi}_M = \|\Psi_M\|^{-1}\Psi_M$ obeys

$$\left\|\left(H_{\text{eff}} - M\epsilon_k\right)\hat{\Psi}_M\right\| \leq C\,|k|^{-1}\sqrt{k_F}M^{\frac{5}{2}}$$

for a constant $C > 0$ depending only on δ and ε. Furthermore, it holds that

$$\epsilon_k = \sqrt{8 \langle v_k, h_k v_k \rangle + 4 \frac{\langle v_k, h_k^3 v_k \rangle}{\langle v_k, h_k v_k \rangle}} + O\left(k_F^{-\frac{1}{2}} |k|^4 \right).$$

Here $O(k_F^{-\frac{1}{2}} |k|^4)$ is a quantity that is bounded in absolute value by $k_F^{-\frac{1}{2}} |k|^4$ times a constant independent of k_F and k.

This theorem shows that we can consider $\hat{\Psi}_M$ to be an "approximate eigenfunction" of H_{eff} with "approximate eigenvalue" $M\epsilon_k$, when M is not too large. Let us give some quick remarks on this theorem:

1. The norm estimate implies both a dynamic and a spectral estimate: Owing to the elementary time evolution estimate $\|(e^{-itH} - e^{-itE})\psi\| \leq \|(H - E)\psi\| t$ this shows that

$$\|(e^{-itH_{\mathrm{eff}}} - e^{-itM\epsilon_k})\hat{\Psi}_M\| \ll 1 \quad \text{for} \quad M\epsilon_k t \ll C \frac{M\epsilon_k}{|k|^{-1}\sqrt{k_F} M^{\frac{5}{2}}} \sim M^{-\frac{3}{2}} k_F |k|; \tag{3.3}$$

note that $(M\epsilon_k)^{-1}$ is the characteristic timescale of the oscillation of $\hat{\Psi}_M$, so this is a non-trivial statement for $M \ll (k_F|k|)^{\frac{2}{3}}$.

Spectrally, thanks to the operator inequality $|\mathbb{1} - \mathbb{1}_{[E-\delta, E+\delta]}(H)| \leq \delta^{-1}|H - E|$, the norm estimate in Theorem 1 implies that

$$\|(\mathbb{1} - \mathbb{1}_{[M\epsilon_k-\delta, M\epsilon_k+\delta]}(H_{\mathrm{eff}}))\hat{\Psi}_M\| \ll 1 \quad \text{for} \quad |k|^{-1} \sqrt{k_F} M^{\frac{5}{2}} \ll \delta, \tag{3.4}$$

namely the state $\hat{\Psi}_M$ is essentially localized in the spectral space $\mathbb{1}_{[M\epsilon_k-\delta, M\epsilon_k+\delta]}(H_{\mathrm{eff}}) \mathcal{H}_N$. This justifies the interpretation that $\hat{\Psi}_M$ is an "approximate eigenfunction" of H_{eff}.

2. The condition $|k| \leq k_F^{\delta}$ with $\delta < 1/2$ is natural since we need $|k| \ll k_F^{1/2}$ to separate the plasmon frequency from other eigenvalues of $2\widetilde{E}_k$. When $|k| \sim k_F^{1/2}$, the plasmon mode merges into the continuum (the interval $[0, 2\lambda_{k,\max}]$ containing the remaining spectrum of $2\widetilde{E}_k$) as argued already by Bohm and Pines. See Fig. 1 for a numerical computation of the plasmon frequency and the continuum spectrum of $2\widetilde{E}_k$ when $|k|$ increases.
3. The estimate for ϵ_k is quite precise. Evidently the error term $k_F^{-1/2}|k|^4$ is much smaller than $k_F^{3/2}$ when $|k| \ll k_F^{1/2}$; moreover

$$\epsilon_k = \sqrt{8 \langle v_k, h_k v_k \rangle + 4 \frac{\langle v_k, h_k^3 v_k \rangle}{\langle v_k, h_k v_k \rangle}} + o(1) \tag{3.5}$$

for $|k| \ll k_F^{\frac{1}{8}}$. To make connections to the physics literature, we note that replacing the underlying Riemann sums by integrals (and keeping only the leading part of $\langle v_k, h_k^3 v_k \rangle$), and setting $g = 4\pi e^2$ and $n = \frac{N}{\mathcal{V}} = \frac{4\pi}{3} k_F^3$ we find (with $\frac{\hbar^2}{2m} = 1$)

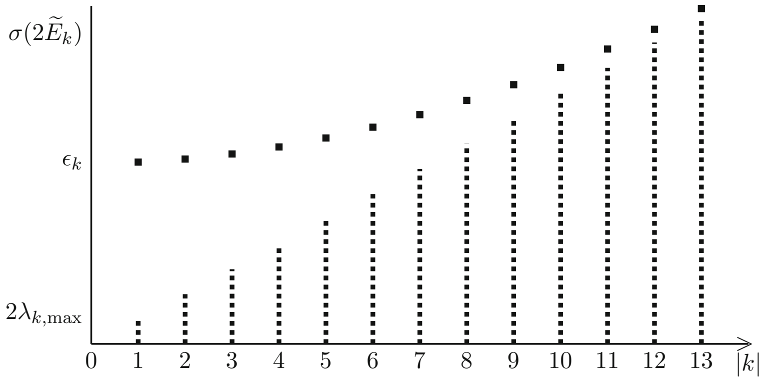

Fig. 1 The spectrum of $2\widetilde{E}_k$ with $\hat{V}_k = 4\pi|k|^{-2}$ and $k_F = 500$

$$\epsilon_k \approx \sqrt{\frac{4\pi e^2}{3\pi^2}k_F^3 + \frac{12}{5}k_F^2\,|k|^2} \approx \hbar\omega_0 + \frac{\frac{6}{5}k_F^2|k|^2}{\hbar\omega_0} = \hbar\left(\omega_0 + \frac{3}{10}\frac{v_F^2}{\omega_0}|k|^2\right), \tag{3.6}$$

where $\omega_0 = \sqrt{\frac{4\pi n e^2}{m}}$ is the plasmon frequency and $v_F = 2\hbar^{-1}k_F$ is the Fermi velocity. This describes a plasmon dispersion relation of

$$\omega(k) \approx \omega_0 + \frac{3}{10}\frac{v_F^2}{\omega_0}|k|^2, \tag{3.7}$$

which is in agreement with [20, Eq. (3.90c)], [19, Eq. (5.19)] and [14, Eq. (15.60)]. See Sect. 6 for a detailed explanation of (3.6).

4. In the mean-field regime, where $V = k_F^{-1}W$ with a fixed potential W, the bosonic collective excitations were discussed in [5] on the dynamics and in [2] on the spectrum (see, e.g., [2, Eq. (3.38)] for an analogue of (3.6)). In this case, the separation of the plasmon frequency holds in a weak sense: Although the largest eigenvalue of $2\widetilde{E}_k$ are within the same order of magnitude of many other eigenvalues, i.e., of order k_F, the distance from the plasmon frequency to the next-highest one is also of order k_F while the gaps between other eigenvalues are at most $O(|k|)$ (recall that we are interested in the case $|k| \ll k_F^{1/2}$). This assertion follows easily from the same argument leading to (2.52).

In contrast, in the present work we focus on the more physical regime where V is independent of k_F. As we go beyond the mean-field regime, the largest eigenvalue is much larger than the others, and the genuinely large gap of the spectrum ensures the almost-delocalization of the eigenfunction, which is important for our estimate.

5. Our analysis can be extended to all potentials satisfying $\hat{V}_k \geq 0$ and $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2 < \infty$. To be precise, for any $k \in \mathbb{Z}_*^3$ and $M \in \mathbb{N}$ such that $\hat{V}_k \gg k_F^{-1}$ and $1 \leq M \ll k_F\,|k|^{\frac{1}{2}}$, if we take ϵ_k, φ_k and $\hat{\Psi}_M$ as in Theorem 1, then we have the norm estimate

$$\left\|(H_{\text{eff}} - M\epsilon_k)\,\hat{\Psi}_M\right\| \le \frac{C}{\sqrt{k_F}\,|k|}\sqrt{\sum_{l\in 2B_F}\min\left\{1, k_F\hat{V}_l\right\}\hat{V}_l\,|l|^2 + Ck_F^3\sum_{l\in\mathbb{Z}^3\setminus 2B_F}\hat{V}_l^2 M^{\frac{5}{2}}}. \tag{3.8}$$

Note that in the case of the Coulomb potential $\hat{V}_k = g\,|k|^{-2}$ we may explicitly estimate

$$\sum_{l\in 2B_F}\min\left\{1, k_F\hat{V}_l\right\}\hat{V}_l\,|l|^2 \sim \int_0^{\sqrt{k_F}} r^2 dr + k_F\int_{\sqrt{k_F}}^{2k_F}\frac{1}{r^2}r^2 dr \le k_F^{\frac{3}{2}} + k_F k_F \le k_F^2,$$
$$k_F^3\sum_{l\in\mathbb{Z}^3\setminus 2B_F}\hat{V}_l^2 \sim k_F^3\int_{2k_F}^{\infty}\frac{1}{r^4}r^2 dr \le k_F^3 k_F^{-1} = k_F^2, \tag{3.9}$$

and hence (3.8) boils down to the norm estimate in Theorem 1. We refer to Sect. 6 for further explanation of (3.8).

Outline of the proof The main mathematical difficulty of the proof lies on the fact that H_{eff} is not a bosonic operator. More precisely, the operators $b_k(\varphi)$ only satisfies the CCR in a weak sense, and controlling the exchange terms (the error terms from the CCR) requires a careful analysis. In particular, estimating the norm of the approximate eigenstate $b_k^*(\varphi)^M\,\psi_{\text{FS}}$ is already nontrivial, and this will be done in Sect. 4, together with an analysis of the action of H_{eff} on this state. Until this point, we keep the analysis general and do not use any properties of the one-body operators $\widetilde{E}_k$ and h_k in the definition of H_{eff}. These one-body operators will be analyzed in detail in Sect. 5. Finally, we conclude the proof of the main theorem in Sect. 6.

4 Analysis of the approximate eigenstates

Let $k \in \mathbb{Z}_*^3$ be given and let $\varphi \in \ell^2(L_k)$ be the normalized eigenstate of $2\widetilde{E}_k$ corresponding to the greatest eigenvalue ϵ_k. For $M \in \mathbb{N}_0$ we define a state $\Psi_M \in \{\mathcal{N}_E = M\}$ by

$$\Psi_M = b_k^*(\varphi)^M\,\psi_{\text{FS}}. \tag{4.1}$$

In this section, we estimate the norms of Ψ_M and $(H_{\text{eff}} - M\epsilon_k)\Psi_M/\|\Psi_M\|$; the main results are stated in Propositions 1 and 2, respectively.

Before going to the two corresponding subsections, let us recall some basic commutator computations. First, we recall that the generalized excitation operators, given by

$$b_k(\varphi) = \sum_{p\in L_k}\langle\varphi, e_p\rangle b_{k,p}, \quad b_k^*(\varphi) = \sum_{p\in L_k}\langle e_p, \varphi\rangle b_{k,p}^*, \tag{4.2}$$

with $b_{k,p} = c^*_{p-k} c_p$, obey the commutation relations

$$\begin{aligned}[b_k(\varphi), b_l(\psi)] &= \left[b^*_k(\varphi), b^*_l(\psi)\right] = 0 \\ \left[b_k(\varphi), b^*_l(\psi)\right] &= \delta_{k,l} \langle \varphi, \psi \rangle + \varepsilon_{k,l}(\varphi; \psi)\end{aligned} \tag{4.3}$$

with

$$\varepsilon_{k,l}(\varphi;\psi) = -\sum_{p\in L_k}\sum_{q\in L_l} \langle \varphi, e_p\rangle \langle e_q, \psi\rangle \left(\delta_{p,q} c_{q-l} c^*_{p-k} + \delta_{p-k,q-l} c^*_q c_p\right). \tag{4.4}$$

For use below we calculate the commutator $\left[\varepsilon_{l,k}(\phi;\varphi), b^*_k(\psi)\right]$: As

$$\begin{aligned}&\left[\delta_{p,q} c_{q-k} c^*_{p-l} + \delta_{p-l,q-k} c^*_q c_p, b^*_{k,r}\right] \\ &= \delta_{p,q}\left[c_{q-k} c^*_{p-l}, c^*_r c_{r-k}\right] + \delta_{p-l,q-k}\left[c^*_q c_p, c^*_r c_{r-k}\right] \\ &= \delta_{p,q} c^*_r c_{q-k}\left\{c^*_{p-l}, c_{r-k}\right\} + \delta_{p-l,q-k} c^*_q \left\{c_p, c^*_r\right\} c_{r-k} \\ &= \delta_{p,q}\delta_{p-l,r-k} c^*_r c_{q-k} + \delta_{p,r}\delta_{p-l,q-k} c^*_q c_{r-k}\end{aligned} \tag{4.5}$$

for $p \in L_l$ and $q, r \in L_k$, we find

$$\begin{aligned}&\left[\varepsilon_{l,k}(\phi;\varphi), b^*_k(\psi)\right] \\ &= -\sum_{p\in L_l}\sum_{q,r\in L_k} \langle \phi, e_p\rangle\langle e_q, \varphi\rangle\langle e_r, \psi\rangle \left[\delta_{p,q} c_{q-k} c^*_{p-l} + \delta_{p-l,q-k} c^*_q c_p, b^*_{k,r}\right] \\ &= -\sum_{p\in L_k\cap L_l}\sum_{r\in L_k} \langle \phi, e_p\rangle\langle e_p, \varphi\rangle\langle e_r, \psi\rangle \delta_{p-l,r-k} c^*_r c_{p-k} \\ &\quad - \sum_{p\in L_k\cap L_l}\sum_{q\in L_k} \langle \phi, e_p\rangle\langle e_q, \varphi\rangle\langle e_p, \psi\rangle \delta_{p-l,q-k} c^*_q c_{p-k} \\ &= -\sum_{p\in L_k\cap L_l}\sum_{q\in L_k} \langle \phi, e_p\rangle \left(\langle e_p, \varphi\rangle\langle e_q, \psi\rangle + \langle e_q, \varphi\rangle\langle e_p, \psi\rangle\right) \delta_{p-l,q-k} c^*_q c_{p-k} \\ &= -\sum_{p\in L_k\cap L_l}\sum_{q\in L_k} \delta_{p-l,q-k} \langle \phi, e_p\rangle \left(\langle e_p, \varphi\rangle\langle e_q, \psi\rangle + \langle e_q, \varphi\rangle\langle e_p, \psi\rangle\right) b^*_{2k-l,q}.\end{aligned} \tag{4.6}$$

In particular

$$\begin{aligned}&\left[\varepsilon_{k,k}(\phi;\varphi), b^*_k(\psi)\right] \\ &= -\sum_{p\in L_k}\sum_{q\in L_k} \delta_{p-k,q-k} \langle \phi, e_p\rangle \left(\langle e_p, \varphi\rangle\langle e_q, \psi\rangle + \langle e_q, \varphi\rangle\langle e_p, \psi\rangle\right) b^*_{k,q} \\ &= -2\sum_{p\in L_k} \langle \phi, e_p\rangle\langle e_p, \varphi\rangle\langle e_p, \psi\rangle b^*_{k,p}.\end{aligned} \tag{4.7}$$

4.1 Estimating the norm of Ψ_M

In this subsection we will prove the following:

Proposition 1 *It holds that*

$$M! \geq \|\Psi_M\|^2 \geq M!\left(1 - \frac{M(M-1)}{2}\|\varphi\|_6^3\right).$$

Below we will see that φ is "almost completely delocalized," i.e., $\left|\langle e_p, \varphi\rangle\right| \sim \dim\left(\ell^2(L_k)\right)^{-\frac{1}{2}} = |L_k|^{-\frac{1}{2}}$, whence

$$\|\varphi\|_6^3 \sim \sqrt{\sum_{p\in L_k} \frac{1}{|L_k|^3}} = |L_k|^{-1} \tag{4.8}$$

and so the proposition implies that

$$\|\Psi_M\|^2 \geq C(M!), \quad M \ll |L_k|^{\frac{1}{2}} \sim k_F |k|^{\frac{1}{2}}. \tag{4.9}$$

We note the following general estimates:

Lemma 1 *Let* $(\phi_k)_{k\in\mathbb{Z}^3_*}$ *be a collection of vectors* $\phi_k \in \ell^2(L_k)$. *Then for any* $\Psi \in \mathcal{H}_N$

$$\left\|\sum_{k\in\mathbb{Z}^3_*} b_k(\phi_k)\Psi\right\| \leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\phi_k\|^2}\left\|\mathcal{N}_E^{\frac{1}{2}}\Psi\right\|,$$
$$\left\|\sum_{k\in\mathbb{Z}^3_*} b_k^*(\phi_k)\Psi\right\| \leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\phi_k\|^2}\left\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\right\|.$$

Proof For $\sum_{k\in\mathbb{Z}^3_*} b_k(\phi_k)\Psi$ we estimate

$$\begin{aligned}
&\left\|\sum_{k\in\mathbb{Z}^3_*} b_k(\phi_k)\Psi\right\| = \left\|\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\langle\phi_k, e_p\rangle c^*_{p-k}c_p\Psi\right\| \\
&= \left\|\sum_{p\in B_F^c}\sum_{k\in\mathbb{Z}^3_*} 1_{L_k}(p)\langle\phi_k, e_p\rangle c^*_{p-k}c_p\Psi\right\| \leq \sum_{p\in B_F^c}\left\|\left(\sum_{k\in\mathbb{Z}^3_*} 1_{L_k}(p)\langle\phi_k, e_p\rangle c^*_{p-k}\right)c_p\Psi\right\| \\
&\leq \sum_{p\in B_F^c}\sqrt{\sum_{k\in\mathbb{Z}^3_*} 1_{L_k}(p)\left|\langle\phi_k, e_p\rangle\right|^2}\|c_p\Psi\| \leq \sqrt{\sum_{p\in B_F^c}\sum_{k\in\mathbb{Z}^3_*} 1_{L_k}(p)\left|\langle\phi_k, e_p\rangle\right|^2}\sqrt{\sum_{p\in B_F^c}\|c_p\Psi\|^2} \\
&= \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\phi_k\|^2}\sqrt{\langle\Psi, \mathcal{N}_E\Psi\rangle} = \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\phi_k\|^2}\left\|\mathcal{N}_E^{\frac{1}{2}}\Psi\right\|
\end{aligned} \tag{4.10}$$

by the usual fermionic estimate $\left\| \sum_p a_p c_p^* \right\|_{\mathrm{Op}} \leq \sqrt{\sum_p \left|a_p\right|^2}$. For the second estimate we note that by the quasi-bosonic commutation relations of Eq. (4.3)

$$\begin{aligned}\left[\sum_{k\in\mathbb{Z}^3_*} b_k\left(\phi_k\right), \sum_{l\in\mathbb{Z}^3_*} b_l^*\left(\phi_l\right)\right] &= \sum_{k,l\in\mathbb{Z}^3_*} \delta_{k,l}\left\langle\phi_k,\phi_l\right\rangle + \sum_{k,l\in\mathbb{Z}^3_*} \varepsilon_{k,l}\left(\phi_k;\phi_l\right)\\ &= \sum_{k\in\mathbb{Z}^3_*}\left\|\phi_k\right\|^2 + \sum_{k,l\in\mathbb{Z}^3_*} \varepsilon_{k,l}\left(\phi_k;\phi_l\right)\end{aligned} \tag{4.11}$$

whence the second estimate will follow from the first provided $\sum_{k,l\in\mathbb{Z}^3_*}\varepsilon_{k,l}\left(\phi_k;\phi_l\right) \leq 0$. This is indeed the case since by definition

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}^3_*} \varepsilon_{k,l}\left(\phi_k;\phi_l\right) = -\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l}\left\langle\phi_k,e_p\right\rangle\left\langle e_q,\phi_l\right\rangle\\ \left(\delta_{p,q}c_{q-l}c_{p-k}^* + \delta_{p-k,q-l}c_q^*c_p\right)\end{aligned} \tag{4.12}$$

which factorizes as the negative of a sum of squares: Firstly

$$\begin{aligned}&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l}\left\langle\phi_k,e_p\right\rangle\left\langle e_q,\phi_l\right\rangle\delta_{p,q}c_{q-l}c_{p-k}^*\\ &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left\langle\phi_k,e_p\right\rangle\left\langle e_p,\phi_l\right\rangle c_{p-l}c_{p-k}^*\\ &= \sum_{p\in B_F^c}\left(\sum_{l\in\mathbb{Z}^3_*}1_{L_l}(p)\left\langle e_p,\phi_l\right\rangle c_{p-l}\right)\left(\sum_{k\in\mathbb{Z}^3_*}1_k(p)\left\langle e_p,\phi_k\right\rangle c_{p-k}\right)^*.\end{aligned} \tag{4.13}$$

Similarly

$$\begin{aligned}&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l}\left\langle\phi_k,e_p\right\rangle\left\langle e_q,\phi_l\right\rangle\delta_{p-k,q-l}c_q^*c_p\\ &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)}\sum_{q\in(L_l-l)}\left\langle\phi_k,e_{p+k}\right\rangle\left\langle e_{q+l},\phi_l\right\rangle\delta_{p,q}c_{q+l}^*c_{p+k}\\ &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)\cap(L_l-l)}\left\langle\phi_k,e_{p+k}\right\rangle\left\langle e_{p+l},\phi_l\right\rangle c_{p+l}^*c_{p+k}\\ &= \sum_{p\in B_F}\left(\sum_{l\in\mathbb{Z}^3_*}1_{L_l-l}(p)\left\langle\phi_l,e_{p+l}\right\rangle c_{p+l}^*\right)^*\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k-k}(p)\left\langle\phi_k,e_{p+k}\right\rangle c_{p+k}\right).\end{aligned} \tag{4.14}$$

□

Proof of Proposition 1 (Upper bound): For any $\Psi \in \mathcal{H}_N$ and $\phi \in \ell^2(L_k)$ it holds that

$$\left\| b_k^*(\phi)\,\Psi \right\| \leq \|\phi\| \sqrt{\langle \Psi, (\mathcal{N}_E + 1)\,\Psi \rangle} \tag{4.15}$$

(this is a special case of Lemma 1). In particular, since $\varphi \in \ell^2(L_k)$ is normalized and $\Psi_M \in \{\mathcal{N}_E = M\}$ we have

$$\|\Psi_M\|^2 = \left\| b^*(\varphi)\,\Psi_{M-1} \right\|^2 \leq \|\varphi\|^2 \langle \Psi_{M-1}, (\mathcal{N}_E + 1)\,\Psi_{M-1} \rangle = M \|\Psi_{M-1}\|^2 \tag{4.16}$$

whence $\|\Psi_M\|^2 \leq M!\,\|\psi_{\mathrm{FS}}\|^2 = M!$. □

Obtaining the lower bound will require some additional work. We note the following:

Lemma 2 *For any $l \in \mathbb{Z}^3_*$ and $\phi \in \ell^2(L_l)$ it holds that*

$$b_l(\phi)\,\Psi_M = \delta_{k,l} M \langle \phi, \varphi \rangle \Psi_{M-1} + \frac{M(M-1)}{2} \left[\varepsilon_{l,k}(\phi; \varphi), b_k^*(\varphi) \right] \Psi_{M-2}.$$

In particular, for $k = l$,

$$b_k(\phi)\,\Psi_M = M \langle \phi, \varphi \rangle \Psi_{M-1} - M(M-1) \left(\sum_{p \in L_k} \langle \phi, e_p \rangle \langle e_p, \varphi \rangle^2 b_{k,p}^* \right) \Psi_{M-2}.$$

Proof We calculate

$$\begin{aligned}
\left[b_l(\phi), b_k^*(\varphi)^M \right] &= \sum_{j=1}^{M} b_k^*(\varphi)^{M-j} \left[b_l(\phi), b_k^*(\varphi) \right] b_k^*(\varphi)^{j-1} \\
&= \delta_{k,l} \langle \phi, \varphi \rangle \sum_{j=1}^{M} b_k^*(\varphi)^{M-1} + \sum_{j=1}^{M} b_k^*(\varphi)^{M-j} \varepsilon_{l,k}(\phi; \varphi)\, b_k^*(\varphi)^{j-1} \\
&= \delta_{k,l} M \langle \phi, \varphi \rangle b_k^*(\varphi)^{M-1} + M b_k^*(\varphi)^{M-1} \varepsilon_{l,k}(\phi; \varphi) \\
&\quad + \sum_{j=1}^{M} \sum_{j'=1}^{j-1} b_k^*(\varphi)^{M-j} b_k^*(\varphi)^{j-1-j'} \left[\varepsilon_{l,k}(\phi; \varphi), b_k^*(\varphi) \right] b_k^*(\varphi)^{j'-1}.
\end{aligned} \tag{4.17}$$

Here the third equation in (4.17) is obtained by iterating the second one and commuting the operator $b_k^*(\varphi)$ to the left. Note that it follows from Eq. (4.6) that $\left[\varepsilon_{l,k}(\phi; \varphi), b_k^*(\varphi) \right]$ commutes with $b_k^*(\varphi)$. As $b_l(\phi)\,\psi_{\mathrm{FS}} = 0 = \varepsilon_{l,k}(\phi; \varphi)\,\psi_{\mathrm{FS}}$ we thus find by applying (4.17) to ψ_{FS} that

$$b_l(\phi)\,\Psi_M = \delta_{k,l} M \langle \phi, \varphi \rangle b_k^*(\varphi)^{M-1} \psi_{\mathrm{FS}} + \left(\sum_{j=1}^{M} \sum_{j'=1}^{j-1} 1 \right)$$

$$\begin{aligned}&\left[\varepsilon_{l,k}\left(\phi;\varphi\right),b_k^*\left(\varphi\right)\right]b_k^*\left(\varphi\right)^{M-2}\psi_{\mathrm{FS}}\\&=\delta_{k,l}M\left\langle\phi,\varphi\right\rangle\Psi_{M-1}+\frac{M\left(M-1\right)}{2}\left[\varepsilon_{l,k}\left(\phi;\varphi\right),b_k^*\left(\varphi\right)\right]\Psi_{M-2}.\end{aligned}\tag{4.18}$$

The $k=l$ case follows by inserting Eq. (4.7). □

Proof of Proposition 1 (Lower bound): We define $\varphi^{(3)}\in\ell^2\left(L_k\right)$ by

$$\varphi^{(3)}=\sum_{p\in L_k}\left|\left\langle e_p,\varphi\right\rangle\right|^2\left\langle e_p,\varphi\right\rangle e_p.\tag{4.19}$$

We then see by Lemma 2 that $\|\Psi_M\|^2$ obeys

$$\begin{aligned}\|\Psi_M\|^2&=\left\langle\Psi_{M-1},b_k\left(\varphi\right)\Psi_M\right\rangle\\&=\left\langle\Psi_{M-1},M\left\langle\varphi,\varphi\right\rangle\Psi_{M-1}-M\left(M-1\right)\left(\sum_{p\in L_k}\left\langle\varphi,e_p\right\rangle\left\langle e_p,\varphi\right\rangle^2b_{k,p}^*\right)\Psi_{M-2}\right\rangle\\&=M\left\|\Psi_{M-1}\right\|^2-M\left(M-1\right)\left\langle\Psi_{M-1},b_k^*\left(\varphi^{(3)}\right)\Psi_{M-2}\right\rangle.\end{aligned}\tag{4.20}$$

From this we can deduce the desired lower bound by induction. For $M=0,1$ we have equality. Suppose that case $M-1$ holds. Then

$$\begin{aligned}\|\Psi_M\|^2&=M\left\|\Psi_{M-1}\right\|^2-M\left(M-1\right)\left\langle\Psi_{M-1},b_k^*\left(\varphi^{(3)}\right)\Psi_{M-2}\right\rangle\\&\geq M\left\|\Psi_{M-1}\right\|^2-M\left(M-1\right)\left\|\Psi_{M-1}\right\|\left\|b_k^*\left(\varphi^{(3)}\right)\Psi_{M-2}\right\|\\&\geq M\left\|\Psi_{M-1}\right\|^2-M\left(M-1\right)^{\frac{3}{2}}\left\|\varphi^{(3)}\right\|\left\|\Psi_{M-1}\right\|\left\|\Psi_{M-2}\right\|\\&\geq M\left(\left(M-1\right)!\left(1-\frac{\left(M-1\right)\left(M-2\right)}{2}\|\varphi\|_6^3\right)\right)-M\left(M-1\right)\|\varphi\|_6^3\left(M-1\right)!\\&=M!\left(\left(1-\frac{\left(M-1\right)\left(M-2\right)}{2}\|\varphi\|_6^3\right)-\left(M-1\right)\|\varphi\|_6^3\right)\\&=M!\left(1-\frac{M\left(M-1\right)}{2}\|\varphi\|_6^3\right)\end{aligned}\tag{4.21}$$

where we recognized that

$$\left\|\varphi^{(3)}\right\|=\sqrt{\sum_{p\in L_k}\left|\left\langle e_p,\varphi\right\rangle\right|^6}=\|\varphi\|_6^3.\tag{4.22}$$

The proof of Proposition 1 is complete. □

4.2 Action of the effective Hamiltonian on Ψ_M

We now consider the action of

$$\begin{aligned} H_{\mathrm{eff}} &= H'_{\mathrm{kin}} + 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \left\langle e_p, \left(\widetilde{E}_k - h_k\right) e_q \right\rangle b^*_{k,p} b_{k,q} \\ &= H'_{\mathrm{kin}} + 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} b^*_k \left(\left(\widetilde{E}_k - h_k\right) e_p\right) b_{k,p} =: H'_{\mathrm{kin}} + H_{\mathrm{QB}} \end{aligned} \tag{4.23}$$

on Ψ_M, and in doing so prove the following:

Proposition 2 *For $\hat{\Psi}_M = \|\Psi_M\|^{-1} \Psi_M$ it holds that*

$$\left\| \left(H_{\mathrm{eff}} - M\epsilon_k\right) \hat{\Psi}_M \right\| \leq \frac{2 \|\varphi\|^2_\infty \sqrt{\sum_{l \in \mathbb{Z}^3_*} \left\| \widetilde{E}_l - h_l \right\|^2_{\mathrm{HS}}}}{\sqrt{1 - M^2 \|\varphi\|^3_6}} M^{\frac{5}{2}}.$$

We start with

Lemma 3 *We have*

$$\left\| \left(H_{\mathrm{eff}} - M\epsilon_k\right) \Psi_M \right\| \leq \frac{M(M-1)}{2} \|\mathcal{E}\Psi_{M-2}\| \tag{4.24}$$

where

$$\mathcal{E} = \sum_{p,q \in L_k} \langle e_p, \varphi \rangle \langle e_q, \varphi \rangle \left(\sum_{l \in \mathbb{Z}^3_*} \delta_{p-l,q-k} 1_{L_l}(p)\, b^*_l\left(A_l e_p\right) \right) c^*_q c_{p-k}. \tag{4.25}$$

with $A_l = 2\left(\widetilde{E}_l - h_l\right)$.

Proof From the first identity of Eq. (2.24) it follows that

$$\left[H'_{\mathrm{kin}}, b^*_k(\varphi)\right] = b^*_k\left(2h_k\varphi\right) \tag{4.26}$$

whence

$$\left[H'_{\mathrm{kin}}, b^*_k(\varphi)^M\right] = M b^*_k\left(2h_k\varphi\right) b^*_k(\varphi)^{M-1}, \tag{4.27}$$

implying that

$$H'_{\mathrm{kin}} \Psi_M = M b^*_k\left(2h_k\varphi\right) \Psi_{M-1}. \tag{4.28}$$

For H_{QB} we have by Lemma 2 that (abbreviating $A_l = 2\left(\widetilde{E}_l - h_l\right)$)

$$
\begin{aligned}
&H_{\mathrm{QB}}\Psi_M \\
&= \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l} b_l^*\left(A_l e_p\right)\left(\delta_{k,l} M\left\langle e_p,\varphi\right\rangle \Psi_{M-1} + \frac{M\left(M-1\right)}{2}\left[\varepsilon_{l,k}\left(e_p;\varphi\right), b_k^*\left(\varphi\right)\right]\Psi_{M-2}\right) \\
&= M\sum_{p\in L_k} b_k^*\left(A_k e_p\right)\left\langle e_p,\varphi\right\rangle \Psi_{M-1} \\
&\quad + \frac{M\left(M-1\right)}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l} b_l^*\left(A_l e_p\right)\left[\varepsilon_{l,k}\left(e_p;\varphi\right), b_k^*\left(\varphi\right)\right]\Psi_{M-2} \\
&= M b_k^*\left(A_k\varphi\right)\Psi_{M-1} + \frac{M\left(M-1\right)}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l} b_l^*\left(A_l e_p\right)\left[\varepsilon_{l,k}\left(e_p;\varphi\right), b_k^*\left(\varphi\right)\right]\Psi_{M-2}.
\end{aligned}
$$

In all then

$$
\begin{aligned}
H_{\mathrm{eff}}\Psi_M = M b_k^*\left(2\widetilde{E}_k\varphi\right)\Psi_{M-1} &+ \frac{M\left(M-1\right)}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l} b_l^*\left(A_l e_p\right) \\
&\times\left[\varepsilon_{l,k}\left(e_p;\varphi\right), b_k^*\left(\varphi\right)\right]\Psi_{M-2},
\end{aligned}
\tag{4.29}
$$

so as φ is an eigenvector of $2\widetilde{E}_k$ with eigenvalue ϵ_k,

$$
\left\|\left(H_{\mathrm{eff}} - M\epsilon_k\right)\Psi_M\right\| \leq \frac{M\left(M-1\right)}{2}\left\|\mathcal{E}\Psi_{M-2}\right\|. \tag{4.30}
$$

Here the error term on the right-hand side is

$$
\begin{aligned}
\mathcal{E} &= -\frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l} b_l^*\left(A_l e_p\right)\left[\varepsilon_{l,k}\left(e_p;\varphi\right), b_k^*\left(\varphi\right)\right] \\
&= \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\sum_{q\in L_k}\delta_{p-l,q-k}\left\langle e_p,\varphi\right\rangle\left\langle e_q,\varphi\right\rangle b_l^*\left(A_l e_p\right) c_q^* c_{p-k},
\end{aligned}
\tag{4.31}
$$

where we inserted the commutator of Eq. (4.6). This term can be rewritten as (4.25). □

We can now estimate the error term $\mathcal{E}$ as follows:

Lemma 4 *It holds that*

$$
\left\|\mathcal{E}\Psi_{M-2}\right\| \leq 2M\sqrt{M-1}\,\|\varphi\|_\infty^2\sqrt{\sum_{l\in\mathbb{Z}^3_*}\left\|\widetilde{E}_l - h_l\right\|_{\mathrm{HS}}^2}\,\left\|\Psi_{M-2}\right\|.
$$

Proof Write

$$
B_{p,q} = \sum_{l\in\mathbb{Z}^3_*}\delta_{p-l,q-k} 1_{L_l}\left(p\right) b_l\left(A_l e_p\right) \tag{4.32}
$$

for brevity, so that $\mathcal{E}$ given by Eq. (4.25) can be written as

$$\mathcal{E} = \sum_{p,q\in L_k} \langle e_p, \varphi\rangle \langle e_q, \varphi\rangle B^*_{p,q} c^*_q c_{p-k}. \tag{4.33}$$

Then $\mathcal{E}^*\mathcal{E}$ is given by

$$\mathcal{E}^*\mathcal{E} = \sum_{p,p',q,q'\in L_k} \langle \varphi, e_p\rangle \langle \varphi, e_q\rangle \langle e_{p'}, \varphi\rangle \langle e_{q'}, \varphi\rangle c^*_{p-k} c_q B_{p,q} B^*_{p',q'} c^*_{q'} c_{p'-k}. \tag{4.34}$$

Note that by Lemma 1, the operators $B^*_{p,q}$ obey

$$\begin{aligned}
\sum_{p,q\in L_k} \left\| B^*_{p,q} \Psi_{M-2} \right\|^2 &\le \sum_{p,q\in L_k} \sum_{l\in\mathbb{Z}^3_*} \delta_{p-l,q-k} 1_{L_l}(p) \left\| A_l e_p \right\|^2 \left\| (\mathcal{N}_E + 1)^{\frac{1}{2}} \Psi_{M-2} \right\|^2 \\
&= (M-1) \sum_{l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} \left(\sum_{q\in L_k} \delta_{p-l,q-k} \right) \left\| A_l e_p \right\|^2 \|\Psi_{M-2}\|^2 \\
&\le (M-1) \sum_{l\in\mathbb{Z}^3_*} \sum_{p\in L_l} \left\| A_l e_p \right\|^2 \|\Psi_{M-2}\|^2 = (M-1) \left(\sum_{l\in\mathbb{Z}^3_*} \|A_l\|^2_{\mathrm{HS}} \right) \|\Psi_{M-2}\|^2 .
\end{aligned} \tag{4.35}$$

Since $\left[c^*_{p-k} c_q, B_{p,q}\right] = 0$ it holds that

$$\begin{aligned}
c^*_{p-k} c_q B_{p,q} B^*_{p',q'} c^*_{q'} c_{p'-k} &= B_{p,q} c^*_{p-k} c_q c^*_{q'} c_{p'-k} B^*_{p',q'} \\
&= B_{p,q} c^*_{q'} c_{p'-k} c^*_{p-k} c_q B^*_{p',q'} + B_{p,q} \left[c^*_{p-k} c_q, c^*_{q'} c_{p'-k} \right] B^*_{p',q'}
\end{aligned} \tag{4.36}$$

so using also that $\left[c^*_{p-k} c_q, c^*_{q'} c_{p'-k}\right] = \delta_{p,p'}\delta_{q,q'} - \delta_{p,p'} c^*_{q'} c_q - \delta_{q,q'} c_{p'-k} c^*_{p-k}$ we find

$$\begin{aligned}
&\|\mathcal{E}\Psi_{M-2}\|^2 \\
&= \sum_{p,p',q,q'\in L_k} \langle \varphi, e_p\rangle \langle \varphi, e_q\rangle \langle e_{p'}, \varphi\rangle \langle e_{q'}, \varphi\rangle \left\langle c^*_{p'-k} c_{q'} B^*_{p,q} \Psi_{M-2}, c^*_{p-k} c_q B^*_{p',q'} \Psi_{M-2} \right\rangle \\
&\quad - \sum_{p,q,q'\in L_k} \left| \langle e_p, \varphi\rangle \right|^2 \langle \varphi, e_q\rangle \langle e_{q'}, \varphi\rangle \left\langle c_{q'} B^*_{p,q} \Psi_{M-2}, c_q B^*_{p,q'} \Psi_{M-2} \right\rangle \\
&\quad - \sum_{p,p',q\in L_k} \left| \langle e_q, \varphi\rangle \right|^2 \langle \varphi, e_p\rangle \langle e_{p'}, \varphi\rangle \left\langle c^*_{p'-k} B^*_{p,q} \Psi_{M-2}, c^*_{p-k} B^*_{p',q} \Psi_{M-2} \right\rangle \\
&\quad + \sum_{p,q\in L_k} \left| \langle e_p, \varphi\rangle \right|^2 \left| \langle e_q, \varphi\rangle \right|^2 \left\| B^*_{p,q} \Psi_{M-2} \right\|^2 =: T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{4.37}$$

We estimate the separate terms. For T_1 we can apply the Cauchy–Schwarz inequality and Eq. (4.35) to bound

$$
\begin{aligned}
|T_1| &\leq \|\varphi\|_\infty^4 \sum_{p,p',q,q' \in L_k} \left\| c^*_{p'-k} c_{q'} B^*_{p,q} \Psi_{M-2} \right\| \left\| c^*_{p-k} c_q B^*_{p',q'} \Psi_{M-2} \right\| \\
&\leq \|\varphi\|_\infty^4 \sum_{p,p',q,q' \in L_k} \left\| c^*_{p-k} c_q B^*_{p',q'} \Psi_{M-2} \right\|^2 \leq \|\varphi\|_\infty^4 \sum_{p',q' \in L_k} \left\| \mathcal{N}_E B^*_{p',q'} \Psi_{M-2} \right\|^2 \\
&\leq (M-1)^2 \|\varphi\|_\infty^4 \sum_{p',q' \in L_k} \left\| B^*_{p',q'} \Psi_{M-2} \right\|^2 \leq (M-1)^3 \|\varphi\|_\infty^4 \\
&\quad \times \left(\sum_{l \in \mathbb{Z}^3_*} \|A_l\|^2_{\mathrm{HS}} \right) \|\Psi_{M-2}\|^2 ,
\end{aligned}
\tag{4.38}
$$

and T_2 is similarly bounded as

$$
\begin{aligned}
|T_2| &\leq \|\varphi\|_\infty^4 \sum_{p,q,q' \in L_k} \left\| c_{q'} B^*_{p,q} \Psi_{M-2} \right\| \left\| c_q B^*_{p,q'} \Psi_{M-2} \right\| \\
&\leq \|\varphi\|_\infty^4 \sum_{p \in L_k} \sqrt{\sum_{q,q' \in L_k} \left\| c_{q'} B^*_{p,q} \Psi_{M-2} \right\|^2} \sqrt{\sum_{q,q' \in L_k} \left\| c_q B^*_{p,q'} \Psi_{M-2} \right\|^2} \\
&\leq \|\varphi\|_\infty^4 \sum_{p,q \in L_k} \left\| \mathcal{N}_E^{\frac{1}{2}} B^*_{p,q} \Psi_{M-2} \right\|^2 \leq (M-1)^2 \|\varphi\|_\infty^4 \left(\sum_{l \in \mathbb{Z}^3_*} \|A_l\|^2_{\mathrm{HS}} \right) \|\Psi_{M-2}\|^2 ,
\end{aligned}
\tag{4.39}
$$

the same estimate holding also for T_3. Finally T_4 is just bounded by

$$
|T_4| \leq (M-1)^2 \|\varphi\|_\infty^4 \left(\sum_{l \in \mathbb{Z}^3_*} \|A_l\|^2_{\mathrm{HS}} \right) \|\Psi_{M-2}\|^2 \tag{4.40}
$$

so combining the estimates we find

$$
\begin{aligned}
\|\mathcal{E} \Psi_{M-2}\| &\leq M \sqrt{M-1} \, \|\varphi\|_\infty^2 \sqrt{\sum_{l \in \mathbb{Z}^3_*} \|A_l\|^2_{\mathrm{HS}}} \, \|\Psi_{M-2}\| \\
&= 2M \sqrt{M-1} \, \|\varphi\|_\infty^2 \sqrt{\sum_{l \in \mathbb{Z}^3_*} \left\| \widetilde{E}_l - h_l \right\|^2_{\mathrm{HS}}} \, \|\Psi_{M-2}\| .
\end{aligned}
\tag{4.41}
$$

□

Proposition 2 now follows by combining Lemma 3 with Lemma 4 and Proposition 1 to see that

$$\begin{aligned}
&\left\| \left(H_{\text{eff}} - M\epsilon_k\right)\hat{\Psi}_M \right\| \leq M\left(M-1\right)\frac{\left\|\mathcal{E}\Psi_{M-2}\right\|}{\left\|\Psi_M\right\|} \\
&\leq 2M^2\left(M-1\right)^{\frac{3}{2}}\left\|\varphi\right\|_\infty^2\sqrt{\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h_l\right\|_{\text{HS}}^2}\frac{\left\|\Psi_{M-2}\right\|}{\left\|\Psi_M\right\|} \\
&\leq 2M^2\left(M-1\right)^{\frac{3}{2}}\left\|\varphi\right\|_\infty^2\sqrt{\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h_l\right\|_{\text{HS}}^2}\sqrt{\frac{(M-2)!}{M!\left(1-\frac{M(M-1)}{2}\left\|\varphi\right\|_6^3\right)}} \\
&= 2M^{\frac{3}{2}}\left(M-1\right)\left\|\varphi\right\|_\infty^2\sqrt{\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h_l\right\|_{\text{HS}}^2}\frac{1}{\sqrt{1-\frac{M(M-1)}{2}\left\|\varphi\right\|_6^3}} \\
&\leq \frac{2\left\|\varphi\right\|_\infty^2\sqrt{\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h_l\right\|_{\text{HS}}^2}}{\sqrt{1-M^2\left\|\varphi\right\|_6^3}}M^{\frac{5}{2}}.
\end{aligned} \tag{4.42}$$

5 Estimation of one-body quantities

To proceed we must now derive some estimates on the one-body quantities involved - we need to verify that φ is indeed "almost delocalized" and bound $\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h_l\right\|_{\text{HS}}^2$. We prove the following:

Proposition 3 *For $|k| \ll \sqrt{k_F}$, it holds that*

$$\left\|\varphi\right\|_\infty^2,\ \left\|\varphi\right\|_6^3 \leq \frac{C}{k_F^2\,|k|},$$

and

$$\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h_l\right\|_{\text{HS}}^2 \leq Ck_F^5$$

for $C > 0$ independent of k and k_F.

Note that the condition $|k| \ll \sqrt{k_F}$, namely $|k|/\sqrt{k_F} \to 0$ as $k_F \to \infty$, holds if $|k| \leq k_F^\delta$ with $\delta < 1/2$.

5.1 General estimates

To avoid unnecessary subscripts we consider instead of $\ell^2\left(L_k\right)$ a general n-dimensional inner product space $(V, \langle\cdot,\cdot\rangle)$, on which a positive symmetric operator $h : V \to V$ acts, with diagonalizing basis $(e_i)_{i=1}^n$ and eigenvalues $(\lambda_i)_{i=1}^n$, and a fixed $v \in V$ such that $\langle e_i, v\rangle > 0$ for all $1 \leq i \leq n$.

Define $\widetilde{E}: V \to V$ by

$$\tilde{E} = \left(h^{\frac{1}{2}}\left(h + 2P_v\right)h^{\frac{1}{2}}\right)^{\frac{1}{2}} = \left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}}. \tag{5.1}$$

Let $\varphi \in V$ be a normalized eigenvector of $\widetilde{E}$ with greatest eigenvalue ϵ ($> \|h\|$) (note that we do not include the factor of 2 in this section), with phase chosen such that $\left\langle h^{\frac{1}{2}}v, \varphi\right\rangle \geq 0$. Then as $\widetilde{E}$ squares to $h^2 + 2P_{h^{\frac{1}{2}}v}$, we have

$$\epsilon^2\varphi = h^2\varphi + 2\left\langle h^{\frac{1}{2}}v, \varphi\right\rangle h^{\frac{1}{2}}v \tag{5.2}$$

whence

$$\varphi = 2\left\langle h^{\frac{1}{2}}v, \varphi\right\rangle\left(\epsilon^2 - h^2\right)^{-1}h^{\frac{1}{2}}v. \tag{5.3}$$

This identity lets us describe the components of φ (with respect to $(e_i)_{i=1}^n$) in terms of the single unknown ϵ: Taking the inner product with e_i yields

$$\langle e_i, \varphi\rangle = 2\left\langle h^{\frac{1}{2}}v, \varphi\right\rangle\left\langle e_i, \left(\epsilon^2 - h^2\right)^{-1}h^{\frac{1}{2}}v\right\rangle = 2\left\langle h^{\frac{1}{2}}v, \varphi\right\rangle\frac{\sqrt{\lambda_i}}{\epsilon^2 - \lambda_i^2}\langle e_i, v\rangle \tag{5.4}$$

and now we may note that $2\left\langle h^{\frac{1}{2}}v, \varphi\right\rangle$ is simply a constant independent of i. As φ is by assumption normalized, we thus have

$$\langle e_i, \varphi\rangle = \frac{1}{\sqrt{\sum_{i=1}^N \frac{\lambda_i}{(\epsilon^2-\lambda_i^2)^2}|\langle e_i, v\rangle|^2}}\frac{\sqrt{\lambda_i}}{\epsilon^2 - \lambda_i^2}\langle e_i, v\rangle, \quad 1 \leq i \leq n. \tag{5.5}$$

Note that by the variational principle $\epsilon^2 \geq \tilde{E}^2 = h^2 + 2P_{h^{\frac{1}{2}}v}$, we have

$$\epsilon^2 \geq \frac{\left\langle h^{\frac{1}{2}}v, \left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)h^{\frac{1}{2}}v\right\rangle}{\left\langle h^{\frac{1}{2}}v, h^{\frac{1}{2}}v\right\rangle} = 2\langle v, hv\rangle + \frac{\left\langle v, h^3v\right\rangle}{\langle v, hv\rangle} \geq 2\langle v, hv\rangle. \tag{5.6}$$

So we immediately obtain the following:

Lemma 5 *Let* $\lambda_{\max} = \max_{1\leq i\leq n}\lambda_i$. *Then provided* $2\langle v, hv\rangle > \lambda_{\max}^2$ *it holds that*

$$|\langle e_i, \varphi\rangle| \leq \frac{2\langle v, hv\rangle}{2\langle v, hv\rangle - \lambda_{\max}^2}\frac{\sqrt{\lambda_i}}{\sqrt{\langle v, hv\rangle}}|\langle e_i, v\rangle|, \quad 1 \leq i \leq n.$$

Proof We simply estimate

$$
\begin{aligned}
|\langle e_i, \varphi\rangle| &= \frac{1}{\sqrt{\sum_{i=1}^N \frac{\lambda_i}{\left(\epsilon^2-\lambda_i^2\right)^2} |\langle e_i, v\rangle|^2}} \frac{\sqrt{\lambda_i}}{\epsilon^2-\lambda_i^2} |\langle e_i, v\rangle| \\
&\leq \frac{1}{\frac{1}{\epsilon^2}\sqrt{\sum_{i=1}^N \lambda_i |\langle e_i, v\rangle|^2}} \frac{1}{\epsilon^2-\lambda_{\max}^2}\sqrt{\lambda_i}\, |\langle e_i, v\rangle| \\
&= \frac{\epsilon^2}{\epsilon^2-\lambda_{\max}^2} \frac{\sqrt{\lambda_i}}{\langle v, hv\rangle} |\langle e_i, v\rangle| \qquad (5.7)
\end{aligned}
$$

and note that by (5.6),

$$
\frac{\epsilon^2}{\epsilon^2-\lambda_{\max}^2} = \frac{1}{1-\frac{\lambda_{\max}^2}{\epsilon^2}} \leq \frac{1}{1-\frac{\lambda_{\max}^2}{2\langle v, hv\rangle}} = \frac{2\langle v, hv\rangle}{2\langle v, hv\rangle - \lambda_{\max}^2}. \qquad (5.8)
$$

□

For the statement of Theorem 1 it is also interesting to bound ϵ from above: We just saw that

$$
\epsilon^2 \geq 2\langle v, hv\rangle + \frac{\left\langle v, h^3 v\right\rangle}{\langle v, hv\rangle} \qquad (5.9)
$$

and the right-hand side is in fact the leading contribution to ϵ^2:

Lemma 6 *Provided* $2\langle v, hv\rangle > \lambda_{\max}^2$ *it holds that*

$$
\epsilon^2 \leq 2\langle v, hv\rangle + \frac{\left\langle v, h^3 v\right\rangle}{\langle v, hv\rangle} + \frac{4\left\langle v, h^3 v\right\rangle \lambda_{\max}^2}{\left(2\langle v, hv\rangle - \lambda_{\max}^2\right)^2}.
$$

Proof By the identity $\varphi = \left\| \left(\epsilon^2-h^2\right)^{-1} h^{\frac{1}{2}} v \right\|^{-1} \left(\epsilon^2-h^2\right)^{-1} h^{\frac{1}{2}} v$ we have that

$$
\begin{aligned}
\left\langle \varphi, h^2\varphi\right\rangle &= \frac{\left\langle v, h^3\left(\epsilon^2-h^2\right)^{-2} v\right\rangle}{\left\langle v, h\left(\epsilon^2-h^2\right)^{-2} v\right\rangle} \leq \frac{\frac{1}{\left(\epsilon^2-\lambda_{\max}^2\right)^2}\left\langle v, h^3 v\right\rangle}{\frac{1}{\epsilon^4}\langle v, hv\rangle} \\
&= \frac{\epsilon^4}{\left(\epsilon^2-\lambda_{\max}^2\right)^2} \frac{\left\langle v, h^3 v\right\rangle}{\langle v, hv\rangle} \leq \frac{\left(2\langle v, hv\rangle\right)^2}{\left(2\langle v, hv\rangle - \lambda_{\max}^2\right)^2} \frac{\left\langle v, h^3 v\right\rangle}{\langle v, hv\rangle} \qquad (5.10)
\end{aligned}
$$

where we estimated as above. Continuing the estimate we then find

$$
\left\langle \varphi, h^2\varphi\right\rangle \leq \frac{\left\langle v, h^3 v\right\rangle}{\langle v, hv\rangle} + \frac{\left(2\langle v, hv\rangle\right)^2 - \left(2\langle v, hv\rangle - \lambda_{\max}^2\right)^2}{\left(2\langle v, hv\rangle - \lambda_{\max}^2\right)^2} \frac{\left\langle v, h^3 v\right\rangle}{\langle v, hv\rangle}
$$

$$= \frac{\left\langle v, h^3 v \right\rangle}{\langle v, hv \rangle} + \frac{4\langle v, hv \rangle \lambda_{\max}^2 - \lambda_{\max}^4}{\left(2\langle v, hv \rangle - \lambda_{\max}^2\right)^2} \frac{\left\langle v, h^3 v \right\rangle}{\langle v, hv \rangle} \leq \frac{\left\langle v, h^3 v \right\rangle}{\langle v, hv \rangle} + \frac{4\left\langle v, h^3 v \right\rangle \lambda_{\max}^2}{\left(2\langle v, hv \rangle - \lambda_{\max}^2\right)^2}. \tag{5.11}$$

From the eigenvalue equation for ϵ^2 we can then conclude that

$$\begin{aligned} \epsilon^2 &= \left\langle \varphi, h^2 \varphi \right\rangle + 2\left|\left\langle h^{\frac{1}{2}} v, \varphi \right\rangle\right|^2 \leq 2\left\| h^{\frac{1}{2}} v \right\|^2 \|\varphi\|^2 + \frac{\left\langle v, h^3 v \right\rangle}{\langle v, hv \rangle} + \frac{4\left\langle v, h^3 v \right\rangle \lambda_{\max}^2}{\left(2\langle v, hv \rangle - \lambda_{\max}^2\right)^2} \\ &= 2\langle v, hv \rangle + \frac{\left\langle v, h^3 v \right\rangle}{\langle v, hv \rangle} + \frac{4\left\langle v, h^3 v \right\rangle \lambda_{\max}^2}{\left(2\langle v, hv \rangle - \lambda_{\max}^2\right)^2}. \end{aligned} \tag{5.12}$$

□

Estimating $\left\|\widetilde{E} - h\right\|_{\mathrm{HS}}$

Finally we consider $\|\widetilde{E} - h\|_{\mathrm{HS}} = \left\| \left(h^2 + 2P_{h^{\frac{1}{2}} v}\right)^{\frac{1}{2}} - h \right\|_{\mathrm{HS}}$. In [11, Eq. (7.22)] we derived the identity

$$\widetilde{E} - h = \frac{4}{\pi} \int_0^\infty \frac{t^2}{1 + 2\left\langle v, h\left(h^2 + t^2\right)^{-1} v \right\rangle} P_{\left(h^2+t^2\right)^{-1} h^{\frac{1}{2}} v}\, dt \tag{5.13}$$

from which it follows that

$$0 \leq \left\langle e_i, \left(\widetilde{E} - h\right) e_j \right\rangle \leq 2\frac{\sqrt{\lambda_i \lambda_j}}{\lambda_i + \lambda_j} \langle e_i, v \rangle \left\langle v, e_j \right\rangle, \quad 1 \leq i, j \leq n. \tag{5.14}$$

This is asymptotically optimal for "small v," but without the mean-field scaling we also need to consider "large v." While a direct elementwise estimate appears to be more involved in this regime, a good Hilbert–Schmidt estimate is in fact simpler. Covering both regimes, we have the following:

Lemma 7 *It holds that*

$$\|\widetilde{E} - h\|_{\mathrm{HS}}^2 \leq \min\left\{2\langle v, hv \rangle, 4\|v\|^4\right\}.$$

Proof We first note that

$$\begin{aligned} \|\widetilde{E} - h\|_{\mathrm{HS}}^2 &= \operatorname{tr}\left(\left(\widetilde{E} - h\right)^2\right) = \operatorname{tr}\left(\widetilde{E}^2 + h^2 - \widetilde{E}h - h\widetilde{E}\right) \\ &= 2\operatorname{tr}\left(h^2 + P_{h^{\frac{1}{2}} v} - h^{\frac{1}{2}} \widetilde{E} h^{\frac{1}{2}}\right) \end{aligned} \tag{5.15}$$

as $\widetilde{E}^2 = h^2 + 2P_{h^{\frac{1}{2}}v}$. Since we may trivially estimate that $\widetilde{E} \geq h$ we can then conclude

$$\left\|\widetilde{E} - h\right\|_{\mathrm{HS}}^2 \leq 2\,\mathrm{tr}\left(h^2 + P_{h^{\frac{1}{2}}v} - h^2\right) = 2\left\|h^{\frac{1}{2}}v\right\|^2 = 2\langle v, hv\rangle. \tag{5.16}$$

For the other estimate we simply apply the elementwise estimate:

$$\begin{aligned}\left\|\widetilde{E} - h\right\|_{\mathrm{HS}}^2 &= \sum_{i,j=1}^{n}\left|\left\langle e_i, \left(\tilde{E} - h\right)e_j\right\rangle\right|^2 \leq 4\sum_{i,j=1}^{n}\frac{\lambda_i\lambda_j}{\left(\lambda_i + \lambda_j\right)^2}\left|\langle e_i, v\rangle\left\langle v, e_j\right\rangle\right|^2 \\ &\leq 4\sum_{i,j=1}^{n}\frac{\lambda_i\lambda_j}{\left(\lambda_i + \lambda_j\right)^2}\left|\langle e_i, v\rangle\left\langle v, e_j\right\rangle\right|^2 \leq 4\left(\sum_{i=1}^{n}|\langle e_i, v\rangle|^2\right)^2 = 4\|v\|^4.\end{aligned} \tag{5.17}$$

□

5.2 Proof of Proposition 3

To prove Proposition 3 we now only need to insert the specific one-body operators of our problem; recall that in this case

$$h_k e_p = \lambda_{k,p}e_p, \quad \lambda_{k,p} = \frac{1}{2}\left(|p|^2 - |p-k|^2\right) \tag{5.18}$$

and

$$v_k = \sqrt{\frac{\hat{V}_k}{2(2\pi)^3}}\sum_{p\in L_k}e_p. \tag{5.19}$$

First, for $\|\varphi\|_6^3$ and $\|\varphi\|_\infty^2$, we trivially have that

$$\|\varphi\|_6^3 = \sqrt{\sum_{p\in L_k}\left|\left\langle e_p, \varphi\right\rangle\right|^6} \leq \sqrt{|L_k|}\,\|\varphi\|_\infty^3 \tag{5.20}$$

and by Lemma 5 we have the estimate

$$\|\varphi\|_\infty \leq \frac{2\langle v_k, h_k v_k\rangle}{2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2}\frac{\sqrt{\lambda_{k,\max}}}{\sqrt{\langle v_k, h_k v_k\rangle}}\max_{p\in L_k}\left|\left\langle e_p, v_k\right\rangle\right| \tag{5.21}$$

provided $2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2 > 0$. Moreover, under the condition that $|k| \ll \sqrt{k_F}$, the k-dependent quantities behave as (recall that $\hat{V}_k = g\,|k|^{-2}$)

$$\lambda_{k,\max} \sim k_F\,|k|\,, \quad \max_{p\in L_k}\left|\langle e_p, v_k\rangle\right| = \sqrt{\frac{\hat{V}_k}{2\,(2\pi)^3}} \sim \sqrt{\hat{V}_k} \sim |k|^{-1} \tag{5.22}$$

and

$$\langle v_k, h_k v_k\rangle = \frac{\hat{V}_k}{2\,(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p} \sim k_F^3. \tag{5.23}$$

Here note that the behavior $\lambda_{k,\max} \sim k_F\,|k|$ can be deduced easily from (2.48). So the estimate on $\|\varphi\|_\infty$ in (5.21) boils down to

$$\|\varphi\|_\infty \le C\frac{k_F^3}{k_F^3 - C' k_F^2\,|k|^2}\frac{\sqrt{k_F\,|k|}}{\sqrt{k_F^3}}\,|k|^{-1} \le C\frac{1}{\sqrt{k_F^2\,|k|}}. \tag{5.24}$$

The estimate on $\|\varphi\|_6$ in (5.20) can be simplified using $\sqrt{|L_k|} \sim \sqrt{k_F^2\,|k|}$. It follows that

$$\|\varphi\|_\infty^2\,,\ \|\varphi\|_6^3 \le \frac{C}{k_F^2\,|k|} \tag{5.25}$$

when $|k| \ll \sqrt{k_F}$, as claimed.

For $\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h\right\|_{\mathrm{HS}}^2$ we note that by Lemma 7, when $|l| \le 2k_F$,

$$\begin{aligned}\left\|\widetilde{E}_l - h_l\right\|_{\mathrm{HS}}^2 &\le \min\left\{2\langle v_l, h_l v_l\rangle\,, 4\,\|v_l\|^4\right\} \le C\min\left\{k_F^3, k_F^4\,|l|^{-2}\right\}\\ &= Ck_F^3\min\left\{1, k_F\,|l|^{-2}\right\}\end{aligned} \tag{5.26}$$

and when $|l| > 2k_F$

$$\left\|\widetilde{E}_l - h_l\right\|_{\mathrm{HS}}^2 \le 4\,\|v_l\|^4 \le C\hat{V}_k^2\,|B_F|^2 \le Ck_F^6\,|l|^{-4}\,, \tag{5.27}$$

whence

$$\begin{aligned}\sum_{l\in\mathbb{Z}_*^3}\left\|\widetilde{E}_l - h\right\|_{\mathrm{HS}}^2 &\le C\sum_{l\in B(0,\sqrt{k_F})\cap\mathbb{Z}^3} k_F^3 + C\sum_{l\in\overline{B}(0,2k_F)\setminus B(0,\sqrt{k_F})\cap\mathbb{Z}^3} k_F^4\,|l|^{-2}\\ &\quad + C\sum_{l\in\mathbb{Z}^3\setminus\overline{B}(0,2k_F)} k_F^6\,|l|^{-4}\\ &\le Ck_F^{4+\frac{1}{2}} + Ck_F^5 + Ck_F^5 \le Ck_F^5\end{aligned} \tag{5.28}$$

as claimed. □

6 Conclusion

We can now conclude the proof of the main result.

Proof of Theorem 1: For the first part of Theorem 1, by Proposition 2 and the estimates of Proposition 3 we have

$$
\begin{aligned}
\left\| (H_{\text{eff}} - M\epsilon_k)\, \hat{\Psi}_M \right\| &\leq C \frac{1}{k_F^2\,|k|} \frac{\sqrt{\sum_{l\in\mathbb{Z}^3_*} \left\| \widetilde{E}_l - h_l \right\|_{\text{HS}}^2}}{\sqrt{1 - \frac{M^2}{k_F^2|k|}}} M^{\frac{5}{2}} \\
&\leq C\,|k|^{-1} \sqrt{k_F^{-4} \sum_{l\in\mathbb{Z}^3_*} \left\| \widetilde{E}_l - h_l \right\|_{\text{HS}}^2} M^{\frac{5}{2}} \\
&\leq C\,|k|^{-1} \sqrt{k_F} M^{\frac{5}{2}}
\end{aligned}
\tag{6.1}
$$

where the assumption $|k| \leq k_F^{\delta}$ ensures the applicability of Proposition 3 and the condition $M \leq k_F^{\varepsilon}$ ensures that $\left(1 - \frac{M^2}{k_F^2|k|}\right)^{-\frac{1}{2}} \leq C$.

For the second part of Theorem 1, concerning ϵ_k, we have by Lemma 6 that (remembering to include a factor of 2)

$$
\epsilon_k \leq \sqrt{8\,\langle v_k, h_k v_k\rangle + 4\frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle} + \frac{16\,\langle v_k, h_k^3 v_k\rangle\,\lambda_{k,\max}^2}{\left(2\,\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2\right)^2}}
\tag{6.2}
$$

and

$$
\epsilon_k \geq \sqrt{8\,\langle v_k, h_k v_k\rangle + 4\frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle}} \geq C\sqrt{\langle v_k, h_k v_k\rangle} \geq C k_F^{\frac{3}{2}}.
\tag{6.3}
$$

As $\sqrt{a+b} - \sqrt{a} \leq \frac{b}{2\sqrt{a}}$ we may then estimate

$$
\begin{aligned}
&\epsilon_k - \sqrt{8\,\langle v_k, h_k v_k\rangle + 4\frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle}} \\
&\leq \frac{1}{2\sqrt{8\,\langle v_k, h_k v_k\rangle + 4\frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle}}} \frac{16\,\langle v_k, h_k^3 v_k\rangle\,\lambda_{k,\max}^2}{\left(2\,\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2\right)^2} \\
&\leq C\frac{1}{k_F^{\frac{3}{2}}} \frac{k_F^5\,|k|^2\,(k_F\,|k|)^2}{k_F^6} = C k_F^{-\frac{1}{2}}\,|k|^4
\end{aligned}
\tag{6.4}
$$

for the claim that

$$\epsilon_k = \sqrt{8\left\langle v_k, h_k v_k\right\rangle + 4\frac{\left\langle v_k, h_k^3 v_k\right\rangle}{\left\langle v_k, h_k v_k\right\rangle}} + O\left(k_F^{-\frac{1}{2}}\left|k\right|^4\right). \tag{6.5}$$

The proof of Theorem 1 is complete. □

Further explanation of (3.6) for ϵ_k in the Thermodynamic Limit

In the thermodynamic limit, in which we replace Riemann sums by the corresponding integrals, we have

$$\left\langle v_k, h_k^\beta v_k\right\rangle = \frac{\hat{V}_k}{2(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^\beta \sim \frac{\hat{V}_k}{2(2\pi)^3}\int_{\mathcal{L}_k}\left(k\cdot p - \frac{1}{2}\left|k\right|^2\right)^\beta dp \tag{6.6}$$

where $\mathcal{L}_k = \left\{p \in \mathbb{R}^3 \mid |p-k| \leq k_F < |p|\right\}$ is now the "solid" lune. By integrating along the $k \cdot p =$ constant planes one may reexpress the integral, when $|k| \leq 2k_F$, as

$$\begin{aligned}\int_{\mathcal{L}_k} f\left(k\cdot p - \frac{1}{2}\left|k\right|^2\right)dp &= 2\pi\left|k\right|\int_{\frac{1}{2}|k|}^{k_F} f\left(\left|k\right|\left(t-\frac{1}{2}\left|k\right|\right)\right)\left(t-\frac{1}{2}\left|k\right|\right)dt\\ &+\pi\int_{k_F}^{k_F+|k|} f\left(\left|k\right|\left(t-\frac{1}{2}\left|k\right|\right)\right)\left(k_F^2-(t-\left|k\right|)^2\right)dt\end{aligned} \tag{6.7}$$

with $f(x) = x^\beta$. It follows that for Coulomb

$$\begin{aligned}\left\langle v_k, h_k v_k\right\rangle &\sim \frac{\hat{V}_k}{2(2\pi)^3}\left(\frac{2\pi}{3}k_F^3\left|k\right|^2\right) = \frac{1}{4}\frac{g}{6\pi^2}k_F^3,\\ \left\langle v_k, h_k^3 v_k\right\rangle &\sim \frac{\hat{V}_k}{2(2\pi)^3}\left(\frac{2\pi}{5}k_F^5\left|k\right|^4 + \frac{\pi}{6}k_F^3\left|k\right|^6\right) \approx \frac{1}{4}\frac{g}{10\pi^2}k_F^5\left|k\right|^2\end{aligned} \tag{6.8}$$

whence

$$\epsilon_k \approx \sqrt{\frac{g}{3\pi^2}k_F^3 + 4\frac{\frac{1}{4}\frac{g}{10\pi^2}k_F^5\left|k\right|^2}{\frac{1}{4}\frac{g}{6\pi^2}k_F^3}} = \sqrt{\frac{g}{3\pi^2}k_F^3 + \frac{12}{5}k_F^2\left|k\right|^2} \tag{6.9}$$

which is the previously mentioned Eq. (3.6).

Further explanation of (3.8) for general potentials

Our analysis can be extended easily to any potential satisfying

$$\hat{V}_k \geq 0, \quad \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 < \infty. \tag{6.10}$$

To be precise, let $k \in \mathbb{Z}^3_*$ and $M \in \mathbb{N}$ satisfy $\hat{V}_k \gg k_F^{-1}$ and $1 \leq M \ll k_F \, |k|^{\frac{1}{2}}$, and let ϵ_k, φ_k and $\hat{\Psi}_M$ be as in Theorem 1. The proof of Proposition 2 remains unchanged and we only need to generalize slightly the one-body estimates in Proposition 3. We can use exactly (5.21), (5.22) and (5.23), without substituting $\hat{V}_k = g|k|^{-2}$, to get

$$\begin{aligned} \|\varphi\|_\infty &\leq C \frac{k_F^3 \, |k|^2 \, \hat{V}_k}{k_F^3 \, |k|^2 \, \hat{V}_k - C k_F^2 \, |k|^2} \frac{\sqrt{k_F \, |k|}}{\sqrt{k_F^3 \, |k|^2 \, \hat{V}_k}} \sqrt{\hat{V}_k} \\ &= C \frac{\hat{V}_k}{\hat{V}_k - C k_F^{-1}} \frac{1}{\sqrt{k_F^2 \, |k|}} \leq C \frac{1}{\sqrt{k_F^2 \, |k|}} \end{aligned} \tag{6.11}$$

under the condition that $\hat{V}_k \gg k_F^{-1}$. Using again (5.20) and $\sqrt{|L_k|} \sim \sqrt{k_F^2 \, |k|}$, we also have

$$\|\varphi\|_6^3 = \sqrt{\sum_{p \in L_k} \left| \langle e_p, \varphi \rangle \right|^6} \leq \sqrt{|L_k|} \, \|\varphi\|_\infty^3 \leq C \frac{1}{k_F^2 \, |k|}. \tag{6.12}$$

Moreover, we can split

$$\sum_{l \in \mathbb{Z}^3_*} \left\| \tilde{E}_l - h_l \right\|_{\mathrm{HS}}^2 = \sum_{l \in 2B_F} \left\| \tilde{E}_l - h_l \right\|_{\mathrm{HS}}^2 + \sum_{l \in \mathbb{Z}^3 \setminus 2B_F} \left\| \tilde{E}_l - h_l \right\|_{\mathrm{HS}}^2 \tag{6.13}$$

and estimate by Lemma 7 that for $|l| \leq 2k_F$

$$\begin{aligned} \left\| \tilde{E}_l - h_l \right\|_{\mathrm{HS}}^2 &\leq \min \left\{ 2 \langle v_l, h_l v_l \rangle, 4 \, \|v_l\|^4 \right\} \leq C \min \left\{ k_F^3 \, |l|^2 \, \hat{V}_l, k_F^4 \, |l|^2 \, \hat{V}_l^2 \right\} \\ &\leq C \min \left\{ 1, k_F \hat{V}_l \right\} k_F^3 \hat{V}_l \, |l|^2 . \end{aligned} \tag{6.14}$$

For $|l| > 2k_F$ we simply estimate

$$\left\| \tilde{E}_l - h_l \right\|_{\mathrm{HS}}^2 \leq 4 \, \|v_l\|^4 \leq C \, |L_l|^2 \, \hat{V}_l^2 \leq C k_F^6 \hat{V}_l^2. \tag{6.15}$$

Thus

$$\sum_{l\in\mathbb{Z}^3_*}\left\|\tilde{E}_l-h_l\right\|^2_{\mathrm{HS}}\le Ck_F^3\left(\sum_{l\in 2B_F}\min\left\{1,k_F\hat{V}_l\right\}\hat{V}_l\,|l|^2+Ck_F^3\sum_{l\in\mathbb{Z}^3\backslash 2B_F}\hat{V}_l^2\right). \tag{6.16}$$

Inserting (6.11), (6.12) and (6.16) in the estimate in Proposition 2, we obtain (3.8).

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