Positivity

Topological duals of locally convex function spaces

Teemu Pennanen[1] · Ari-Pekka Perkkiö[2]

Abstract

This paper studies topological duals of locally convex function spaces that are natural generalizations of Fréchet and Banach function spaces. The dual is identified with the direct sum of another function space, a space of purely finitely additive measures and the annihilator of L^∞. This allows for quick proofs of various classical as well as new duality results e.g. in Lebesgue, Musielak–Orlicz, Orlicz–Lorentz space and spaces associated with convex risk measures. Beyond Banach and Fréchet spaces, we obtain completeness and duality results in general paired spaces of random variables.

Keywords Banach function spaces · Topological duals · Finitely additive measures

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1 Introduction

Banach function spaces (BFS) provide a convenient set up for functional analysis in spaces of measurable functions. Many well known properties of e.g. Lebesgue spaces and Orlicz spaces extend to BFS with minor modifications; see e.g. [1, 20, 22, 38]. Extensions to Fréchet function spaces have been studied e.g. in [4]. This paper studies topological duals of more general locally convex function spaces where the topology is generated by an arbitrary collection of seminorms satisfying the usual BFS axioms.

Building on the classical result of Yosida and Hewitt [37,Section 2] on the dual of L^∞, we identify the topological dual as the direct sum of another space of random variables (Köthe dual), a space of purely finitely additive measures and the annihilator

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of L^∞. The last two components have a singularity property that has been found useful, e.g., in the analysis of convex integral functionals by Rockafellar [32] in the case of L^∞ and by Kozek [19] in the case of Orlicz spaces. In the case of L^∞, the last component in the dual vanishes while in other Orlicz spaces, the second one vanishes; see [28,Chapter IV]. Our result thus unifies the two seemingly complementary cases.

The main result is illustrated first by simple derivations of various existing as well as new duality results in Musielak–Orlicz, Marcinkiewicz, Lorentz and Orlicz–Lorentz spaces. We go beyond the existing BFS settings by identifying topological duals of the space of random variables with finite moments, generalized Musielak–Orlicz spaces as well as spaces of random variables associated with convex risk measures. The last instance has attracted attention in the recent literature of mathematical finance and insurance; see e.g. [21, 27] and [11].

Section 5 is concerned with dual pairs of linear spaces of random variables that play a central role e.g. in stochastic optimization and optimal control; see e.g. [36] and the examples in [33]. We show that the corresponding Mackey and strong topologies are generated by (uncountable) collections of seminorms satisfying the usual BFS axioms. We then obtain duality and completeness results as corollaries of the main results of the paper.

The rest of the paper is organized as follows. Section 2 reviews the duality theory for L^∞. Section 3 extends the notion of an integral with respect to a finitely additive measure to measurable not necessarily bounded random variables. Section 4 defines a general locally convex space of random variables and gives the main result of the paper by characterizing the topological dual of a space. Section 5 studies spaces of random variables in separating duality. Section 6 applies the main result to characterize the topological dual in various known and new settings.

2 Topological dual of L^∞

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a σ-algebra $\mathcal{F}$ and a countably additive probability measure P. This section reviews the topological dual of the Banach space L^∞ of equivalence classes of essentially bounded measurable functions on a probability space $(\Omega, \mathcal{F}, P)$. We consider $\mathbb{R}^n$-valued functions and endow L^∞ with the norm

$$\|u\|_{L^\infty} := |(\|u_1\|_{L^\infty}, \ldots, \|u_n\|_{L^\infty})|,$$

where $|\cdot|$ is a norm on $\mathbb{R}^n$. The dual norm on $\mathbb{R}^n$ is denoted by $|\cdot|^*$.

Let $\mathcal{M}$ be the set of P-absolutely continuous finitely additive $\mathbb{R}^n$-valued measures on $(\Omega, \mathcal{F})$ and let $\mathcal{M}_s$ be set of those $m \in \mathcal{M}$ which are *singular* ("purely finitely additive" in the terminology of [37]; see [37,Theorem 1.22]) in the sense that there is a decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$ and $|m|^*(\Omega \setminus A^\nu) = 0$. Given $m \in \mathcal{M}$, the set function $|m|^* : \mathcal{F} \to \mathbb{R}$ is defined by

$$|m|^*(A) := |m^+(A) + m^-(A)|^*,$$

where ith components of $m^+ \in \mathcal{M}$ and $m^- \in \mathcal{M}$ are the positive and negative parts, respectively, of the ith component m_i of m; see [37,Theorem 1.12].

Recall that the space $\mathcal{E}$ of $\mathbb{R}^n$-valued simple random variables (i.e. piecewise constant with a finite range) is dense in L^∞. Given $m \in \mathcal{M}$, the integral of a $u \in \mathcal{E}$ is defined by

$$\int_\Omega u\,dm := \sum_{j=1}^J \alpha^j m(A^j),$$

where $A^j \in \mathcal{F}$ and $\alpha^j \in \mathbb{R}^n$, $j = 1, \ldots, m$ are such that $u = \sum_{j=1}^m \alpha^j 1_{A^j}$ On L^∞, the integral is defined as the unique norm continuous linear extension from $\mathcal{E}$ to L^∞.

The following is from [37,Theorem 2.3] except that we do not assume that the underlying measure space is complete; see also [2,Sections 4.7 and 10.2]. The proof uses [9,Theorem 20.35] which does not rely on the completeness but identifies the dual of L^∞ with the space of finitely additive measures that are absolutely continuous with respect to P. Results of [37,Section 1] on decomposition of finitely additive measures then complete the proof. The above are concerned with real-valued random variables but the extension to the vector-valued case is straightforward; see [35,Lemma 1] for an extension to Banach space-valued random variables. Throughout this paper, the expectation of a random variable $z \in L^1$ is denoted by $E[z]$. The inner product of two vectors $\xi, \eta \in \mathbb{R}^n$ is denoted by $\xi \cdot \eta$.

Theorem 1 (Yosida–Hewitt) *The topological dual $(L^\infty)^*$ of L^∞ can be identified with $\mathcal{M}$ in the sense that for every $u^* \in (L^\infty)^*$ there exists a unique $m \in \mathcal{M}$ such that*

$$\langle u, u^* \rangle = \int_\Omega u\,dm,$$

where the integral is defined componentwise. The dual norm is given by

$$\|m\|_{L^\infty}^* = |m|^*(\Omega).$$

Moreover, $\mathcal{M} = L^1 \oplus \mathcal{M}_s$ in the sense that for every $m \in \mathcal{M}$ there exist unique $y \in L^1$ and $m^s \in \mathcal{M}_s$ such that

$$\int_\Omega u\,dm = E[u \cdot y] + \int_\Omega u\,dm^s.$$

We have $m^s = 0$ if and only if $\langle u1_{A^\nu}, u^* \rangle \to 0$ for every $u \in L^\infty$ and every decreasing $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ such that $P(A^\nu) \searrow 0$.

Proof Assume first that $n = 1$. By [9,Theorem 20.35], the dual of L^∞ can be identified with the linear space of finitely additive P-absolutely continuous measures m in the sense that every $u^* \in (L^\infty)^*$ can be expressed as

$$\langle u, u^* \rangle = \int_\Omega u\,dm$$

and, conversely, any such integral belongs to $(L^\infty)^*$. By [37,Theorem 1.24], there is a unique decomposition $m = m^a + m^s$, where m^a is countably additive and m^s is purely finitely additive. The construction in [37] also shows that m^a and m^s are absolutely continuous with respect to m and thus, absolutely continuous with respect to P as well. By [37,Theorem 1.22], there is a decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ such that $P(A^\nu) \searrow 0$ and $m^s(\Omega \setminus A^\nu) = 0$. The functional $y^s \in (L^\infty)^*$ given by

$$\langle u, y^s \rangle := \int_\Omega u \, dm^s$$

then has the property in the statement. By Radon-Nikodym, there exists a $y \in L^1$ such that

$$\langle u, u^* \rangle := E[u \cdot y] + \int_\Omega u \, dm^s.$$

To prove the last claim, it is clear that the given condition holds if $m^s = 0$. To prove the converse, let $u^* \in (L^\infty)^*$ and consider the representation in terms of $y \in L^1$ and $m^s \in \mathcal{M}_s$ given by the second claim. Let A^ν be the sets in the characterization of the singularity of m^s. By [37,Theorems 1.12 and 1.17], $m^s = m^{s+} - m^{s-}$ for nonnegative purely finitely additive m^{s+} and m^{s-}. Given $\epsilon > 0$, [37,Theorem 1.21] gives the existence of $A \in \mathcal{F}$ such that $m^{s+}(\Omega \setminus A) < \epsilon$ and $m^{s-}(A) < \epsilon$. We have

$$\langle u 1_A 1_{A^\nu}, u^* \rangle = E[1_A 1_{A^\nu} u \cdot y] + m^s(A \cap A^\nu) \to m^s(A) > m^{s+}(\Omega) - 2\epsilon.$$

Under the given condition, the left side converges to zero. Since $\epsilon > 0$ was arbitrary, $m^{s+} = 0$. By symmetry, $m^{s-} = 0$ so $m^s = 0$.

By [37,Theorem 2.3], the dual norm of $\| \cdot \|_{L^\infty}$ is given by $\|m\|_{TV} := m^+(\Omega) + m^-(\Omega)$. This completes the proof of the case $n = 1$. The general case follows from the fact that the dual of a Cartesian product of Banach spaces is the Cartesian product of the dual spaces with the norm

$$\|u\|_{L^\infty}^* = |(\|m_1\|_{TV}, \dots, \|m_n\|_{TV})|^*,$$

which completes the proof. $\square$

3 Extension of the integral

In [37] and in Sect. 2, integrals with respect to an $m \in \mathcal{M}$ were defined only for elements of L^∞ as norm-continuous extensions of integrals of simple functions. Weakening the topology, it is possible to extend the definition of the integral to a larger space of measurable functions using Daniell's construction much as in [3,Chapter II] which considered countably additive integrals of arbitrary (not necessarily $\mathcal{F}$-measurable) functions.

Another approach to integration of unbounded functions with respect to finitely additive measures is that of Dunford; see Dunford and Schwartz [6] or Luxemburg [23].

A benefit of the Daniell extension adopted here is that it gives rise to a simpler definition of integrability that is easier to verify for larger classes of measurable functions.

Given $m \in \mathcal{M}$, we define $\rho_m : L^0 \to \overline{\mathbb{R}}$ by

$$\rho_m(u) := \sup_{u' \in L^\infty} \left\{ \int_\Omega u' dm \ \middle| \ |u'_j| \le |u_j| \ \forall j = 1, \ldots, n \right\}$$

$$= \sum_{j=1}^n \rho_{m_j}(u_j),$$

where

$$\rho_{m_j}(u_j) = \sup_{u' \in L^\infty(\mathbb{R})} \left\{ \int_\Omega u' dm_j \ | \ |u'| \le |u_j| \right\}.$$

We denote

$$\text{dom } \rho_m := \{ u \in L^0 \ | \ \rho_m(u) < \infty \}.$$

The extension of the one-dimensional integral in Theorem 29 gives the following.

Theorem 2 *Given $m \in \mathcal{M}$, the integral*

$$u \mapsto \int_\Omega u \, dm$$

has a unique linear extension from L^∞ to dom ρ_m such that

$$\left| \int_\Omega u \, dm \right| \le \rho_m(u) \quad \forall u \in \text{dom } \rho_m.$$

If m is purely finitely additive, there exists a decreasing $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$ and $\int_\Omega u 1_{\Omega \setminus A^\nu} dm = 0$ for all $u \in \text{dom } \rho_m$.

Proof The extension is given by

$$\int_\Omega u \, dm := \sum_{j=1}^m \int_\Omega u_j dm_j,$$

where the integrals on the right are the extensions of the one-dimensional integrals given in Theorem 29. We get

$$\left| \int_\Omega u \, dm \right| \le \sum_{j=1}^m \left| \int_\Omega u_j dm_j \right| \le \sum_{j=1}^m \rho_{m_j}(u_j) = \rho_m(u),$$

where the second inequality comes from Theorem 29. The sets A^ν can be taken as the unions of the componentwise sets given by Theorem 29.   □

We call the extension in Theorem 2 the *m-integral* of *u* and denote it by

$$\int_\Omega u \, dm.$$

The elements of dom ρ_m will be said to be *m-integrable*. If *m* is countably additive, then, by e.g. [34,Theorem 14.60],

$$\rho_m(u) = \sum_{j=1}^n \int_\Omega |u_j| d|m_j| = \sum_{j=1}^n E[|u_j||y_j|],$$

where *y* is the density of *m*, and thus,

$$\text{dom } \rho_m = \{u \mid u_j \in L^1(\Omega, \mathcal{F}, |m_j|) \; \forall j = 1, \ldots, n\}.$$

In this case, the integral is the Lebesgue integral.

4 Topological duals of spaces of random variables

This section contains the main result of the paper. The setup extends that of Banach function spaces by replacing the norm by an arbitrary collection of seminorms thus covering more general locally convex spaces of random variables. The main result identifies the topological dual of the space with the direct sum of a space of random variables and two spaces of singular functionals, the first of which is represented by finitely additive measures while the second is the orthogonal complement of L^∞.

Let L^0 be the linear space of $\mathbb{R}^n$-valued random variables. Let $\mathcal{P}$ be a collection of sublinear (i.e. convex and positively homogeneous) functions $p : L^0 \to \overline{\mathbb{R}}$ with $p(u) = p(-u)$ for all $u \in L^0$, define

$$L_\mathcal{P} := \bigcap_{p \in \mathcal{P}} \text{dom } p,$$

and endow $L_\mathcal{P}$ with the locally convex topology generated by $\mathcal{P}$. Recall that dom $p := \{u \in L^0 \mid p(u) < \infty\}$. Our aim is to characterize the topological dual $L_\mathcal{P}^*$ of $L_\mathcal{P}$. To this end, we will assume that, for each $p \in \mathcal{P}$,

(A1) there exists a constant $c > 0$ such that $p(u) \le c\|u\|_{L^\infty}$ for all $u \in L^\infty$,
(A2) $p(u') \le p(u)$ for every $u', u \in L^0$ with $|u'_j| \le |u_j| \; \forall j = 1, \ldots, n$ almost surely.

Occasionally, we will also assume the following

(A3) $p(u1_{A^\nu}) \searrow 0$ for all $u \in L^\infty$ and decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$.
(A4) $p(u1_{A^\nu}) \searrow 0$ for all $u \in L_\mathcal{P}$ and decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$.

It is clear that (A3) and (A4) are implied by the following

$p(u^\nu) \searrow 0$ for all $(u^\nu) \in L^\infty$ such that $|u^\nu| \searrow 0$,
$p(u^\nu) \searrow 0$ for all $(u^\nu) \in L_\mathcal{P}$ such that $|u^\nu| \searrow 0$,

respectively. The case where $\mathcal{P}$ is a singleton has been extensively studied ever since the publication of [22]; see e.g. the monographs [1, 20, 38, 39]. When $\mathcal{P}$ is a singleton satisfying merely (A2), $L^\mathcal{P}$ is usually called a *Banach function space* provided it is complete. A sufficient condition for completeness in the general case is given in Remark 9 below.

Our approach relies on (A1) which is assumed also e.g. in [1] but avoided in many other treatments of Banach function spaces. The assumption allows for simple proofs building on Theorem 1 and it provides more information on the structure of the topological dual; see Theorem 6 below. Under a mild condition, spaces satisfying (A2) alone are isomorphic to spaces satisfying both (A1) and (A2).

Remark 3 Let $\tilde{\mathcal{P}}$ be a collection of seminorms satisfying (A2) but not necessarily (A1). If there exists $\bar{u} \in L_{\tilde{\mathcal{P}}}$ such that $\bar{u}_j > 0$ almost surely for all j, then $L_{\tilde{\mathcal{P}}}$ is linearly isomorphic to a space $L_\mathcal{P}$ satisfying (A1) and (A2). The elements of $L^*_{\tilde{\mathcal{P}}}$ can thus be expressed as

$$\tilde{u} \mapsto \langle i(\tilde{u}), u^* \rangle,$$

where i is the isomorphism and $u^* \in L^*_\mathcal{P}$.

Proof Define $i : L^0 \to L^0$ by $i(u)_j := u_j / \bar{u}_j$ and, for every $\tilde{p} \in \tilde{\mathcal{P}}$, let $p(u) := \tilde{p}(i^{-1}(u))$. By (A2),

$$p(u) = \tilde{p}((\bar{u}_j u_j)_{j=1}^n) \leq \tilde{p}((\|u_j\|_{L^\infty} \bar{u}_j)_{j=1}^n) \leq \tilde{p}(\|u\|_{L^\infty} \bar{u}) = \tilde{p}(\bar{u}) \|u\|_{L^\infty},$$

so $\mathcal{P} := \{\tilde{p} \circ i^{-1} \mid \tilde{p} \in \tilde{\mathcal{P}}\}$ satisfies (A1) and (A2). The restriction of A to $L_{\tilde{\mathcal{P}}}$ is an isomorphism between $L_{\tilde{\mathcal{P}}}$ and $L_\mathcal{P}$. $\square$

A seminorm p is said to be *rearrangement invariant* (or symmetric) if $p(u) = p(\tilde{u})$ whenever u and $\tilde{u}$ have the same distribution in the sense that $P(|u| > \tau) = P(|\tilde{u}| > \tau)$ for all $\tau \geq 0$.

Remark 4 Consider the scalar case $n = 1$. Given $p \in \mathcal{P}$, let

$$\hat{\phi}_p(t) := \sup_{A \in \mathcal{F}} \{p(1_A) \mid P(A) \leq t\} \quad \text{and} \quad \check{\phi}_p(t) := \inf_{A \in \mathcal{F}} \{p(1_A) \mid P(A) \geq t\}.$$

Under (A2), condition (A3) is equivalent to $\lim_{t \searrow 0} \hat{\phi}_p(t) = 0$. If $\lim_{t \searrow 0} \check{\phi}_p(t) > 0$, then $\text{dom } p = L^\infty$. If p is rearrangement invariant, then, for any $A \in \mathcal{F}$ with $P(A) = t$,

$$\hat{\phi}_p(t) = \check{\phi}_p(t) = p(1_A),$$

where the common value is known as the *fundamental function*; see e.g. [1]. In this case, $L_\mathcal{P} = L^\infty$ unless (A3) holds.

Proof Assuming (A3), let $t^\nu \searrow 0$. There exists $(A^\nu)_{\nu=1}^\infty$ such that $P(A^\nu) \le t^\nu$ and $\hat{\phi}_p(t^\nu) \le p(1_{A^\nu}) + 1/\nu$. Passing to a subsequence if necessary, $1_{A^\nu} \to 0$ almost surely. Defining $\hat{A}^\nu := \bigcup_{\nu' \ge \nu} A^\nu$, $(\hat{A}^\nu)_{\nu=1}^\infty$ is decreasing with $A^\nu \subset \hat{A}^\nu$ and $P(\hat{A}^\nu) \searrow 0$, so, by (A3)

$$\hat{\phi}(t^\nu) \le p(1_{\hat{A}^\nu}) + 1/\nu \searrow 0.$$

For the converse, let $u \in L^\infty$ and $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $t^\nu := P(A^\nu) \searrow 0$. By (A2),

$$p(u1_{A^\nu}) \le p(\|u\|_{L^\infty} 1_{A^\nu}) \le \|u\|_{L^\infty} \hat{\phi}(t^\nu) \searrow 0.$$

If $\inf_{t>0} \check{\phi}_p(t) > 0$, then

$$p(u) \ge p(\nu 1_{|u| \ge \nu}) \ge \nu \check{\phi}_p(P(|u| \ge \nu)) \ge \nu \inf_{t>0} \check{\phi}_p(t) \quad \forall \nu$$

unless $u \in L^\infty$.   □

For each $p \in \mathcal{P}$, we define a sublinear symmetric function p° on the space $\mathcal{M}$ of P-absolutely continuous finitely additive measures by

$$p^\circ(m) := \sup_{u \in L^\infty} \left\{ \int_\Omega u \, dm \mid p(u) \le 1 \right\}.$$

The following shows, in particular, that every $m \in \operatorname{dom} p^\circ$ decomposes uniquely into the sum of countably additive and singular measures both in $\operatorname{dom} p^\circ$.

Lemma 5 *Assume that $p \in \mathcal{P}$ satisfies (A2) and let $m \in \operatorname{dom} p^\circ$. Every $u \in \operatorname{dom} p$ is m-integrable and*

$$\int_\Omega u \, dm \le p(u) p^\circ(m).$$

There exist unique $y \in L^1 \cap \operatorname{dom} p^\circ$ and $m^s \in \mathcal{M}_s \cap \operatorname{dom} p^\circ$ such that

$$\int_\Omega u \, dm = E[u \cdot y] + \int_\Omega u \, dm^s \quad \forall u \in \operatorname{dom} p.$$

Given $m^s \in \mathcal{M}_s \cap \operatorname{dom} p^\circ$, there exists a decreasing $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ such that $P(A^\nu) \searrow 0$ and

$$\int u 1_{\Omega \setminus A^\nu} \, dm^s = 0 \quad \forall u \in \operatorname{dom} p.$$

Under (A3), $\mathcal{M}_s \cap \operatorname{dom} p^\circ = \{0\}$.

Proof By (A2),

$$\rho_m(u) = \sup_{u' \in L^\infty} \left\{ \int_\Omega u' dm \mid |u'_j| \le |u_j| \ j = 1, \dots, n \right\}$$

$$\le \sup_{u' \in L^\infty} \left\{ \int_\Omega u' dm \mid p(u') \le p(u) \right\} = p(u) p^\circ(m),$$

so every $u \in \mathrm{dom}\, p$ is m-integrable. Theorem 2 now gives

$$\left| \int_\Omega u dm \right| \le p(u) p^\circ(m)$$

and the existence of the sequence $(A^\nu)_{\nu=1}^\infty$ for very $m^s \in \mathcal{M}_s \cap \mathrm{dom}\, p^\circ$.

By Theorem 1, there exist unique $y \in L^1$ and $m^s \in (L^\infty)^s$ such that $m = yP + m^s$. Let $\alpha < p^\circ(y)$ and $\alpha^s < p^\circ(m^s)$ and $u, u^s \in L^\infty$ such that $p(u), p(u^s) \le 1$ and

$$\int_\Omega u y dP \ge \alpha \quad \text{and} \quad \int_\Omega u^s dm^s \ge \alpha^s.$$

Let $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ be decreasing with $P(A^\nu) \searrow 0$ and $m^s(\Omega \setminus A^\nu) = 0$ and let $u^\nu = \lambda u 1_{\Omega \setminus A^\nu} + (1 - \lambda) u^s 1_{A^\nu}$, where $\lambda \in (0, 1)$. By convexity and (A2),

$$p(u^\nu) \le \lambda p(u 1_{\Omega \setminus A^\nu}) + (1 - \lambda) p(u^s 1_{A^\nu}) \le \lambda p(u) + (1 - \lambda) p(u^s) \le 1$$

while

$$\limsup \int_\Omega u^\nu dm \ge \lambda \alpha + (1 - \lambda) \alpha^s.$$

Thus, $p^\circ(m) \ge \lambda \alpha + (1 - \lambda) \alpha^s$. Since $\alpha < p^\circ(y)$ and $\alpha^s < p^\circ(m^s)$ were arbitrary, $p^\circ(m) \ge \lambda p^\circ(y) + (1 - \lambda) p^\circ(m^s)$. Since $\lambda \in (0, 1)$ was arbitrary, we get $p^\circ(y) \le p^\circ(m)$ and $p^\circ(m^s) \le p^\circ(m)$. Thus, $y \in \mathrm{dom}\, p^\circ$ and $m^s \in \mathrm{dom}\, p^\circ$.

To prove the last claim, let $m^s \in \mathcal{M}_s \cap \mathrm{dom}\, p^\circ$. By the first claim,

$$\int_\Omega u 1_A dm^s \le p(u 1_A) p^\circ(m^s) \quad \forall u \in L^\infty, A \in \mathcal{F}$$

so, by the last claim of Theorem 1, condition (A3) implies $m^s = 0$. □

Let

$$\mathcal{M}^{\mathcal{P}^\circ} := \bigcup_{p \in \mathcal{P}} \mathrm{dom}\, p^\circ,$$

the set of P-absolutely continuous finitely additive measures $m \in \mathcal{M}$ such that $p^\circ(m) < \infty$ for some $p \in \mathcal{P}$. The set of purely finitely additive elements of $\mathcal{M}^{\mathcal{P}^\circ}$ will be denoted by $\mathcal{M}_s^{\mathcal{P}^\circ}$. The set of densities $y = dm/dP$ of countably additive

$m \in \mathcal{M}^{\mathcal{P}^\circ}$ will be denoted by $L^{\mathcal{P}^\circ}$. By Lemma 5, $\mathcal{M}^{\mathcal{P}^\circ} = L^{\mathcal{P}^\circ} \oplus \mathcal{M}_s^{\mathcal{P}^\circ}$. In the setting of Banach function spaces where $\mathcal{P}$ is a singleton, $L^{\mathcal{P}^\circ}$ is often called the "associate space" or the Köthe dual of $L_{\mathcal{P}}$; see e.g. [1, 8, 39] and Lemma 8 below.

The following is the main result of this section. It identifies the topological dual $L_{\mathcal{P}}^*$ of $L_{\mathcal{P}}$ with the direct sum of $\mathcal{M}^{\mathcal{P}^\circ} = L^{\mathcal{P}^\circ} \oplus \mathcal{M}_s^{\mathcal{P}^\circ}$ and the annihilator

$$(L^\infty)^\perp := \{w \in L_{\mathcal{P}}^* \mid \langle u, w \rangle = 0 \; \forall u \in L^\infty\}$$

of L^∞.

Theorem 6 *We have*

$$L_{\mathcal{P}}^* = L^{\mathcal{P}^\circ} \oplus \mathcal{M}_s^{\mathcal{P}^\circ} \oplus (L^\infty)^\perp$$

in the sense that for every $u^* \in L_{\mathcal{P}}^*$ there exist unique $y \in L^{\mathcal{P}^\circ}$, $m^s \in \mathcal{M}_s^{\mathcal{P}^\circ}$ and $w \in (L^\infty)^\perp$ such that

$$\langle u, u^* \rangle = E[u \cdot y] + \int_\Omega u \, dm^s + \langle u, w \rangle.$$

For every $u \in L_{\mathcal{P}}$ and $m \in \mathcal{M}^{\mathcal{P}^\circ}$,

$$\int_\Omega u \, dm \le p(u) p^\circ(m).$$

In particular, restricted to $\mathcal{M}^{\mathcal{P}^\circ}$, p° coincides with the polar seminorm of p, i.e.

$$p^\circ(m) = \sup_{u \in L_{\mathcal{P}}} \{\langle u, m \rangle \mid p(u) \le 1\}.$$

Given $w \in (L^\infty)^\perp$ and $u \in L_{\mathcal{P}}$, there exists a decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$ and

$$\langle u, w \rangle = \langle u 1_{A^\nu}, w \rangle \quad \forall \nu = 1, 2, \ldots.$$

Under (A3), $\mathcal{M}_s^{\mathcal{P}^\circ} = \{0\}$ so

$$L_{\mathcal{P}}^* = L^{\mathcal{P}^\circ} \oplus (L^\infty)^\perp,$$

and under (A4), $\mathcal{M}_s^{\mathcal{P}^\circ} = \{0\}$ and $(L^\infty)^\perp = \{0\}$ so

$$L_{\mathcal{P}}^* = L^{\mathcal{P}^\circ}.$$

Proof By Lemma 5, $\mathcal{M}^{\mathcal{P}^\circ} \subset L_{\mathcal{P}}^*$, so $\mathcal{M}^{\mathcal{P}^\circ} \oplus (L^\infty)^\perp \subseteq L_{\mathcal{P}}^*$. To prove the opposite inclusion, let $u^* \in L_{\mathcal{P}}^*$. There exists $p \in \mathcal{P}$ and $\gamma > 0$ such that $u^* \le \gamma p$. Assumption (A1) implies that u^* is continuous in L^∞. By Theorem 1, there exists

a unique $m \in \mathcal{M}$ such that $\langle u, u^* \rangle = \int_\Omega u \, dm$ for all $u \in L^\infty$. Since $u^* \leq \gamma p$, we have $m \in \operatorname{dom} p^\circ$, so m is continuous on $L_\mathcal{P}$ by Lemma 5. Now $w := u^* - m$ belongs to $(L^\infty)^\perp$, so u^* has the required decomposition. Given another decomposition $u^* = \tilde{m} + \tilde{w}$ with $\tilde{w} \in (L^\infty)^\perp$ and $\tilde{m} \in \mathcal{M}^{\mathcal{P}^\circ}$, we have $(m - \tilde{m}) + (w - \tilde{w}) = 0$. Thus $\int_\Omega u \, d(m - \tilde{m}) = 0$ for all $u \in L^\infty$, so $m - \tilde{m} = 0$ and hence also $w - \tilde{w} = 0$, so the decomposition is unique.

The inequality follows directly from that of Lemma 5. Let $u \in L_\mathcal{P}$ and $A^v := \{|u| > v\}$. Clearly $P(A^v) \searrow 0$ and $u 1_{\Omega \setminus A^v} \in L^\infty$, so $\langle u 1_{\Omega \setminus A^v}, w \rangle = 0$ and thus w is singular. That $\mathcal{M}^{\mathcal{P}^\circ} = L^{\mathcal{P}^\circ}$ under (A3) is the last claim of Lemma 5. Under (A4), the truncations $u^v := u 1_{\{|u| \geq v\}}$ of any $u \in L_\mathcal{P}$ converge to u so L^∞ is dense in $L_\mathcal{P}$ and thus, $(L^\infty)^\perp = \{0\}$. □

When $\mathcal{P}$ is a singleton, we are in the setting of [38,Theorem 15.70.2], where $L_\mathcal{P}^*$ is decomposed into the direct sum of $L^{\mathcal{P}^\circ}$ and "singular elements". Theorem 6 gives a more precise description of the singular elements as the direct sum of $\mathcal{M}_s^{\mathcal{P}^\circ}$ and $(L^\infty)^\perp$. Applications will be given in the following sections.

Let $M_\mathcal{P}$ be the closure of L^∞ in $L_\mathcal{P}$.

Corollary 7 *We have*

$$M_\mathcal{P}^* = L^{\mathcal{P}^\circ} \oplus \mathcal{M}_s^{\mathcal{P}^\circ}$$

in the sense that for every $u^* \in M_\mathcal{P}^*$ there exist unique $y \in L^{\mathcal{P}^\circ}$ and $m \in \mathcal{M}_s^{\mathcal{P}^\circ}$ such that

$$\langle u, u^* \rangle = E[u \cdot y] + \int_\Omega u \, dm \quad \forall u \in M_\mathcal{P}$$

If (A3) holds, then $\mathcal{M}_s^{\mathcal{P}^\circ} = \{0\}$ and $M_\mathcal{P}^* = L^{\mathcal{P}^\circ}$. If (A4) holds, then $M_\mathcal{P} = L_\mathcal{P}$.

Proof By Hahn-Banach, a continuous linear functional on $M_\mathcal{P}$ is a restriction to $M_\mathcal{P}$ of a continuous linear functional on $L_\mathcal{P}$. The first two claims thus follow from Theorem 6. To prove the last claim, take any $u \in L_\mathcal{P}$ and define $u^v \in L^\infty$ as the pointwise projection of u on the Euclidean ball of radius v. Under (A4), $p(u - u^v) \searrow 0$ for all $p \in \mathcal{P}$, so L^∞ is dense in $L_\mathcal{P}$. □

The we end this section by giving some basic properties of $L^{\mathcal{P}^\circ}$. The *Köthe dual* of $L_\mathcal{P}$ is the linear space

$$L_\mathcal{P}' := \{y \in L^0 \mid u \cdot y \in L^1 \; \forall u \in L_\mathcal{P}\}.$$

By definition, $L^{\mathcal{P}^\circ} \subseteq L_\mathcal{P}'$. Lemma 8 below gives sufficient conditions for the converse. Recall that a locally convex space is *barreled* if every closed convex absorbing set is a neighborhood of the origin. By the Baire category theorem, Banach and Fréchet spaces are barreled.

Lemma 8 *For each $p \in \mathcal{P}$,*

1. *there is a constant $c > 0$ such that $c\|y\|_{L^1} \leq p^\circ(y)$ for all $y \in L^1$,*
2. *$p^\circ(y') \leq p^\circ(y)$ for every $y', y \in L^1$ with $|y'_j| \leq |y_j|$ for $j = 1, \dots, n$,*
3. *$E[u \cdot y] \leq p(u)p^\circ(y)$ for every $u \in L^0$ and $y \in L^1$ (Hölder's inequality).*

If the $\mathcal{P}$-topology is barreled and stronger than that of L^0, then $L^{\mathcal{P}^\circ} = L'_{\mathcal{P}}$.

Proof Assumption (A1) implies 1. Under (A2),

$$
\begin{aligned}
p^\circ(y') &= \sup_{u' \in L^\infty, u \in L^\infty} \left\{ E[u' \cdot y'] \,\middle|\, |u'_j| \leq |u_j|,\ p(u) \leq 1 \right\} \\
&= \sup_{u \in L^\infty} \left\{ E\left[\sum_{j=1}^n |u_j||y'_j| \right] \,\middle|\, p(u) \leq 1 \right\} \\
&\leq \sup_{u \in L^\infty} \left\{ E\left[\sum_{j=1}^n |u_j||y_j| \right] \,\middle|\, p(u) \leq 1 \right\} \\
&= \sup_{u' \in L^\infty, u \in L^\infty} \left\{ E[u' \cdot y] \,\middle|\, |u'_j| \leq |u_j|,\ p(u) \leq 1 \right\} \\
&= p^\circ(y),
\end{aligned}
$$

so 2 holds. The inequality in 3 follows from the inequality in Lemma 5.

To prove the last claim, let $y \in L^0$ be such that $E[u \cdot y] < \infty$ for all $u \in L_{\mathcal{P}}$. Then $p_y(u) := E[|u||y|] < \infty$ for all $u \in L_{\mathcal{P}}$. By Fatou's lemma, p_y is lsc in the L^0-topology. By assumption, p_y is lsc also in the $\mathcal{P}$-topology. When the $\mathcal{P}$-topology is barreled, p_y is continuous (see e.g. [33,Corollary 8B]), so $y \mapsto E[u \cdot y]$ is continuous as well. By Theorem 6, $y \in L^{\mathcal{P}^\circ}$. □

In the setting of Banach function spaces where $\mathcal{P}$ is a singleton, the last claim of Lemma 8 recovers [1,Lemma 1.2.6]. The following gives sufficient conditions for the space $L_{\mathcal{P}}$ to be complete and thus, barreled, when the topology is metrizable.

Remark 9 If the $\mathcal{P}$-topology is stronger than that of L^0 and $p \in \mathcal{P}$ are lower semi-continuous on L^0, then $L_{\mathcal{P}}$ is complete. In this case, $L_{\mathcal{P}}$ is a Banach/Fréchet (and, in particular, barreled) if $\mathcal{P}$ is a singleton/countable.

If $p(u) = \rho(|u|)$ for an nondecreasing $\rho : L^0 \to \overline{\mathbb{R}}$, the function p is lsc in probability if and only if ρ has the Fatou property: for any sequence $(\eta^\nu)_{\nu=1}^\infty \subset L^0_+$ with $\eta \nearrow \eta \in L^0$, $\lim \rho(\eta^\nu) = \rho(\eta)$.

Proof If (u^ν) is a Cauchy net in $L_{\mathcal{P}}$, it is Cauchy also in L^0 so, by completeness of L^0, it L^0-converges to an $u \in L^0$. Being Cauchy in $L_{\mathcal{P}}$ means that for every $\epsilon > 0$ and $p \in \mathcal{P}$, there is a $\bar{\nu}$ such that

$$
p(u^\nu - u^\mu) \leq \epsilon \quad \forall \nu, \mu \geq \bar{\nu}.
$$

The lower semicontinuity then gives

$$p(u^\nu - u) \le \epsilon \quad \forall \nu \ge \bar\nu$$

so $u \in L_\mathcal{P}$, by triangle inequality, and (u^ν) converges in $L_\mathcal{P}$ to u. Thus $L_\mathcal{P}$ is complete.

If p is lsc, $\liminf \rho(\eta^\nu) \ge \rho(\eta)$ while $\rho(\eta^\nu) \le \rho(\eta)$ since ρ is nondecreasing. If Fatou property holds and $u^\nu \to u$ in probability, then, passing to a subsequence if necessary, $u^\nu \to u$ almost surely. Then $\eta^\nu := \inf_{\nu' \ge \nu} |u^{\nu'}|$ increases pointwise to $|u|$, so $p(u) = \liminf \rho(\eta^\nu) \le \liminf p(u^\nu)$. □

5 Solid spaces of random variables

Axiom (A1) implies that $L_\mathcal{P}$ contains L^∞ while axiom (A2) implies that it is *solid* in the sense that it contains every $u' \in L^0$ for which there exists $u \in L_\mathcal{P}$ with $|u'_j| \le |u_j|$ for all $j = 1, \ldots, n$. By Lemma 8, $L^{\mathcal{P}^\circ}$ is solid as well. This section starts with an arbitrary pair $(\mathcal{U}, \mathcal{Y})$ of solid spaces of random variables in separating duality under the bilinear form

$$\langle u, y \rangle := E[u \cdot y].$$

We assume that both $\mathcal{U}$ and $\mathcal{Y}$ contain L^∞ and show that the Mackey and the strong topologies arise from (uncountable) collections of seminorms on L^0 satisfying the axioms of Sect. 4. We then obtain completeness and duality results as corollaries of the results there.

The *weak topology* generated by $\mathcal{Y}$ on $\mathcal{U}$ will be denoted by $\sigma(\mathcal{U}, \mathcal{Y})$. Similarly on $\mathcal{Y}$. The *Mackey topology* $\tau(\mathcal{U}, \mathcal{Y})$ on $\mathcal{U}$ is generated by the collection of seminorms defined as the *support functions*

$$\sigma_C(u) := \sup_{y \in C} E[u \cdot y]$$

of symmetric $\sigma(\mathcal{Y}, \mathcal{U})$-compact sets $C \subset \mathcal{Y}$. The *strong topology* on $\mathcal{U}$ is defined similarly by all the symmetric $\sigma(\mathcal{Y}, \mathcal{U})$-bounded sets in $\mathcal{Y}$, i.e. symmetric sets $C \subset \mathcal{Y}$ such that $\sigma_C(u) < \infty$ for all $u \in \mathcal{U}$. By the bipolar theorem, this is the topology generated by all $\tau(\mathcal{U}, \mathcal{Y})$-lower semicontinuous seminorms on $\mathcal{U}$. By the Mackey–Arens theorem, the Mackey topology on $\mathcal{U}$ is the finest topology under which the topological dual of $\mathcal{U}$ coincides with $\mathcal{Y}$. Since compact sets are bounded, the strong topology is stronger than $\tau(\mathcal{U}, \mathcal{Y})$.

Lemma 11 below does not require solidity but merely *decomposability* in the sense that $u1_A + \bar{u}1_{\Omega \setminus A} \in \mathcal{U}$ for every $u \in \mathcal{U}$, $\bar{u} \in L^\infty$ and $A \in \mathcal{F}$.

Example 10 Solid spaces containing L^∞ are decomposable but there are decomposable spaces that are not solid. Indeed, let $\Omega = [0, 1]$, $\mathcal{F}$ the Borel sigma algebra and P the Lebesgue measure. Let $u(\omega) := \omega^{-\frac{1}{4}} + \omega^{-\frac{1}{2}}$ and $\mathcal{U} := L^\infty + L$, where L is the linear span of functions of the form $u1_A$ with $A \in \mathcal{F}$. Then $\mathcal{U}$ is decomposable, by construction, but not solid, since it does not contain $\bar{u}(\omega) = \omega^{-\frac{1}{4}}$ while $0 < \bar{u} < u$.

The following is Lemma 6 from [26].

Lemma 11 *If $\mathcal{U}$ and $\mathcal{Y}$ are decomposable, then $L^\infty \subseteq \mathcal{U} \subseteq L^1$ and*

$$\sigma(L^1, L^\infty)|_{\mathcal{U}} \subseteq \sigma(\mathcal{U}, \mathcal{Y}), \quad \sigma(\mathcal{U}, \mathcal{Y})|_{L^\infty} \subseteq \sigma(L^\infty, L^1),$$
$$\tau(L^1, L^\infty)|_{\mathcal{U}} \subseteq \tau(\mathcal{U}, \mathcal{Y}), \quad \tau(\mathcal{U}, \mathcal{Y})|_{L^\infty} \subseteq \tau(L^\infty, L^1).$$

Lemma 12 *If $\mathcal{U}$ is solid, then, for every $u \in \mathcal{U}$,*

1. *$y \mapsto u \cdot y$ is continuous from $(\mathcal{Y}, \sigma(\mathcal{Y}, \mathcal{U}))$ to $(L^1, \sigma(L^1, L^\infty))$,*
2. *$\eta \mapsto \eta u$ is continuous from $(L^\infty, \tau(L^\infty, L^1))$ to $(\mathcal{U}, \tau(\mathcal{U}, \mathcal{Y}))$.*

Proof For any $u \in \mathcal{U}$, $y \in \mathcal{Y}$ and $\eta \in L^\infty$,

$$E[(u \cdot y)\eta] = E[(\eta u) \cdot y].$$

Solidity implies that $\eta u \in \mathcal{U}$ so 1 holds. Let $K \subset \mathcal{Y}$ be $\sigma(\mathcal{Y}, \mathcal{U})$-compact. We have

$$\sup_{y \in K} \langle y, \eta u \rangle = \sup_{y \in K} \langle u \cdot y, \eta \rangle_{L^\infty} = \sup_{\xi \in D} \langle \xi, \eta \rangle_{L^\infty},$$

where $D = \{u \cdot y \mid y \in K\}$ is $\sigma(L^1, L^\infty)$-compact since $y \mapsto u \cdot y$ is continuous, by 1. $\qquad\square$

Lemma 12 implies, in particular, that axiom (A3) is necessary for the second claim of Corollary 7.

Corollary 13 *In the setting of Corollary 7, (A3) holds if and only if $M_{\mathcal{P}}^* = L^{\mathcal{P}^\circ}$.*

Proof By Corollary 7, (A3) implies $M_{\mathcal{P}}^* = L^{\mathcal{P}^\circ}$. On the other hand, if $M_{\mathcal{P}}^* = L^{\mathcal{P}^\circ}$, the topology of $M_{\mathcal{P}}$ cannot be stronger than $\tau(M_{\mathcal{P}}, L^{\mathcal{P}^\circ})$. In that case, Lemma 12 implies that $p(u\eta^\nu) \to 0$ if $\eta^\nu \to 0$ in $\tau(L^\infty, L^1)$. Since $1_{A^\nu} \to 0$ in $\tau(L^\infty, L^1)$ if $P(A^\nu) \to 0$, assumption (A3) holds. $\qquad\square$

The following characterization of $\sigma(\mathcal{U}, \mathcal{Y})$-compact sets will be useful. In the case of Orlicz spaces, a similar characterization of relative compactness can be found e.g. in [5, 28].

Lemma 14 *Given $C \subset \mathcal{U}$, the following are equivalent.*

1. *C is $\sigma(\mathcal{U}, \mathcal{Y})$-precompact,*
2. *$\{u \cdot y \mid u \in C\}$ is $\sigma(L^1, L^\infty)$-precompact for every $y \in \mathcal{Y}$,*
3. *$\{u_j y_j \mid u \in C\}$ is $\sigma(L^1, L^\infty)$-precompact for every $y \in \mathcal{Y}$ and $j = 1, \ldots, n$.*

Proof Since continuous images of precompact sets are precompact, 3 implies 2, and, by Lemma 12, 1 implies 3. Clearly, 2 implies 3, so it suffices to show that 2 and 3 imply 1. Let (u^ν) be a net in C. Since $\mathcal{Y}$ contains constants, the sets $\{u_j \mid u \in C\}$ are $\sigma(L^1, L^\infty)$-precompact by 3. Thus there is a subnet and $u \in C$ such that $u^\nu \to u$ in $\sigma(L^1, L^\infty)$. Let $y \in \mathcal{Y}$ and $\epsilon > 0$. By the Dunford–Pettis theorem, 2 implies that $\{u \cdot y \mid u \in C\}$ is

uniformly integrable, so there exists n such that $|E[(u^\nu - u) \cdot y 1_{|y|>n}]| < \epsilon$ for every ν. Since $u^\nu \to u$ in $\sigma(L^1, L^\infty)$, there exists ν' such that $|E[(u^\nu - u) \cdot y 1_{|y| \le n}]| < \epsilon$ for all $\nu \ge \nu'$. Thus, for all $\nu \ge \nu'$,

$$|E[(u^\nu - u) \cdot y]| \le 2\epsilon,$$

which proves that $u^\nu \to u$ in $\sigma(\mathcal{U}, \mathcal{Y})$. □

The *solid hull* $s(C)$ of a set $C \subseteq \mathcal{U}$ is the smallest solid set containing C. Clearly,

$$s(C) = \{u \in \mathcal{U} \mid \exists \bar{u} \in C : |u_j| \le |\bar{u}_j| \ \forall j \ a.s.\}.$$

Corollary 15 *The solid hull of a $\sigma(\mathcal{U}, \mathcal{Y})$-bounded set is $\sigma(\mathcal{U}, \mathcal{Y})$-bounded and the solid hull of a $\sigma(\mathcal{U}, \mathcal{Y})$-precompact set is $\sigma(\mathcal{U}, \mathcal{Y})$-precompact.*

Proof We have $E[u \cdot y] < \infty$ if and only if $E \sum_j |u_j||y_j| < \infty$, which implies the first claim. By Lemma 14 and the Dunford–Pettis theorem, a set $C \subseteq \mathcal{U}$ is $\sigma(\mathcal{U}, \mathcal{Y})$-precompact if and only if $C_{y,j} := \{u_j y_j \mid u \in C\}$ is uniformly integrable for every $y \in \mathcal{Y}$ and $j = 1, \ldots, n$. Uniform integrability of $C_{y,j}$ means that, for every $\epsilon > 0$, there exists $M > 0$ such that $E|1_{|\bar{u}_j y_j| \ge M} \bar{u}_j y_j| < \epsilon$ for every $\bar{u} \in C$. Clearly, uniform integrability of $C_{y,j}$ implies that of $s(C)_{y,j}$. Thus, if C is precompact, then $s(C)$ is precompact. □

Let $\mathcal{C}_s$ be the collection of solid hulls of $\sigma(\mathcal{Y}, \mathcal{U})$-bounded sets. We define $\mathcal{P}_s$ as the collection of functions p_C on L^0 defined by

$$p_C(u) := \sup_{y \in C} E[u \cdot y],$$

where $C \in \mathcal{C}_s$ and the expectation is defined as $+\infty$ unless the positive part of $u \cdot y$ is integrable. Analogously, we define $\mathcal{C}_\tau$ as the collection of solid hulls of $\sigma(\mathcal{Y}, \mathcal{U})$-compact sets and $\mathcal{P}_\tau$ as the collection of functions p_C with $C \in \mathcal{C}_\tau$. By Corollary 15, the restrictions of $\mathcal{P}_s$ and $\mathcal{P}_\tau$ to $\mathcal{U}$ generate the strong and the Mackey topologies, respectively. Note that solid hulls of convex sets in $\mathcal{U}$ need not be convex. For subsets of L^0_+, however, taking the solid hull and convex hull commute; see [18,Proposition 1.1].

Lemma 16 *The members of $\mathcal{P}_s$ satisfy (A1) and (A2) while the members of $\mathcal{P}_\tau$ satisfy (A1)–(A4). Both $\mathcal{P}_s$ and $\mathcal{P}_\tau$ contain the L^1-norm and their members are L^0-lsc.*

Proof Since $\sigma(\mathcal{Y}, \mathcal{U})$-bounded sets are L^1-bounded, the functions p_C are dominated by the L^∞-norm. Thus, $\mathcal{P}_s$ satisfies (A1). Since the sets $C \subset \mathcal{C}_s$ are solid, [34,Theorem 14.60] gives

$$p_C(u) = \sup_{y \in C, y' \in L^0} \left\{ E[u \cdot y'] \,\middle|\, |y'_j| \le |y_j| \ \forall j \right\}$$

$$= \sup_{y \in C} E \sum_{j=1}^{n} |u_j||y_j|,$$

so p_C satisfies (A2). By Fatou's lemma, each supremand is L^0-lsc so p_C is L^0-lsc as well.

Since $\mathcal{P}_\tau \subset \mathcal{P}_s$, axioms (A1) and (A2) are again satisfied by $\mathcal{P}_\tau$ and its elements are L^0-lsc. Given $C \in \mathcal{C}_\tau$ and $u \in \operatorname{dom} p_C$, Lemma 14 and the Dunford–Pettis theorem imply that the set $\{u \cdot y \mid y \in C\}$ is uniformly integrable so $p_C(u1_{A^\nu}) \searrow 0$ whenever $(A^\nu)_{\nu=1}^\infty$ is a decreasing sequence with $P(A^\nu) \searrow 0$. Thus, $\mathcal{P}_\tau$ satisfies (A4).

By Banach–Alaoglu, the unit ball B of L^∞ is $\sigma(L^\infty, L^1)$-compact so, by Lemma 11, it is $\sigma(\mathcal{Y}, \mathcal{U})$-compact as well. Thus, $B \in \mathcal{C}_\tau$, so $\mathcal{P}_\tau$ contains the L^1-norm and thus $\mathcal{P}_s$ does as well. □

Recall that the *Köthe dual* of a space $\mathcal{U}$ of measurable functions is the linear space

$$\mathcal{U}' := \{y \in L^0 \mid u \cdot y \in L^1 \ \forall u \in \mathcal{U}\}.$$

Our assumptions on $\mathcal{U}$ and $\mathcal{Y}$ imply that they are contained in each other's Köthe duals. The following shows, in particular, that if $\mathcal{U}$ is equal to the Köthe dual of $\mathcal{Y}$, then $\mathcal{U}$ and $\mathcal{Y}$ arise from the construction of Sect. 4 with the Mackey-seminorms $\mathcal{P}_\tau$.

Theorem 17 *We have $\mathcal{Y} = L^{\mathcal{P}_\tau^\circ}$, $\mathcal{U} \subset L^{\mathcal{P}_\tau} \subset \mathcal{Y}'$ and the following are equivalent*

1. *$\mathcal{U} = \mathcal{Y}'$,*
2. *$\mathcal{U} = L_{\mathcal{P}_\tau}$,*
3. *$\mathcal{U}$ is $\tau(\mathcal{U}, \mathcal{Y})$-complete,*
4. *$\mathcal{U}$ is sequentially $\sigma(\mathcal{U}, \mathcal{Y})$-complete.*

Proof Given $C \in \mathcal{C}_\tau$, let $D \subset \mathcal{Y}$ be its $\sigma(\mathcal{Y}, \mathcal{U})$-closed convex hull. By definition, $p_D = p_C$ on $\mathcal{U}$ and, in particular, on L^∞. The $\sigma(\mathcal{Y}, \mathcal{U})$-precompactness of C implies that D is $\sigma(\mathcal{Y}, \mathcal{U})$-compact. By Lemma 11, D is $\sigma(L^1, L^\infty)$-compact as well. An application of the bipolar theorem in the duality pairing (L^1, L^∞) thus gives

$$p_C^\circ(y) = \inf\{\gamma > 0 \mid y/\gamma \in D\} \quad \forall y \in L^1,$$

so $\operatorname{dom} p_C^\circ \subset \mathcal{Y}$. Thus, $L^{\mathcal{P}_\tau^\circ} \subseteq \mathcal{Y}$. Any $y \in \mathcal{Y}$ belongs to some $C \in \mathcal{C}_\tau$, so $y \in \operatorname{dom} p_C^\circ$. Thus, $\mathcal{Y} \subseteq L^{\mathcal{P}_\tau^\circ}$.

Each $p \in \mathcal{P}_\tau$ is finite on $\mathcal{U}$, so $\mathcal{U} \subset L_{\mathcal{P}_\tau}$. Let $u \in L_{\mathcal{P}_\tau}$. For every $y \in \mathcal{Y}$, there is a $p \in \mathcal{P}_\tau$ such that $p^\circ(y) < \infty$ so, by the Hölder's inequality in Lemma 8,

$$E[u \cdot y] \leq p(u)p^\circ(y) < \infty.$$

This proves the second claim.

By the second claim, 1 implies 2. Lemma 16 and Remark 9 imply that $L_{\mathcal{P}_\tau}$ is complete. Thus, 2 implies 3. On the other hand, by Lemma 16, $\mathcal{P}_\tau$ satisfies (A1)–(A4) so L^∞ is dense in $L_{\mathcal{P}_\tau}$. Since $\mathcal{U}$ is decomposable, it contains L^∞. Thus, if $\mathcal{U}$ is complete in the relative topology of $\mathcal{P}_\tau$, it has to coincide with $L_{\mathcal{P}_\tau}$. Thus, 3 implies 2.

We next show that 4 implies 1. Let $u \in L^1$ be in the Köthe dual and let $u^\nu \in L^\infty$ be the pointwise projection of u to the Euclidean ball with radius ν. By dominated convergence, $E[u^\nu \cdot y] \to E[u \cdot y]$ for all $y \in \mathcal{Y}$. Thus, $(u^\nu)_{\nu=1}^\infty$ is weakly Cauchy so 4 implies that it has a $\sigma(\mathcal{U}, \mathcal{Y})$-limit $u' \in \mathcal{U}$. It follows that $E[u \cdot y] = E[u' \cdot y]$ for all $y \in \mathcal{Y}$ so $u = u'$.

It remains to show that 2 implies 4. Let $(u^\nu)_{\nu=1}^\infty$ be a $\sigma(\mathcal{U}, \mathcal{Y})$-Cauchy sequence. Since $\sigma(\mathcal{U}, \mathcal{Y})$ is stronger than $\sigma(L^1, L^\infty)$ which, by [6,Theorem IV.8.6], is sequentially complete, there exists $u \in L^1$ such that $u^\nu \to u$ in $\sigma(L^1, L^\infty)$. Since $\sigma(\mathcal{U}, \mathcal{Y})$-Cauchy sequences are bounded in any topology compatible with the pairing, the sequence is also bounded in the $\mathcal{P}_\tau$-topology. Thus, for any $p \in \mathcal{P}_\tau$, there exists $\gamma \in \mathbb{R}$ such that $p(u^\nu) \le \gamma$. Since level-sets of p are closed in L^1 and $\mathcal{U} = L_{\mathcal{P}_\tau}$, we get $u \in \mathcal{U}$. It suffices to show that $u^\nu \to u$ in $\sigma(\mathcal{U}, \mathcal{Y})$.

Let $y \in \mathcal{Y}$. By Lemma 12, $(u^\nu \cdot y)_{\nu=1}^\infty$ is Cauchy in $\sigma(L^1, L^\infty)$, so by sequential completeness of L^1 again, it converges in $\sigma(L^1, L^\infty)$ to some $\xi \in L^1$. By Mazur's theorem, there is a subsequence of convex combinations $\bar{u}^\nu$ such that $\bar{u}^\nu \to u$ in L^1-norm, and thus $\bar{u}^\nu \cdot y \to u \cdot y$ in probability. Clearly, $\bar{u}^\nu \cdot y \to \xi$ in $\sigma(L^1, L^\infty)$, so we must have $\xi = u \cdot y$. Thus, $E[u^\nu \cdot y] \to E[u \cdot y]$ which completes the proof. □

The following shows, in particular, that if $\mathcal{U}$ and $\mathcal{Y}$ are Köthe duals of each other, then they arise from the construction of Sect. 4 with the strong seminorms $\mathcal{P}_s$. Theorem 6 then yields a characterization of the strong dual of $\mathcal{U}$.

Theorem 18 *If $\mathcal{U} = \mathcal{Y}'$, then $\mathcal{U} = L_{\mathcal{P}_s}$. If $\mathcal{Y} = \mathcal{U}'$, then $\mathcal{Y} = L^{\mathcal{P}_s^\circ}$. If $\mathcal{U} = L_{\mathcal{P}_s}$ and $\mathcal{Y} = L^{\mathcal{P}_s^\circ}$, then the strong dual of $\mathcal{U}$ may be identified with $\mathcal{Y} \oplus \mathcal{M}_s^{\mathcal{P}_s^\circ} \oplus (L^\infty)^\perp$, in the sense that for every $u^* \in \mathcal{U}^*$, there exist $u \in \mathcal{Y}$, $m^s \in \mathcal{M}_s^{\mathcal{P}_s^\circ}$ and $w \in (L^\infty)^\perp$ such that*

$$\langle u, u^* \rangle = E[u \cdot y] + \int_\Omega u \, dm^s + \langle u, w \rangle \quad \forall u \in \mathcal{U}.$$

Proof Since $\mathcal{U} \subset L_{\mathcal{P}_s} \subset L_{\mathcal{P}_\tau}$, the first claim follows from Theorem 17. Since $L^{\mathcal{P}_\tau^\circ} \subseteq L^{\mathcal{P}_s^\circ}$, Theorem 17 implies $\mathcal{Y} \subseteq L^{\mathcal{P}_s^\circ}$. On the other hand, since $\mathcal{U} \subseteq L_{\mathcal{P}_s}$, the Hölder's inequality in Lemma 8 implies

$$E[u \cdot y] < \infty \quad \forall u \in \mathcal{U}, \ \forall y \in L^{\mathcal{P}_s^\circ}$$

so $L^{\mathcal{P}_s^\circ} \subseteq \mathcal{U}'$. Thus, if $\mathcal{Y} = \mathcal{U}'$, we have $L^{\mathcal{P}_s^\circ} \subseteq \mathcal{Y}$. When $\mathcal{U} = L_{\mathcal{P}_s}$ and $\mathcal{Y} = L^{\mathcal{P}_s^\circ}$, we are in the setting of Sect. 4. By Lemma 16, $\mathcal{P}_s$ satisfies (A1) and (A2), so the last claim follows from Theorem 6. □

In the setting of Theorem 6, one may wonder what is the strong topology generated by $L^{\mathcal{P}^\circ}$ on $L_\mathcal{P}$.

Theorem 19 *If $L_\mathcal{P}$ is barreled and $p \in \mathcal{P}$ are $\sigma(L_\mathcal{P}, L^{\mathcal{P}^\circ})$-lsc, then the strong topology generated by $L^{\mathcal{P}^\circ}$ on $L_\mathcal{P}$ coincides with the $\mathcal{P}$-topology.*

Proof If p is $\sigma(L_\mathcal{P}, L^{\mathcal{P}^\circ})$-lsc, Theorem 6 and the bipolar theorem imply that

$$p(u) = \sup_{y \in L^{\mathcal{P}^\circ}} \{ E[u \cdot y] \mid p^\circ(y) \le 1 \}.$$

It follows that the level sets of the functions p° generate the $L_\mathcal{P}$-topology. Since the level sets are bounded, the $L_\mathcal{P}$-topology is weaker than the strong topology generated by $L^{\mathcal{P}^\circ}$. On the other hand, if $L_\mathcal{P}$ is barreled, then the elements of $\mathcal{P}_s$ are $L_\mathcal{P}$-continuous.   □

6 Applications

This section applies the results of the previous sections to more specific situations. We obtain quick proofs of many well known as well as new results.

6.1 Random variables with finite moments

Given an increasing sequence $S \subset [1, \infty)$, let

$$L^S := \{u \in L^1 \mid \forall p \in S : \ \|u\|_{L^p} < \infty\},$$

If S contains its supremum $\bar{p} := \sup S$, then L^S coincides with the classical Lebesgue space $L^{\bar{p}}$. If $\bar{p} \notin S$, L^S is the space of random variables with finite pth moments $m_p(u) := E[|u|^p]$ for p strictly less than $\bar{p}$. When S is unbounded, L^S is the space of random variables with finite moments. Let S' be the set of conjugate exponents of S, i.e.

$$S' := \left\{ q \in (1, \infty] \mid \exists p \in S : \ \frac{1}{p} + \frac{1}{q} = 1 \right\}$$

and define

$$L_{S'} := \{y \in L^1 \mid \exists q \in S' : \ \|y\|_{L^q} < \infty\}.$$

The L^p-norms with $p < \infty$ satisfy (A1)–(A4). The following example is thus a direct consequence of Corollary 7.

Theorem 20 *Equipped with the collection of seminorms $\| \cdot \|_{L^p}$, $s \in S$, the space L^S is Fréchet and its dual may be identified with $L_{S'}$ under the bilinear form $\langle u, y \rangle := E[u \cdot y]$.*

6.2 Marcinkiewicz and Lorentz spaces

Given a random variable $u \in L^0$, we will denote the *distribution function* of $|u|$ by $n_u(\tau) := P(|u| > \tau)$ and its *quantile function* by

$$q_u(t) := \inf\{\tau \in \mathbb{R} \mid n_u(\tau) \le t\}.$$

In the terminology of Banach function spaces, the quantile function is usually called the "decreasing rearrangement of u; see e.g. [1]. Given a nonnegative concave increasing

function ϕ on $[0, 1]$ with $\phi(0) = 0$, the associated *Marcinkiewicz space* is the set M_ϕ of $u \in L^0$ with

$$\|u\|_\phi := \sup_{t \in (0,1]} \left\{ \frac{1}{\phi(t)} \int_0^t q_u(s)ds \right\} < \infty.$$

Recall that a probability space is *resonant* if it is atomless or completely atomic with all atoms having equal measure.

Theorem 21 *Assume that $(\Omega, \mathcal{F}, P)$ is resonant. Endowed with the norm $\| \cdot \|_\phi$, the space M_ϕ is a Banach. If $\lim_{t \searrow 0} t/\phi(t) > 0$, we have $M_\phi = L^\infty$. Assume now that $\lim_{t \searrow 0} t/\phi(t) = 0$. The topological dual of M_ϕ is*

$$M_\phi^* = \Lambda_\Phi \oplus (L^\infty)^\perp,$$

where Λ_Φ is the Lorentz space

$$\Lambda_\phi := \{y \in L^1 \mid \|y\|_\phi^* < \infty\},$$

where

$$\|y\|_\phi^* := \int_0^1 q_y(t)d\phi(t).$$

The closure M_ϕ^0 of L^∞ in M_ϕ can be expressed as

$$M_\phi^0 = \{u \in L^1 \mid \lim_{t \searrow 0} \frac{1}{\phi(t)} \int_0^t q_u(s)ds = 0\}.$$

The topological dual of M_ϕ^0 is Λ_Φ and the topological dual of Λ_ϕ is M_ϕ.

Proof We apply Theorem 6 with $\mathcal{P} = \{p\}$ where $p(u) = \|u\|_\phi$. Since

$$\|u\|_\phi \geq \frac{1}{\phi(1)} \int_0^1 q_u(s)ds = \phi(1)E[|u|],$$

we have $M_\phi \subset L^1$ and its topology is stronger than the L^0-topology. By Lemma 30,

$$u \mapsto \int_0^t q_u(t)dt$$

is the infimal projection of a sublinear function of s and u and thus, sublinear in u. It is also continuous in L^1. It follows that $\| \cdot \|_\phi$ is sublinear, symmetric and lsc in L^1.

By Remark 9, M_ϕ is Banach. Since $q_u \leq \|u\|_{L^\infty}$, we have

$$\|u\|_\phi \leq \sup_{t \in (0,1]} \frac{t}{\phi(t)} \|u\|_{L^\infty},$$

where $\sup_{t \in (0,1]} \frac{t}{\phi(t)} < \infty$ since ϕ is concave and strictly positive for $t > 0$. Thus, (A1) holds. Property (A2) is clear. Given $A \in \mathcal{F}$,

$$\|1_A\|_\phi = \sup_t \frac{1}{\phi(t)} \min\{t, P(A)\} = \frac{P(A)}{\phi(P(A))},$$

since $t \mapsto \frac{t}{\phi(t)}$ is increasing by concavity. Thus $\hat{\phi}_p(t) := \frac{t}{\phi(t)}$ is the fundamental function of M_ϕ. By Remark 4, $M_\phi = L^\infty$ if $\lim_{t \searrow 0} t/\phi(t) > 0$ while (A3) holds if $\lim_{t \searrow 0} t/\phi(t) = 0$. We have

$$
\begin{aligned}
\|y\|_\phi^* &= \sup_{u \in L^1} \{E[uy] \mid \|u\|_\phi \leq 1\} \\
&= \sup_{u \in L^1} \left\{ \int_0^1 q_u(t) q_y(t) dt \mid \int_0^t q_u(s) ds \leq \phi(t) \; \forall t \in [0,1] \right\} \\
&= \int_0^1 q_y(t) \phi'(t) dt \\
&= \int_0^1 q_y(t) d\phi(t),
\end{aligned}
$$

where the second equality follows from [1,Corollary 2.4.4] and the third from Hardy's lemma [1,Proposition 2.3.6]. The representation of the topological dual of M_ϕ now follows from Theorem 6.

If $u \in L^\infty$, q_u is bounded, so

$$\lim_{t \searrow 0} \frac{1}{\phi(t)} \int_0^t q_u(s) ds = \lim_{t \searrow 0} \frac{t}{\phi(t)} \frac{1}{t} \int_{[0,t]} q_u(s) ds = 0,$$

by assumption. Thus, $L^\infty \subset M_\phi^0$. Let $u \in M_\phi$ and M_ϕ^0. We have $q_{u+\tilde{u}}(s^1 + s^2) \leq q_u(s^1) + q_{\tilde{u}}(s^2)$, so

$$
\begin{aligned}
\lim_{t \searrow 0} \frac{1}{\phi(t)} \int_0^t q_u(s) ds &\leq \lim_{t \searrow 0} \frac{1}{\phi(t)} \int_0^t (q_{u-\tilde{u}}(s/2) + q_{\tilde{u}}(s/2)) ds \\
&= \lim_{t \searrow 0} \frac{1}{\phi(t)} \int_0^t q_{u-\tilde{u}}(s/2) ds \\
&= \lim_{t \searrow 0} \frac{2}{\phi(t)} \int_0^{2t} q_{u-\tilde{u}}(s) ds \\
&\leq \lim \frac{1}{\phi(2t)} \int_0^{2t} q_{u-\tilde{u}}(s) ds \\
&\leq \|u - \tilde{u}\|_\phi,
\end{aligned}
$$

where the second last inequality follows from concavity of ϕ. Thus, M_ϕ^0 is closed in M_ϕ so M_ϕ^0 contains the closure of L^∞. To prove the converse, let $u \in M_\phi^0$ and $u^\nu =$

$u1_{\{|u|\leq v\}}$. We have $q_{u-u^v}(t) = 0$ for $t \geq t^v := P(|u| \geq v)$ while $q_{u-u^v}(t) = q_u(t)$ for $t < t^v$. Thus,

$$\|u - u^v\|_\phi = \sup_{t \in [0,1]} \left\{ \frac{1}{\phi(t)} \int_0^t q_{u-u^v}(s)ds \right\} = \sup_{t \in [0,t^v]} \left\{ \frac{1}{\phi(t)} \int_0^t q_u(s)ds \right\}.$$

Since $u \in M_\phi^0$, this converges to 0 as $v \to \infty$. Thus, M_ϕ^0 is the closure of L^∞ in M_ϕ.

To prove the last claim, we apply Theorem 6 to p°. By Lemma 8, the Lorentz seminorm satisfies (A1) and (A2). If $y^v \searrow 0$ with $\|y^v\|_\phi^* < \infty$, we have $q_{y^v} \searrow 0$, so by monotone convergence, $\|y^v\|_\phi^* \searrow 0$. Thus, the Lorenz norm satisfies (A4) as well. The fact that the topological dual of Λ_ϕ is M_ϕ now follows from Theorem 6 and the fact that, by the bipolar theorem, p is the polar of p°. □

Except for the the characterization of M_ϕ^*, the statements of Theorem 21 can be found in Theorem 5.2, Theorem 5.4 and Lemma 5.4 of [20,Chapter II]; see also [13,Section 1] and the proof of [13,Theorem 1.5]. The general results of Sect. 4 simplifies the proofs and provide the additional characterization of M_ϕ^*.

Much like in Theorem 20, one could also characterize topological duals of locally convex (resp. Fréchet) spaces obtained by intersecting Marcinkiewicz spaces associated with a (resp countable) collection of nonnegative concave increasing functions ϕ. One could also use the results of Sect. 4 to study the abstract Lorentz spaces recently introduced in [17].

6.3 Modular spaces and Luxemburg norms

This section studies a general class of Banach spaces that arise from a positive symmetric convex function (a *convex modular* in the terminology of [25]) on L^0 much like Orlicz spaces arise from the Luxemburg norm associated with a given Young functional; see Sect. 6.4 below. Theorem 22 below allows for quick proofs and various extensions of existing results in the theory of Banach function spaces.

Given a set C in a linear space, we will use the notation

$$\text{pos } C := \bigcup_{\beta > 0} (\beta C) \quad \text{and} \quad C^\infty := \bigcap_{\beta > 0} (\beta C).$$

Let $H : L^0 \to \overline{\mathbb{R}}_+$ be a L^0-lsc symmetric convex function such that $H(0) = 0$ and

(H1) $L^\infty \subset \text{pos}(\text{dom } H)$ and $\{u \in L^0 \mid H(u) \leq 1\}$ is bounded in probability,
(H2) $H(u') \leq H(u)$ whenever $|u'_j| \leq |u_j|$ for $j = 1, \ldots, n$.

Consider the space

$$L^H := \text{pos dom } H = \{u \in L^1 \mid \exists \beta > 0 : H(u/\beta) < \infty\}$$

endowed with the *Luxemburg norm*

$$\|u\|_H := \inf\{\beta > 0 \mid H(u/\beta) \leq 1\}.$$

Define $H^* : \mathcal{M} \to \overline{\mathbb{R}}$ by

$$H^*(m) := \sup_{u \in L^\infty} \left\{ \int_\Omega u\,dm - H(u) \right\}.$$

and

$$\mathcal{M}^{H^*} := \mathrm{pos}\,\mathrm{dom}\,H^* = \{m \in \mathcal{M} \mid \exists \beta > 0 : H^*(m/\beta) < \infty\}.$$

The set of purely finitely additive elements of $\mathcal{M}^{H^*}$ is denoted by $\mathcal{M}_s^{H^*}$ and the set of densities of countably additive measures of $\mathcal{M}^{H^*}$ by L^{H^*}. Note that

$$(\mathrm{dom}\,H)^\infty = \{u \in L^0 \mid H(u/\beta) < \infty \; \forall \beta > 0\}.$$

Theorem 22 *Equipped with the norm $\|\cdot\|_H$, the space L^H is Banach, its dual may be identified with*

$$L^{H^*} \oplus \mathcal{M}_s^{H^*} \oplus (L^\infty)^\perp$$

and L^{H^*} coincides with the Köthe dual of L^H. For any $m \in \mathcal{M}^{H^*}$, the dual norm can be expressed as

$$\|m\|_H^* = \sup_{u \in L^\infty} \left\{ \int_\Omega u\,dm \mid H(u) \le 1 \right\} = \inf_{\beta > 0}\{\beta H^*(m/\beta) + \beta\},$$

where the infimum is attained. Moreover,

$$\|m\|_{H^*} \le \|m\|_H^* \le 2\|m\|_{H^*},$$

where

$$\|m\|_{H^*} := \inf\{\beta > 0 \mid H^*(m/\beta) \le 1\}.$$

The dual of the closure M^H of L^∞ in L^H can be identified with $L^{H^*} \oplus \mathcal{M}_s^{H^*}$. If

(H3) $H(u^\nu) \searrow 0$ whenever $(u^\nu)_{\nu=1}^\infty \subset L^\infty$ with $|u^\nu| \searrow 0$ almost surely,

then $\mathcal{M}_s^{H^*} = \{0\}$, $(L^H)^* = L^{H^*} \oplus (L^\infty)^\perp$ and $(M^H)^* = L^{H^*}$. If $L^\infty \subseteq \mathrm{dom}\,H$ and

(H4) $H(u^\nu) \searrow 0$ whenever $(u^\nu)_{\nu=1}^\infty \subset \mathrm{dom}\,H$ with $|u^\nu| \searrow 0$ almost surely,

then $M^H = (\mathrm{dom}\,H)^\infty$ and, in particular, $L^H = M^H$ if $\mathrm{dom}\,H$ is a cone.

Proof We apply Theorem 6 in the case $\mathcal{P} = \{p\}$, where

$$p(u) := \inf\{\beta > 0 \mid H(u/\beta) \le 1\}.$$

By (H1), p is finite on L^∞. Since p is lsc on L^0, it is lsc on L^∞. Thus, by [33,Corollary 8B], p is continuous in L^∞ and thus (A1) holds. Assumption (A2) is clear from (H2).

Let $u^\nu \to u$ in L^0 be such that $p(u^\nu) \le \alpha$. This means that $H(u^\nu/\alpha) \le 1$, so the L^0-lower semicontinuity of H implies that of p. Let O be an L^0-neighborhood of the origin. By the boundedness assumption in (H1), there exists $\lambda > 0$ such that $\lambda O \supset \{u \in L^0 \mid H(u) \le 1\} = \{u \in L^H \mid p(u) \le 1\}$. Thus $O \cap L^H$ is a neighborhood in L^H, so the topology of L^H is no weaker than the relative L^0-topology. By Remark 9, L^H is Banach. By Lemma 8, L^{H^*} is the Köthe dual of L^H.

Let $m \in \mathcal{M}$. Since the infimum in the definition of the p is attained,

$$p^\circ(m) = \sup_{u \in L^\infty} \left\{ \int_\Omega u \, dm \mid p(u) \le 1 \right\} = \sup_{u \in L^\infty} \left\{ \int_\Omega u \, dm \mid H(u) \le 1 \right\}.$$

Lagrangian duality (see e.g. [33,Example 1"]) gives

$$p^\circ(m) = \inf_{\beta > 0} \sup_{u \in L^\infty} \left\{ \int_\Omega u \, dm - \beta H(u) + \beta \right\} = \inf_{\beta > 0} \{\beta H^*(m/\beta) + \beta\},$$

where the infimum is attained. It follows that $\operatorname{dom} p^\circ = \operatorname{pos} \operatorname{dom} H^*$. The first claim thus follows from Theorem 6.

Clearly, $p^\circ(m) = \|m\|_H^*$ for all $m \in \mathcal{M}^{H^*}$ and

$$p^\circ(m) \le \inf_{\beta > 0} \{\beta H^*(m/\beta) + \beta \mid H^*(m/\beta) \le 1\} \le 2 \inf\{\beta > 0 \mid H^*(m/\beta) \le 1\}.$$

On the other hand, we have

$$p^\circ(m) = \inf_{\beta > 0} \{\beta H^*(m/\beta) + \beta\} = \inf_{\alpha > 0} \frac{g(\alpha m)}{\alpha},$$

where $g(m) = H^*(m) + 1$. Since $H^* \ge 0$, we have $g(m) \ge \|m\|_{H^*}$ when $\|m\|_{H^*} \le 1$. When $\|m\|_{H^*} > 1$, convexity and the fact that $H^*(0) = 0$ give

$$H^*(m/\|m\|_{H^*}) \le H^*(m)/\|m\|_{H^*}.$$

By definition of $\|m\|_{H^*}$, the left side equals 1 so $\|m\|_{H^*} \le H^*(m) \le g(m)$. Thus,

$$p^\circ(m) \ge \inf_{\alpha > 0} \frac{\|\alpha m\|_{H^*}}{\alpha} = \|m\|_{H^*}.$$

The expression for the dual of M^H follows from Corollary 7. If (H3) holds and $|u^\nu| \searrow 0$ almost surely in L^∞, then for all $\beta > 0$,

$$H(u^\nu/\beta) \searrow 0$$

so $p(u^\nu) \searrow 0$. In particular, (A3) holds so, by Corollary 7, $\mathcal{M}_s^{H^*} = \{0\}$.

To prove the last claim, let $u \in (\text{dom } H)^\infty$, $u^\nu := u\mathbb{1}_{|u| \leq \nu}$ and $\beta > 0$. By (H2), $u - u^\nu = u1_{\Omega \setminus \{|u| \leq \nu\}} \in \beta \text{ dom } H$ so (H4) implies

$$H((u - u^\nu)/\beta) \searrow 0.$$

Since $\beta > 0$ was arbitrary, we get $p(u - u^\nu) \searrow 0$ so $(\text{dom } H)^\infty \subseteq M^H$. To prove the converse, note first that the assumption $L^\infty \subset \text{dom } H$ implies $L^\infty \subset (\text{dom } H)^\infty$. It thus suffices to show that $(\text{dom } H)^\infty$ is closed in L^H. If (u^ν) is in $(\text{dom } H)^\infty$ and converges to $u \in M^H$, we have for any $\beta > 0$,

$$H(u/(2\beta)) \leq \frac{1}{2}H(u^\nu/\beta) + \frac{1}{2}H((u - u^\nu)/\beta) \leq \frac{1}{2}H(u^\nu/\beta) + \frac{1}{2}$$

for ν large enough, so $H(u/2\beta) < \infty$ and thus $u \in (\text{dom } H)^\infty$. $\qquad\square$

6.4 Musielak–Orlicz spaces

Let $\Phi : \mathbb{R} \times \Omega \to \overline{\mathbb{R}}$ be a random Young function in the sense that

$$\omega \mapsto \{(\xi, \alpha) \mid \Phi(\xi, \omega) \leq \alpha\}$$

is a measurable mapping (see [34,Chapter 14]) and for each $\omega \in \Omega$ the function $\Phi(\cdot, \omega)$ is convex, symmetric, vanishes at the origin and, for P-almost every ω, $\text{dom } \Phi(\cdot, \omega) \neq \{0\}$ and $\Phi(\cdot, \omega)$ is nonconstant. If Φ only takes finite real values, the measurability condition holds if and only if $\Phi(\xi, \cdot)$ is measurable for every $\xi \in \mathbb{R}$; see [34,Example 14.29].

The associated Musielak–Orlicz space is the normed space

$$L^\Phi := \{u \in L^0 \mid \|u\|_\Phi < \infty\},$$

where

$$\|u\|_\Phi := \inf\{\beta > 0 \mid E\Phi(|u|/\beta) \leq 1\}.$$

Theorem 22 gives a quick proof of the following characterization of the topological dual of L^Φ. The characterization involves the conjugate function

$$\Phi^*(\eta, \omega) := \sup_{\xi \in \mathbb{R}}\{\xi\eta - \Phi(\xi, \omega)\}.$$

By [34,Theorem 14.50], Φ^* is a random Young function.

Theorem 23 *Assume that* $\Phi(a, \cdot) \in L^1$ *for some constant* $a > 0$. *The space* L^Φ *is Banach and its dual* $(L^\Phi)^*$ *can be identified with*

$$L^{\Phi^*} \oplus \mathcal{M}_s^{\Phi^*} \oplus (L^\infty)^\perp,$$

where L^{Φ^*} coincides with the Köthe dual of L^Φ and

$$\mathcal{M}_s^{\Phi^*} = \{m \in \mathcal{M}_s \mid \sigma_\Phi(m) < \infty\}$$

with $\sigma_\Phi(m) := \sup_{u \in L^\infty}\{\int_\Omega u\, dm \mid E\Phi(|u|) < \infty\}$. For any $y + m^s \in L^{\Phi^*} \oplus \mathcal{M}_s^{\Phi^*}$, the dual norm can be expressed as

$$\begin{aligned}
\|y + m^s\|_\Phi^* &= \sup_{u \in L^\infty}\left\{E[u \cdot y] + \int_\Omega u\, dm^s \mid E\Phi(|u|) \le 1\right\} \\
&= \inf_{\beta > 0}\{\beta E\Phi^*(|y|^*/\beta) + \beta\} + \sigma_\Phi(m^s),
\end{aligned}$$

where the infimum is attained. Moreover,

$$\|y + m^s\|_{\Phi^*} \le \|y + m^s\|_\Phi^* \le 2\|y + m^s\|_{\Phi^*} \quad \forall y \in L^{\Phi^*}, \ m^s \in \mathcal{M}_s^{\Phi^*},$$

where

$$\|y + m^s\|_{\Phi^*} := \inf\{\beta > 0 \mid E\Phi^*(|y|/\beta) + \sigma_\Phi(m^s/\beta) \le 1\}.$$

The dual of the closure M^Φ of L^∞ in L^Φ is

$$(M^\Phi)^* = L^{\Phi^*} \oplus \mathcal{M}_s^{\Phi^*}.$$

If $\Phi(a, \cdot) \in L^1$ for all $a > 0$, then $\mathcal{M}_s^{\Phi^*} = \{0\}$ and

$$M^\Phi = \{\xi \in L^1 \mid E\Phi(|\xi|/\beta) < \infty \ \ \forall \beta > 0\}.$$

If $\operatorname{dom} E\Phi$ is a cone, then $L^\Phi = M^\Phi$ and their dual is L^{Φ^*}. The set $\operatorname{dom} E\Phi$ is a cone, in particular, if Φ satisfies Δ_2-condition: there exists $x_0 \in \operatorname{dom} E\Phi$ and $K > 0$ such that $\Phi(2x) \le K\Phi(x)$ for all $x \ge x_0$.

Proof We apply Theorem 22 to $H(u) := E\Phi(|u|)$. The assumption $\Phi(a) \in L^1$ implies that $H(u) < \infty$ when $\|u\|_{L^\infty} \le a$ so $L^\infty \subseteq \operatorname{pos}(\operatorname{dom} H)$. Defining

$$\eta(\omega) := \sup\{r > 0 \mid \Phi^*(r, \omega) \le 1\},$$

we have $E\Phi^*(\eta) \le 1$. Since $\sup_a \Phi(a) > 0$ almost surely, $\eta > 0$ almost surely. By Fenchel's inequality,

$$\Phi(|u|) \ge \eta|u| - \Phi^*(\eta),$$

so

$$\{u \in L^0 \mid H(u) \le 1\} \subset \{u \in L^0 \mid E[\eta|u|] \le 1 + E\Phi^*(\eta)\}.$$

Thus, the left is side is bounded in probability, since the right side is so. Thus, (H1) holds.

Condition (H2) holds since Φ is increasing. By [32,Theorem 1] and [31,Theorem 15.3],

$$H^*(m) = \sup_{u \in L^\infty} \left\{ \int u \, dm - E\Phi(u) \right\} = E\Phi^*(|y|^*) + \sigma_\Phi(m^s).$$

If $\Phi(a) \in L^1$ for all $a > 0$, then $L^\infty \subset \text{dom } H$ and (H3) and (H4) hold by monotone convergence theorem. If dom $E\Phi$ is a cone, then $\Phi(a, \cdot) \in L^1$ for all $a > 0$. All the claims except for the last one thus follow from Theorem 22. Assume the Δ_2-condition and let $x \in \text{dom } E\Phi$. We have $E\Phi(2x) \leq E\Phi(2(x \vee x_0)) \leq KE\Phi(x \vee x_0) = KE[\Phi(x) \vee \Phi(x_0)] < \infty$ so $2x \in \text{dom } E\Phi$. Since dom $E\Phi$ is a convex set, this implies that it is a cone. $\square$

If Φ is nonrandom, we recover the classical Orlicz spaces and the last part of Theorem 23 implies that, if Φ is finite, then $\mathcal{M}_s^{\Phi^*} = \{0\}$, while otherwise, $L^\Phi = L^\infty$ so $(L^\infty)^\perp = \{0\}$. Extensions to Banach space-valued functions have been studied in [12]. In [25], the assumption $\Phi(a, \cdot) \in L^1$ for all $a > 0$ is called "local integrability". Thus we recover [25,Theorem 13.17] for probability spaces without assuming local integrability of Φ^*; see also [24,Theorem 2.4.4]. Our characterization of the dual without local integrability seems new.

Proposition 24 *Assume that the measure P is atomless. In the setting of Theorem 23, $\mathcal{M}_s^{\Phi^*} = \{0\}$ if and only if $\Phi(a, \cdot) \in L^1$ for all $a > 0$.*

Proof By Theorem 23, $\Phi(a, \cdot) \in L^1$ for all $a > 0$ implies $\mathcal{M}_s^{\Phi^*} = \{0\}$, where $\mathcal{M}_s^{\Phi^*} = \text{dom } \sigma_\Phi \cap \mathcal{M}_s \neq \{0\}$. Assume now that $\Phi(a, \cdot) \notin L^1$ for some $a > 0$. It suffices to show that dom $\sigma_\Phi \cap \mathcal{M}_s \neq \{0\}$. There exists $a' > a$ and $\epsilon > 0$ such that $E\Phi(u) = +\infty$ for every $u \in \mathbb{B}_\epsilon(a')$, so the L^∞-strong closure of dom $E\Phi$ is strictly smaller than L^∞. Since the strong closure is the domain of the $\sigma(L^\infty, \mathcal{M})$-conjugate of σ_Φ, dom $\sigma_\Phi \neq \{0\}$.

By Lemma 31, the $\sigma(L^\infty, L^1)$-closure of dom $E\Phi$ equals

$$\{u \in L^\infty \mid |u(\omega)| \leq \rho(\omega) \text{ a.s.}\},$$

where

$$\rho(\omega) := \inf\{a > 0 \mid \Phi(a, \omega) = \infty\}.$$

If $\rho = \infty$ almost surely, then $\sigma(L^\infty, L^1)$-closure of dom $E\Phi$ is the whole L^∞, so $\sigma_\Phi(y) = +\infty$ for every nonzero $y \in L^1$, and thus dom $\sigma_\Phi \cap \mathcal{M}_s \neq \{0\}$.

If $P(\rho < \infty) > 0$, there exists a constant $\bar{a} > 0$ such that $P(A) > 0$, where $A := \{\rho \leq \bar{a}\}$. Since P is atomless, there exists $A^\nu \subset A$ such that $P(A^\nu) > 0$ and $P(A^\nu) \searrow 0$. There exists a nonzero $m^s \in \mathcal{M}_s$ with $m^s(\Omega \setminus A^\nu) = 0$ for every ν (e.g., take m^s as an accumulation point of the $\sigma(\mathcal{M}, L^\infty)$-precompact net $(y^\nu)_{\nu=1}^\infty \subset L^1$, where $y^\nu := 1_{A^\nu}/P(A^\nu)$). Every $u \in L^\infty \cap \text{dom } E\Phi$ satisfies $|u| \leq \rho \leq \bar{a}$ almost

surely on A^ν, so every such u satisfies $\langle u, m^s \rangle \leq \bar{a}|m^s|^*$. Thus $\sigma_\Phi(m^s) < \infty$ and dom $\sigma_\Phi \cap \mathcal{M}_s \neq \{0\}$. □

General Musielak–Orlicz spaces are isomorphic to Musielak–Orlicz spaces whose Young function satisfies the the assumption in Theorem 23.

Remark 25 Given an arbitrary random Young function $\tilde{\Phi}$,

$$\rho(\omega) := \sup\{r \in \mathbb{R} \mid \tilde{\Phi}(r, \omega) \leq 1\}$$

is measurable and takes values in $(0, \infty)$ almost surely. The space $L^{\tilde{\Phi}}$ is isomorphic to L^Φ, where

$$\Phi(u, \omega) := \tilde{\Phi}(\rho(\omega)u, \omega)$$

satisfies the first assumption of Theorem 23. The dual of $L^{\tilde{\Phi}}$ is thus isomorphic to that of L^Φ characterized in Theorem 23. Indeed, the isomorphism is $(A\tilde{u})(\omega) = \tilde{u}(\omega)/\rho(\omega)$ so the elements of $(L^{\tilde{\Phi}})^*$ can be expressed as

$$\tilde{u} \mapsto \langle A\tilde{u}, u^* \rangle,$$

where $u^* \in (L^\Phi)^*$.

6.5 Generalized Musielak–Orlicz spaces

Let r be an lsc norm on L^0 satisfying (A1) and (A2) such that the r-topology is stronger than that of L^0. By Remark 9, the space L_r is Banach. Let Φ be as in Sect. 6.4 and define

$$L^{\Phi,r} := \{u \in L^0 \mid \|u\|_{\Phi,r} < \infty\},$$

where

$$\|u\|_{\Phi,r} := \inf\{\beta > 0 \mid r(\Phi(|u|/\beta)) \leq 1)\}.$$

Note that, if r is the L^1-norm, then $L^{\Phi,r}$ is the Musielak–Orlicz space studied in Sect. 6.4. If Φ is nonrandom and r is the Lorentz-norm

$$r(x) = \int_0^1 q_x(t)d\phi(t)$$

associated with a concave function ϕ (see Sect. 6.2), we get

$$r(\Phi(|u|)) = \int_0^1 q_{\Phi(|u|)}(t)d\phi(t) = \int_0^1 \Phi(q_u(t))d\phi(t)$$

so $L^{\Phi,r}$ becomes the Orlicz–Lorentz space; see e.g. [10] and its references.

Consider the positively homogeneous function

$$\Sigma(x^*, m) := \sup_{x \in L_r, u \in L^\infty} \{\langle x, x^* \rangle + \langle u, m \rangle \mid \Phi(|u|) + x \leq 0 \ a.s.\}$$

on $L_r^* \times \mathcal{M}$ and let

$$\mathcal{M}^\Sigma := \{m \in \mathcal{M} \mid \exists x^* \in L_r^* : \ \Sigma(x^*, m) < \infty\}.$$

Denote the absolutely continuous and singular elements of $\mathcal{M}^\Sigma$ by L^Σ and $\mathcal{M}_s^\Sigma$, respectively. Let

$$\sigma(v, y, \omega) := \sup_{x \in \mathbb{R}, u \in \mathbb{R}^n} \{xv + u \cdot y \mid \Phi(|u|, \omega) + x \leq 0\}$$

$$= \begin{cases} v\Phi^*(|y|/v, \omega) & \text{if } v > 0 \\ (\Phi^*)^\infty(|y|, \omega) & \text{otherwise.} \end{cases}$$

The second expression above comes from [31,Theorem 13.3]. It involves the *recession function* of the conjugate Φ^* defined as

$$(\Phi^*)^\infty(\eta, \omega) := \lim_{v \searrow 0} v\Phi^*(\eta/v, \omega).$$

Theorem 26 *Assume that* $\Phi(|u|) \in L_r$ *for all* $u \in L^\infty$ *and that* $\{x \in L^0 \mid r(x) \leq 1\}$ *is bounded in* L^0. *Endowed with the norm* $\|u\|_{\Phi,r}$, *the space* $L^{\Phi,r}$ *is a Banach, its dual may be identified with*

$$L^\Sigma \oplus \mathcal{M}_s^\Sigma \oplus (L^\infty)^\perp$$

and L^Σ *is the Köthe dual of* $L^{\Phi,r}$. *For any* $m \in \mathcal{M}^\Sigma$, *the dual norm can be expressed as*

$$\|m\|_{\Phi,r}^* = \inf_{x^* \in L_r^*} \{\Sigma(x^*, m) + r^\circ(x^*)\},$$

where the infimum is attained. Moreover,

$$\|m\|_{\Sigma,r^\circ} \leq \|m\|_{\Phi,r}^* \leq 2\|m\|_{\Sigma,r^\circ},$$

where

$$\|m\|_{\Sigma,r^\circ} := \inf_{x^* \in L_r^*} \max\{\Sigma(x^*, m), r^\circ(x^*)\}.$$

The dual of the closure $M^{\Phi,r}$ *of* L^∞ *in* $L^{\Phi,r}$ *can be identified with* $L^\Sigma \oplus \mathcal{M}_s^\Sigma$.

If (a) r *satisfies (A4) or (b)* r *satisfies (A3) and* $\Phi(u) \in L^\infty$ *for every* $u \in L^\infty$, *then* $\mathcal{M}_s^\Sigma = \{0\}$,

$$L^\Sigma = \{y \in L^1 \mid \exists v \in L^{r^\circ} : E\sigma(v, y) < \infty\},$$

and the dual of $M^{\Phi,r}$ is L^Σ. If r satisfies (A4), then

$$M^{\Phi,r} = \{u \in L^0 \mid r(\Phi(|u|/\beta)) < \infty \quad \forall \beta > 0\},$$

and, $M^{\Phi,r} = L^{\Phi,r}$ if there exists $x_0 \in L^0$ and $K > 0$ such that $r(\Phi(x_0)) < \infty$ and $\Phi(2x) \le K\Phi(x)$ for all $x \ge x_0$.

Proof This fits Theorem 22 with

$$H(u) := \begin{cases} r(\Phi(|u|)) & \text{if } \Phi(|u|) \in L^0, \\ +\infty & \text{otherwise.} \end{cases}$$

Given an L^0-converging sequence $u^\nu \to u$, the pointwise lower semicontinuity of Φ gives $\liminf \Phi(u^\nu) \ge \Phi(u)$ so the lower semicontinuity and (A2) of r give

$$\liminf H(u^\nu) \ge r(\liminf \Phi(u^\nu)) \ge H(u),$$

so H is L^0-lsc. As in the proof of Theorem 23, there exists an $\eta \in L^0$ strictly positive such that $\Phi(|u|) \ge |u|\eta - \Phi^*(\eta)$ almost surely. We have

$$\{u \in L^0 \mid H(u) \le 1\} \subseteq \{u \in L^0 \mid r(|u|\eta - \Phi^*(\eta)) \le 1\},$$

where the right side is L^0-bounded since $\{\xi \in L^0 \mid r(\xi) \le 1\}$ is L^0-bounded by assumption. Since $\Phi(u) \in L_r$ for all $u \in L^\infty$, we have $L^\infty \subset \text{dom } H$. Thus, H satisfies (H1)–(H2).

We compute the conjugate H by employing conjugate duality; see [33]. The function

$$r_+(x) := \inf_{x' \ge x} r(x')$$

is convex and increasing in the partial order of L^0 so the function $F(x, u) := r_+(\Phi(|u|) + x)$ is convex on $L_r \times L^\infty$. Clearly, $F(0, u) = H(u)$. The conjugate F^* on $L_r^* \times \mathcal{M}$ has the expression

$$
\begin{aligned}
F^*(x^*, m) &:= \sup_{x \in L_r, u \in L^\infty} \{\langle x, x^* \rangle + \langle u, m \rangle - r_+(\Phi(|u|) + x)\} \\
&= \sup_{x, x' \in L_r, u \in L^\infty} \{\langle x, x^* \rangle + \langle u, m \rangle - r(x') \mid \Phi(|u|) + x \le x'\} \\
&= \sup_{x'', x' \in L_r, u \in L^\infty} \{\langle x' + x'', x^* \rangle + \langle u, m \rangle - r(x') \mid \Phi(|u|) + x'' \le 0\} \\
&= \Sigma(x^*, m) + r^*(x^*),
\end{aligned}
$$

where, by positive homogeneity of r,

$$r^*(x^*) = \sup_{x \in L_r} \{\langle x, x^* \rangle - r(x)\} = \begin{cases} 0 & \text{if } r^\circ(x^*) \leq 1, \\ +\infty & \text{otherwise,} \end{cases}$$

where $r^\circ(x^*) := \sup\{\langle x, x^* \rangle \mid r(x) \leq 1\}$. For any $u \in L^\infty$, the function $F(\cdot, u)$ is continuous on L_r, so [33,Theorem 17] gives

$$H^*(m) = \inf_{x^* \in L_r^*} F^*(x^*, m) = \inf_{x^* \in L_r^*} \{\Sigma(x^*, m) \mid r^\circ(x^*) \leq 1\},$$

where the infimum is attained. Thus, by Theorem 22 and positive homogeneity of Σ,

$$\begin{aligned} \|m\|_{\Phi,r}^* &= \inf_{\beta > 0} \{\beta H^*(m/\beta) + \beta\} \\ &= \inf_{\beta > 0, x^* \in L_r^*} \{\beta \Sigma(x^*, m/\beta) + \beta \mid r^\circ(x^*) \leq 1\} \\ &= \inf_{\beta > 0, x^* \in L_r^*} \{\Sigma(x^*, m) + \beta \mid r^\circ(x^*) \leq \beta\} \\ &= \inf_{x^* \in L_r^*} \{\Sigma(x^*, m) + r^\circ(x^*)\}. \end{aligned}$$

The dual Luxemburg norm can be expressed as

$$\begin{aligned} \|m\|_{H^*} &= \inf\{\beta > 0 \mid H^*(m/\beta) \leq 1\} \\ &= \inf\{\beta > 0 \mid \exists x^* \in L_r^* : \Sigma(x^*, m/\beta) \leq 1, \ r^\circ(x^*) \leq 1\} \\ &= \inf\{\beta > 0 \mid \exists x^* \in L_r^* : \Sigma(x^*, m) \leq \beta, \ r^\circ(x^*) \leq \beta\} \\ &= \inf_{x^* \in L_r^*} \max\{\Sigma(x^*, m), r^\circ(x^*)\}. \end{aligned}$$

Since Σ is positively homogeneous, we have

$$\text{pos dom } H^* = \{m \in \mathcal{M} \mid \exists x^* \in L_r^* : \ \Sigma(x^*, m) < \infty\}.$$

Thus, the claims concerning the dual space and its norm follow from Theorem 22.

By the first assumption, $L^\infty \subset \text{dom } H$ so both (a) and (b) imply (H3) and then, $\mathcal{M}_s^\Sigma = \{0\}$, by Theorem 22. Let $x^* \in L_r^*$ and $y \in L^\Sigma$. By Theorem 6, (b) implies the existence $v \in L^1$ and $x_s^* \in (L^\infty)^\perp$ such that $x^* = v + x_s^*$. Under (a), $x_s^* = 0$. We have $\Sigma(x^*, y) = +\infty$ unless $x^* \geq 0$. Otherwise, the supremum in the definition of Σ is attained with $x = \Phi(|u|)$ which belongs to L^∞ under (b). We, thus have $\Sigma(x^*, y) = \Sigma(v, y)$ for all $x^* \geq 0$. By [34,Theorem 14.60],

$$\begin{aligned} \Sigma(v, y) &= \sup_{x \in L_r, u \in L^\infty} \{E[xv + u \cdot y] \mid \Phi(|u|) + x \leq 0 \ a.s.\} \\ &= E\left[\sup_{x \in \mathbb{R}, u \in \mathbb{R}^n} \{xv + u \cdot y \mid \Phi(|u|) + x \leq 0\} \right] = E\sigma(v, y). \end{aligned}$$

If r satisfies (A4) then both (H3) and (H4) hold, so Theorem 22 gives

$$M^\Sigma = (\text{dom } H)^\infty.$$

The set on the right can be written as $\{u \in L^0 \mid r(\Phi(|u|/\beta)) < \infty \ \forall \beta > 0\}$. Let $x \in L^0_+$ be such that $r(\Phi(x)) < \infty$. Under the last condition,

$$\Phi(2x) \le \Phi(2(x \vee x_0)) \le K\Phi(x \vee x_0) = K[\Phi(x) \vee \Phi(x_0)]$$

so

$$r(\Phi(2x)) \le Kr(\Phi(x) \vee \Phi(x_0)) \le Kr(\Phi(x) + \Phi(x_0)) \le K[r(\Phi(x)) + r(\Phi(x_0))]$$

and thus, $r(\Phi(2x)) < \infty$. Since $\{u \in L^0 \mid r(\Phi(|u|)) < \infty\}$ is a convex set, it has to be a cone so $L^{r,\Phi} = M^{r,\Phi}$. □

The proof of Theorem 26 gives also the expression

$$\|m\|_{\Phi,r^\circ} = \inf\{\beta > 0 \mid H^*(m/\beta) \le 1\},$$

where

$$H^*(m) = \inf_{x^* \in L^*_r} \{\Sigma(x^*, m) \mid r^\circ(x^*) \le 1\},$$

and the infimum is attained. Under condition (b) in the theorem, we have $\mathcal{M}^\Sigma_s = \{0\}$ and

$$L^\Sigma = \{y \in L^1 \mid h^*(y/\beta) \le 1\},$$

where

$$h^*(y) = \inf_{v \in L^{r^\circ}} \{E\sigma(v, y) \mid r^\circ(v) \le 1\}.$$

If r is the Lorentz-norm associated with a concave increasing function ϕ, then, by Theorem 21, r° is the Marcinkiewicz-norm so

$$h^*(y) = \inf_{v \in L^{r^\circ}} \{E\sigma(v, y) \mid \int_0^t q_v \le \phi(t) \ \forall t, \ v \ge 0\}.$$

The above characterization of the Köthe dual thus extends that in [16,Corollary 4.12] and [15,Theorem 2.2] to random Φ in the case of a finite underlying measure. The singular components of the dual have been analyzed in the recent article [14].

6.6 Risk measures

This section studies scalar-valued random variables so $n = 1$. A function $\rho : L^0 \to \overline{\mathbb{R}}$ is a *convex risk measure* if it is convex, nondecreasing, $\rho(0) = 0$ and $\rho(\xi + r) = \rho(\xi) + r$ for all $\xi \in L^0$ and $r \in \mathbb{R}$; see e.g. [7]. As in [21], we consider the associated normed space

$$L^\rho := \{u \in L^0 \mid \|u\|_\rho < \infty\},$$

where

$$\|u\|_\rho := \inf\{\beta > 0 \mid \rho(|u|/\beta) \le 1\}.$$

When ρ is positively homogeneous, we simply get $\|u\|_\rho = \rho(|u|)$ as e.g. in [27].

Theorem 22 gives a quick proof of the following.

Theorem 27 *Assume that ρ is L^0-lsc and that $\{u \in L^0 \mid \rho(|u|) \le 1\}$ is bounded in L^0. The space L^ρ is a Banach and its dual can be identified with*

$$\mathcal{M}^\alpha \oplus (L^\infty)^\perp,$$

where

$$\mathcal{M}^\alpha = \{m \in \mathcal{M} \mid \exists \beta > 0 : \alpha(|m|/\beta) < \infty\}$$

with $\alpha : \mathcal{M} \to \overline{\mathbb{R}}$ defined by

$$\alpha(m) := \sup_{\xi \in L^\infty_+} \left\{ \int_\Omega \xi\, dm - \rho(\xi) \right\}.$$

For any $m \in \mathcal{M}^\alpha$, the dual norm can be expressed as

$$\|m\|_\rho^* = \sup_{u \in L^\infty} \left\{ \int_\Omega u\, dm \mid \rho(u) \le 1 \right\} = \inf_{\beta > 0} \left\{ \beta\alpha(|m|/\beta) + \beta \right\},$$

where the infimum is attained. Moreover,

$$\|m\|_\alpha \le \|m\|_\rho^* \le 2\|m\|_\alpha,$$

where

$$\|m\|_\alpha := \inf\{\beta > 0 \mid \alpha(|m|/\beta) \le 1\}.$$

If ρ *has the Lebesgue property on* L^∞: $\rho(\xi^\nu) \searrow 0$ *for any decreasing sequence* $(\xi^\nu) \subset L^\infty$ *with* $\xi^\nu \searrow 0$ *almost surely, then the dual of the closure* M^ρ *of* L^∞ *in* L^ρ *can be identified with*

$$L^\alpha := \{y \in L^1 \mid \exists \beta > 0 : \alpha(|y|/\beta) < \infty\}$$

and $(L^\rho)^* = L^\alpha \oplus (L^\infty)^\perp$. If ρ has the Lebesgue property on $\mathrm{dom}\,\rho$: $\rho(\xi^\nu) \searrow 0$ for any decreasing sequence $(\xi^\nu) \subset \mathrm{dom}\,\rho$ with $\xi^\nu \searrow 0$ almost surely, then

$$M^\rho = \{u \in L^1 \mid \rho(|u|/\beta) < \infty \;\forall \beta > 0\},$$

and, in particular, $L^\rho = M^\rho$ if $\mathrm{dom}\,\rho$ is a cone.

Proof We apply Theorem 22 to the function $H(u) := \rho(|u|)$. By assumption, (H1) and (H2) hold. Indeed, by monotonicity and translation invariance, $\rho(|u|) \le \rho(\|u\|_{L^\infty}) = \|u\|_{L^\infty}$, so $L^\infty \subset \mathrm{dom}\,H$. Conditions (H3) and (H4) in Theorem 22 translate directly to 1 and 2. Thus the claims follow from Theorem 22, since here

$$
\begin{aligned}
H^*(m) &:= \sup_{u \in L^\infty} \left\{ \int u\,dm - \rho(|u|) \right\} \\
&= \sup_{u \in L^\infty, \xi \in L^\infty_+} \left\{ \int u\xi\,dm - \rho(\xi) \mid |u| = 1 \right\} \\
&= \sup_{\xi \in L^\infty_+} \left\{ \int \xi\,d|m| - \rho(\xi) \right\} \\
&= \alpha(|m|),
\end{aligned}
$$

where the second last equality follows from [37,Theorem 2.3] and the fact that $\nu(A) := \int_A \xi\,dm$ is a finitely additive measure with $|\nu|(A) = \int_A \xi\,d|m|$.   □

The fact that $(L^\rho)^* = L^\alpha \oplus (L^\infty)^\perp$ under the Lebesgue property sharpens [21,Theorem 4.12] which states that each $u^* \in (L^\rho)^*$ can be expressed uniquely as $u^* = y + u^s$ for some $y \in L^1$ and $u^s \in (L^\infty)^\perp$. The other statements seem new.

Appendix

The first part of this appendix studies integration of measurable not-necessarily bounded functions with respect to a real-valued finitely additive measure m. Define

$r_m : L^0_+ \to \overline{\mathbb{R}}$ by

$$r_m(\eta) := \sup_{u' \in L^\infty} \left\{ \int_\Omega u' dm \mid |u'| \le \eta \right\}.$$

Lemma 28 *Let m be a real-valued finitely additive measure.*

1. For any $\eta \in L^\infty_+$,

$$r_m(\eta) = \sup_{u' \in L^\infty} \left\{ \int_\Omega \eta(u' dm) \,\middle|\, |u'| \le 1 \right\} \le ||\eta||_{L^\infty} ||m||_{TV}.$$

In particular, r_m is L^∞-norm continuous and sublinear relative to L^∞_+.
2. For every $\eta \in L^0_+$,

$$r_m(\eta) = \lim_{v \nearrow \infty} r_m(\eta \wedge v)$$

3. r_m is subadditive in the sense that

$$r_m(\eta^1 + \eta^2) \le r_m(\eta^1) + r_m(\eta^2) \quad \forall \eta^1, \eta^2 \in \operatorname{dom} r_m.$$

Proof The expression in 1 follows from the change of variables $\tilde{u} = \eta u'$. As to 2, the inequality $r_m(\eta) \ge \lim_v r_m(\eta \wedge v)$ is clear. To prove the opposite inequality, let $\alpha \in \mathbb{R}$ with $r_m(\eta) > \alpha$. There exists $u' \in L^\infty$ with $|u'| \le \eta$ and $r_m(|u'|) > \alpha$. Then $|u'| \wedge v \to |u'|$ in L^∞-norm, so monotonicity and 1 give

$$\lim r_m(\eta \wedge v) \ge \lim r_m(|u'| \wedge v) > \alpha.$$

This proves 2. Given $\eta^1, \eta^2 \in \operatorname{dom} p$ and $\eta > 0$, we have $(\eta^1 + \eta^2) \wedge v \le \eta^1 \wedge v + \eta^2 \wedge v$. Indeed, a concave function vanishing at the origin is subadditive on the positive reals. Thus, by 2 and 1,

$$
\begin{aligned}
r_m(\eta^1 + \eta^2) &= \limsup_v r_m((\eta^1 + \eta^2) \wedge v) \\
&\le \limsup_v (r_m(\eta^1 \wedge v) + r_m(\eta^2 \wedge v)) \\
&\le \limsup_v r_m(\eta^1 \wedge v) + \limsup_v r_m(\eta^2 \wedge v) \\
&= r_m(\eta^1) + r_m(\eta^2),
\end{aligned}
$$

which completes the proof. □

Define $\rho_m : L^0 \to \overline{\mathbb{R}}$ by

$$\rho_m(u) := r_m(|u|).$$

Theorem 29 *For any real-valued finitely additive measure m,*

1. ρ_m is symmetric and sublinear, and $\rho_m(u') \leq \rho_m(u)$ whenever $|u'| \leq |u|$,
2. for any $u \in \text{dom } \rho_m$ and $\epsilon > 0$, there exists $u' \in L^\infty$ with $\rho_m(u - u') < \epsilon$,
3. $\int_\Omega u\, dm$ has a unique ρ_m-continuous linear extension from L^∞ to $\text{dom } \rho_m$,
4. if m is purely finite additive, there exists a decreasing $(A^v)_{v=1}^\infty \subset \mathcal{F}$ with $P(A^v) \searrow 0$ and $\int_\Omega u 1_{\Omega \setminus A^v}\, dm = 0$ for all $u \in \text{dom } \rho_m$.

Proof Properties in 1 are clear. To prove 2, assume first that m is nonnegative. Given $u^i \in \text{dom } \rho_m \cap L^0_+$ and $\epsilon > 0$, let $\tilde{u}^i \in L^\infty$ be such that $0 \leq \tilde{u}^i \leq u^i$ and $\rho_j(u^i) \leq \langle \tilde{u}^i, m \rangle + \epsilon$. Then $\tilde{u}^1 + \tilde{u}^2 \leq u^1 + u^2$ and

$$\rho_m(u^1) + \rho_m(u^2) \leq \langle \tilde{u}^1 + \tilde{u}^2, m \rangle + 2\epsilon \leq \rho_m(u^1 + u^2) + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, ρ_m is superlinear on $\text{dom } \rho_m \cap L^0_+$. Given $u \in \text{dom } \rho_m$ and $\epsilon > 0$, Lemma 28 gives $\rho_m(u^+) \leq \rho_m(u^+ \wedge v) + \epsilon$ for v large enough. By superlinearity,

$$\rho_m(u^+ - u^+ \wedge v) + \rho_m(u^+ \wedge v) \leq \rho_m(u^+) \leq \rho_m(u^+ \wedge v) + \epsilon.$$

Similarly, $\rho_m(u^- - u^- \wedge v) \leq \epsilon$, so $\rho_m(u - \pi_{v\mathbb{B}} u) \leq 2\epsilon$ by sublinearity of ρ_m. By [37,Theorem 1.12], general $m \in \mathcal{M}$ can be written as $m = m^+ - m^-$ for nonnegative $m^+, m^- \in \mathcal{M}$, so

$$\rho_m(u - \pi_{v\mathbb{B}} u) \leq \rho_{m^+}(u - \pi_{v\mathbb{B}} u) + \rho_{m^-}(u - \pi_{v\mathbb{B}} u) \leq 4\epsilon$$

for v large enough.

We have $\int_\Omega u\, dm \leq \rho_m(u)$ on L^∞, so, by Hahn-Banach, there exists a ρ_m-continuous linear extension of m to $\text{dom } \rho_m$. Since L^∞ is dense in $\text{dom } \rho_m$, the extension is unique. If m is purely finitely additive, there exists $(A^v)_{v=1}^\infty \subset \mathcal{F}$ with $P(A^v) \searrow 0$ and $\int_\Omega u 1_{\Omega \setminus A^v}\, dm = 0$ for all $u \in L^\infty$. Note that r_m inherits this property so that ρ_m and the integral does as well. $\qquad\square$

The following was used in the proof of Theorem 21. Its proof was given as an exercise on page 89 of [1].

Lemma 30 *We have*

$$\int_0^t q_u(t)\, dt = \inf_{s \in \mathbb{R}^+} \{ ts + E[|u| - s]^+ \}.$$

Proof By Theorems 23.5 and 24.2 of [31], the functions

$$f(t) := \int_0^t q_u(s)\, ds$$

and

$$f^*(s) = \int_0^s n_u(\tau)\, d\tau - \int_0^\infty n_u(\tau)\, d\tau = -\int_s^\infty n_u(\tau)\, d\tau$$

are concave and conjugate to each other. By Fubini,

$$f^*(s) = -E \int_s^\infty 1_{\{|u|>\tau\}} d\tau = -E[|u| - s]^+$$

so

$$\int_0^t q_u(s) ds = \inf_{s \in \mathbb{R}_+} \{ts - f^*(s)\} = \inf_{s \in \mathbb{R}_+} \{ts + E[|u| - s]^+\},$$

by the biconjugate theorem (see e.g. [31,Theorem 12.2]). $\qquad\square$

The following lemma was used in the proof of Proposition 24. Let $\Phi : \mathbb{R} \times \Omega \to \overline{\mathbb{R}}$ be a convex normal integrand in the sense that

$$\omega \mapsto \{(\xi, \alpha) \mid \Phi(\xi, \omega) \leq \alpha\}$$

is a convex-valued measurable mapping (see [34,Chapter 14]). The associated integral functional $E\Phi : L^0 \to \overline{\mathbb{R}}$ is defined by

$$E\Phi(u) := \int_\Omega \Phi(u(\omega), \omega) dP(\omega).$$

By [34,Theorem 14.50], $\Phi^*(y, \omega) = \sup_u \{u \cdot y - \Phi(u, \omega)\}$ is a normal integrand. The integral functional $E\Phi^* : L^0 \to \overline{\mathbb{R}}$ is defined similarly.

Lemma 31 *Assume that $\mathcal{U}$ and $\mathcal{Y}$ are decomposable spaces in separating duality under the bilinear form*

$$\langle u, y \rangle := E[u \cdot y].$$

If $E\Phi : \mathcal{U} \to \overline{\mathbb{R}}$ and $E\Phi^* : \mathcal{Y} \to \overline{\mathbb{R}}$ are finite somewhere, then the $\sigma(\mathcal{U}, \mathcal{Y})$-closure of $\{u \in \mathcal{U} \mid u \in \text{dom } E\Phi\}$ is $\{u \in \mathcal{U} \mid u \in \text{cl dom } \Phi \text{ a.s.}\}$.

Proof By [30,Theorem 2], $(E\Phi)^*(y) = E\Phi^*(y)$. Thus, by [29,Corollary 3D], $\sigma_{\text{cl dom } E\Phi} = (E\Phi^*)^\infty$. By monotone convergence theorem, $(E\Phi^*)^\infty = E(\Phi^*)^\infty$, where, by [29,Corollary 3D] again, $(\Phi^*)^\infty = \sigma_{\text{cl dom } \Phi}$. By [30,Theorem 2] again, $(E\sigma_{\text{cl dom } \Phi})^* = E\delta_{\text{cl dom } \Phi}$, so the claim follows from the biconjugate theorem. $\qquad\square$

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