On Bose–Einstein condensates in the Thomas–Fermi regime

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Abstract

We study a system of N trapped bosons in the Thomas–Fermi regime with an interacting pair potential of the form $g_N N^{3\beta-1} V(N^\beta x)$, for some $\beta \in (0, 1/3)$ and g_N diverging as $N \to \infty$. We prove that there is complete Bose–Einstein condensation at the level of the ground state and, furthermore, that, if $\beta \in (0, 1/6)$, condensation is preserved by the time evolution.

Keywords Quantum gases · Bose–Einstein condensates · Quantum many-body systems

1 Introduction and Main Results

In recent years, the analysis of quantum systems of interacting particles, in particular bosons, in suitable scaling regimes (mean-field, Gross–Pitaevskii, thermodynamic, etc.) has represented a very flourishing line of research in mathematical physics. Starting from the pioneering works [27, 30, 53], several results have been obtained along this research line about the ground state behavior of interacting bosons and Bose–Einstein condensation (BEC) [38, 40–42] in both the mean-field (see [28, 34–36, 49, 52] and references therein), in the Gross–Pitaevskii (GP) regimes (see [6, 7, 29, 43–45] and references therein) and in the thermodynamic limit (see [3, 25, 54] and references therein), as well as about the many-body dynamics in the mean-field (see [22, 33, 47] and references therein) and in the GP (see [5, 11, 23, 48] and references therein) settings. As we are going to see in more details later, in both these regimes, the effective behavior of the many-body system is shown to be suitably approximated studying an effective one-particle nonlinear problem.

In the typical setting, a large number N of bosonic particles is confined in a box or, more generally, by a trapping potential, and the bottom of the spectrum of the

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corresponding Hamiltonian, or its unitary evolution, is investigated as $N \to \infty$. The many-body Schrödinger operator has the form

$$H_N^{\text{trap}} := \sum_{j=1}^{N} \left(-\Delta_j + V_{\text{ext}}(x_j) \right) + g_N N^{3\beta-1} \sum_{1 \leqslant j < k \leqslant N} v(N^\beta(x_j - x_k)), \quad (1.1)$$

and it is supposed to act on $L_s^2(\mathbb{R}^{3N})$, which is the subspace of $L^2(\mathbb{R}^{3N})$ consisting of all functions which are symmetric with respect to permutations of the N particles in 3 dimensions. Here, V_{ext} is the trapping potential, which for simplicity is assumed to be homogeneous of order[1] s (see also Remark 1.2), i.e.,

$$V_{\text{ext}}(x) = \lambda|x|^s, \quad \text{with } s \geqslant 2 \text{ and } \lambda > 0, \quad (1.2)$$

while $N^{3\beta}v(N^\beta x)$ (with $\beta > 0$) is the pair interaction potential, which is suitably rescaled by the prefactor g_N/N. In the following we use the notation $v_N(x) := N^{3\beta}v(N^\beta x)$. The parameter $\beta > 0$ varies in $[0, 1]$ and, with $g_N = g = \text{const.}$, identifies different scaling regimes: mean-field for $\beta = 0$, mean-field and zero-range for $\beta \in (0, 1)$ and GP for $\beta = 1$. In fact, β is also a measure of the *diluteness* of the system: indeed, denoting by $\bar{\rho}$ the average density, we have $\bar{\rho} \simeq N \left\| \rho_N^{\text{GP}} \right\|_\infty$ (see [16]), where ρ_N^{GP} is the density associated to the condensate wave function.

In this paper we will investigate the Thomas–Fermi (TF) limit, where $g_N \to \infty$. In the TF regime, one thus has (see Remark 1.3)

$$\bar{\rho} R_N^3 \propto \frac{N^{1-3\beta}}{g_N^{\frac{3}{s+3}}},$$

where $R_N \sim N^{-\beta}$ is the effective range of the interaction potential and $g_N^{\frac{1}{s+3}}$ is the size of the TF radius, i.e., the characteristic length scale of the condensate (see Remark 1.3). Hence, since $g_N \to +\infty$ as $N \to \infty$, the above quantity certainly tends to 0 for $\beta > 1/3$, so that the gas is dilute and only binary scattering events are relevant; on the opposite, for $\beta < 1/3$, the quantity may diverge, depending on the how fast g_N tends to $+\infty$, so that any particle in the gas can interact with all the others, as in the typical mean-field picture.

The final outcome in terms of effective description is physically different but it is remarkable that, for any $\beta \in (0, 1]$, a one-particle Schrödinger operator with cubic nonlinearity is recovered as $N \to \infty$, giving rise to the GP functional

$$\mathcal{E}_{\text{GP}}^{\text{trap}}(u) := \langle u, hu \rangle + 4\pi g \int_{\mathbb{R}^3} dx \, |u(x)|^4 \quad (1.3)$$

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[1] The condition $s \geqslant 2$ is actually assumed only to simplify the presentation of the result, which holds true with different (more involved) estimates of the error terms also for $s \geqslant 0$.

in the stationary setting, where h is the one-body operator

$$h := -\Delta + V_{\text{ext}}, \tag{1.4}$$

and to the GP time-dependent equation

$$i\partial_t \psi_t = (-\Delta + 8\pi g |\psi_t|^2)\psi_t, \tag{1.5}$$

in the dynamical one: note the absence of the trapping potential in the above evolution equation, which is usually switched-off after the occurrence of BEC to observe the gas dynamics. Only the coefficient g of such a nonlinear term changes for different values of β: for $\beta \in (0, 1)$ the coefficient is proportional to the integral of the interaction potential (which has then to be assumed finite), while, for $\beta = 1$, its scattering length appears. In both cases the coefficient g is assumed to be constant as $N \to \infty$.

The physical motivation behind the choice of letting g_N grow with N is that in physical heuristics, the effective coefficient g is very often assumed to be large and the asymptotics $g \to +\infty$ considered, which goes under the name of *Thomas–Fermi regime*, because of a similarity at the level of the one-particle density with the Thomas–Fermi theory of electrons. The main reason behind such a choice is that in real experiments $g \propto aN$, a being the measured scattering length of the interaction potential, and therefore, even in presence of few thousands of atoms, the quantity might be quite large compared to the other characteristic lengths of the system.

Another important motivation to let $g = g_N$ diverge is to take into account other settings and, in particular, the thermodynamic limit, i.e., the regime in which $N, L \to \infty$, but $\rho := N/L^3$ is kept fixed: after rescaling the lengths this limit can be recast in the form above and recovered for $\beta = 1/3$, $g_N = N^{2/3}$. Finally, it is remarkable that a quite consistent literature is available about the ground state behavior of the GP functional in the asymptotic regime $g \to +\infty$, in particular in presence of rotation (see [17, 18] and references therein), when vortices might appear as a manifestation of the superfluidity of the system.

In this very same setting, the effective dynamics of vortices can also be studied [31] and shown to be driven by the trapping potential: if at initial time a trapped bosonic system is set in a condensate state with finitely many vortices (e.g., by putting it under sufficiently fast rotation) and then it is let to evolve (switching off the rotation), it is possible to follow the trajectories of vortex points and prove that they are determined by an ODE depending only on the external trap. We are going to comment further on this result below but we point out here that two key ingredients are needed in its proof: the system is assumed to be in the TF regime and the effective evolution equation for the condensate is given by the time-dependent GP equation. Here, we are precisely concerned with proving that such assumptions make sense. More precisely, we show that

1. in the TF regime (for $\beta < 1/3$, $g_N \ll N$) Bose–Einstein condensation takes place at the level of the ground state of a system of interacting bosons;
2. in the same TF limit, the effective time-evolution of a state with sufficiently low energy is governed by the time-dependent GP equation.

A similar regime has indeed been studied so far at the many-body level only in [2, 13, 16]. However, in [16] the focus is different: the object of the study is the interplay between the TF regime and the presence of a rapid rotation and its effect to the ground state energy is investigated to leading order. We comment later on the similarities between [2, 13] and our work.

Let us now provide some more details about the setting we are going to consider. The main object under investigation is the many-body Hamiltonian (1.1) and its time evolution, acting on bosonic (i.e., symmetric) wave functions in $L^2_s(\mathbb{R}^{3N})$. We restrict our analysis to interaction potentials satisfying the assumptions below.

Assumption 1 The potential $v \neq 0$ is of positive type, i.e., $\widehat{v} \geqslant 0$ and spherically symmetric. Furthermore, we assume that $|x|v \in L^1(\mathbb{R}^3)$, $v \in L^2(\mathbb{R}^2)$.

Let us comment further about the characteristic length scales of the system. The most important one is indeed the range of the interaction potential $\propto N^{-\beta}$ and, if g_N is not too large, we claim that it is the only relevant one to be compared with the mean free path of the particles $\propto N^{-1/3}$. This yields the threshold at $\beta = 1/3$ mentioned above. Another important length scale is provided by the *scattering length* associated to v_N: for a potential v the scattering length is defined through the solution of the zero energy scattering equation, i.e.,

$$\left(-\Delta + \tfrac{1}{2}v\right) f = 0, \tag{1.6}$$

with the boundary condition $f(x) \to 1$ as $|x| \to \infty$. The scattering length a is then given by

$$8\pi a = \int_{\mathbb{R}^3} dx\, v(x) f(x). \tag{1.7}$$

If we denote by a_N the scattering length of the potential $g_N N^{3\beta-1} v(N^\beta x) = g_N v_N(x) N^{-1}$ then one can go through the Born approximation of a_N to write

$$a_N = \frac{g_N}{8\pi N} \int_{\mathbb{R}^3} dx\, v_N(x) f_N(x) \sim \frac{g_N}{8\pi N} \int_{\mathbb{R}^3} dx\, v(x), \tag{1.8}$$

where f_N is the solution of the related scattering equation, i.e., $-\Delta f_N + (g_N/2)v_N f_N = 0$. In order for the expansion to make sense, one has to require that v is integrable and

$$g_N \ll N^{1-\beta}, \tag{1.9}$$

which we always do, but at the same this implies that

$$a_N \ll R_N \sim N^{-\beta}, \tag{1.10}$$

for any $\beta \in [0, 1)$. In other words, a_N is always much smaller than the range of the potential in the regime we are going to consider and plays no role in the picture. A

transition is expected to occur when $g_N \sim N^{1-\beta}$, for $\beta \geqslant 1/3$, which is precisely the case considered[2] in [13]. The last characteristic length of the problem is the *healing length* ξ_N, which is given by (see [24, Sect. 4.1])

$$\xi_N = \frac{1}{\sqrt{N \|\rho_N^{\text{GP}}\|_\infty}} \sim N^{-\frac{1}{2}} g_N^{-\frac{3}{2(s+3)}}. \tag{1.11}$$

Hence, one can readily check that under the conditions of next Theorem 1.1

$$a_N \ll \xi_N \ll R_N. \tag{1.12}$$

We denote by $E_0(N)$ the ground state energy of the system, i.e.,

$$E_0(N) := \inf \sigma \left(H_N^{\text{trap}} \right) \equiv \inf \left\{ \left\langle \psi, H_N^{\text{trap}} \psi \right\rangle \mid \psi \in L_s^2(\mathbb{R}^{3N}), \ \|\psi\|_{L_s^2(\mathbb{R}^{3N})} = 1 \right\}. \tag{1.13}$$

Moreover, $\Psi_0 \in L_s^2(\mathbb{R}^{3N})$ is the unique (up to an overall phase) ground state of H_N and we denote by γ_{Ψ_0} the associated reduced density matrix, i.e.,

$$\gamma_{\Psi_0} := \text{Tr}_{2,\dots,N} |\psi_0\rangle\langle\psi_0|, \tag{1.14}$$

where with $\text{Tr}_{2,\dots,N}$ we denoted the partial trace on all particles but one.

It is well known that in the mean-field regime ($\beta = 0$, $g_N = g$ constant), the ground state energy per particle is well approximated, in the limit $N \to \infty$, by the infimum of the Hartree functional, which approximately coincides with the restriction of the quadratic form associated to H_N to functions of the form $\psi = u^{\otimes N}$, with $u \in L^2(\mathbb{R}^3)$, $\|u\|_{L^2(\mathbb{R}^3)} = 1$, i.e.,

$$\mathcal{E}_{\text{H}}^{\text{trap}}(u) := \langle u, hu \rangle + \tfrac{1}{2}\langle u \otimes u, vu \otimes u \rangle. \tag{1.15}$$

More precisely, in [34], it is shown that

$$\lim_{N\to\infty} \frac{E_0(N)}{N} = \inf \left\{ \mathcal{E}_{\text{H}}^{\text{trap}}(u) \mid \|u\|_{L^2(\mathbb{R}^3)} = 1 \right\} =: E^{\text{H}}. \tag{1.16}$$

For trapped bosons in the mean field regime, it has been proven that the system exhibits BEC and its excitation spectrum can be approximated via the well-known Bogoliubov approximation [28], with corrections that can be explicitly identified [9, 10]. Analogous results are proven for a system of bosons in a box in the same regime [19, 35, 52].

Similarly, it is also known that in the GP regime ($\beta = 1$, $g_N = g$ constant), the ground state energy per particle can be approximated, in the limit $N \to \infty$, by the

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[2] In the notation of [13], $\beta = 1 - \kappa$ and $g_N = N^\kappa$, so that $g_N = N^{1-\beta}$.

infimum of the GP effective energy (1.3). Indeed, in [43] it is proven that, in the limit $N \to \infty$, it holds (recall (1.3))

$$\lim_{N \to \infty} \frac{E_0(N)}{N} = \inf \left\{ \mathcal{E}_{\text{GP}}^{\text{trap}}(u) \mid \|u\|_{L^2(\mathbb{R}^3)} = 1 \right\} =: E_{\text{GP}}^{\text{trap}}. \tag{1.17}$$

The occurrence of BEC in this case has been proven [15, 44] as well as the validity of the Bogoliubov approximation [14, 45]. Similar results both on BEC and the excitation spectrum are available in (see [7, 29, 30] and references therein) for N interacting bosons in a box.

Concerning the intermediate regimes, we have to distinguish between two different cases. The first interesting regime is studied in [8] and it is characterized by taking $\beta \in (0, 1)$ and $g_N = g > 0$ small enough. In [8] the validity of the Bogoliubov theory is proven in the aforementioned regime, studying both the ground state energy and the excitation spectrum of the system. More recently another regime has been investigated, which can be thought as an interpolation between the GP regime and the thermodynamic limit and it is described by $\beta \in (0, 1)$ small enough and $g_N = N^{1-\beta}$. In [2, 13] it is proven that, in such regime, the low-energy states exhibit BEC and the low-energy excitation spectrum is studied. Note that, according to the discussion above on the characteristic length scales of the problem, this choice corresponds to a scattering length of the interaction of the same order of its effective range.

In the present work, on the other hand, we are mostly interested in the time-evolution of condensates in the TF regime ($g_N \gg 1$). For the sake of completeness, we also show that BEC occurs in the same setting, at least at the level of the ground state or of any approximate ground state, as it can be easily obtained by a suitable adaptation of the arguments in [28, 52] and with a much simpler proof (thanks to stronger assumption on β and g_N) than the ones in [2, 13]. More precisely, we prove that the ground state energy of (1.1), as long as $\beta < 1/3$, exhibits BEC in the (unique) minimizer of the GP functional, which in our setting reads as

$$\mathcal{E}_{\text{GP},N}^{\text{trap}}(\psi) := \langle \psi, h\psi \rangle + \frac{g_N}{2}(\int v) \int_{\mathbb{R}^3} dx \, |\psi(x)|^4. \tag{1.18}$$

Theorem 1.1 (BEC in the ground state) *Let H_N be the Hamiltonian defined in* (1.1) *and let v_N be an interaction potential such that v satisfies Assumption* 1. *Assume that*

$$\beta < \frac{1}{3}, \quad \text{and} \quad g_N \ll \min \left\{ N^{\frac{(1-3\beta)(s+3)}{s+5}}, N^{\frac{(s+3)\beta}{2(s+1)}} \right\}. \tag{1.19}$$

Let Ψ be any normalized many-body state such that $\langle \Psi, H_N \Psi \rangle \leqslant E_N + o(N)$ and let γ_ψ be the associated reduced density matrix defined as in (1.14). *Then,*

$$1 - \langle \varphi_N^{\text{GP}}, \gamma_\Psi \varphi_N^{\text{GP}} \rangle \leqslant C g_N^{\frac{s+5}{s+3}} N^{3\beta - 1} + C g_N^{\frac{2(s+1)}{s+3}} N^{-\beta}, \tag{1.20}$$

where $\varphi_N^{\text{GP}} \in L^2(\mathbb{R}^3)$ is the unique (up to a phase) normalized minimizer of (1.18).

Remark 1.2 (Homogeneous trapping potential) For the sake of simplicity, we assumed that the trapping potential V_{ext} in (1.2) is homogeneous of order $s > 0$. As it is customary in the TF regime, such an assumption can be relaxed and replaced with the weaker condition that V_{ext} is asymptotically homogeneous, i.e., $V_{\text{ext}}(x) \sim |x|^s$, as $|x| \to +\infty$, because only the behavior at large $|x|$ matters to leading order. This however makes the estimates of the error terms as well as the conditions on the parameters more implicit.

Remark 1.3 (GP minimizer) We point out that, since the GP functional in (1.18) depends on N, the same holds for its minimizer φ_N^{GP}. Indeed, it is very well known that the density $|\varphi_N^{\text{GP}}|^2$ is pointwise close as $N \to \infty$ to the minimizer of the TF functional

$$\mathcal{E}_{\text{TF},g_N}^{\text{trap}}(\rho) = \int_{\mathbb{R}^3} dx \left\{ V_{\text{ext}}(x)\rho + \frac{g_N}{2}(\smallint v)\rho^2 \right\} \tag{1.21}$$

among positive densities ρ normalized in L^1. More precisely, one has (see, e.g., [41, Thm. 2.2])

$$\inf_{\|\psi\|_2=1} \mathcal{E}_{\text{GP},N}^{\text{trap}}(\psi) = \inf_{\rho \geqslant 0, \|\rho\|_1=1} \mathcal{E}_{\text{TF},g_N}^{\text{trap}}(\rho)\,(1+o(1))$$

$$= g_N^{\frac{s}{s+3}} \inf_{\rho \geqslant 0, \|\rho\|_1=1} \mathcal{E}_{\text{TF},1}^{\text{trap}}(\rho) + O\left(g_N^{-\frac{2}{s+3}} \log g_N \right), \tag{1.22}$$

as $g_N \to +\infty$, and, denoting by $\rho_{g_N}^{\text{TF}}$ the unique TF minimizer,

$$g_N^{\frac{3}{s+3}} \rho_{g_N}^{\text{TF}}\left(g_N^{\frac{1}{s+3}} x \right) = \rho_1^{\text{TF}}(x), \qquad \left\| g_N^{\frac{3}{s+3}} \left| \varphi_N^{\text{GP}}\left(g_N^{\frac{1}{s+3}} \cdot \right) \right|^2 - \rho_1^{\text{TF}} \right\|_\infty = o(1), \tag{1.23}$$

i.e., both the GP and TF minimizers spread over a length scale $g_N^{\frac{1}{s+3}}$ (TF radius) and tend pointwise to 0 as $N \to +\infty$.

Remark 1.4 (Excitation spectrum) In the present paper we do not investigate the excitation spectrum in the TF regime, but we remark that for a system of N bosons confined in a box, it is possible to study it with similar techniques as in [52]. The extension to the case of trapped bosons is however not completely trivial and beyond the scope of the present paper.

Remark 1.5 (Rotating systems) It would be interesting to generalize our result to rotating systems. It is indeed well-know that if the rotational velocity Ω_{rot} of the condensate is not too large, i.e., it is below a critical threshold $\Omega_c(N)$, the GP minimizer is unique. In this case, the proof of BEC is expected to hold with minor modifications. For faster rotation however the GP functional is known to have more than one minimizer. In that case the proof of BEC is more involved and one expects BEC in the mixed state given by convex combination of the GP minimizers (see [39, 43]).

As already anticipated above, after having proven that there is BEC in the ground state, a natural question that arises is whether BEC is preserved or not by time evolution. More explicitly, the question we would like to answer is whether we can find an *effective dynamics* for a single particle state that well describes the evolution of the condensed state. We consider a system of N trapped identical bosons, as described before, and we suppose that at the initial time the state is condensed. After a while, we remove the trap, thus the Hamiltonian of the system becomes

$$H_N = \sum_{j=1}^{N}(-\Delta_j) + g_N N^{3\beta-1} \sum_{1 \leqslant j < k \leqslant N} v(N^\beta(x_j - x_k)),$$

with domain $D(H_N) := L_s^2(\mathbb{R}^{3N}) \cap H^2(\mathbb{R}^{3N})$. The dynamics of the system is then encoded in the fact that the state at time t, $\psi_t \in L_s^2(\mathbb{R}^{3N})$, is a solution of the Schrödinger equation

$$\begin{cases} i\partial_t \psi_t = H_N \psi_t, \\ \psi_t|_{t=0} = \psi_0. \end{cases} \tag{1.24}$$

We prove that, up to some conditions on β and g_N, we can find an effective equation, i.e., the time-dependent GP equation in the TF limit, which reads

$$i\partial_t \varphi_t = \left(-\Delta + g_N |\varphi_t|^2\right)\varphi_t \tag{1.25}$$

such that ψ_t converges to the solution of the effective problem. Although we need some technical assumptions on the solution ψ_t, we are able to provide an explicit rate of convergence.

In the case of dilute limits, this question already has some answers. For example, it is well known [21, 22, 26, 47, 51] that in the mean-field limit, if the initial state ψ_0 exhibits BEC on φ_0, then ψ_t exhibits BEC on a family of single particle states $\{\varphi_t\}_{t\in\mathbb{R}}$ which solve the *nonlinear Hartree equation*:

$$i\partial_t \varphi_t = \left(-\Delta + v * |\varphi_t|^2\right)\varphi_t. \tag{1.26}$$

A similar result for the GP regime is well know (see [4, 11, 22, 23, 48] and references therein). Meaning that if the initial datum exhibits BEC, then BEC is preserved also at later times on a family of single particle states $\{\varphi_t\}_{t\in\mathbb{R}}$ which solve the *nonlinear GP equation*:

$$i\partial_t \varphi_t = \left(-\Delta + 8\pi g |\varphi_t|^2\right)\varphi_t, \tag{1.27}$$

where g is a constant which is defined, in the limit as $N \to \infty$, as N times the scattering length of the system, i.e., $Na \to g$ as $N \to \infty$.

As we already discussed for the ground state energy, also for the dynamical framework there are different intermediate regimes corresponding to $\beta \in (0, 1)$. Also in

this case [48] if there is BEC at the initial state, there is still BEC in the many-body state of the system at later times on a family of single particle states $\{\varphi_t\}_{t\in\mathbb{R}}$ which solve the following nonlinear equation:

$$i\partial_t \varphi_t = \left(-\Delta + (\smallint v) |\varphi_t|^2\right)\varphi_t. \tag{1.28}$$

Moreover, for the same regime, a norm approximation for the evolution of an initial state exhibiting BEC is also available (see [12]). This last nonlinear equation is rather similar to the GP equation, with the major difference being that instead of having the constant g here we find $\smallint v$. This is what one should expect since in this regime, N times the scattering length of the pair potential goes to $\smallint v$ in the limit $N \to \infty$.

Thus, if we now look at our setting in which $Na = \mathcal{O}(g_N)$, we can expect that the effective dynamics is a cubic nonlinear Schrödinger equation. Since the effective scattering length of our potential grows as g_N as $N \to \infty$, such an equation can not be independent of N.

Moreover, we have that, in the sense of distributions, $N^{3\beta} v \left(N^\beta x\right) \to \left(\smallint v\right) \delta_0(x)$ as $N \to \infty$. Using this, one could also see equation (1.28) as a limit for $N \to \infty$ of an Hartree equation as in (1.26) with the substitution $v \to N^{3\beta} v \left(N^\beta \cdot\right)$. Notice also that this sort-of intermediate equation is still N dependent, but has now the advantage of being a good candidate for the equation governing the dynamics of the single particle state for our system.

Our discussion above led us to guess that we can expect condensation at later times only on a state which is still $N-$dependent. For this purpose we will need to consider two different differential equations. In analogy to the GP limit, from now on we define the GP equation as

$$i\partial_t \varphi_t^{\mathrm{GP}} = \left(-\Delta + g_N \left(\smallint v\right) \left|\varphi_t^{\mathrm{GP}}\right|^2\right)\varphi_t^{\mathrm{GP}}. \tag{1.29}$$

In what follows we will need to work with the associated GP functional. Since to study the dynamics we switch off the trap, to distinguish this functional from the one introduced before, we define

$$\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(u) := \int_{\mathbb{R}^3} dx \, |\nabla u(x)|^2 + \frac{g_N}{2} (\smallint v) \int_{\mathbb{R}^3} dx \, |u(x)|^4. \tag{1.30}$$

Theorem 1.6 (Dynamics) *Let $\beta \in (0, 1/6)$ and $\lambda \in (3\beta, 1-3\beta)$. Let ψ_t be the solution of (1.24) and φ_t^{GP} be the solution of (1.29) with initial data respectively ψ_0 and φ_0, both with norm equal to 1. Suppose that v satisfies Assumption 1. It holds true that*

$$\left\|\gamma_{\psi_t}^{(1)} - |\varphi_t^{\mathrm{GP}}\rangle\langle\varphi_t^{\mathrm{GP}}|\right\| \leqslant \sqrt{2}\left(N^{\frac{1-\lambda}{2}} \left\|\gamma_{\psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right\|^{\frac{1}{2}} + N^{\frac{3\beta-\lambda}{2}}\right) e^{C_v C_N(\varphi_0, t) g_N |t|}$$

$$+ C\sqrt{g_N N^{-\beta}} \left(1 + \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + g_N N^{-\beta} \|\varphi_0\|_{L^\infty}^2\right) e^{C_v C_N(\varphi_0, t)^2 g_N |t|}, \tag{1.31}$$

where $\|\cdot\|$ denotes the operator norm and the constant C_v only depends on the potential v and $C_N(\varphi_0, t)$ is given by

$$C_N(\varphi_0, t) := \|\varphi_0\|_{H^2(\mathbb{R}^3)} \, e^{Cg_N^2 \left([\mathcal{E}_{GP}^{free}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4 \right)|t|}. \qquad (1.32)$$

Remark 1.7 (Time-scale of the GP approximation) The initial datum determines the maximal time-scale on which the GP approximation makes sense through the quantity $C_N(\varphi_0, t)$ and in particular the exponential appearing in (1.32). This in turn depends on the GP energy of the initial datum via (1.32), which, according to (1.22), can not be smaller than $Cg_N^{s/(s+3)}$. We stress that, although we switch off the trap in studying the dynamics, heuristically φ_0 is the initial state of the trapped system: this is why here we suppose that $\mathcal{E}_{GP}^{free}(\varphi_0) \sim Cg_N^{s/(s+3)}$. Now, the exponential on the right hand side of (1.32) is bounded by a constant as long as

$$t < t_\star \sim g_N^{-\frac{2(2s+3)}{s+3}}. \qquad (1.33)$$

Note indeed that, assuming the H^2 norm of φ_0 to be bounded uniformly, the second summand in the exponent in (1.32) is much smaller than the first, i.e., of order $g_N^2 N^{-2\beta}$, which vanishes because of the factor $N^{-2\beta}$ (see also below). For a time shorter than $t_\star$ now, $C_N(\varphi_0, t)$ can be bounded by a constant. Plugging this bound into (1.31) and using similar arguments as above, one gets

$$\left\| \gamma_{\psi_t}^{(1)} - |\varphi_t^{GP}\rangle\langle\varphi_t^{GP}| \right\| \leq C \left(N^{\frac{1-\lambda}{2}} \left\| \gamma_{\psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0| \right\|^{\frac{1}{2}} + N^{\frac{3\beta-\lambda}{2}} + g_N^{\frac{2s+3}{2(s+3)}} N^{-\frac{\beta}{2}} \right), \qquad (1.34)$$

which tends to 0 as $N \to +\infty$ provided the initial datum is not too far from a condensate on φ_0 and assuming

$$g_N \ll N^{\frac{\beta(s+3)}{2s+3}}. \qquad (1.35)$$

Note that, due to the kinetic energy being sub-leading with respect to the potential energy in the Thomas Fermi regime, to determine the optimal time scale for the validity of the GP approximation is not an easy task.

Remark 1.8 (Hartree vs GP approximation) The estimate (1.31) is obtained via the following inequality

$$\left\| \gamma_{\psi_t}^{(1)} - |\varphi_t^{GP}\rangle\langle\varphi_t^{GP}| \right\| \leq \left\| \gamma_{\psi_t}^{(1)} - |\varphi_t^{H}\rangle\langle\varphi_t^{H}| \right\| + 2 \left\| \varphi_t^{GP} - \varphi_t^{H} \right\|_2,$$

where we denote by φ_t^{H} the solution of the Hartree equation (see (3.1)). More precisely, the first line in the right hand side of (1.31) is the estimate of $\left\| \gamma_{\psi_t}^{(1)} - |\varphi_t^{H}\rangle\langle\varphi_t^{H}| \right\|$ and the second line is a bound for $2 \left\| \varphi_t^{GP} - \varphi_t^{H} \right\|_2$. Note then that our result suggests

that $\beta = 1/7$ is the threshold from Hartree to GP. More precisely, if we assume condensation on the initial state φ_0 and $g_N N^{-\beta} \ll 1$ (see Remarks 1.7 and 1.9), then for $\beta < 1/7$ we see that for values of λ arbitrarily close to $1 - 3\beta$, Hartree gives a better estimates. However, if $\beta > 1/7$, the term $\left\| \gamma_{\psi_t}^{(1)} - |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| \right\|$ gives a worse approximation than $\left\| \varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}} \right\|_2$. Note also that the interval of times we can cover in Theorem 1.6 is the same for both the Hartree and the GP approximations since the constant $C_N(\varphi_0, t)$ in (1.32) is appearing in both of them.

Remark 1.9 (Time-scale of the vortex dynamics) As anticipated, it would be interesting to compare our result with the findings of [31], concerning the effective dynamics of vortices.[3]

However, a first major difference with the setting considered there is the absence of the trapping potential, which is mostly due to a technical obstruction for the propagation in time of suitable bounds along the nonlinear dynamics. This difference makes a naive comparison (i.e., for $V_{\mathrm{ext}} = 0$) not so meaningful, but in absence of an external trap, $t_\star$ in (1.33) is always much smaller than the characteristic time-scale of the vortex dynamics in [31], i.e., $t_{\mathrm{vortex}} \sim g_N^{2/(s+3)} \log g_N$. However, if g_N is chosen to grow very slowly with respect to N, i.e. $g_N^3 \ll \log\log N$, then the time of the vortices can be achieved.

1.1 Outline and Sketch of the Proofs

The paper is organized as follows. In Sect. 2 we prove Theorem 1.1, whose goal is to prove BEC of the ground state Ψ_0 of H_N on the one-particle state φ_N^{GP}. The proof's strategy is very similar to the one about the mean-field scaling studied in [28, 52]. More precisely,

- We first approximate the GP interaction with respect to the Hartree one, using some uniform bounds on φ_N^{GP} (see Proposition 2.4).
- Then, we follow the ideas already developed in [28, 52] to get a bound on the expected number of particles in the ground state which are not in φ_N^{GP} (see Proposition 2.5).
- Finally, we use the aforementioned bound to prove the condensation estimate. To do that we need to study, in the limit $N \to +\infty$, the bottom of the spectrum of the effective GP Hamiltonian as well as its spectral gap, this is done in Proposition 2.1.

In Sect. 3 we prove Theorem 1.6. The main ideas are the following:

- First we approximate the many body dynamics with the Hartree evolution. In this part we follow the ideas developed in [47] using also some Sobolev bounds on the Hartree state at time t which are proved in Proposition 3.1.
- Then, we compare the Hartree dynamics with the GP one using Grönwall's lemma (see Proposition 3.5).

---

[3] Note that, differently to the one of our paper, the setting in [31] is two-dimensional. This is indeed a natural framework to study the motion of vortices as two-dimensional objects. However, this setting can be thought as a system with a cylindrical trap. This justifies our comparison.

In general, we consider a scaling limit in which the coupling constant, g_N, is N-dependent. This choice requires a careful analysis to keep track of the N-dependence in all the bounds we need to prove both Theorems 1.1 and 1.6.

2 Ground State Energy

The aim of this section is to prove Theorem 1.1. We start by recalling that, under our assumptions, the GP functional in (1.18) admits a unique positive minimizer, which for short we denote in this section by φ^{GP} (in place of φ_N^{GP}) and which satisfies the Euler-Lagrange equation

$$(-\Delta + V_{\mathrm{ext}})\varphi^{\mathrm{GP}} + g_N(\smallint v)|\varphi^{\mathrm{GP}}|^2\varphi^{\mathrm{GP}} = \mu_{\mathrm{GP}}\varphi^{\mathrm{GP}}, \tag{2.1}$$

where

$$\mu_{\mathrm{GP}} := E_{\mathrm{GP},N}^{\mathrm{trap}} + \frac{g_N}{2}(\smallint v)\|\varphi^{\mathrm{GP}}\|_{L^4(\mathbb{R}^3)}^4. \tag{2.2}$$

Because of the uniqueness of the solution to (2.1), φ^{GP} is also the ground state of the GP operator h^{GP}, defined as

$$h^{\mathrm{GP}} := -\Delta + V_{\mathrm{ext}} + g_N(\smallint v)|\varphi^{\mathrm{GP}}|^2. \tag{2.3}$$

One can find a complete set of normalized eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ for the GP operator h^{GP}, where we identify $\varphi_0 \equiv \varphi^{\mathrm{GP}}$. Given that the spectrum of h^{GP} is discrete, we can assume that the corresponding eigenvalues are ordered in such a way that $\mu_{\mathrm{GP}} \equiv \mu_0 < \mu_1$ and $\mu_j \leq \mu_{j+1}$ for any $j \geq 1$. In what follows it is very important that the inequality $\mu_{\mathrm{GP}} < \mu_1$ is strict, which is due to the fact that h^{GP} has a unique ground state (see also Proposition 2.1).

To prove Theorem 1.1 it is convenient to introduce an operator, which we call $\mathcal{N}_+$, that counts the number of particles outside φ^{GP}, i.e., the number of excited particles. For short, we now set $P := |\varphi^{\mathrm{GP}}\rangle\langle\varphi^{\mathrm{GP}}|$ and $Q := 1 - P = \sum_{n\neq 0}|\varphi_n\rangle\langle\varphi_n|$. Furthermore, P_j and Q_j will denote copies of the operators P and Q acting on the j−th particle. The operator $\mathcal{N}_+$ can then be explicitly written as

$$\mathcal{N}_+ := \sum_{j=1}^{N} Q_j. \tag{2.4}$$

We now briefly explain the strategy of the proof of Theorem 1.1. In order to prove BEC of the ground state Ψ_0 of H_N in the one-particle state φ^{GP}, we prove in Proposition 2.5 an upper bound to $\langle\Psi_0, \mathcal{N}_+\Psi_0\rangle$. To do that, we need to prove a lower bound on the energy gap $\mu_1 - \mu_{\mathrm{GP}}$.

2.1 Preliminary Estimates

In this section we prove a lower bound for the energy gap $\mu_{\text{GP}} - \mu_1$ and we estimate $\|\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}$ and $\|\nabla\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}$ in terms of g_N.

2.1.1 Energy Gap

Acting through the unitary transformation defined by

$$\varphi(x) = \varepsilon^{\frac{3}{s+2}} \psi\left(\varepsilon^{\frac{2}{s+2}} x\right), \quad \varepsilon := (g_N(\int v))^{-\frac{s+2}{2(s+3)}} \tag{2.5}$$

one gets

$$\langle \varphi, h^{\text{GP}} \varphi \rangle = \varepsilon^{-\frac{2s}{s+2}} \left\langle \psi, \left(-\varepsilon^2\Delta + V_{\text{ext}} + |\psi_0^\varepsilon|^2\right) \psi \right\rangle =: \varepsilon^{-\frac{2s}{s+2}} \langle \psi, h_\varepsilon \psi \rangle, \tag{2.6}$$

where ψ_0^ε is the ground state of h_ε with energy μ_0^ε, which is explicitly given by

$$\varphi^{\text{GP}}(x) =: \varepsilon^{\frac{3}{s+2}} \psi_0^\varepsilon\left(\varepsilon^{\frac{2}{s+2}} x\right). \tag{2.7}$$

Analogously, we denote by $\mu_1^\varepsilon > \mu_0^\varepsilon$ the second lowest eigenvalue[4] of h_ε, and by ψ_1^ε the associated normalized eigenstate. It easily follows from (2.6) that

$$\mu_1 - \mu_{\text{GP}} = \varepsilon^{-\frac{2s}{(s+2)}} \left(\mu_1^\varepsilon - \mu_0^\varepsilon\right). \tag{2.8}$$

To study the energy gap, it is then sufficient to find a bound for $\mu_1^\varepsilon - \mu_0^\varepsilon$, that is what we do in the next proposition.

Proposition 2.1 (Energy gap) *There exists a constant $C > 0$ depending only on v and V_{ext} such that*

$$\mu_1^\varepsilon - \mu_0^\varepsilon > C\varepsilon^2. \tag{2.9}$$

As a consequence, there exists a constant $C > 0$ depending only on v and V_{ext} such that

$$\mu_1 - \mu_{\text{GP}} > C g_N^{-\frac{2}{s+3}}. \tag{2.10}$$

A key ingredient for the proof of Proposition 2.1 is an integral estimate for ψ_1^ε. This is the content of the next proposition, whose proof is very similar to [46, Proposition 3.5].

---

[4] Note that the strict inequality $\mu_1^\varepsilon > \mu_0^\varepsilon$ follows from the uniqueness of the ground state of h_ε.

Proposition 2.2 (Integral decay estimates) *Let $A(x)$ be the Agmon distance associated to $V_{\text{ext}}(x) = k|x|^s$ $(s \geqslant 2)$, i.e.,*

$$A(x) = \sqrt{k}\,\frac{|x|^{1+\frac{s}{2}}}{1 + s/2}. \tag{2.11}$$

Then

$$\int_{\mathbb{R}^3} dx\, e^{\frac{A(x)}{\varepsilon^2}}|\psi_1^\varepsilon(x)|^2 \leqslant (1 + \mu_1^\varepsilon)e^{\frac{C}{\varepsilon^2}(1+\mu_1^\varepsilon)}, \tag{2.12}$$

where $C > 0$ is a positive constant which depends only on the trapping potential V_{ext}.

In the proof we use the following lemma.

Lemma 2.1 *Let $W(x) := V_{\text{ext}}(x) + |\psi_0^\varepsilon(x)|^2 - \mu_1^\varepsilon$ and let $\Phi(x) := A(x)/2\varepsilon^2$, where $A(x)$ is as in (2.11). It holds true that*

$$\int_{\mathbb{R}^3} dx\, \left(W(x) - \varepsilon^2|\nabla\Phi(x)|^2 \right) e^{2\Phi(x)}|\psi_1^\varepsilon(x)|^2 \leqslant 0. \tag{2.13}$$

The proof of the lemma above is an adaptation of [46, Lemma 3.6].

Proof of Proposition 2.2 We set

$$\Omega := \left\{ x \in \mathbb{R}^3 \mid V_{\text{ext}}(x) - 4\mu_1^\varepsilon < 4 \right\} \tag{2.14}$$

and

$$\Phi(x) := \frac{1}{2\varepsilon^2}A(x), \tag{2.15}$$

so that

$$|\nabla\Phi(x)|^2 = \frac{1}{4\varepsilon^2}V_{\text{ext}}(x). \tag{2.16}$$

We then have that for each $x \in \Omega^c \equiv \mathbb{R}^3 \setminus \Omega$,

$$\left(V_{\text{ext}}(x) + |\psi_0^\varepsilon(x)|^2 - \mu_1^\varepsilon \right) - \varepsilon^2|\nabla\Phi(x)|^2 \geqslant 1. \tag{2.17}$$

In what follows we use $W(x)$ to denote the potential $W(x) := V_{\text{ext}}(x) + |\psi_0^\varepsilon(x)|^2 - \mu_1^\varepsilon$. By (2.17), we get

$$\int_{\Omega^c} dx\, e^{2\Phi(x)}|\psi_1^\varepsilon(x)|^2 \leqslant \int_{\Omega^c} dx\, \left(W(x) - \varepsilon^2|\nabla\Phi(x)|^2 \right) e^{2\Phi(x)}|\psi_1^\varepsilon(x)|^2$$

$$\leqslant -\int_{\Omega} dx \left(W(x) - \varepsilon^2 |\nabla \Phi(x)|^2 \right) e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2,$$

$$(2.18)$$

where in the last inequality we used that

$$\int_{\mathbb{R}^3} dx \left(W(x) - \varepsilon^2 |\nabla \Phi(x)|^2 \right) e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2 \leqslant 0, \qquad (2.19)$$

which is what is stated in Lemma 2.1. Using now that $W(x) - \varepsilon^2 |\nabla \Phi|^2 \geqslant -\mu_1^{\varepsilon}$, from (2.18), we have

$$\int_{\Omega^c} dx \, e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2 \leqslant \mu_1^{\varepsilon} \int_{\Omega} dx \, e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2, \qquad (2.20)$$

which implies

$$\int_{\mathbb{R}^3} dx \, e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2 \leqslant (\mu_1^{\varepsilon} + 1) \int_{\Omega} dx \, e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2. \qquad (2.21)$$

To conclude, we bound the integral in the r.h.s. above. It is then enough to use that there exists a constant $C > 0$ which depends only on the trapping potential such that

$$\sup_{x \in \Omega} e^{\frac{\Phi(x)}{\varepsilon^2}} \leqslant e^{\frac{C}{\varepsilon^2}(1 + \mu_1^{\varepsilon})}. \qquad (2.22)$$

We then get that

$$\int_{\mathbb{R}^3} dx \, e^{2\Phi(x)} |\psi_1^{\varepsilon}(x)|^2 \leqslant (\mu_1^{\varepsilon} + 1) e^{\frac{C}{\varepsilon^2}(1 + \mu_1^{\varepsilon})}. \qquad (2.23)$$

$\square$

Proposition 2.3 (Boundedness of μ_j^{ε}) *There exist a constants $C > 0$ depending only on v and V_{ext} such that, for $j = 0, 1$,*

$$0 < \mu_j^{\varepsilon} \leqslant C. \qquad (2.24)$$

Proof We first prove that μ_0^{ε} is bounded. It is trivial to see that the energy $\mathcal{E}^{\varepsilon}(\psi)$, where we set

$$\mathcal{E}^{\varepsilon}(\psi) := \langle \psi, (-\varepsilon^2 \Delta + V_{\text{ext}}) \psi \rangle + \frac{1}{2} \int_{\mathbb{R}^3} dx \, |\psi(x)|^4, \qquad (2.25)$$

is uniformly bounded, by evaluating it on a suitable trial state. Therefore, being $\mathcal{E}^\varepsilon$ the energy functional associated to h^ε and being ψ_0^ε the corresponding ground state, one gets

$$\mu_0^\varepsilon = \mathcal{E}^\varepsilon(\psi_0^\varepsilon) + \frac{1}{2}\|\psi_0^\varepsilon\|_4^4 \leqslant 2\mathcal{E}^\varepsilon(\psi_0^\varepsilon) \leqslant C. \tag{2.26}$$

We now bound μ_1^ε. By the min-max principle

$$\mu_1^\varepsilon = \inf_{\substack{\|\psi\|_{L^2(\mathbb{R}^3)}=1 \\ \psi \perp \psi_0^\varepsilon}} \langle \psi, h_\varepsilon \psi \rangle. \tag{2.27}$$

Notice that given that V_{ext} is even, then ψ_0^ε is even. Therefore, for any normalized odd trial state $\psi_{\text{trial}} \perp \psi_0^\varepsilon$, and as a consequence we get

$$\mu_1^\varepsilon \leqslant \langle \psi_{\text{trial}}, h_\varepsilon \psi_{\text{trial}} \rangle \leqslant C(1 + \|\psi_0^\varepsilon\|_\infty^2). \tag{2.28}$$

Furthermore, a direct application of the maximum principle to the eigenvalue equation

$$\varepsilon^2 \Delta \psi_0^\varepsilon = \left(V_{\text{ext}} + |\psi_0^\varepsilon|^2 - \mu_0^\varepsilon \right) \psi_0^\varepsilon \tag{2.29}$$

provides an upper bound on $\|\psi_0^\varepsilon\|_\infty$: since by elliptic theory ψ_0^ε is smooth, at any maximum point $\bar{x} \in \mathbb{R}^3$, one has

$$0 \geqslant \varepsilon^2 \Delta \psi_0^\varepsilon(\bar{x}) \geqslant \left(|\psi_0^\varepsilon(\bar{x})|^2 - \mu_0^\varepsilon \right) \psi_0^\varepsilon(\bar{x}), \tag{2.30}$$

so that (recall that $\psi_0^\varepsilon > 0$)

$$|\psi_0^\varepsilon(\bar{x})|^2 \leqslant \mu_0^\varepsilon \leqslant C, \tag{2.31}$$

which in turn implies the uniform boundedness of μ_1^ε via (2.28). $\qquad\square$

Proof of Proposition 2.1 We first prove (2.9) and then deduce from it the estimate (2.10).

Bound for $\mu_1^\varepsilon - \mu_0^\varepsilon$. Let $u(x) := \psi_1^\varepsilon(x)/\psi_0^\varepsilon(x)$, which is well posed since $\psi_0^\varepsilon(x) > 0$. We have that

$$\mu_1^\varepsilon = \mu_0^\varepsilon + \varepsilon^2 \int_{\mathbb{R}^3} dx\, |\nabla u(x)|^2 |\psi_0^\varepsilon(x)|^2,$$

since, by the eigenvalue equations for ψ_0^ε,

$$\int_{\mathbb{R}^3} dx\, \varepsilon^2 |\nabla \psi_1^\varepsilon(x)|^2 = \int_{\mathbb{R}^3} dx\, \varepsilon^2 |u(x)\nabla \psi_0^\varepsilon(x) + (\nabla u(x))\psi_0^\varepsilon(x)|^2$$

$$= \int_{\mathbb{R}^3} dx \left\{ \varepsilon^2 |\nabla u(x)|^2 |\psi_0^\varepsilon(x)|^2 - \varepsilon^2 |u(x)|^2 \psi_0^\varepsilon(x) \Delta \psi_0^\varepsilon(x) \right\}$$

$$= \int_{\mathbb{R}^3} dx \left\{ \varepsilon^2 |\nabla u(x)|^2 |\psi_0^\varepsilon(x)|^2 - V_{\text{ext}}(x) |u(x)|^2 |\psi_0^\varepsilon(x)|^2 \right.$$

$$\left. - |u(x)|^2 |\psi_0^\varepsilon(x)|^4 + \mu_0^\varepsilon |u(x)|^2 |\psi_0^\varepsilon(x)|^2 \right\}. \tag{2.32}$$

To find a lower bound for $\mu_1^\varepsilon - \mu_0^\varepsilon$, we then use Poincaré's inequality. More precisely, let $B_R := \left\{ x \in \mathbb{R}^3 \,|\, |x| < R \right\}$ for some $R > 0$ to be fixed later and let

$$u_R := \int_{B_R} u(x) |\psi_0^\varepsilon(x)|^2. \tag{2.33}$$

We can bound

$$\mu_1^\varepsilon - \mu_0^\varepsilon \geqslant C_R \varepsilon^2 \int_{B_R} dx \, |u(x) - u_R|^2 |\psi_0^\varepsilon(x)|^2,$$

by the weighted Poincaré's inequality (see [50] and references therein) and where $C_R > 0$ is a finite constant depending only on R. Now, recalling that $\psi_1^\varepsilon \perp \psi_0^\varepsilon$, we get

$$\int_{B_R} dx \, |u(x) - u_R|^2 |\psi_0^\varepsilon(x)|^2 = \int_{B_R} dx \, |\psi_1^\varepsilon(x)|^2 + \left| \int_{B_R} dx \, \psi_0^\varepsilon(x) \psi_1^\varepsilon(x) \right|^2$$

$$\times \left(\int_{B_R} dx \, |\psi_0^\varepsilon(x)|^2 - 2 \right)$$

$$= 1 - \int_{B_R^c} dx \, |\psi_1^\varepsilon(x)|^2 - \left| \int_{B_R^c} dx \, \psi_0^\varepsilon(x) \psi_1^\varepsilon(x) \right|^2$$

$$\times \left(1 + \int_{B_R^c} dx \, |\psi_0^\varepsilon(x)|^2 \right), \tag{2.34}$$

where we also used that $\|\psi_j^\varepsilon\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 0, 1$ Using now Cauchy-Schwarz, we obtain

$$\mu_1^\varepsilon - \mu_0^\varepsilon \geqslant C_R \varepsilon^2 \left(1 - 3 \int_{B_R^c} dx \, |\psi_1^\varepsilon(x)|^2 \right). \tag{2.35}$$

We then have to bound

$$\int_{B_R^c} dx \, |\psi_1^\varepsilon(x)|^2, \tag{2.36}$$

to do that we use Proposition 2.2. Indeed, from (2.12), we get that there exist two constants $C, C' > 0$ such that

$$\int_{B_R^c} dx \, |\psi_1^\varepsilon(x)|^2 \leqslant (1 + \mu_1^\varepsilon) e^{-\frac{C}{\varepsilon^2}(C'R^{1+\frac{s}{2}} - 1 - \mu_1^\varepsilon)}. \tag{2.37}$$

Given that μ_1^ε is uniformly bounded, there exists a constant $C'' > 0$ independent of ε such that $\mu_1^\varepsilon \leq C''$. We can then fix R so that

$$R^{1+\frac{s}{2}} > \frac{1 + C''}{C'} \geqslant \frac{1 + \mu_1^\varepsilon}{C'}, \tag{2.38}$$

to get

$$\mu_1^\varepsilon - \mu_0^\varepsilon > C\varepsilon^2. \tag{2.39}$$

Bound for $\mu_{\text{GP}} - \mu_1$. By (2.9) and (2.8)

$$\mu_{\text{GP}} - \mu_1 \geqslant C g_N^{-\frac{2}{s+3}}, \tag{2.40}$$

which implies the result. $\qquad\square$

2.1.2 Estimates for $\|\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}$, $\|\nabla\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}$

Proposition 2.4 (L^∞ bounds) *Let φ^{GP} be the normalized ground state of the operator h^{GP} defined in (2.3). There exists a positive constant $C > 0$ such that*

$$\|\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)} \leqslant C g_N^{-\frac{3}{2(s+3)}}, \quad \|\nabla\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)} \leqslant C g_N^{\frac{2s-3}{2(s+3)}}. \tag{2.41}$$

Proof Recall that φ^{GP} is such that $\langle \varphi^{\text{GP}}, h^{\text{GP}}\varphi^{\text{GP}} \rangle = \mu_{\text{GP}}$, it then follows that φ^{GP} satisfies the following variational equation

$$\Delta\varphi^{\text{GP}} = \left(V_{\text{ext}} + g_N(\smallint v)|\varphi^{\text{GP}}|^2 - \mu_{\text{GP}}\right)\varphi^{\text{GP}}. \tag{2.42}$$

Proceeding similarly as in Proposition 2.3, one can prove that

$$|\varphi^{\text{GP}}(x_0)|^2 \leqslant \frac{\mu_{\text{GP}}}{g_N(\smallint v)}. \tag{2.43}$$

From (2.8) together with (2.24), it follows that

$$\mu^{\mathrm{GP}} \leqslant C g_N^{\frac{s}{s+3}}, \tag{2.44}$$

which immediately implies

$$\|\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)} \leqslant C g_N^{-\frac{3}{2(s+3)}}, \qquad s \geqslant 2. \tag{2.45}$$

We now estimate the L^∞-norm of $\nabla\varphi^{\mathrm{GP}}$. For short, we set

$$f(x) := V_{\mathrm{ext}}(x)\varphi^{\mathrm{GP}}(x) + g_N(\smallint v)|\varphi^{\mathrm{GP}}(x)|^2\varphi^{\mathrm{GP}}(x) - \mu_{\mathrm{GP}}\varphi^{\mathrm{GP}}(x), \tag{2.46}$$

so that φ^{GP} satisfies $\Delta\varphi^{\mathrm{GP}} = f$. The bound on the L^∞-norm of $\nabla\varphi^{\mathrm{GP}}$ can then be proven by standard elliptic estimates. More precisely, there exists a constant $\gamma > 0$ such that for all $x \in \mathbb{R}^3$ it holds

$$\|\nabla\varphi^{\mathrm{GP}}\|_{L^\infty(B_{\rho/2}(x_0))} \leqslant \gamma \|f\|_{L^\infty(B_\rho(x_0))} + \frac{16\gamma}{\rho^4}\|\varphi^{\mathrm{GP}}\|_{L^1(B_\rho(x_0))}, \tag{2.47}$$

where we denote by $B_r(x_0) := \{y \in \mathbb{R}^3 \,|\, |y - x_0| < r\}$. For a proof of (2.47) we refer to (Proposition 11.2, [20]). By Cauchy–Schwarz inequality, we get

$$\begin{aligned}
|\nabla\varphi^{\mathrm{GP}}(x_0)| &\leqslant \|\nabla\varphi^{\mathrm{GP}}\|_{L^\infty(B_{\rho/2}(x_0))} \leqslant C\left(\|f\|_{L^\infty(\mathbb{R}^3)} + \rho^{-\frac{5}{2}}\|\varphi^{\mathrm{GP}}\|_{L^2(\mathbb{R}^3)}\right) \\
&\leqslant C\|f\|_{L^\infty(\mathbb{R}^3)},
\end{aligned} \tag{2.48}$$

where in the last inequality we fix the radius ρ to be $\rho = \|f\|_{L^\infty(\mathbb{R}^3)}^{-2/5}$ and used that $\|\varphi^{\mathrm{GP}}\|_{L^2(\mathbb{R}^3)} = 1$. We then have to estimate the L^∞ norm of f. To do that, we use the bounds (2.44), (2.45) and we get

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{R}^3)} &= \|V_{\mathrm{ext}}(x)\varphi^{\mathrm{GP}}(x) + g_N(\smallint v)|\varphi^{\mathrm{GP}}(x)|^2\varphi^{\mathrm{GP}}(x) - \mu_{\mathrm{GP}}\varphi^{\mathrm{GP}}(x)\|_{L^\infty(\mathbb{R}^3)} \\
&\leqslant C g_N^{\frac{2s-3}{2(s+3)}} + \|V_{\mathrm{ext}}\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}.
\end{aligned} \tag{2.49}$$

Using the exponential decay of φ^{GP}, which can be deduced from Proposition 2.2 together with the maximum principle for subharmonic functions and which implies that

$$|\varphi^{\mathrm{GP}}(x)| \leqslant C g_N^{-\frac{3}{2(s+3)}} e^{-C g_N^{\frac{s+2}{s+3}}(|g_N^{-\frac{1}{s+3}}x|-C')} \qquad \text{for } |x| \geqslant g_N^{\frac{1}{s+3}}R, \tag{2.50}$$

for some $R > 0$ big enough, it follows that

$$\|V_{\mathrm{ext}}\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)} \leqslant C g_N^{\frac{2s-3}{2(s+3)}}. \tag{2.51}$$

Inserting (2.49) and (2.51) in (2.48), we get

$$\|\nabla\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)} \leqslant C g_N^{\frac{2s-3}{2(s+3)}}. \tag{2.52}$$

$\square$

2.2 Bound for $\mathcal{N}_+$

Before proving Theorem 1.1, we need a bound on $\mathcal{N}_+$. This is the content of the next Proposition. The proof is very similar to the one of [28, Lemma 1]. In the proof we will use Lemma 2.2, which is proven below.

Proposition 2.5 (Bound for $\mathcal{N}_+$) *Let v satisfy Assumption 1. Then, as $N \to \infty$, there exists a constant C depending only on v such that*

$$\langle\Psi_0, \mathcal{N}_+\Psi_0\rangle \leqslant C\left(g_N^{\frac{s+5}{s+3}} N^{3\beta} + g_N^{\frac{2(s+1)}{s+3}} N^{1-\beta}\right). \tag{2.53}$$

Proof We first prove an upper bound for $E_0(N)$. As a trial state we choose the one in which all the particles occupy the GP minimizer, i.e., $(\varphi^{\mathrm{GP}})^{\otimes N}$, obtaining

$$E_0(N) \leqslant N\langle\varphi^{\mathrm{GP}}, (-\Delta + V_{\mathrm{ext}})\varphi^{\mathrm{GP}}\rangle + g_N \frac{(N-1)}{2} \int_{\mathbb{R}^3} dx\, (v_N * |\varphi^{\mathrm{GP}}|^2)(x)|\varphi^{\mathrm{GP}}(x)|^2$$

$$\leqslant N\langle\varphi^{\mathrm{GP}}, (-\Delta + V_{\mathrm{ext}})\varphi^{\mathrm{GP}}\rangle + g_N(\smallint v)\frac{N-1}{2} \int_{\mathbb{R}^3} dx\, |\varphi^{\mathrm{GP}}(x)|^4$$

$$+ Cg_N \frac{N-1}{N^\beta}\|\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}$$

$$\leq N E_{\mathrm{GP},N}^{\mathrm{trap}} + Cg_N N^{1-\beta}\|\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}, \tag{2.54}$$

where in the second inequality we used Lemma 2.2.

We now prove the lower bound. For any $m \in \mathbb{N}$ we set $\psi_m(x) := |\varphi^{\mathrm{GP}}|^2(x) - \frac{1}{N}\sum_{j=1}^N \eta_m(x - x_j)$, where $\eta_m(x) \in C_c^\infty(\mathbb{R}^3)$ is a sequence of mollifiers, such that $\|\eta_m\|_2 = 1$ for any $m \in \mathbb{N}$ and $(v_N * \eta_m)(x) \to v_N(x)$ a.e. as $m \to \infty$. We then have

$$0 \leqslant \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \overline{\psi}_m(x)v_N(x-y)\psi_m(y) = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, |\varphi^{\mathrm{GP}}(x)|^2 v_N(x-y)|\varphi^{\mathrm{GP}}(y)|^2$$

$$- \frac{2}{N}\sum_{j=1}^N \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, |\varphi^{\mathrm{GP}}(x)|^2 v_N(x-y)\eta_m(y-x_j)$$

$$+ \frac{1}{N^2}\sum_{j,k=1}^N \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \eta_m(x-x_j)v_N(x-y)\eta_m(y-x_k), \tag{2.55}$$

where the first inequality follows from the fact that $\hat{v} \geqslant 0$. Taking the limit $m \to \infty$, by the monotone convergence theorem, we get

$$
0 \leqslant \int_{\mathbb{R}^3} dx \, (v_N * |\varphi^{\mathrm{GP}}|^2)(x)|\varphi^{\mathrm{GP}}(x)|^2 - \frac{2}{N} \sum_{j=1}^{N} (v_N * |\varphi^{\mathrm{GP}}|^2)(x_j)
$$

$$
+ \frac{1}{N^2} \sum_{j,k=1}^{N} v_N(x_j - x_k), \tag{2.56}
$$

which implies that

$$
\frac{g_N}{N} \sum_{1 \leqslant j < k \leqslant N} v_N(x_j - x_k) \geqslant -\frac{g_N N}{2} \int_{\mathbb{R}^3} dx \, (v_N * |\varphi^{\mathrm{GP}}|^2)(x)|\varphi^{\mathrm{GP}}(x)|^2
$$

$$
+ g_N \sum_{j=1}^{N} (v_N * |\varphi^{\mathrm{GP}}|^2)(x_j) - g_N N^{3\beta} v(0). \tag{2.57}
$$

From (2.57), we then get

$$
H_N^{\mathrm{trap}} \geqslant \sum_{j=1}^{N} \left[-\Delta_j + V_{\mathrm{ext}}(x_j) + g_N(v_N * |\varphi^{\mathrm{GP}}|^2)(x_j) \right]
$$

$$
- \frac{g_N N}{2} \int_{\mathbb{R}^3} dx \, (v_N * |\varphi^{\mathrm{GP}}|^2)(x)|\varphi^{\mathrm{GP}}(x)|^2 - g_N N^{3\beta} v(0).
$$

Now, using Lemma 2.2 twice and exploiting the normalization of φ^{GP}, we can replace both convolutions, obtaining

$$
H_N^{\mathrm{trap}} - N E_{\mathrm{GP},N}^{\mathrm{trap}} \geqslant \sum_{j=1}^{N} (h_j^{\mathrm{GP}} - \mu_{\mathrm{GP}}) - g_N N^{3\beta} v(0) \tag{2.58}
$$

$$
- C g_N N^{1-\beta} \|\varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)} \|\nabla \varphi^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}. \tag{2.59}
$$

Hence, recalling the eigenfunctions $\{\varphi_n\}_{n \in \mathbb{N}}$ of h^{GP} with eigenvalues $\{\mu_n\}_{n \in \mathbb{N}}$ (with $\varphi_0 \equiv \varphi^{\mathrm{GP}}$, $\mu_0 \equiv \mu_{\mathrm{GP}}$), we have that

$$
\sum_{j=1}^{N} (h_j^{\mathrm{GP}} - \mu_{\mathrm{GP}}) = \sum_{j \geqslant 0} (\mu_j - \mu_0)|\varphi_j\rangle\langle\varphi_j| \geqslant (\mu_1 - \mu_0)\mathcal{N}_+ \geqslant 0 \tag{2.60}
$$

that we plug into (2.58), so obtaining in combination with the upper bound (2.54)

$$(\mu_1 - \mu_{\text{GP}})\langle\Psi_0, \mathcal{N}_+\Psi_0\rangle \leqslant g_N N^{3\beta} v(0) + C g_N N^{1-\beta}\|\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}.$$

(2.61)

Applying Propositions 2.1 and 2.4, we finally get

$$C g_N^{-\frac{2}{s+3}}\langle\Psi_0, \mathcal{N}_+\Psi_0\rangle \leqslant (\mu_1 - \mu_{\text{GP}})\langle\Psi_0, \mathcal{N}_+\Psi_0\rangle \leqslant C'\left(g_N N^{3\beta} + g_N^{\frac{2s}{s+3}} N^{1-\beta}\right),$$

(2.62)

for two positive constant C, C'. The result immediately follows. $\qquad\square$

We now prove a Lemma on the comparison between the Hartree and GP interactions.

Lemma 2.2 (Hartree and GP interactions) *Let v satisfy Assumption 1. Then, there exists a constant $C > 0$ such that*

$$\left\|v_N * |\varphi^{\text{GP}}|^2(x) - (\smallint v)\,|\varphi^{\text{GP}}|^2(x)\right\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{C}{N^\beta}\|\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}$$

(2.63)

Proof A direct computation yields

$$\left\|v_N * |\varphi^{\text{GP}}|^2(x) - (\smallint v)\,|\varphi^{\text{GP}}|^2(x)\right\|_{L^\infty(\mathbb{R}^3)}$$

$$= \sup_{x\in\mathbb{R}^3}\left|\int_{\mathbb{R}^3} dy\, v(y)\left(\left|\varphi^{\text{GP}}(x - N^{-\beta}y)\right|^2 - |\varphi^{\text{GP}}|^2(x)\right)\right| \qquad (2.64)$$

$$= \sup_{x\in\mathbb{R}^3}\left|\int_{\mathbb{R}^3} dy\, v(y)\int_0^1 d\lambda\, \frac{2y}{N^\beta}|\varphi^{\text{GP}}(x - \lambda N^{-\beta}y)||\nabla\varphi^{\text{GP}}(x - \lambda N^{-\beta}y)|\right|$$

$$\leqslant \frac{2}{N^\beta}\left(\int_{\mathbb{R}^3} dy\, |y||v(y)|\right)\|\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\varphi^{\text{GP}}\|_{L^\infty(\mathbb{R}^3)}. \qquad (2.65)$$

$\qquad\square$

2.3 Proof of Theorem 1.1

By Proposition 2.5

$$\langle\Psi_0, \mathcal{N}_+\Psi_0\rangle \leqslant C\left(g_N^{\frac{s+5}{s+3}} N^{3\beta} + g_N^{\frac{2(s+1)}{s+3}} N^{1-\beta}\right), \qquad (2.66)$$

so that

$$1 - \langle \varphi^{\mathrm{GP}}, \gamma_{\Psi_0} \varphi^{\mathrm{GP}} \rangle = \frac{1}{N} \left[N - \langle \Psi_0, a^*(\varphi^{\mathrm{GP}}) a(\varphi^{\mathrm{GP}}) \Psi_0 \rangle \right] = \frac{1}{N} \langle \Psi_0, \mathcal{N}_+ \Psi_0 \rangle$$

$$\leqslant C g_N^{\frac{s+5}{s+3}} N^{3\beta - 1} + C g_N^{\frac{2(s+1)}{s+3}} N^{-\beta}, \tag{2.67}$$

which completes the proof.

3 Dynamics

This section is devoted to the proof of Theorem 1.6. We summarize here the main steps.

- First, we use the techniques introduced in [47] to approximate the many-body state at time t, ψ_t, in terms of φ_t^{H}, i.e., in terms of the solution of the nonlinear Hartree equation which reads as

$$i \partial_t \varphi_t^{\mathrm{H}} = \left(-\Delta + g_N v_N * \left| \varphi_t^{\mathrm{H}} \right|^2 \right) \varphi_t^{\mathrm{H}}, \tag{3.1}$$

  where we recall $v_N(x) = N^{3\beta} v(N^{\beta} x)$.
- Afterwards, we estimate the distance between φ_t^{H} and the solution φ_t^{GP} of the GP equation introduced in (1.29), i.e.,

$$i \partial_t \varphi_t^{\mathrm{GP}} = (-\Delta + g_N (\smallint v) |\varphi_t^{\mathrm{GP}}|^2) \varphi_t^{\mathrm{GP}}.$$

In the following we will need to work with the Hartree energy functional, we set here the notation. We define

$$\mathcal{E}_{\mathrm{H}}^{\mathrm{free}}(u) = \int_{\mathbb{R}^3} dx \, |\nabla u(x)|^2 + \frac{g_N}{2} \int_{\mathbb{R}^3} dx \, (v_N * |u|^2)(x) |u(x)|^2. \tag{3.2}$$

3.1 Sobolev Norms of the Solutions

An important role in the proof of Theorem 1.6 is played the L^∞ norms of φ_t^{GP} and φ_t^{H}. We then adapt the proof of [4, Proposition 3.1 (ii)] to our framework to prove a bound for the H^2-norm of φ^{GP} and φ^{H}, from which we deduce the estimates for the L^∞-norms.

Proposition 3.1 (Sobolev bound for $\varphi_t^{\mathrm{H}}, \varphi_t^{\mathrm{GP}}$) *Let φ_t^{GP} and φ_t^{H} be solutions of, respectively, (1.29) and (3.1), both with initial datum $\varphi_0 \in H^n(\mathbb{R}^3)$. Assume that v satisfies Assumption 1.*

(i) *There exists a positive constant $C > 0$ which depends only on the interaction v such that*

$$\sup_{t \in \mathbb{R}} \left\| \varphi_t^{\mathrm{GP}} \right\|_{H^1(\mathbb{R}^3)} \leqslant \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0), \quad \sup_{t \in \mathbb{R}} \| \nabla \varphi_t^{\mathrm{H}} \|_2$$
$$\leqslant C \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + C g_N N^{-\beta} \| \varphi_0 \|_{L^\infty(\mathbb{R}^3)}^2. \tag{3.3}$$

(ii) *There exists a positive constant C such that for any $t \in \mathbb{R}$*

$$\left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} \leq C \, \| \varphi_0 \|_{H^2(\mathbb{R}^3)} \, e^{C g_N^2 [\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 |t|}, \tag{3.4}$$

$$\left\| \varphi_t^{\mathrm{H}} \right\|_{H^2(\mathbb{R}^3)} \leq C \, \| \varphi_0 \|_{H^2(\mathbb{R}^3)} \, e^{C g_N^2 \left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \| \varphi_0 \|_{L^\infty(\mathbb{R}^3)}^4 \right) |t|}. \tag{3.5}$$

In the proof of Proposition 3.1, we use some Strichartz estimates. In particular we use that for any function $f \in L_t^\infty L_x^{\frac{6}{5}}$, one has

$$\sup_{t \in [0,T]} \left\| \int_0^t ds \, e^{i(t-s)\Delta} f(s, \cdot) \right\|_{L^2(\mathbb{R}^3)} \leq \sqrt{T} \sup_{t \in [0,T]} \| f(t, \cdot) \|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{3.6}$$

For a proof of (3.6), see [32, Theorem 1.2].

Proof of Proposition 3.1 We start by showing the bounds in (3.3). The estimate for φ_t^{GP} directly follows from the fact that $\widehat{v} \geqslant 0$ and that the energy is preserved by the GP dynamics. We now look at φ_t^{H}. Notice that, proceeding similarly as in Lemma 2.2 and using Assumption 1 we get

$$|\langle \varphi_0, (v_N * |\varphi_0|^2 - |\varphi_0|^2) \varphi_0 \rangle| \leq C g_N N^{-\beta} \| \nabla \varphi_0 \|_{L^2(\mathbb{R}^3)} \| \varphi_0 \|_{L^\infty(\mathbb{R}^3)}^2. \tag{3.7}$$

Using now Cauchy–Schwartz and the fact that the energy is preserved by the Hartree dynamics, we can write

$$\| \nabla \varphi_t^{\mathrm{H}} \|_{L^2(\mathbb{R}^3)}^2 \leqslant \langle \varphi_0, (-\Delta + \frac{g_N}{2} v_N * |\varphi_0|^2) \varphi_0 \rangle \leqslant \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + C \| \nabla \varphi_0 \|_{L^2(\mathbb{R}^3)}^2$$
$$+ C g_N^2 N^{-2\beta} \| \varphi_0 \|_{L^\infty(\mathbb{R}^3)}^4. \tag{3.8}$$

We then get

$$\sup_{t \in \mathbb{R}} \| \nabla \varphi_t^{\mathrm{H}} \|_2 \leqslant C \sqrt{\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)} + C g_N N^{-\beta} \| \varphi_0 \|_{L^\infty(\mathbb{R}^3)}^2. \tag{3.9}$$

We now prove (3.5). Being φ_t^{GP} the solution of (1.29), we can write

$$\varphi_t^{\mathrm{GP}} = e^{it\Delta} \varphi_{t_0}^{\mathrm{GP}} - i g_N \int_{t_0}^t ds \, e^{i(t-s)\Delta} \left(\left| \varphi_t^{\mathrm{GP}} \right|^2 \varphi_t^{\mathrm{GP}} \right). \tag{3.10}$$

Consider now $\alpha \in \mathbb{N}^3$ with $|\alpha| = 2$; we use the notation $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$. We have

$$\partial^\alpha \varphi_t^{\mathrm{GP}} = e^{it\Delta} \partial^\alpha \varphi_{t_0}^{\mathrm{GP}} - ig_N \int_{t_0}^t ds \, e^{i(t-s)\Delta} \partial^\alpha \left(\left| \varphi_t^{\mathrm{GP}} \right|^2 \varphi_t^{\mathrm{GP}} \right)$$

$$= e^{it\Delta} \partial^\alpha \varphi_{t_0}^{\mathrm{GP}} - ig_N \int_{t_0}^t ds \, e^{i(t-s)\Delta} \sum_{\gamma \leq \beta \leq \alpha} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \left(\partial^{(\alpha-\beta)} \overline{\varphi_t^{\mathrm{GP}}} \partial^{(\beta-\gamma)} \varphi_t^{\mathrm{GP}} \partial^\gamma \varphi_t^{\mathrm{GP}} \right).$$

$$(3.11)$$

By (3.6), we get

$$\sup_{t \in [t_0, t_0+T]} \left\| \partial^\alpha \varphi_t^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \partial^\alpha \varphi_{t_0}^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)}$$

$$+ g_N \sup_{t \in [t_0, t_0+T]} \left\| \int_{t_0}^t ds \, e^{i(t-s)\Delta} \sum_{\gamma \leq \beta \leq \alpha} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \left(\partial^{(\alpha-\beta)} \overline{\varphi_t^{\mathrm{GP}}} \partial^{(\beta-\gamma)} \varphi_t^{\mathrm{GP}} \partial^\gamma \varphi_t^{\mathrm{GP}} \right) \right\|_{L^2(\mathbb{R}^3)}$$

$$\leq \left\| \partial^\alpha \varphi_{t_0}^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)} + 64 g_N \sqrt{T} \sum_{\gamma \leq \beta \leq \alpha} \sup_{t \in [t_0, t_0+T]} \left\| \partial^{(\alpha-\beta)} \varphi_t^{\mathrm{GP}} \partial^{(\beta-\gamma)} \varphi_t^{\mathrm{GP}} \partial^\gamma \varphi_t^{\mathrm{GP}} \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}.$$

$$(3.12)$$

As in [4, Proposition 3.1], via Hölder inequality and Sobolev embeddings, we get that there exists a constant C (which depends only on the Sobolev embeddings) such that

$$\sup_{t \in [t_0, t_0+T]} \left\| \partial^\alpha \varphi_t^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \partial^\alpha \varphi_{t_0}^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)}$$

$$+ C g_N \sqrt{T} \sup_{t \in [t_0, t_0+T]} \left\| \varphi_t^{\mathrm{GP}} \right\|_{H^1(\mathbb{R}^3)}^2 \left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)}.$$

$$(3.13)$$

This means, that up to increasing C and using (3.3), we get that for any $t \in [t_0, t_0 + T]$,

$$\left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} \leq \left\| \varphi_{t_0}^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} + C E g_N \sqrt{T} \sup_{t \in [t_0, t_0+T]} \left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)},$$

where we denoted by $E \equiv \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)$. We can now fix T small enough so that $C E g_N \sqrt{T} = 1/2$, as a consequence we get for any $t \in [t_0, t_0 + T]$

$$\left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} \leq 2 \left\| \varphi_{t_0}^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)}.$$

$$(3.14)$$

Repeating the argument, we get that for any $t > 0$

$$\left\| \varphi_{t+T}^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} \leq 2 \left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)}.$$

$$(3.15)$$

Fix now $t > 0$ and let $k \in \mathbb{N}$ such that $t \in ((k-1)T, kT]$; then

$$
\left\| \varphi_t^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} \leq 2 \left\| \varphi_{(k-1)T}^{\mathrm{GP}} \right\|_{H^2(\mathbb{R}^3)} \leq 2^k \|\varphi_0\|_{H^2(\mathbb{R}^3)} \leq 2^{\frac{t}{T}+1} \|\varphi_0\|_{H^2(\mathbb{R}^3)}
$$
$$
= 2^{4C^2 E^2 g_N^2 t + 1} \|\varphi_0\|_{H^2(\mathbb{R}^3)}, \tag{3.16}
$$

which implies (3.4).

The proof for (3.5) can be done in the same way. Proceeding as above, one gets

$$
\sup_{t \in [t_0, t_0+T]} \left\| \partial^\alpha \varphi_t^{\mathrm{H}} \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \partial^\alpha \varphi_{t_0}^{\mathrm{H}} \right\|_{L^2(\mathbb{R}^3)} + g_N \sqrt{T} \sum_{\gamma \leq \beta \leq \alpha} \binom{\alpha}{\beta} \binom{\beta}{\gamma}
$$
$$
\sup_{t \in [t_0, t_0+T]} \left\| v_N * \left(\partial^{(\alpha-\beta)} \varphi_t^{\mathrm{H}} \partial^{(\beta-\gamma)} \overline{\varphi_t^{\mathrm{H}}} \right) \partial^\gamma \varphi_t^{\mathrm{H}} \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{3.17}
$$

The only difference with respect to before is that to bound the interaction term one has to use Young's inequality for the convolution (see [37, Theorem 4.2]). □

3.2 From the Many-Body Problem to Hartree

In this section we want to compare the many-body problem with the Hartree approximation, the main result is in the proposition below.

Proposition 3.2 (*From Many-Body to Hartree*) *Let* $\beta \in \left(0, \frac{1}{6}\right)$ *and* $\lambda \in (3\beta, 1 - 3\beta)$. *Let* ψ_t *and* φ_t^{H} *be the normalized solution of* (1.24) *and* (3.1) *with initial data* ψ_0 *and* φ_0, *respectively. Suppose that* v *satisfies Assumption* 1. *We have*

$$
\left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\|
$$
$$
\leq \sqrt{2} \left(N^{\frac{1-\lambda}{2}} \left\| |\varphi_0\rangle\langle\varphi_0| - \gamma_{\psi_0}^{(1)} \right\|^{\frac{1}{2}} + N^{\frac{3\beta}{2}} \right) e^{C_v(C_N(\varphi_0,t) + C_N(\varphi_0,t)^2) g_N |t|}, \tag{3.18}
$$

where C_v *is a constant depending on the* L^1 *and the* L^2 *norms of* v, *and* $C_N(\varphi_0, t)$ *is given by*

$$
C_N(\varphi_0, t) = \|\varphi_0\|_{H^2(\mathbb{R}^3)} \, e^{C g_N^2 \left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4 \right) |t|}. \tag{3.19}
$$

The key idea for the proof of Proposition 3.2 is to control the number of *bad particles* in the many-body system (i.e. the particles not in the state φ_t^{H}), we do that using methods introduced in [47].

We now fix the notation for the projectors on the spaces of good and bad particles. We collect and prove the main general properties of these operators in Appendix 1.

Definition 3.1 Let $\varphi \in L^2(\mathbb{R}^3)$ and $\psi \in L_s^2(\mathbb{R}^{3N})$.

1. For any $1 \leq j \leq N$, the projectors $p_j^\varphi : L^2\left(\mathbb{R}^{3N}\right) \to L^2\left(\mathbb{R}^{3N}\right)$ represent the probability of the j-th particle of the state ψ of being in the state φ, i.e.

$$p_j^\varphi \psi\left(x_1, \ldots, x_N\right) := \left(|\varphi\rangle\langle\varphi|\right)_j \psi\left(x_1, \ldots, x_N\right)$$
$$\equiv \varphi\left(x_j\right) \int_{\mathbb{R}^3} dz \, \varphi^*\left(z\right) \psi\left(x_1, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_N\right).$$
(3.20)

Analogously, $q_j^\varphi := 1 - p_j^\varphi$ represents the probability of the j-th particle of the state ψ not being in the state φ.

2. For any $0 \leq k \leq j \leq N$ we set

$$\mathcal{A}_k^j := \left\{ a := \left(a_1, a_2, \ldots, a_j\right) \in \{0, 1\}^j : \sum_{l=1}^{j} a_l = k \right\}$$
(3.21)

and define the orthogonal projector $P_{j,k}^\varphi$ on $L^2\left(\mathbb{R}^{3N}\right)$ as

$$P_{j,k}^\varphi := \sum_{a \in \mathcal{A}_k^j} \prod_{l=N-j+1}^{N} \left[\left(p_l^\varphi\right)^{1-a_l} \left(q_l^\varphi\right)^{a_l} \right].$$
(3.22)

Note that the rank of $P_{j,k}^\varphi$ is the space of states that in the last j particles have exactly k that are not in the state φ (we sum over $\mathcal{A}_k^j$ to make the state symmetric). In the following we denote by P_k^φ the projector given by $P_{N,k}^\varphi$.

3. For any function $f : \{0, \ldots, N\} \to \mathbb{R}$ we define the operator $\widehat{f}^\varphi : L^2\left(\mathbb{R}^{3N}\right) \to L^2\left(\mathbb{R}^{3N}\right)$ as

$$\widehat{f}^\varphi := \sum_{k \in \mathbb{Z}} f\left(k\right) P_k^\varphi.$$
(3.23)

where $f\left(k\right) = 0$ for any $k \in \mathbb{Z} \setminus \{0, \ldots, N\}$.
Similarly, if $f_d\left(k\right) := f\left(k+d\right)$ for any $d, k \in \mathbb{Z}$, we define the operator $\widehat{f}_d^\varphi : L^2\left(\mathbb{R}^{3N}\right) \to L^2\left(\mathbb{R}^{3N}\right)$ associated to f_d as

$$\widehat{f}_d^\varphi := \sum_{k \in \mathbb{Z}} f\left(k+d\right) P_k^\varphi.$$
(3.24)

Throughout the following Section, the hat $\widehat{\cdot}$ will solely be used in the sense of Definition 3.1.

Following [47], we now introduce a functional which encodes the fraction of condensation within itself.

Definition 3.2 For any $\lambda \in (0, 1)$ we define the weight function $\mu^\lambda : \{0, \ldots, N\} \to \mathbb{R}$ as

$$\mu^\lambda (k) := \begin{cases} \frac{k}{N^\lambda}, & \text{for } k \leq N^\lambda \\ 1, & \text{otherwise.} \end{cases}, \tag{3.25}$$

Moreover, for any $N \in \mathbb{N}$, we define the functional $\alpha_N^\lambda : L_s^2 \left(\mathbb{R}^{3N}\right) \times L^2 \left(\mathbb{R}^{3N}\right) \to \mathbb{R}$ as[5]

$$\alpha_N^\lambda (\psi, \varphi) := \left\langle \psi, \widehat{\mu}^{\lambda,\varphi} \psi \right\rangle = \left\| \left(\widehat{\mu}^{\lambda,\varphi}\right)^{1/2} \psi \right\|_{L_s^2(\mathbb{R}^{3N})}^2. \tag{3.26}$$

The proof of Proposition 3.2 is based on some Grönwall estimate on the quantity α_N^λ, which we prove in the next proposition.

Proposition 3.3 *(Grönwall estimate on α_N^λ) Let $\beta \in \left(0, \frac{1}{6}\right)$ and $\lambda \in (3\beta, 1 - 3\beta)$. Let ψ_t and φ_t^{H} be the normalized solution of (1.24) and (3.1) with initial data ψ_0 and φ_0, respectively. Suppose that υ satisfies Assumption 1; we then have*

$$\alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}}\right) \leq \left(\alpha_N^\lambda (\psi_0, \varphi_0) + \frac{1}{N^{\lambda - 3\beta}}\right) e^{C_\upsilon (C_N(\varphi_0, t) + C_N(\varphi_0, t)^2) g_N |t|}, \tag{3.27}$$

where C_υ is a constant depending on the L^1 and the L^2 norms of υ, and $C_N(\varphi_0, t)$ is given by

$$C_N(\varphi_0, t) = \|\varphi_0\|_{H^2(\mathbb{R}^3)} \, e^{C g_N^2 \left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4\right) |t|}. \tag{3.28}$$

3.2.1 Proof of Proposition 3.3

To prove Proposition 3.3 we need some preliminary results which are stated in Lemma 3.1 and in Proposition 3.4. In the proof of these two intermediate results, we use some operator estimates proven in Appendix 1. To simplify notation we will use the following definition.

Definition 3.3 We denote by $U_{j,k}$ the difference between the pair interaction and the mean-field interactions for two particles j and k, i.e.

$$U_{j,k} := (N - 1) \upsilon_N \left(x_j - x_k\right) - N \upsilon_N * |\varphi_t^{\mathrm{H}}|^2 \left(x_j\right) - N \upsilon_N * |\varphi_t^{\mathrm{H}}|^2 \left(x_k\right). \tag{3.29}$$

Lemma 3.1 *Let $\lambda \in (0, 1)$. Let ψ_t and φ_t^{H} be the normalized solution of (1.24) and (3.1), respectively. Then we have,*

$$\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}}\right) = \Gamma_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}}\right), \tag{3.30}$$

---

[5] For ease of notation we will write $\widehat{\mu}^{\lambda,\varphi}$ instead of $\widehat{\mu^{\lambda\,\varphi}}$.

where $\Gamma_N^\lambda : L_s^2 \left(\mathbb{R}^{3N} \right) \times L^2 \left(\mathbb{R}^3 \right) \to \mathbb{R}$ *is defined as*

$$\Gamma_N^\lambda \left(\psi, \varphi \right) := 2 g_N \mathrm{Im} \left(\left\langle \psi, \left(\widehat{\mu}^{\lambda, \varphi} - \widehat{\mu}_1^{\lambda, \varphi} \right) p_1^\varphi p_2^\varphi U_{1,2} p_1^\varphi q_2^\varphi \psi \right\rangle \right)$$

$$+ g_N \mathrm{Im} \left(\left\langle \psi, \left(\widehat{\mu}^{\lambda, \varphi} - \widehat{\mu}_2^{\lambda, \varphi} \right) p_1^\varphi p_2^\varphi U_{1,2} q_1^\varphi q_2^\varphi \psi \right\rangle \right)$$

$$+ 2 g_N \mathrm{Im} \left(\left\langle \psi, \left(\widehat{\mu}^{\lambda, \varphi} - \widehat{\mu}_1^{\lambda, \varphi} \right) p_1^\varphi q_2^\varphi U_{1,2} q_1^\varphi q_2^\varphi \psi \right\rangle \right). \qquad (3.31)$$

Proof First notice that for any $1 \leq j \leq N$ if we differentiate $p_j^{\varphi_t^{\mathrm{H}}}$ we get

$$\partial_t p_j^{\varphi_t^{\mathrm{H}}} = \left(|\varphi_t^{\mathrm{H}}\rangle \langle \partial_t \varphi_t^{\mathrm{H}}| \right)_j + \left(|\partial_t \varphi_t^{\mathrm{H}}\rangle \langle \varphi_t^{\mathrm{H}}| \right)_j = i \left[p_j^{\varphi_t^{\mathrm{H}}}, -\Delta_j + g_N v_N * |\varphi_t^{\mathrm{H}}|^2 (x_j) \right],$$
$$\qquad (3.32)$$

and similarly for $q_j^{\varphi_t^{\mathrm{H}}}$

$$\partial_t q_j^{\varphi_t^{\mathrm{H}}} = i \left[q_j^{\varphi_t^{\mathrm{H}}}, -\Delta_j + g_N v_N * \left| \varphi_t^{\mathrm{H}} \right|^2 (x_j) \right]. \qquad (3.33)$$

It is then useful to introduce H_{H}^φ defined as

$$H_{\mathrm{H}}^\varphi := \sum_{j=1}^N \left[-\Delta_j + g_N v_N * |\varphi|^2 (x_j) \right]. \qquad (3.34)$$

As a consequence, for any $1 \leq j \leq N$, we can write

$$\partial_t p_j^{\varphi_t^{\mathrm{H}}} = i \left[p_j^{\varphi_t^{\mathrm{H}}}, H_{\mathrm{H}}^{\varphi_t^{\mathrm{H}}} \right], \qquad \partial_t q_j^{\varphi_t^{\mathrm{H}}} = i \left[q_j^{\varphi_t^{\mathrm{H}}}, H_{\mathrm{H}}^{\varphi_t^{\mathrm{H}}} \right]. \qquad (3.35)$$

Given that for any $0 \leq k \leq j \leq N$ the operator $P_{j,k}^{\varphi_t^{\mathrm{H}}}$ is just a linear combination of $p_l^{\varphi_t^{\mathrm{H}}}$'s and $q_l^{\varphi_t^{\mathrm{H}}}$'s, we deduce that for any weight $f : \{0, \ldots, N\} \to \mathbb{R}^+$ we get

$$\partial_t \widehat{f}^{\varphi_t^{\mathrm{H}}} = i \left[\partial_t \widehat{f}^{\varphi_t^{\mathrm{H}}}, H_{\mathrm{H}}^{\varphi_t^{\mathrm{H}}} \right]. \qquad (3.36)$$

In order to simplify the notation, for the rest of the proof we will drop the labels φ_t^{H} and λ in the formulas involving the weights and the effective many-body nonlinear operator. If now we differentiate $\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)$, using (3.36) and the symmetry of ψ_t and $\widehat{\mu} \psi_t$, we get

$$\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) = -i \left\langle \psi_t, [\widehat{\mu}, H_N - H_{\mathrm{H}}] \psi_t \right\rangle = -i \frac{g_N}{2} \left\langle \psi_t, \left[\widehat{\mu}, U_{1,2} \right] \psi_t \right\rangle$$

$$= g_N \mathrm{Im} \left(\left\langle \psi_t, \widehat{\mu} \, U_{1,2} \psi_t \right\rangle \right). \qquad (3.37)$$

We want to now decompose the action of the operator $\widehat{\mu}\, U_{1,2}$ in terms of p_1, p_2, q_1 and q_2. To do so, recall that $P_{j,k}$ projects on the space of states in which in the last j particles there are exactly k particles that are not in the state φ_t^{H}; we can then write

$$P_k \equiv P_{N,k} = p_1 p_2 P_{N-2,k} + (p_1 q_2 + q_1 p_2)\, P_{N-2,k-1} + q_1 q_2 P_{N-2,k-2}.$$

As a consequence, for any weight $f : \{0, \ldots, N\} \to \mathbb{R}$ we can use the equality above to get

$$
\begin{aligned}
\widehat{f} &= \sum_{k \in \mathbb{Z}} f(k)\, P_k = \sum_{k \in \mathbb{Z}} f(k) \left(p_1 p_2 P_{N-2,k} + (p_1 q_2 + q_1 p_2)\, P_{N-2,k-1} + P_{N-2,k-2} \right) \\
&\quad - p_1 p_2 \sum_{k \in \mathbb{Z}} f(k+2)\, P_{N-2,k} - (p_1 q_2 + q_1 p_2) \sum_{k \in \mathbb{Z}} f(k+1)\, P_{N-2,k-1} \\
&= p_1 p_2 (\widehat{f} - \widehat{f}_2) + (p_1 q_2 + q_1 p_2)(\widehat{f} - \widehat{f}_1) + \sum_{k \in \mathbb{Z}} f(k)\, P_{N-2,k-2}.
\end{aligned}
\tag{3.38}
$$

We can then insert (3.38) in (3.37) and we get

$$
\begin{aligned}
\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) &= g_N \mathrm{Im} \left(\langle \psi_t, (p_1 p_2\, (\widehat{\mu} - \widehat{\mu}_2) + 2 p_1 q_2\, (\widehat{\mu} - \widehat{\mu}_1))\, U_{1,2} \psi_t \rangle \right) \\
&= \sum_{\substack{\sharp_1 = p_1, q_1 \\ \sharp_2 = p_2, q_2}} \Big[g_N \mathrm{Im} \left(\langle \psi_t, p_1 p_2 (\hat{\mu} - \hat{\mu}_2) U_{1,2} \sharp_1 \sharp_2 \psi_t \rangle \right) \\
&\quad + g_N \mathrm{Im} \left(\langle \psi_t, p_1 q_2 (\hat{\mu} - \hat{\mu}_1) U_{1,2} \sharp_1 \sharp_2 \psi_t \rangle \right) \Big],
\end{aligned}
\tag{3.39}
$$

where we used that the last term in (3.38) is self-adjoint and commutes with $U_{1,2}$. It turns out that many of those terms are either identical or vanishing. In particular, we can use (4.3) to have that $p_1 p_2\, (\widehat{\mu} - \widehat{\mu}_2)\, U_{1,2} p_1 p_2 = \left(p_1 p_2\, (\widehat{\mu} - \widehat{\mu}_2)\, U_{1,2} p_1 p_2 \right)^\dagger$, which implies that $\mathrm{Im} \left(\langle \psi_t, p_1 p_2 (\hat{\mu} - \hat{\mu}_2) U_{1,2} p_1 p_2 \psi_t \rangle \right) = 0$ and similarly we get that $\mathrm{Im} \left(\langle \psi_t, p_1 q_2 (\hat{\mu} - \hat{\mu}_1) U_{1,2} p_1 q_2 \psi_t \rangle \right) = 0$. Moreover, using the symmetry of ψ_t, an easy computation shows that $\langle \psi_t, p_1 q_2 (\hat{\mu} - \hat{\mu}_1) U_{1,2} q_1 p_2 \psi_t \rangle \in \mathbb{R}$, therefore the imaginary part is vanishing. For the remaining terms, we can use again (4.3) and after some manipulations we can combine them to get $g_N \mathrm{Im} \langle \psi_t, p_1 p_2 (\hat{\mu} - \hat{\mu}_1) U_{1,2} p_1 q_2 \psi_t \rangle$. We then obtain

$$
\begin{aligned}
\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) &= 2 g_N \mathrm{Im} \left(\langle \psi_t, p_1 p_2\, (\widehat{\mu} - \widehat{\mu}_1)\, U_{1,2} p_1 q_2 \psi_t \rangle \right) \\
&\quad + g_N \mathrm{Im} \left(\langle \psi_t, p_1 p_2\, (\widehat{\mu} - \widehat{\mu}_2)\, U_{1,2} q_1 q_2 \psi_t \rangle \right) \\
&\quad + 2 g_N \mathrm{Im} \left(\langle \psi_t, p_1 q_2\, (\widehat{\mu} - \widehat{\mu}_1)\, U_{1,2} q_1 q_2 \psi_t \rangle \right),
\end{aligned}
\tag{3.40}
$$

which concludes the proof. $\qquad\square$

The next step is to estimate separately the three terms appearing in $\Gamma_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)$ in terms of $\alpha_N^{\lambda} \lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)$ in order to prove a Grönwall-type estimate for $\alpha_N^\lambda \lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)$. This is the aim of the next Proposition.

Proposition 3.4 *Let* $\lambda \in (0,1)$, *let* ψ_t *and* φ_t *be normalized solutions of (1.24) and (3.1), respectively. The following estimates hold true.*

$$\left| \mathrm{Im} \left(\left\langle \psi_t, \left(\widehat{\mu}_1^{\lambda, \varphi_t^{\mathrm{H}}} - \widehat{\mu}^{\lambda, \varphi_t^{\mathrm{H}}} \right) p_1^{\varphi_t^{\mathrm{H}}} p_2^{\varphi_t^{\mathrm{H}}} U_{1,2} p_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} \psi_t \right\rangle \right) \right|$$

$$\leq \frac{\|v\|_{L^1(\mathbb{R}^3)}}{N^{\frac{1+\lambda}{2}}} \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}^2 \sqrt{\alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)}, \tag{3.41}$$

$$\left| \mathrm{Im} \left(\left\langle \psi_t, \left(\widehat{\mu}_2^{\lambda, \varphi_t^{\mathrm{H}}} - \widehat{\mu}^{\lambda, \varphi_t^{\mathrm{H}}} \right) p_1^{\varphi_t^{\mathrm{H}}} p_2^{\varphi_t^{\mathrm{H}}} U_{1,2} q_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} \psi_t \right\rangle \right) \right| \tag{3.42}$$

$$\leq 2 \max \left\{ \|v\|_{L^1(\mathbb{R})}, \|v\|_{L^2(\mathbb{R})} \right\} \max \left\{ \left\| \varphi_t^{\mathrm{H}} \right\|_{L^4(\mathbb{R}^3)}, \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)} \right\}^2$$

$$\times \left(\alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) + \frac{N^{3\beta - \lambda}}{2} \right), \tag{3.43}$$

$$\left| \mathrm{Im} \left(\left\langle \psi_t, \left(\widehat{\mu}_1^{\lambda, \varphi_t^{\mathrm{H}}} - \widehat{\mu}^{\lambda, \varphi_t^{\mathrm{H}}} \right) p_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} U_{1,2} q_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} \psi_t \right\rangle \right) \right| \tag{3.44}$$

$$\leq \frac{2 \max \left\{ \|v\|_{L^1(\mathbb{R}^3)}, \|v\|_{L^2(\mathbb{R}^3)} \right\}}{N^{\frac{1-\lambda}{2}}} \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}$$

$$\times \left(N^{\frac{3\beta}{2}} + \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)} \right) \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right). \tag{3.45}$$

Proof In order to simplify the notation, during the proof we shall drop the labels λ and φ_t^{H} for short. To start we note that for any function we can rewrite the operator $p_1 v_N (x_1 - x_2) p_1$ in the following more convenient form:

$$p_1 v_N (x_1 - x_2) p_1 = \left(|\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| \right)_1 v_N (x_1 - x_2) \left(|\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| \right)_1$$

$$= \left(v_N * |\varphi_t^{\mathrm{H}}|^2 \right) (x_2) p_1. \tag{3.46}$$

Proof of (3.41). Using the fact $p_j q_j = q_j p_j = 0$ for any $1 \leq j \leq N$, by (3.46), we can rewrite

$$p_1 p_2 U_{1,2} p_1 q_2 = p_1 p_2 \left[(N-1) v_N (x_1 - x_2) - N v_N * |\varphi_t^{\mathrm{H}}|^2 (x_1) - N v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \right] p_1 q_2$$

$$\quad - N p_1 p_2 \, v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \, p_1 q_2$$

$$= - p_1 p_2 \, v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \, p_1 q_2. \tag{3.47}$$

As a consequence, using (4.2) and (4.3), we are able to bound the left-hand term in (3.41) as

$$\left| \mathrm{Im} \left(\langle \psi_t, (\widehat{\mu}_1 - \widehat{\mu}) p_1 p_2 \, v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \, p_1 q_2 \psi_t \rangle \right) \right| \leq \left\| v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \, p_2 \right\| \left\| (\widehat{\mu} - \widehat{\mu}_{-1}) q_2 \psi_t \right\|$$

$$\leq \left\| v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \, p_2 \right\| \left\| (\widehat{\mu} - \widehat{\mu}_{-1}) \widehat{v} \psi_t \right\|. \tag{3.48}$$

We then bound separately these last two terms. To bound the first term, one can easily show that

$$\left\| v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2)\, p_2 \right\|^2 \leq \left\langle \varphi_t^{\mathrm{H}}, \left(v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \right)^2 \varphi_t^{\mathrm{H}} \right\rangle \leq \left\| \varphi_t^{\mathrm{H}} \right\|_{L^6(\mathbb{R}^3)}^2 \left\| v_N * |\varphi_t^{\mathrm{H}}|^2 (x_2) \right\|_{L^3(\mathbb{R}^3)}^2$$

$$\leq \left\| \varphi_t^{\mathrm{H}} \right\|_{L^6(\mathbb{R}^3)}^6 \left\| v \right\|_{L^1(\mathbb{R}^3)}^2 \lesssim \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}^4 \left\| v \right\|_{L^1(\mathbb{R}^3)}^2. \tag{3.49}$$

To bound the second term we want to use the explicit definition of μ. In particular, we have that $|\mu(k) - \mu_{-1}(k)| \lesssim C N^{-\lambda}$. Moreover, for any $k \leq N^\lambda$, we have

$$\left| (\mu(k) - \mu_{-1}(k))^2\, \nu^2(k) \right| \leq \frac{k}{N^{1+2\lambda}} = \frac{1}{N^{1+\lambda}} \mu(k). \tag{3.50}$$

Given that $\mu(k) - \mu_{-1}(k) = 0$ if $k \geq N^\lambda + 1$, we get that

$$\left\| (\widehat{\mu} - \widehat{\mu}_{-1})\, \widehat{\nu} \psi_t \right\| = \left\langle \psi_t, (\widehat{\mu} - \widehat{\mu}_{-1})^2 \widehat{\nu}^2 \psi_t \right\rangle^{\frac{1}{2}} \leq \frac{1}{N^{\frac{1+\lambda}{2}}} \left\langle \psi_t, \widehat{\mu} \psi_t \right\rangle^{\frac{1}{2}}$$

$$= \frac{1}{N^{\frac{1+\lambda}{2}}} \sqrt{\alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)}. \tag{3.51}$$

The estimates (3.49) and (3.51) imply (3.41).

Proof of (3.42). To prove (3.42), we first look directly at the operator $p_1 p_2 U_{1,2} q_1 q_2$. We have

$$p_1 p_2 U_{1,2} q_1 q_2 = (N-1)\, p_1 p_2 v_N (x_1 - x_2)\, q_1 q_2. \tag{3.52}$$

We act similarly as before. In order to do so, notice first that the function $\mu_2 - \mu$ is positive and corresponds still to a weight, so we can consider the square root of the operator $\widehat{\mu}_2 - \widehat{\mu}$ and using (4.3), we get

$$\left| \mathrm{Im} \left(\left\langle \psi_t, \left(\widehat{\mu}_2^{\lambda, \varphi_t^{\mathrm{H}}} - \widehat{\mu}^{\lambda, \varphi_t^{\mathrm{H}}} \right) p_1^{\varphi_t^{\mathrm{H}}} p_2^{\varphi_t^{\mathrm{H}}} U_{1,2} q_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} \psi_t \right\rangle \right) \right|$$

$$\leq (N-1) \left| \left\langle \psi_t, (\widehat{\mu}_2 - \widehat{\mu})^{\frac{1}{2}} p_1 p_2 v_N (x_1 - x_2)\, q_1 q_2 (\widehat{\mu} - \widehat{\mu}_{-2})^{\frac{1}{2}} \psi_t \right\rangle \right|.$$

We now use that $v_N(x_1 - x_2)$ is nonzero only in a small region where $x_1 \approx x_2$, because of the quick decay of v_N. To exploit this fact is then convenient to symmetrize $(N-1)\, v_N(x_1 - x_2)$ replacing it with $\sum_{k=2}^{N} v_N(x_1 - x_k)$ to get, using Cauchy-Schwarz,

$$(N-1) \left| \left\langle \psi_t, (\widehat{\mu}_2 - \widehat{\mu})^{\frac{1}{2}} p_1 p_2 v_N (x_1 - x_2)\, q_1 q_2 (\widehat{\mu} - \widehat{\mu}_{-2})^{\frac{1}{2}} \psi_t \right\rangle \right|$$

$$\leq \left\| \sum_{j=2}^{N} q_j v_N (x_1 - x_j)\, p_1 p_j (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \psi_t \right\| \left\| (\widehat{\mu} - \widehat{\mu}_{-2})^{1/2} q_1 \psi_t \right\|. \tag{3.53}$$

We now bound those terms separately. For the second term, we can again write explicitly the difference $\widehat{\mu} - \widehat{\mu}_{-2}$:

$$\widehat{\mu}(k) - \widehat{\mu}_{-2}(k) = \begin{cases} \frac{2}{N^\lambda} & \text{for } k \leq N^\lambda, \\ \frac{N^\lambda + 2 - k}{N^\lambda} & \text{for } N^\lambda < k \leq N^\lambda + 2, \\ 0 & \text{otherwise}, \end{cases} \tag{3.54}$$

and by (4.2) we then get

$$\left\| (\widehat{\mu} - \widehat{\mu}_{-2})^{1/2} q_1 \psi_t \right\| = \left\langle \psi_t, (\widehat{\mu} - \widehat{\mu}_{-2}) \widehat{v}^2 \psi_t \right\rangle^{\frac{1}{2}} \leq \sqrt{\frac{2}{N}} \langle \psi_t, \widehat{\mu} \psi_t \rangle^{\frac{1}{2}}$$

$$= \sqrt{\frac{2}{N}} \sqrt{\alpha_N^\lambda (\psi_t, \varphi_t^H)}. \tag{3.55}$$

We now take into account the first term in the r.h.s of (3.53). Explicitly, it is

$$\left\| \sum_{j=2}^{N} q_j v_N (x_1 - x_j) p_1 p_j (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \psi_t \right\|^2$$

$$= \sum_{j=2}^{N} \sum_{\ell=2}^{N} \left\langle \psi_t, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} p_1 p_j v_N (x_1 - x_j) q_j q_\ell v_N (x_1 - x_\ell) p_1 p_\ell (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \psi_t \right\rangle, \tag{3.56}$$

and now we bound it as a sum of two different terms, one in which $j = \ell$ and one for the case $j \neq \ell$. For the case $j \neq \ell$, by (4.9) and (4.10), we can write

$$N^2 \left\langle \psi_t, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} p_1 p_2 v_N (x_1 - x_2) q_2 q_3 v_N (x_1 - x_3) p_1 p_3 (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \psi_t \right\rangle$$

$$\leq N^2 \left\| \sqrt{|v_N (x_1 - x_2)|} \sqrt{|v_N (x_1 - x_3)|} p_1 p_3 (\widehat{\mu}_2 - \widehat{\mu})^{1/2} q_2 \psi_t \right\|^2$$

$$\leq N^2 \left\| \sqrt{|v_N (x_1 - x_2)|} p_2 \right\|^4 \left\| (\widehat{\mu}_2 - \widehat{\mu})^{1/2} q_2 \psi_t \right\|^2.$$

Moreover, by (4.10), we get

$$\left\| \sqrt{|v_N (x_1 - x_2)|} p_2 \right\|^4 = \| p_2 |v_N (x_1 - x_2)| p_2 \|^2 \leq \left\| \varphi_t^H \right\|_{L^\infty(\mathbb{R}^3)}^4 \| v \|_{L^1(\mathbb{R}^3)}^2,$$

and on the other hand proceeding similarly as in (3.55) we can estimate

$$\left\| (\widehat{\mu}_2 - \widehat{\mu})^{1/2} q_2 \psi_t \right\|^2 \leq \frac{2}{N} \alpha_N^\lambda (\psi_t, \varphi_t^H). \tag{3.57}$$

We then get

$$\left| N^2 \left\langle \psi_t, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \, p_1 p_2 v_N \, (x_1 - x_2) \, q_2 q_3 v_N \, (x_1 - x_3) \, p_1 p_3 \, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \, \psi_t \right\rangle \right|$$
$$\leq 2N \, \|v\|_{L^1(\mathbb{R}^3)}^2 \, \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}^4 \, \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right). \tag{3.58}$$

We then look at the second term, i.e., the case $j = \ell$, in the right hand side of (3.56): for this term we cannot bring any q close to the state ψ_t, and is therefore not possible to exploit the state ψ_t. We then bound every operator in it directly in norm using (4.11), we get

$$N \left\langle \psi_t, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \, p_1 p_2 v_N \, (x_1 - x_2) \, q_2 v_N \, (x_1 - x_2) \, p_1 p_2 \, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \, \psi_t \right\rangle$$
$$\leq N \, \|v_N \, (x_1 - x_2) \, p_1 p_2\|^2 \, \left\| (\widehat{\mu}_2 - \widehat{\mu})^{\frac{1}{2}} \right\|^2 \leqslant 2N^{1+3\beta-\lambda} \, \left\| \varphi_t^{\mathrm{H}} \right\|_{L^4(\mathbb{R}^3)}^4 \, \|v\|_{L^2(\mathbb{R}^3)}^2. \tag{3.59}$$

Combining (3.58) and (3.59), we can conclude that

$$\left\| \sum_{j=2}^N q_j v_N \, (x_1 - x_j) \, p_1 p_j \, (\widehat{\mu}_2 - \widehat{\mu})^{1/2} \, \psi_t \right\|$$
$$\leq \sqrt{ 2N \, \|v\|_{L^1(\mathbb{R})}^2 \, \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}^4 \, \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) + 2N^{1+3\beta-\lambda} \, \left\| \varphi_t^{\mathrm{H}} \right\|_{L^4(\mathbb{R}^3)}^4 \, \|v\|_{L^2(\mathbb{R}^3)}^2 }. \tag{3.60}$$

Moreover, the estimates (3.55) and (3.60), imply (3.42).
Proof of (3.44). We start the proof of (3.44) by writing

$$\left| \mathrm{Im} \left(\left\langle \psi_t, \left(\widehat{\mu}_1^{\lambda, \varphi_t^{\mathrm{H}}} - \widehat{\mu}^{\lambda, \varphi_t^{\mathrm{H}}} \right) p_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} U_{1,2} q_1^{\varphi_t^{\mathrm{H}}} q_2^{\varphi_t^{\mathrm{H}}} \psi_t \right\rangle \right) \right|$$
$$\leq \left\| (\widehat{\mu}_1 - \widehat{\mu})^{\frac{1}{2}} q_2 \psi_t \right\| \, \|U_{1,2} p_1\| \, \left\| (\widehat{\mu} - \widehat{\mu}_{-1})^{1/2} q_1 q_2 \psi_t \right\|$$
$$\leq \sqrt{\frac{N}{N-1}} \, \left\| (\widehat{\mu}_1 - \widehat{\mu})^{\frac{1}{2}} \widehat{v} \psi_t \right\| \, \|U_{1,2} p_1\| \, \left\| (\widehat{\mu} - \widehat{\mu}_{-1})^{\frac{1}{2}} \widehat{v}^2 \psi_t \right\|. \tag{3.61}$$

Proceeding as above we can estimate directly $(\widehat{\mu}_1 - \widehat{\mu}) \widehat{v}^2 \leq \frac{1}{N} \widehat{\mu}$ and $(\widehat{\mu} - \widehat{\mu}_{-1}) \widehat{v}^4 \leq \frac{1}{N^{2-\lambda}} \widehat{\mu}$.
We now bound the potential term. We have, using (4.9), that

$$\|U_{1,2} p_1\| \leq N \, \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)} \max \left\{ \|v\|_{L^1(\mathbb{R}^3)}, \|v\|_{L^2(\mathbb{R}^3)} \right\} \left(N^{\frac{3\beta}{2}} + \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)} \right). \tag{3.62}$$

Inserting (3.62) in (3.61), we get (3.44). □

Combining Lemma 3.1 and Proposition 3.4, we can prove Proposition 3.3.

Proof of Proposition 3.3 As stated before, the key idea of the proof is to exploit a Grönwall-type argument. In order to do that, we use Lemma 3.1 and Proposition 3.4 to write

$$
\begin{aligned}
\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) &= \Gamma_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) \\
&\leqslant C_v g_N \left(\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)} \left[1 + \frac{1}{N^{\frac{1-\lambda-3\beta}{2}}} \right] + \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2 \left[1 + \frac{1}{N^{\frac{1-\lambda}{2}}} \right] \right) \alpha_N^\lambda(\psi_y, \varphi_t^{\mathrm{H}}) \\
&\quad + C_v g_N \frac{\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2}{N^{\frac{1+\lambda}{2}}} \sqrt{\alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right)} \\
&\quad + \frac{C_v g_N}{N^{\lambda-3\beta}} \left(\left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)} + \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}^2 \right),
\end{aligned}
\tag{3.63}
$$

where C_v is a constant that depends only on the L^1 and L^2 norms of v. We can now do Cauchy-Schwarz in the next to the last term and discard some terms which are subleading for any $\lambda \in (0, 1 - 3\beta)$, to get

$$
\begin{aligned}
\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) &\leqslant C_v g_N \left(\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)} + \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2 \right) \alpha_N^\lambda(\psi_y, \varphi_t^{\mathrm{H}}) \\
&\quad + \frac{C_v g_N}{N^{1+\lambda}} \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2 + \frac{C_v g_N}{N^{\lambda-3\beta}} \left(\left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)} + \left\| \varphi_t^{\mathrm{H}} \right\|_{L^\infty(\mathbb{R}^3)}^2 \right).
\end{aligned}
\tag{3.64}
$$

Being $\lambda \in (3\beta, 1 - 3\beta)$, we get

$$
\partial_t \alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) \leqslant C_v g_N \left(\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)} + \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2 \right) \left(\alpha_N^\lambda(\psi_y, \varphi_t^{\mathrm{H}}) + \frac{1}{N^{\lambda-3\beta}} \right).
\tag{3.65}
$$

To bound $\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}$, we use Proposition 3.1. We find

$$
\begin{aligned}
\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)} &\leqslant \|\varphi_t^{\mathrm{H}}\|_{H^2(\mathbb{R}^3)} \leqslant \|\varphi_0\|_{H^2(\mathbb{R}^3)} \, e^{C g_N^2 \left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4 \right) |t|} \\
&= C_N(\varphi_0, t).
\end{aligned}
\tag{3.66}
$$

By Grönwall, we can then conclude that

$$
\alpha_N^\lambda \left(\psi_t, \varphi_t^{\mathrm{H}} \right) \leq \left(\alpha_N^\lambda \left(\psi_0, \varphi_0 \right) + \frac{1}{N^{\lambda-3\beta}} \right) e^{C_v(C_N(\varphi_0,t) + C_N(\varphi_0,t)^2) g_N |t|}.
\tag{3.67}
$$

$\square$

3.2.2 Proof of Proposition 3.2

The proof of Proposition 3.2 is split in two parts:

- First, we prove that

$$\left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\|^2 \leqslant 2\alpha_N^\lambda(\psi_t, \varphi_t^{\mathrm{H}}). \tag{3.68}$$

- Secondly, we prove that for some $\lambda \in (0, 1)$, for any $\psi \in L_s^2(\mathbb{R}^{3N})$ with $\|\psi\|_{L_s^2(\mathbb{R}^{3N})} = 1$ and for any normalized $\varphi \in L^2(\mathbb{R}^3)$, one has

$$\alpha_N^\lambda(\psi, \varphi) \leqslant N^{1-\lambda} \left\| |\varphi\rangle\langle\varphi| - \gamma_\psi^{(1)} \right\|. \tag{3.69}$$

Combining the estimates in (3.68) and in (3.69) with the result in Proposition 3.3 we get the result.

Proof of Proposition 3.2 We start by proving (3.68). Being $|\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \in \mathfrak{S}_2(L^2(\mathbb{R}^3))$, where with $\mathfrak{S}_2$ we denote the p Schatten space with $p = 2$, we have,

$$\left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\| \leq \left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\|_{\mathfrak{S}_2(L^2(\mathbb{R}^3))}. \tag{3.70}$$

Moreover, using (4.2), we can write

$$\left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\|_{\mathfrak{S}_2(L^2(\mathbb{R}^3))}^2 \leq 2\left(1 - \mathrm{tr}\left[|\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}|\gamma_{\psi_t}^{(1)}\right]\right) = 2\left(1 - \left\|p_1^{\varphi_t^{\mathrm{H}}}\psi_t\right\|_{L_s^2(\mathbb{R}^{3N})}^2\right)$$

$$= 2\left\|q_1^{\varphi_t^{\mathrm{H}}}\psi_t\right\|_{L_s^2(\mathbb{R}^{3N})}^2 = 2\left\|\widehat{\nu}^{\varphi_t^{\mathrm{H}}}\psi_t\right\|_{L_s^2(\mathbb{R}^{3N})}^2. \tag{3.71}$$

Given that $\nu^2 \leq \mu^\lambda$ we can write

$$\left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\| \leq \sqrt{2}\left\|\widehat{\nu}^{\varphi_t^{\mathrm{H}}}\psi_t\right\|_{L_s^2(\mathbb{R}^{3N})} \leq \sqrt{2\alpha_N^\lambda(\psi_t, \varphi_t^{\mathrm{H}})}. \tag{3.72}$$

Consider now $\alpha_N^\lambda(\psi, \varphi) = \langle\psi\,\widehat{\mu}^{\lambda,\varphi}\psi\rangle$, we have

$$\alpha_N^\lambda(\psi, \varphi) \leq N^{1-\lambda}\left\langle\psi, \left(\widehat{\nu}^\varphi\right)^2\psi\right\rangle = N^{1-\lambda}\left\|q_1^\varphi\psi\right\|_{L_s^2(\mathbb{R}^{3N})}^2 \leq N^{1-\lambda}\left\| |\varphi\rangle\langle\varphi| - \gamma_\psi^{(1)} \right\|, \tag{3.73}$$

where we used that $\mu^\lambda(k) \leqslant N^{1-\lambda}\nu^2$ and where the last inequality follows from the definition of q_1^φ. If we now apply (3.69) to $\psi_0 \in L^2(\mathbb{R}^{3N})$ and $\varphi_0 \in L^2(\mathbb{R}^3)$, by (3.68) and Proposition 3.3, we get

$$\left\| |\varphi_t^{\mathrm{H}}\rangle\langle\varphi_t^{\mathrm{H}}| - \gamma_{\psi_t}^{(1)} \right\|^2 \leqslant 2 \left(N^{1-\lambda} \left\| |\varphi_0\rangle\langle\varphi_0| - \gamma_{\psi_0}^{(1)} \right\| + \frac{1}{N^{\lambda-3\beta}} \right) e^{C_v(C_N(\varphi_0,t)+C_N(\varphi_0,t)^2)g_N|t|}.$$

(3.74)

□

3.3 From Hartree to Gross–Pitaevskii

The main goal of this section is to prove the following proposition.

Proposition 3.5 *(From Hartree to Gross–Pitaevskii) Let $\beta \in (0, 1/3)$. Let φ_t^{GP} and φ_t^{H} be the normalized solution of* (1.29) *and* (3.1)*, respectively, both with initial data $\varphi_0 \in L^2(\mathbb{R}^3)$ Suppose that v satisfies Assumption* 1*. We have*

$$\|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)} \leqslant C\sqrt{g_N} \frac{1 + [\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N N^{-\beta}\|\varphi_0\|_{L^\infty}^2}{N^{\frac{\beta}{2}}} e^{C_v C_N(\varphi_0,t)^2 g_N|t|},$$

(3.75)

where C_v is a constant which depends only on v and $C_N(\varphi_0, t)$ is given by

$$C_N(\varphi_0, t) = \|\varphi_0\|_{H^2(\mathbb{R}^3)} e^{Cg_N^2\left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta}\|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4\right)|t|}.$$

(3.76)

Proof We prove (3.75) using Grönwall estimate. To do that we want to bound the following quantity

$$\begin{aligned}
\partial_t \|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2 &= 2\mathrm{Re}\,\langle\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}, \partial_t(\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}})\rangle \\
&= 2g_N(\textstyle\int v)\mathrm{Im}\,\langle\varphi_t^{\mathrm{H}}, (|\varphi_t^{\mathrm{H}}|^2 - |\varphi_t^{\mathrm{GP}}|^2)\varphi_t^{\mathrm{GP}}\rangle \\
&\quad + 2g_N(\textstyle\int v)\mathrm{Im}\,\langle\varphi_t^{\mathrm{GP}}, (|\varphi_t^{\mathrm{H}}|^2 - v_N * |\varphi_t^{\mathrm{H}}|^2)\varphi_t^{\mathrm{H}}\rangle.
\end{aligned}$$

We now estimate the two quantities above separately. For the first one, we have

$$\begin{aligned}
&\left| 2g_N \mathrm{Im}\,\langle\varphi_t^{\mathrm{H}}, (|\varphi_t^{\mathrm{H}}|^2 - |\varphi_t^{\mathrm{GP}}|^2)\varphi_t^{\mathrm{GP}}\rangle \right| \\
&\leqslant Cg_N \|\varphi_t^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}(\|\varphi_t^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)} + \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)})\|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2 \\
&\leqslant Cg_N(\|\varphi_t^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}^2 + \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2)\|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

(3.77)

We now estimate the second one, we have

$$\begin{aligned}
|2g_N \mathrm{Im}\,\langle\varphi_t^{\mathrm{GP}}&, (|\varphi_t^{\mathrm{H}}|^2 - v_N * |\varphi_t^{\mathrm{H}}|^2)\varphi_t^{\mathrm{H}}\rangle| \\
&\leqslant 2g_N \int_{\mathbb{R}^3} dx\, |\varphi_t^{\mathrm{GP}}(x)||\varphi_t^{\mathrm{H}}(x)| \int_{\mathbb{R}^3} dy\, v(y)\left(|\varphi_t^{\mathrm{H}}(x)|^2 - \left|\varphi_t^{\mathrm{H}}\left(x - \frac{y}{N^\beta}\right)\right|^2\right) \\
&= 2g_N \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, v(y)|\varphi_t^{\mathrm{GP}}(x)||\varphi_t^{\mathrm{H}}(x)| \int_0^1 d\lambda \frac{d}{d\lambda}\left|\varphi_t^{\mathrm{H}}\left(x - \frac{\lambda y}{N^\beta}\right)\right|^2
\end{aligned}$$

(3.78)

$$
\leqslant 4 \frac{g_N}{N^\beta} \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \int_0^1 d\lambda \, |y| v(y) |\varphi_t^{\mathrm{GP}}(x)| |\varphi_t^{\mathrm{H}}(x)| \left| \varphi_t^{\mathrm{H}} \left(x - \frac{\lambda y}{N^\beta} \right) \right| \left| \nabla \varphi_t^{\mathrm{H}} \left(x - \frac{\lambda y}{N^\beta} \right) \right|
$$

$$
\leqslant C \frac{g_N}{N^\beta} \|xv\|_{L^1(\mathbb{R}^3)} \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2 (1 + \|\nabla \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2). \tag{3.79}
$$

Then we find

$$
\partial_t \|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2 \leqslant C g_N \left(\|\varphi_t^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)}^2 + \|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)}^2 \right)
$$
$$
\times \left(\|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1 + \|\nabla \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)}^2}{N^\beta} \right). \tag{3.80}
$$

We can bound for any time t the L^∞ norms of φ_t^{GP}, φ_t^{H}, using the bound for the H^2 norms proved in Proposition 3.1, i.e.,

$$
\|\varphi_t^{\mathrm{GP}}\|_{L^\infty(\mathbb{R}^3)} \leqslant \|\varphi_0\|_{H^2(\mathbb{R}^3)} e^{C g_N^2 [\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 |t|}, \tag{3.81}
$$

$$
\|\varphi_t^{\mathrm{H}}\|_{L^\infty(\mathbb{R}^3)} \leqslant C \|\varphi_0\|_{H^2(\mathbb{R}^3)} e^{C g_N^2 \left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4 \right) |t|}. \tag{3.82}
$$

Moreover, by Proposition 3.1, we also can bound uniformly in time

$$
\|\nabla \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)} \leqslant C \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + C g_N N^{-\beta} \|\varphi_0\|_{L^\infty}^2. \tag{3.83}
$$

Thus, by Grönwall, we find

$$
\|\varphi_t^{\mathrm{GP}} - \varphi_t^{\mathrm{H}}\|_{L^2(\mathbb{R}^3)} \leqslant C \sqrt{g_N} \frac{1 + \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + g_N N^{-\beta} \|\varphi_0\|_{L^\infty}^2}{N^{\frac{\beta}{2}}} e^{C_v C_N(\varphi_0,t)^2 g_N |t|}, \tag{3.84}
$$

where C_v is a constant which depends on $\|xv\|_{L^1(\mathbb{R}^3)}$ and $C(\varphi_0)$ is defined as in (3.66), i.e.,

$$
C_N(\varphi_0, t) = \|\varphi_0\|_{H^2(\mathbb{R}^3)} e^{C g_N^2 \left([\mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0)]^2 + g_N^2 N^{-2\beta} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^4 \right) |t|}. \tag{3.85}
$$

$\square$

3.4 Proof of Theorem 1.6

In this last section we combine the results of Proposition 3.3 and Proposition 3.5 to prove (1.31). We start by writing

$$
\left\| \gamma_{\psi_t}^{(1)} - P_{\varphi_t^{\mathrm{GP}}} \right\| \leq \left\| \gamma_{\psi_t}^{(1)} - P_{\varphi_t^{\mathrm{H}}} \right\| + \left\| P_{\varphi_t^{\mathrm{H}}} - P_{\varphi_t^{\mathrm{GP}}} \right\|
$$
$$
\leq \left\| \gamma_{\psi_t}^{(1)} - P_{\varphi_t^{\mathrm{H}}} \right\| + 2 \left\| \varphi_t^{\mathrm{H}} - \varphi_t^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)}. \tag{3.86}
$$

By Proposition 3.2, we get

$$
\left\| \gamma_{\psi_t}^{(1)} - P_{\varphi_t^{\mathrm{H}}} \right\| \leq \sqrt{2} \left(N^{\frac{1-\lambda}{2}} \left\| \gamma_{\psi_0}^{(1)} - P_{\varphi_0} \right\|^{\frac{1}{2}} + \frac{1}{N^{\frac{\lambda-3\beta}{2}}} \right) e^{C_v(C_N(\varphi_0,t)+C_N(\varphi_0,t)^2)g_N|t|}.
$$
(3.87)

On the other hand, from Proposition 3.5 we have

$$
2 \left\| \varphi_t^{\mathrm{H}} - \varphi_t^{\mathrm{GP}} \right\|_{L^2(\mathbb{R}^3)} \leq C \sqrt{g_N} \frac{1 + \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + g_N N^{-\beta} \|\varphi_0\|_{L^\infty}^2}{N^{\frac{\beta}{2}}} e^{C_v C_N(\varphi_0,t)^2 g_N|t|}.
$$
(3.88)

Combining the two previous results we then get

$$
\left\| \gamma_{\psi_t}^{(1)} - P_{\varphi_t^{\mathrm{GP}}} \right\| \leq \sqrt{2} \left(N^{\frac{1-\lambda}{2}} \left\| \gamma_{\psi_0}^{(1)} - P_{\varphi_0} \right\|^{\frac{1}{2}} + \frac{1}{N^{\frac{\lambda-3\beta}{2}}} \right) e^{C_v(C_N(\varphi_0,t)+C_N(\varphi_0,t)^2)g_N|t|}
$$
$$
+ C \sqrt{g_N} \frac{1 + \mathcal{E}_{\mathrm{GP}}^{\mathrm{free}}(\varphi_0) + g_N N^{-\beta} \|\varphi_0\|_{L^\infty}^2}{N^{\frac{\beta}{2}}} e^{C_v C_N(\varphi_0,t)^2 g_N|t|},
$$
(3.89)

where C_v now is a constant which depends on $\|v\|_{L^1(\mathbb{R}^3)}$, $\|v\|_{L^2(\mathbb{R}^3)}$ and $\|xv\|_{L^1(\mathbb{R}^3)}$.

Appendix A: Properties of the projectors p_j^φ, q_j^φ

Proposition A.1 *Let p_j^φ and q_j^φ be defined as in* (3.20).

(i) *Let $v : \{0, 1, \ldots, N\} \to \mathbb{R}$ be given by $v(k) := \sqrt{\frac{k}{N}}$. Then the square of $\widehat{v}^{\varphi}$ is the fraction of particles not in the state φ, i.e.,*

$$(\widehat{v}^{\varphi})^2 = \frac{1}{N} \sum_{j=1}^{N} q_j^{\varphi}. \tag{4.1}$$

(ii) *For any $f : \{0, 1, \ldots, N\} \to \mathbb{R}$ and any symmetric state $\psi \in L_s^2 (\mathbb{R}^{3N})$*

$$\left\| \widehat{f}^{\varphi} q_1^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)} = \left\| \widehat{f}^{\varphi} \widehat{v}^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)},$$

$$\left\| \widehat{f}^{\varphi} q_1^{\varphi} q_2^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)} \leq \sqrt{\frac{N}{N-1}} \left\| \widehat{f}^{\varphi} (\widehat{v}^{\varphi})^2 \psi \right\|_{L_s^2(\mathbb{R}^3)}. \tag{4.2}$$

(iii) *For any $f : \{0, 1, \ldots, N\} \to \mathbb{R}$, $v : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and $j, k = 0, 1, 2$, we have*

$$\widehat{f}^{\varphi} Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} = Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} \widehat{f}_{j-k}^{\varphi}, \tag{4.3}$$

where $Q_0^{\varphi} := p_1^{\varphi} p_2^{\varphi}$, $Q_1^{\varphi} := p_1^{\varphi} q_2^{\varphi}$ and $Q_2^{\varphi} := q_1^{\varphi} q_2^{\varphi}$.

Proof To prove (i) note that $\bigcup_{k=0}^{N} \mathcal{A}_k = \{0, 1\}^N$, so that $1 = \sum_{k=0}^{N} P_k^{\varphi}$. Using also $\left(q_j^{\varphi} \right)^2 = q_j^{\varphi}$ and $q_j^{\varphi} p_j^{\varphi} = 0$, we get

$$q_j^{\varphi} \prod_{l=1}^{N} \left[(p_l^{\varphi})^{1-a_l} (q_l^{\varphi})^{a_l} \right] = \begin{cases} \prod_{l=1}^{N} \left[(p_l^{\varphi})^{1-a_l} (q_l^{\varphi})^{a_l} \right] & \text{if } a_j = 1, \\ 0. & \text{else} \end{cases}$$

$$= a_j \prod_{l=1}^{N} \left[(p_l^{\varphi})^{1-a_l} (q_l^{\varphi})^{a_l} \right]. \tag{4.4}$$

Given that for any $\mathbf{a} \in \mathcal{A}_k$ the sum of the a_j's is k we deduce that

$$\frac{1}{N} \sum_{j=1}^{N} q_j^{\varphi} = \frac{1}{N} \sum_{j=1}^{N} q_j^{\varphi} \sum_{k=0}^{N} P_k^{\varphi} = \frac{1}{N} \sum_{k=0}^{N} \sum_{j=1}^{N} q_j^{\varphi} P_k^{\varphi} = \frac{1}{N} \sum_{k=0}^{N} k P_k^{\varphi} = \sum_{k=0}^{N} (v(k))^2 P_k^{\varphi} \tag{4.5}$$

and (i) follows.

We now prove (4.2). We can use the symmetry of ψ and the previous point to get

$$\left\| \widehat{f}^{\varphi} \widehat{v}^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)}^2 = \left\langle \psi, (\widehat{f}^{\varphi})^2 (\widehat{v}^{\varphi})^2 \psi \right\rangle = \left\langle \psi, (\widehat{f}^{\varphi})^2 q_1^{\varphi} \psi \right\rangle = \left\| \widehat{f}^{\varphi} q_1^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)}^2. \tag{4.6}$$

Similarly, for the second inequality in (4.2), we have

$$\left\| \widehat{f}^{\varphi} \left(\widehat{v}^{\varphi}\right)^2 \psi \right\|_{L_s^2(\mathbb{R}^3)}^2 = \frac{1}{N^2} \sum_{j,k=1}^{N} \left\langle \psi, \left(\widehat{f}^{\varphi}\right)^2 q_j^{\varphi} q_k^{\varphi} \psi \right\rangle = \frac{N-1}{N} \left\| \widehat{f}^{\varphi} q_1^{\varphi} q_2^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)}^2$$
$$+ \frac{1}{N} \left\| \widehat{f}^{\varphi} \widehat{v}^{\varphi} \psi \right\|_{L_s^2(\mathbb{R}^3)}^2. \tag{4.7}$$

To prove (4.3) we use the definition of the Q_j's and the previous points to get

$$\widehat{f}^{\varphi} Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} = \sum_{l \in \mathbb{Z}} f(l) P_l^{\varphi} Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} = \sum_{l \in \mathbb{Z}} f(l) Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} P_{N-2, l-j}^{\varphi}$$
$$= \sum_{l \in \mathbb{Z}} f(l) Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} P_{l-j+k}^{\varphi} = Q_j^{\varphi} v(x_1, x_2) Q_k^{\varphi} \widehat{f}_{j-k}^{\varphi}. \tag{4.8}$$

$\square$

Proposition A.2 *Let p_j^{φ} and q_j^{φ} be defined as in* (3.20). *Let $a, a' \in [1, +\infty]$ such that $1/a + 1/a' = 1$ and let $u \in L^{2a'}\left(\mathbb{R}^3\right)$, $\varphi \in L^{2a}\left(\mathbb{R}^3\right)$. It holds true that*

$$\left\| u(x_1 - x_2) p_1^{\varphi} \right\| \leq \|\varphi\|_{L^{2a}(\mathbb{R}^3)} \|u\|_{L^{2a'}(\mathbb{R}^3)}, \tag{4.9}$$
$$\left\| p_1^{\varphi} u(x_1 - x_2) p_1^{\varphi} \right\| \leq \|\varphi\|_{L^{2a}(\mathbb{R}^3)}^2 \|u\|_{L^{a'}(\mathbb{R}^3)}. \tag{4.10}$$

Moreover, if $a \in [1, 2]$, one has

$$\left\| u(x_1 - x_2) p_1^{\varphi} p_2^{\varphi} \right\| \leq \|\varphi\|_{L^{2a}(\mathbb{R}^3)}^2 \|u\|_{L^{a'}(\mathbb{R}^3)}. \tag{4.11}$$

Proof The estimate (4.9) directly follows from Hölder inequality,

$$\left\| u(x_1 - x_2) p_1^{\varphi} \right\|^2 = \sup_{\|\psi\|=1} \left\| u(x_1 - x_2) p_1^{\varphi} \psi \right\|^2 \leqslant \sup_{\|\psi\|=1} \|\psi\| \sup_{x_2 \in \mathbb{R}^3} \left\langle \varphi, u^2(\cdot - x_2)\varphi \right\rangle$$
$$\leq \|\varphi\|_{L^{2a}(\mathbb{R}^3)}^2 \|u\|_{L^{2a'}(\mathbb{R}^3)}^2. \tag{4.12}$$

To prove (4.10), we use (4.9). Indeed,

$$\left\| p_1^{\varphi} u(x_1 - x_2) p_1^{\varphi} \right\| \leq \left\| \sqrt{|u(x_1 - x_2)|} p_1^{\varphi} \right\|^2 \leq \|\varphi\|_{L^{2a}(\mathbb{R}^3)}^2 \left\| \sqrt{u} \right\|_{L^{2a'}(\mathbb{R}^3)}^2$$
$$\leq \|\varphi\|_{L^{2a}(\mathbb{R}^3)}^2 \|u\|_{L^{a'}(\mathbb{R}^3)}. \tag{4.13}$$

Finally, the estimate (4.11) is a consequence of the Young inequality for the convolution (see [37, Theorem 4.2]). More precisely, we have

$$\|u(x_1 - x_2) p_1 p_2\|^2 = \sup_{\|\psi\|=1} \left\langle \psi, (|\varphi\rangle\langle\varphi|)_1 (|\varphi\rangle\langle\varphi|)_2 u^2(x_1 - x_2) (|\varphi\rangle\langle\varphi|)_1 (|\varphi\rangle\langle\varphi|)_2 \psi \right\rangle$$
$$\tag{4.14}$$

$$\leq \int_{\mathbb{R}^6} dx_1 dx_2 \, u^2 \, (x_1 - x_2) \, |\varphi \, (x_1)|^2 \, |\varphi \, (x_2)|^2 = \left\langle |\varphi|^2 \, , u^2 * |\varphi|^2 \right\rangle$$

$$\leq \|\varphi\|^2_{L^{2p}(\mathbb{R}^3)} \, \|\varphi\|^2_{L^{2q}(\mathbb{R}^3)} \, \|u\|^2_{L^{2r}(\mathbb{R}^3)} \, . \tag{4.15}$$

$\square$

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