

RESEARCH ARTICLE

Sasaki structures distinguished by their basic Hodge numbers

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Abstract

In all odd dimensions at least 5 we produce examples of manifolds admitting pairs of Sasaki structures with different basic Hodge numbers. In dimension 5 we prove more precise results, for example, we show that on connected sums of copies of $S^2 \times S^3$ the number of Sasaki structures with different basic Hodge numbers within a fixed homotopy class of almost contact structures is unbounded. All the Sasaki structures we consider are negative in the sense that the basic first Chern class is represented by a negative definite form of type (1,1). We also discuss the relation of these results to contact topology.

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1 | INTRODUCTION

A Sasaki structure consists of a contact form together with a compatible Riemannian metric and a transverse complex structure, which define a transverse Kähler structure for the Reeb foliation. These structures are the odd-dimensional analogs of Kähler or projective-algebraic complex structures on even-dimensional manifolds. The purpose of this paper is to exhibit families of closed manifolds which support two or more Sasaki structures distinguished by their transverse Hodge numbers. Our most interesting examples are in dimension 5, but we also prove results in arbitrary higher dimensions.

There are several motivations for looking for manifolds with multiple Sasaki structures, some of them coming from contact topology and others coming from transverse Kähler geometry, involving not only the Hodge numbers but also the type or sign of the basic first Chern class. Before reviewing these motivations, we briefly discuss the topological (non-)invariance of Hodge numbers on Kähler manifolds, particularly on complete intersections. This is of independent interest, and will be useful for building examples in the Sasaki context.

1.1 | Hodge numbers of Kähler manifolds

For compact Kähler manifolds the Hodge decomposition of the cohomology shows that certain linear combinations of Hodge numbers are topological invariants. The question whether there are additional linear combinations that are topological invariants was asked first by Hirzebruch in his famous collection of open problems from 1954, but was only resolved less than ten years ago by the first named author and Schreieder [23]. Note that for questions like these the intuition gained from small dimensions may be misleading, since in complex dimension 1 all Hodge numbers are determined by Betti numbers, and in complex dimension 2 they are determined by the Betti numbers and the signature. However, starting in dimension 3, there are always more Hodge numbers than there are obvious topological invariants. Indeed the answer to Hirzebruch's question obtained in [23] is that the only linear combinations of Hodge numbers which are invariant under orientation-preserving diffeomorphisms between compact Kähler manifolds are those that can be expressed through the Betti numbers and the signature.

After the results of [23] the question arose whether more Hodge numbers or combinations thereof might be invariant if one restricts the class of Kähler manifolds one considers, for example, by looking only at manifolds with a simple enough cohomology structure. In this direction, Chataur [7] asked whether diffeomorphic complete intersections must always have the same Hodge numbers. The first dimension for which this question is interesting is dimension 3. It is not hard to see from the Hirzebruch–Riemann–Roch theorem that two three-dimensional complete intersections have different Hodge numbers if and only if they have different first Chern classes; see Lemma 8. Since examples with distinct first Chern classes were found by Wang and Du [32], this answers Chataur's question in the negative. After this work was completed, we found that this conclusion, based on the same examples, was also reached recently by Wang, Yu and Wang [33].

1.2 | Basic Hodge numbers of Sasaki manifolds

For a Sasaki structure the orbits of the Reeb flow are the leaves of a Riemannian foliation, for which one can consider the basic cohomology, giving rise to the notion of basic Betti numbers. The properties of the basic cohomology were studied by El Kacimi-Alaoui, Hector and Nicolau [11–13]. One of their conclusions is that the basic Betti numbers are finite, and are topological invariants of the underlying manifold, so they do not depend on the choice of Sasaki structure; compare [4, Theorem 7.4.14].

Since the Reeb foliation is transversely Kähler, the basic cohomology has a Hodge decomposition mimicking precisely what happens for Kähler manifolds. Once again there are linear combinations of Hodge numbers which can be expressed through the Betti numbers, and are therefore topological invariants. One can now ask for these basic Hodge numbers of Sasaki manifolds the analog of Hirzebruch's question for compact Kähler manifolds which we discussed above.

Concerning this question, it has been known for some time that the basic Hodge numbers are not always topological invariants. However, up to now, there has been only a single example of two Sasaki structures on the same manifold but with different basic Hodge numbers. This example was alluded to by Boyer and Galicki in their monograph [4, p. 234], but did not appear in print until the work of Goertsches, Nozawa and Töben [15]. In that example, see [15, Example 3.4], one of the Sasaki structures is positive, in the sense that its basic first Chern class is represented by a positive (1,1)-form, which implies the vanishing of certain Hodge numbers. The other Sasaki structure is null rather than positive, and has different Hodge numbers. This raises the further question whether different Hodge numbers can only occur for pairs of Sasaki structures with different type.

Very recently, Ražny [29] proved that the basic Hodge numbers of Sasaki manifolds are constant in arbitrary smooth deformations of the Sasaki structure. This completes earlier results of Boyer and Galicki [4] and of Goertsches, Nozawa and Töben [15] who proved the invariance for special types of deformations. Ražny's result gives further impetus to the study of basic Hodge numbers, since it shows they provide a tool, independent of contact topology, to distinguish deformation classes of Sasaki structures.

We show in this paper that in every odd dimension at least 5 there are manifolds which have Sasaki structures with different Hodge numbers. Such examples exist in infinitely many distinct homotopy types in each dimension. Moreover, unlike in the example from [4, 15] mentioned above, they can be chosen to be of the same type, so that the difference in Hodge numbers is not explained by the type or sign of the first Chern class. We use Sasaki structures of negative type, which are of course plentiful as they can be constructed from complex projective manifolds with ample canonical bundle.

Given the results of [23] discussed above, it should not be surprising that the basic Hodge numbers of diffeomorphic Sasaki manifolds may disagree. Indeed, whenever we have two diffeomorphic closed Kähler manifolds with integral Kähler classes, we can try to build diffeomorphic Sasaki manifolds by the so-called Boothby–Wang construction [2], using circle bundles whose Euler classes are the Kähler classes. For these special Sasaki structures the basic Hodge numbers agree with the usual Hodge numbers of the base manifolds, so if the two Kähler manifolds one starts with have different Hodge numbers, then so do the Sasaki structures on the circle bundles. The difficulty in turning this heuristic into a rigorous argument is that the diffeomorphism of Kähler manifolds one starts with will only lift to a diffeomorphism of the total spaces of the circle bundles if it maps one Euler class to the other, that is, it sends one Kähler class to the other. For manifolds with large second Betti number this is a condition which can be hard to check, since one has to compare the Kähler cones of distinct complex structures which are completely unrelated to each other. However, if the Kähler manifolds have $b_2 = 1$, then this difficulty does not arise. Therefore, the examples of complex three-dimensional diffeomorphic complete intersections with distinct Hodge numbers mentioned above do indeed give examples of Sasaki structures on diffeomorphic 7-manifolds but with distinct Hodge numbers.

This kind of argument does not produce five-dimensional examples, because for complex surfaces the Hodge numbers are diffeomorphism invariants; cf. [22, Theorem 2]. However, the classification of simply connected 5-manifolds due to Smale [30] and Barden [1] shows that we can build diffeomorphic 5-manifolds as circle bundles over 4-manifolds which need not be diffeomorphic. In particular, we will see that we can take the 4-manifolds to be algebraic surfaces with different Hodge numbers.

1.3 | Symplectic topology

Eliashberg [8] introduced a dichotomy of three-dimensional contact structures into overtwisted and tight ones, and proved that any homotopy class of almost contact structures contains a unique isotopy class of overtwisted contact structures. After many attempts to extend the notion of overtwistedness to higher dimensions, a satisfactory definition was finally given by Borman, Eliashberg and Murphy [3], who also proved the classification via an h -principle in all dimensions.

The contrast between overtwisted and tight structures is encountered, for instance, when considering symplectic fillability. Both in dimension 3 and higher the existence of a symplectic filling gives an obstruction to overtwistedness [9, 26]. As a consequence of a result of Niederkrüger and Pasquotto [27] the contact structures underlying Sasaki structures are symplectically fillable, hence tight. Therefore they do not fall into the classification given by an h -principle, so that it may happen that two Sasaki structures on the same manifold can have homotopic underlying almost contact structures but non-isotopic contact structures. Examples of this phenomenon have been exhibited by Boyer, Macarini and van Koert [5], who showed that it occurs for positive Sasaki structures on connected sums of copies of $S^2 \times S^3$.

Note that the deformations of Sasaki structures considered by Ražny [29] preserve the isotopy class of the underlying contact structure. It is not clear how the basic Hodge numbers of a Sasaki structure relate to the underlying contact structure; cf. the discussion in Remark 13.

1.4 | Statement of results

The first dimension in which the above questions are interesting for Sasaki manifolds is dimension 5, since in dimension 3 the basic Hodge numbers are determined by the Betti numbers. For simply connected 5-manifolds we have the classification due to Smale [30] and Barden [1], which in particular allows us to find the same 5-manifold as the total space of Boothby–Wang fibrations over very different algebraic surfaces. In this way many interesting examples can be constructed. Instead of stating comprehensive results for many different manifolds, we focus on the simplest case $M_n = n(S^2 \times S^3)$, the connected sum of n copies of $S^2 \times S^3$. For these manifolds we prove the following result which exhibits the topological non-invariance of the basic Hodge numbers:

Theorem 1.

- (A) *For every k there is an n such that M_n has at least k negative Sasaki structures whose underlying almost contact structures are homotopic, but whose contact structures are pairwise inequivalent.*
- (B) *For every k there is an n such that M_n has at least k negative Sasaki structures whose underlying almost contact structures are homotopic, but which have pairwise distinct Hodge numbers.*
- (C) *For every $n = 80m + 76$, the manifold M_n has two negative Sasaki structures whose underlying almost contact structures are homotopic, but whose contact structures are inequivalent, and whose Hodge numbers disagree.*

Negativity means that the basic first Chern class is represented by a negative (1,1)-form. It is a result of Gomez [16] that M_n admits a negative Sasaki structure for every n . By construction, the first Chern class of the underlying contact structure vanishes for all our examples. This means that the underlying almost contact structure is the same as the one for the positive Sasaki structures considered by Boyer, Macarini and van Koert [5].

A few more remarks on this theorem are in order. While part (A) is a straightforward application of known results, it does not give examples with distinct basic Hodge numbers. To prove part (B), involving the Hodge numbers, we actually construct a convenient supply of algebraic surfaces to be used as bases for the Boothby–Wang fibrations, and this construction is of independent interest. It is clear that in statement (B) the number n must grow as we increase k . While it would be possible to tweak the proof of (B) so as to control particular invariants of the contact structure, it seems impossible to actually prove that there are such examples with equivalent contact structures; cf. Remark 13. In both (A) and (C) the inequivalence of the contact structures arises from a result of Hamilton [18], who implemented an idea of Eliashberg, Givental and Hofer [10].

The phenomena exhibited in Theorem 1 also occur on non-spin manifolds. The simplest examples of these are $N_n = M_n \# (S^2 \tilde{\times} S^3)$, where $S^2 \tilde{\times} S^3$ denotes the non-trivial orientable S^3 -bundle over S^2 . Concerning the basic Hodge numbers we will prove the following result:

Theorem 2. *For every k there is an n such that N_n has at least k negative Sasaki structures with pairwise distinct Hodge numbers.*

Finally we show that the topological non-invariance of basic Hodge numbers is true in higher dimensions as well.

Theorem 3. *For all $n > 1$ there exist $(2n + 1)$ -dimensional manifolds which have pairs of negative Sasaki structures with different basic Hodge numbers. Moreover, in each dimension manifolds with such pairs range over infinitely many homotopy types.*

Except for the cases $n = 3$ and 4 , the proof will show that all these manifolds may be taken to be simply connected.

We have taken care to arrange all the examples in the proofs of these theorems to have negative Sasaki structures. Some of the constructions would be more flexible if one did not pay attention to the sign of the first Chern class, and in particular if one were willing to use indefinite Sasaki structures.

2 | SASAKI STRUCTURES AND THEIR TRANSVERSE GEOMETRY

We begin with some definitions and known results, for a more exhaustive treatment we refer to the monograph by Boyer and Galicki [4]. All manifolds are assumed to be smooth, connected, closed and oriented.

A *contact form* η on a manifold M of dimension $2n + 1$ is a 1-form with the property that $\eta \wedge (d\eta)^n$ is a volume form, positive with respect to the given orientation. This means that the *contact structure* $\ker(\eta)$ is a hyperplane field endowed with the positive symplectic form $d\eta$. Therefore, a contact form defines a reduction of the structure group of the tangent bundle from $SO(2n + 1)$ to $\{1\} \times U(n)$, and we refer to (the homotopy class of) this reduction as the underlying *almost contact structure*.

A *K-contact structure* (η, ϕ, R, g) on M consists of a contact form η and an endomorphism ϕ of the tangent bundle TM satisfying the following properties:

- $\phi^2 = -\text{Id} + R \otimes \eta$ where R is the Reeb vector field of η ;
- $\phi|_{\mathcal{D}}$ is an almost complex structure compatible with the symplectic form $d\eta$ on $\mathcal{D} = \ker \eta$;

- the Reeb vector field R is Killing with respect to the metric $g(\cdot, \cdot) = d\eta(\phi \cdot, \cdot) + \eta(\cdot)\eta(\cdot)$.

Given such a structure one can consider the almost complex structure I on the Riemannian cone $(M \times (0, \infty), t^2g + dt^2)$ given by

- $I = \phi$ on $D = \ker \eta$; and
- $I(R) = t\partial_t$.

A *Sasaki structure* is a K -contact structure (η, ϕ, R, g) such that the associated almost complex structure I is integrable. We call a manifold M *Sasakian* if it admits a Sasaki structure. A *Sasaki manifold* is a manifold equipped with a Sasaki structure.

A Sasaki manifold is called *regular* (respectively, *quasi-regular*, *irregular*) if its Reeb foliation has the corresponding property. Every regular Sasaki manifold is a *Boothby–Wang fibration* M over a projective manifold (X, ω) with ω representing an integral class ([2, 4]), that is, the principal S^1 -bundle $\pi : M \rightarrow X$ with Euler class $[\omega]$ and connection 1-form η such that $\pi^*(\omega) = d\eta$. All the Sasaki structures we consider in our examples will be of this type.

The Reeb foliation \mathcal{F} of a Sasaki structure is transversally Kähler. Cohomological properties of the transverse space, that is, the so-called basic cohomology of the foliation, were studied by El Kacimi-Alaoui, Hector and Nicolau [11–13]. They proved many transverse analogues of classical properties of Kähler manifolds. Namely, one can define a basic Dolbeault double complex $\Omega_B^{*,*}(\mathcal{F})$ and prove that it satisfies the Hodge decomposition theorem and Poincaré and Serre dualities. Moreover, in parallel with the standard case one can define basic Chern classes $c_i(\mathcal{F})$ with the use of a transverse connection. It is then natural to call a Sasaki structure *positive*, respectively, *negative* or *null*, if $c_1(\mathcal{F})$ can be represented by a positive definite, respectively, negative definite or null, basic (1,1)-form. It is called indefinite otherwise. Again in analogy with the Kähler case one defines the *basic Betti and Hodge numbers* $b_B^r(\mathcal{F})$ and $h_B^{p,q}(\mathcal{F})$ to be the dimensions of basic de Rham and Dolbeault cohomology groups, respectively.

If M is a regular Sasaki manifold, basic forms on M corresponds exactly to forms on the base X of the Boothby–Wang fibration $\pi : M \rightarrow X$. Moreover, the basic Dolbeault complex $\Omega_B^{*,*}(\mathcal{F})$ is the usual Dolbeault complex $\Omega^{*,*}(X)$ of X . Thus, the basic Betti and Hodge numbers $b_B^r(\mathcal{F})$ and $h_B^{p,q}(\mathcal{F})$ are the Betti and Hodge numbers $b^r(X)$ and $h^{p,q}(X)$ of X . Also, the basic Chern classes $c_i(\mathcal{F})$ are the Chern classes $c_i(X)$ of X . Therefore, the Sasaki structure on M is positive, respectively, negative or null, if and only if the first Chern class $c_1(X)$ can be represented by a positive definite, respectively, negative definite or null, form of type (1,1).

The following result shows that basic Betti numbers are topological invariants of Sasakian manifolds.

Theorem 4 [4, Theorem 7.4.14]. *Let (M, η, ϕ, R, g) be a compact Sasaki manifold. The basic cohomology $H_B^*(\mathcal{F})$ only depends on the topology of M . In particular, the basic Betti numbers of any two Sasaki structures on M agree.*

There can be no such result for the basic Hodge numbers, which are geometric invariants and can distinguish Sasaki structures. The first example of this was given in [15, Example 3.4].

Example 5. Consider M_{21} , the connected sum of 21 copies of $S^2 \times S^3$. On the one hand, this manifold can be endowed with the null Sasaki structure S_1 given by the Boothby–Wang fibration over a $K3$ surface. Since this structure is regular, the basic Hodge numbers of S_1 are the Hodge numbers of the $K3$ surface. In particular $h^{2,0}(\mathcal{F}_1) = 1$. On the other hand, M supports a positive

Sasaki structure S_2 arising as follows; see [4, p. 356]. The connected sum $\#21(S^2 \times S^3)$ can be realized as the link $L_f = V_f \cap S^7 \subset \mathbb{C}^4$, where

$$f(z_0, z_1, z_2, z_3) = z_0^{22} + z_1^{22} + z_2^{22} + z_0 z_3$$

is a weighted homogeneous polynomial of degree $d = 22$ with weight $w = (1, 1, 1, 21)$. This Sasaki structure is positive because $\sum w_i - d = 2 > 0$, see [4, Proposition 9.2.4]. Then a vanishing result proved independently by Nozawa [28] and by Goto [17] guarantees that $h^{2,0}(\mathcal{F}_2) = 0$.

3 | SOME COMPLEX PROJECTIVE MANIFOLDS WITH DISTINCT HODGE NUMBERS

Since for algebraic surfaces the Hodge numbers are diffeomorphism invariants [22], there is no hope for proving Theorem 1(B) using the Boothby–Wang construction on diffeomorphic algebraic surfaces. However, in order to obtain diffeomorphic 5-manifolds via the Boothby–Wang construction it is enough to ask that the surfaces we start with are simply connected and have the same Euler characteristic. A convenient supply of examples is provided by the following theorem.

Theorem 6. *For every positive integer k there are k -tuples of simply connected algebraic surfaces with ample canonical bundles which have the same Euler characteristic, but have pairwise distinct Hodge numbers.*

In terms of Chern numbers, the conditions are that all the surfaces in such a tuple have the same c_2 , but have pairwise distinct c_1^2 , equivalently, pairwise distinct signatures.

Proof. Our examples will be generic smooth hypersurfaces of bidegree $(p, 3q)$ in $\mathbb{C}P^1 \times \mathbb{C}P^2$. For $p > 2$ and $q > 1$ they have ample canonical bundles. Moreover, they are simply connected by the Lefschetz hyperplane theorem. By the adjunction formula the first Chern class of such a hypersurface is

$$c_1 = (2 - p)x_1 + (3 - 3q)x_2, \tag{1}$$

where the x_i are the generators of the second cohomology coming from the two factors of the product $\mathbb{C}P^1 \times \mathbb{C}P^2$. A straightforward calculation yields the following Chern numbers:

$$c_1^2 = 9(q - 1)(3pq - p - 4q), \tag{2}$$

$$c_2 = 3(p(3q - 1)^2 - 6q(q - 1)). \tag{3}$$

The sequence of integers $3q - 1$ contains infinitely many primes, so *a fortiori* it is possible to choose arbitrarily large sets of such numbers which are pairwise coprime. Let us start with a set of $3k$ values of q for which the numbers $3q - 1$ are pairwise coprime. For such a set we want to find an integer n so that $c_2 = 3n$ will be realized by all our choices of q . This means that n must satisfy

$$n \equiv 6q(1 - q) \pmod{(3q - 1)^2} \tag{4}$$

for every q . Since we chose the numbers $3q - 1$ to be pairwise coprime, the Chinese remainder theorem guarantees the existence of solutions to this system of congruences for all our q simultaneously.

For an n solving all the congruences (4), we can solve the Equation (3) with $c_2 = 3n$ for every q to obtain a unique positive integer p such that the hypersurface of bidegree $(p, 3q)$ in $CP^1 \times CP^2$ has Euler number $3n$. This p is given by

$$p = \frac{n + 6q(q - 1)}{(3q - 1)^2},$$

and substituting this into (2) we obtain

$$c_1^2 = \frac{9(q - 1)}{3q - 1}(n - 2q(3q + 1)).$$

For a fixed value of c_1^2 this becomes a cubic equation for q , showing that among our $3k$ surfaces at least k distinct values for c_1^2 are realized. This finishes the proof of the theorem. □

Example 7. To illustrate the theorem, let us start with $q \in \{2, 3, 4, 6, 8\}$. Then the $3q - 1$ range over 5, 8, 11, 17, 23 and so are indeed pairwise coprime. The smallest positive n solving the system of five congruences is

$$n = 21.740.924.188,$$

from which one computes the values of p and c_1^2 for every q . The results are shown in Table 1. Here there are no coincidences among the values of c_1^2 , so we obtain five surfaces with the same c_2 and with pairwise distinct Hodge numbers. The final column in the table shows the divisibility of the first Chern class.

TABLE 1 A five-tuple of surfaces as in Theorem 6

q	p	c_1^2	$d(c_1)$
2	869.636.968	39.133.663.488	1
3	339.701.941	48.917.079.288	1
4	179.677.060	53.364.086.388	1
6	75.228.112	57.549.504.600	5
8	41.098.156	59.551.226.028	1

In higher dimensions the Hodge numbers are not always diffeomorphism-invariant [23], so other constructions are possible.

Let X be a complete intersection of complex dimension 3. As a consequence of the Lefschetz hyperplane theorem X is simply connected and $h^{p,q}(X) = h^{p,q}(CP^3)$ for $p + q < 3$. Moreover, as observed by Hirzebruch a long time ago, the Hodge numbers are actually determined by the Chern numbers in this case; compare [19, Appendix One]. The following includes a strong converse to this observation.

Lemma 8. *Two diffeomorphic complete intersections X and Y of complex dimension 3 have different Hodge numbers if and only if $c_1(X) \neq c_1(Y)$.*

Proof. In this case the only interesting Hodge numbers are $h^{0,3} = h^{3,0}$ and $h^{1,2} = h^{2,1}$. They determine each other by the equation

$$2h^{0,3} + 2h^{1,2} = b_3 = 4 - c_3. \tag{5}$$

By the Hirzebruch–Riemann–Roch Theorem [19] we have

$$h^{0,3} = 1 - \chi = 1 - \frac{1}{24}c_1c_2, \tag{6}$$

where $\chi = \chi(\mathcal{O})$ is the holomorphic Euler characteristic.

Let X be a smooth complete intersection of codimension r with multi-degree (d_1, \dots, d_r) . Without loss of generality we assume $d_i \geq 2$ for all i . We have $c_1(X) = k \cdot x$ and $p_1(X) = m \cdot x^2$, where x is the positive generator of the second cohomology, and

$$k = 4 + r - \sum_{i=1}^r d_i$$

and

$$m = 4 + r - \sum_{i=1}^r d_i^2.$$

The last equation shows that m is negative, unless $r = 1$ and X is a quadric in $\mathbb{C}P^4$. So there can be no other complete intersection diffeomorphic to a quadric.

If Y is another complete intersection that is diffeomorphic to X , then, up to conjugating the complex structure, we may assume that the diffeomorphism pulls back the positive generator in cohomology to the positive generator. Therefore, X and Y have the same first Chern class if and only if they have the same k . Moreover, they have the same m just because they are diffeomorphic. For a fixed k we have

$$2c_2 = c_1^2 - p_1 = (k^2 - m) \cdot x^2$$

and

$$2c_1c_2 = k(k^2 - m) \cdot x^3.$$

Together with Equations (5) and (6), this shows that if X and Y are diffeomorphic and have the same first Chern class, then their Hodge numbers agree.

Conversely, suppose that X and Y are diffeomorphic, and so have the same m , and also have the same Hodge numbers. Then by (6) they have the same $2c_1c_2$ which is given by the formula

$$f(k) = k(k^2 - m)$$

as a function of the first Chern class $k \cdot x$. The derivative of this function is $3k^2 - m$. Since by our earlier observation about quadrics we may assume that m is negative, the derivative of f is strictly

positive, showing that the function is strictly monotonically increasing. In other words, if X and Y have the same Hodge numbers, then they have the same first Chern classes. \square

By Wall’s classification the diffeomorphism type of a three-dimensional complete intersection is determined by the Euler number, the first Pontryagin class and the parity of the first Chern class, see [21, 31]. Libgober and Wood [24] conjectured the existence of diffeomorphic complete intersections with different Chern classes. However, the first such pairs were only found comparatively recently by Wang and Du [32], we list them in Table 2. By Lemma 8 above, these pairs have distinct Hodge numbers. The characteristic classes of a complete intersection X are multiples of the generators x, x^2 of the groups $H^2(X) \cong H^4(X) \cong \mathbb{Z}$. The values in Table 2 are the coefficients that determine the characteristic classes as multiples of x and x^2 .

TABLE 2 Diffeomorphic three-dimensional complete intersections with different c_1

\underline{d}	d	p_1	$\chi = d \cdot c_3$	c_1
(70,16,16,14,7,6)	$7^3 \cdot 5 \cdot 3 \cdot 2^{11}$	-5683	-7.767.425.433.600	-119
(56,49,8,6,5,4,4)	$7^3 \cdot 5 \cdot 3 \cdot 2^{11}$	-5683	-7.767.425.433.600	-121
(88,28,19,14,6,6)	$19 \cdot 11 \cdot 7^2 \cdot 3^2 \cdot 2^8$	-9147	-35.445.749.391.360	-151
(76,56,11,7,6,6,2)	$19 \cdot 11 \cdot 7^2 \cdot 3^2 \cdot 2^8$	-9147	-35.445.749.391.360	-153
(84,29,25,25,18,7)	$29 \cdot 7^2 \cdot 5^4 \cdot 3^3 \cdot 2^3$	-9510	-384.536.710.530.000	-178
(60,58,49,9,5,5,5)	$29 \cdot 7^2 \cdot 5^4 \cdot 3^3 \cdot 2^3$	-9510	-384.536.710.530.000	-180

Remark 9. The conclusion that the diffeomorphic complete intersection threefolds with different first Chern classes from [32] have distinct Hodge numbers was also reached recently in [33]. There, the Hodge numbers are computed by brute force, without a general result along the lines of Lemma 8. The paper [33] also contains a pair of five-dimensional diffeomorphic complete intersections with distinct Hodge numbers.

4 | FIVE-DIMENSIONAL SASAKI MANIFOLDS

In this section we discuss five-dimensional Sasaki manifolds obtained as Boothby–Wang fibrations over algebraic surfaces. In particular, we prove Theorems 1 and 2.

Let X be a smooth simply connected algebraic surface, and $[\omega] \in H^2(X, \mathbb{Z})$ an integral Kähler class. We consider the Sasaki manifold M obtained as the total space of the Boothby–Wang fibration $\pi : M \rightarrow X$ with Euler class $[\omega]$. Then the first Chern class of the almost contact structure underlying the Sasaki structure is $\pi^*(c_1(X))$.

Recall that the *divisibility* $d(\alpha)$ of a class $\alpha \in H^2(X; \mathbb{Z})$ is the maximum number $n \in \mathbb{Z}$ such that $\alpha = n\beta$ for some $0 \neq \beta \in H^2(X; \mathbb{Z})$. A class α is called *indivisible*, or *primitive*, if $d(\alpha) = 1$. For our construction we always scale the Kähler form ω so that it represents a primitive cohomology class. We can then use the following standard result, proved for example in the paper of Hamilton [18].

Lemma 10. *If X is simply connected and $[\omega]$ is primitive, then the total space M of the Boothby–Wang fibration $\pi : M \rightarrow X$ is simply connected with torsion-free cohomology. It is spin if and only if X is spin or $c_1(X) \equiv [\omega] \pmod{2}$.*

Proof. The statements about the fundamental group and the cohomology follow from the exact homotopy sequence and the Gysin sequence for π , see [18, Section 4] for the details. Moreover, $w_2(M)$ is the mod 2 reduction of $\pi^*(c_1(X))$. This vanishes if X is spin, since then $c_1(X)$ has trivial mod 2 reduction, or, more generally, if and only if the mod 2 reduction of $c_1(X)$ is in the kernel of π^* , which is spanned by the Euler class $[\omega]$; compare [18, Lemma 26]. \square

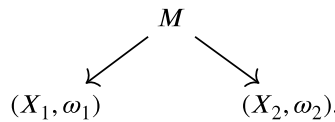
Combining this with the classification of simply connected 5-manifolds and of almost contact structures on them, we obtain:

Proposition 11. *Suppose that $c_1(X)$ is a (positive or negative) multiple of $[\omega]$. Then the total space M is diffeomorphic to $n(S^2 \times S^3)$ with $n = b_2(X) - 1$. The almost contact structure underlying the Sasaki structure is the unique one with trivial first Chern class.*

Proof. The assumption that $c_1(X)$ is a multiple of the Euler class $[\omega]$ implies that M is spin. The classification of simply connected spin 5-manifolds due to Smale [30] implies that M is diffeomorphic to $n(S^2 \times S^3)$ with $n = b_2(X) - 1$. Moreover, $c_1(X)$ is in the kernel of π^* , and so the first Chern class of the (almost) contact structure vanishes. It is a result of Geiges [14] that in this case the first Chern class is a complete invariant of almost contact structures. \square

Recall that two contact structures are called *equivalent* if they can be related by a series of diffeomorphisms and isotopies. Within the same homotopy class of almost contact structures we can sometimes distinguish inequivalent contact structures using the following special case of a theorem of Hamilton [18, Corollary 43].

Proposition 12 [18]. *Let M be a simply connected 5-manifold admitting two different regular Sasaki structures $(\eta_i, \phi_i, R_i, g_i)$ for $i = 1, 2$ with indivisible Euler classes $[\omega_i]$, namely*



If the contact structures defined by η_1 and η_2 have trivial first Chern class and are equivalent, then $d(c_1(X_1)) = d(c_1(X_2))$.

We now prove Theorem 1.

Proof of Theorem 1(A). In [6] Braungardt and the first author constructed arbitrarily large tuples X_1, \dots, X_k of homeomorphic branched covers of the projective plane $\mathbb{C}P^2$. These are pairwise non-diffeomorphic projective surfaces distinguished by the divisibilities of their first Chern classes. Moreover, they all have ample canonical bundles; cf. [6, Corollary 1].

On each of the surfaces in such a k -tuple we perform the Boothby–Wang construction using as Euler classes the primitive integral cohomology classes which are rational multiples of the canonical classes. By Proposition 11 this yields k -tuples of negative Sasaki structures on $M_n = n(S^2 \times S^3)$ with $n = b_2(X_i) - 1$, which are homotopic as almost contact structures.

However, since the first Chern classes $c_1(X_i)$ have pairwise different divisibilities, the contact structures are inequivalent by Proposition 12. \square

Proof of Theorem 1(B). We start with a k -tuple of surfaces X_1, \dots, X_k as in Theorem 6, so they are simply connected with ample canonical bundles, with the same Euler characteristics, but with pairwise distinct Hodge numbers.

Again we perform the Boothby–Wang construction on each surface using as Euler classes the primitive integral cohomology classes which are rational multiples of the canonical classes. By Proposition 11 this yields k -tuples of negative Sasaki structures on $M_n = n(S^2 \times S^3)$ with $n = b_2(X_i) - 1$, which are homotopic as almost contact structures. Their basic Hodge numbers are the usual Hodge numbers of the X_i , so for $i \neq j$ the Sasaki structures obtained from X_i and from X_j have different Hodge numbers. \square

Remark 13. In general we do not have any control over the contact structures in these examples. For instance, in the five-tuple considered in Example 7, two different divisibilities for the first Chern classes occur, so by Proposition 12 there are at least two different contact structures underlying these Sasaki structures. If one wants to prove a version of Theorem 1(B) where one has tuples of Sasaki structures with pairwise distinct Hodge numbers, but such that all the contact structures are equivalent, then one could try to sharpen our proof above, to ensure that all the divisibilities are the same, so that the contact structures could not be distinguished using Proposition 12. However, the vanishing of this particular obstruction does not prove that the contact structures really are equivalent.

Proof of Theorem 1(C). These structures are obtained as Boothby–Wang fibrations over a family of complete intersections and a family of Horikawa surfaces, respectively.

On the one hand, let X_k be a generic complete intersection in $\mathbb{C}P^1 \times \mathbb{C}P^3$ given by intersecting hypersurfaces of bidegree $(2,5)$ and $(k, 1)$. By the adjunction formula we have $c_1(X_k) = -kx_1 - 2x_2$ where x_1 and x_2 are the generators of the cohomology rings of $\mathbb{C}P^1$ and $\mathbb{C}P^3$, respectively. The Chern numbers and holomorphic Euler characteristic of X_k are

$$c_1^2(X_k) = 40k + 8, \quad c_2(X_k) = 80k + 76, \quad \chi(\mathcal{O}_{X_k}) = 10k + 7.$$

Moreover, X_k is simply connected by the Lefschetz hyperplane theorem, and the second Betti number is given by $b_2(X_k) = c_2(X_k) - 2 = 80k + 74$.

On the other hand, consider the following family of Horikawa surfaces Y_i from [20]. Let Σ_i be the Hirzebruch surface of degree i , that is the $\mathbb{C}P^1$ -bundle over $\mathbb{C}P^1$ whose zero section Δ has self-intersection $-i$. Let F denote the class of the fiber of the fibration. Then we can construct the Horikawa surface Y_i as the double cover $pr : Y_i \rightarrow \Sigma_i$ with branch locus homologous to $B = 6\Delta + 2(2i + 3)F$. Note that these surfaces have ample canonical bundle K_{Y_i} since $K_{Y_i} = pr^*(K_{\Sigma_i} + \frac{1}{2}B) = pr^*(\Delta + (i + 1)F)$ and $\Delta + (i + 1)F$ is an ample bundle. Moreover, Y_i is simply connected because the branch locus B is ample. Now the characteristic numbers of Y_i are

$$c_1^2(Y_i) = 2i + 4, \quad c_2(Y_i) = 10i + 56, \quad \chi(\mathcal{O}_{Y_i}) = i + 5.$$

Hence $b_2(Y_i) = 10i + 54$ so $b_2(X_k) = b_2(Y_i)$ for $i = 8k + 2$.

From now on we denote Y_{8k+2} by Z_k . For this we have

$$c_1^2(Z_k) = 16k + 8, \quad c_2(Z_k) = 80k + 76, \quad \chi(\mathcal{O}_{Z_k}) = 8k + 7.$$

Both X_k and Z_k have ample canonical line bundle. Hence, by Proposition 11 the Boothby–Wang construction gives us two negative Sasaki structures with homotopic almost contact structures on the $(80k + 73)$ -fold connected sum $\#(80k + 73)(S^2 \times S^3)$.

Since the Sasaki structures are regular, their basic Hodge numbers are the Hodge numbers of the base surfaces. Therefore, we have

$$\begin{aligned} h^{0,2}(X_k) &= 10k + 6, & h^{1,1}(X_k) &= 60k + 62, \\ h^{0,2}(Z_k) &= 8k + 6, & h^{1,1}(Z_k) &= 64k + 62. \end{aligned}$$

Note, on the one hand, that $d(c_1(X_k)) = \gcd\{k, 2\}$. On the other hand, the main result of [25] implies that Z_k is spin if and only if $B/2$ is the Poincaré dual of the second Stiefel–Whitney class $w_2(\Sigma_{8k+2})$. In other words, Z_k is spin if and only if $B/2$ is divisible by 2 because Σ_{8k+2} is spin. Now the intersection number of $B/2$ with F equals 3 and this implies that Z_k is not spin. In particular, $d(c_1(Z_k))$ is always odd, and so the two Sasaki structures do not have equivalent contact structures by Proposition 12 whenever k is even. \square

We now move on to the case of non-spin 5-manifolds considered in Theorem 2.

Proof of Theorem 2. Again we start with a k -tuple of surfaces X_1, \dots, X_k as in Theorem 6, so they are simply connected with ample canonical bundles, all with the same Euler characteristics, but with pairwise distinct Hodge numbers. Moreover, we now want all the X_i to be non-spin. Inspecting the formula (1) for c_1 we see that it is enough to assume that $3 - 3q$ is always odd, for this ensures that c_1 is not divisible by 2. Now the only constraint we imposed on q was that the different values of $3q - 1$ should be pairwise coprime. We may at the same time assume that all the $3q - 1$ are odd, and so $3 - 3q$ is odd as well.

On $\mathbb{C}P^1 \times \mathbb{C}P^2$ let x_1 and x_2 be the generators of the cohomology coming from the two factors. All classes of the form $e = ax_1 + bx_2$ with positive a and b are Kähler classes. We fix such a class e with b even and a and b relatively prime. On every X_i we perform the Boothby–Wang construction using as Euler classes the restrictions of e to the X_i . This yields negative Sasaki structures on the total spaces. The Euler class is primitive because a and b are relatively prime, and so the total spaces are all simply connected with torsion-free cohomology by Lemma 10. Moreover, the assumption that all the $3q - 1$ are odd but b is even ensures e and $c_1(X_i)$ have different reductions modulo 2, and so the total spaces are not spin. Now Barden’s classification [1] implies that all the total spaces are diffeomorphic to some $N_n = n(S^2 \times S^3) \# (S^2 \widetilde{\times} S^3)$, where $S^2 \widetilde{\times} S^3$ is the non-trivial orientable S^3 -bundle over S^2 . Moreover, n is the same for all our total spaces because all the X_i have the same second Betti number. However, the fact that the X_i have pairwise distinct Hodge numbers means that the basic Hodge numbers of the Sasaki structures on the total spaces are also pairwise distinct. This completes the proof. \square

5 | HIGHER DIMENSIONS

In this section we give examples of manifolds in dimensions at least 7 admitting pairs of Sasaki structures with different basic Hodge numbers. In particular we prove Theorem 3.

Proposition 14. *There exist closed 7-manifolds admitting two negative Sasaki structures with different basic Hodge numbers. Moreover, manifolds with such pairs range over infinitely many homotopy types.*

Proof. Let X_1 and X_2 be two diffeomorphic three-dimensional complete intersections with different first Chern classes as given in Table 2. After conjugating one of the complex structures if necessary we may assume that the diffeomorphism matches up the positive generators x in $H^2(X_1)$ and $H^2(X_2)$. This generator is of course a Kähler class. We perform the Boothby–Wang construction with Euler class $k \cdot x$ on both X_i . The diffeomorphism between X_1 and X_2 then lifts to a diffeomorphism of the resulting negative Sasaki manifolds $(M_1, \eta_1, \phi_1, R_1, g_1)$ and $(M_2, \eta_2, \phi_2, R_2, g_2)$. Lemma 8 and Table 2 imply that these two Sasaki structures are distinguished by their basic Hodge numbers which coincide with the Hodge numbers of X_1 and X_2 . As k varies these examples range over infinitely many homotopy types since the fundamental group of M_i is cyclic of order k by the homotopy exact sequence of the Boothby–Wang fibration. \square

For the remaining dimensions we give a uniform argument in the following theorem.

Theorem 15. *For all $n \geq 4$ there exist closed $(2n + 1)$ -dimensional manifolds admitting two negative Sasaki structures with different basic Hodge numbers. Moreover, in each dimension manifolds with such pairs range over infinitely many homotopy types.*

Proof. Let X_1 and X_2 be two diffeomorphic three-dimensional complete intersections with different first Chern classes given in Table 2. Recall that X_1 and X_2 have different Hodge numbers by Lemma 8.

For any smooth projective-algebraic variety P of complex dimension $n - 3$ with ample canonical bundle the products $X_1 \times P$ and $X_2 \times P$ are diffeomorphic, and the diffeomorphism matches up the positive primitive integral classes $x + K_P$, where x is the positive generator of $H^2(X_i)$. Therefore, the diffeomorphism lifts to the total spaces of the Boothby–Wang fibrations with these primitive classes as Euler classes. These total spaces carry negative Sasaki structures whose basic Hodge numbers are the usual Hodge numbers of the bases $X_i \times P$. The compatibility of the Hodge and Künneth decompositions shows that these Hodge numbers are different because those of the X_i are.

Finally, we can let P range over infinitely many homotopy types, for example, using hypersurfaces of different degrees in $\mathbb{C}P^{n-2}$. This ensures that we obtain infinitely many homotopy types of Sasaki manifolds. \square

We have now proved Theorem 3, since the case $n = 2$ was covered in Theorems 1 and 2, the case $n = 3$ was covered in Proposition 14 and the case $n \geq 4$ in Theorem 15. Moreover, the manifolds considered are simply connected unless $n = 3$, where the fundamental groups are finite cyclic, and $n = 4$ where the fundamental groups are those of algebraic curves of general type.

JOURNAL INFORMATION

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