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RESEARCH ARTICLE

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Unbounded Pontryagin numbers on nonnegatively curved spin manifolds

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Abstract
We prove that any rational linear combination of Pontryagin numbers that does not factor through the universal elliptic genus is unbounded on connected closed spin manifolds of nonnegative sectional curvature.

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1 | INTRODUCTION

One of the most important results in the study of Riemannian manifolds with nonnegative curvature is Gromov's Betti number theorem [2], which gives, in every dimension, a universal upper bound for the Betti numbers of connected nonnegatively curved manifolds, thus bounding the size of such manifolds. As a corollary, any invariant that can be bounded in terms of Betti numbers is bounded on nonnegatively curved manifolds. This applies in particular to the signature or L-genus. In sharp contrast with this, all other linear combinations of Pontryagin numbers are not bounded on nonnegatively curved manifolds. It was proved in [5] that, up to taking multiples, the L-genus is characterized among all linear combinations of Pontryagin numbers by its boundedness on connected manifolds of nonnegative curvature.

It is clear that the result of [5] cannot remain true if one restricts the discussion to spin manifolds, since by the Lichnerowicz argument [6] the $\hat{A}$-genus vanishes on nonnegatively curved spin manifolds, and, starting in dimension 8, it is linearly independent of the L-genus. However, both the L-genus and the $\hat{A}$-genus are specializations of Ochanine's elliptic genus [7], and this led Herrmann and Weisskopf [3] to ask whether linear combinations of Pontryagin numbers which do not arise from the elliptic genus are unbounded on connected spin manifolds of nonnegative curvature. It was shown in [3] that the answer to this question is positive up to dimension 20, using various case by case calculations on suitably chosen examples, which however do not provide a path towards extending the result to arbitrary dimensions. The question posed by Herrmann and

Weiskopf is part of a general philosophy trying to find index-theoretic obstructions for the existence of positively or nonnegatively curved Riemannian metrics, and suggests that the only such obstructions should come from the elliptic genus, cf. Dessai [1].

In this paper, we prove a result in the spirit of [5, Theorem 1] for spin manifolds, which in particular provides a positive answer to [3, Question 1.2]:

Theorem 1. *Any rational linear combination of Pontryagin numbers that does not factor through the elliptic genus is unbounded on connected closed oriented spin manifolds of nonnegative sectional curvature.*

The proof uses the structure of the rational spin bordism ring. We construct sequences of ring generators with the property that each generator of dimension $\geqslant 12$ belongs to a family of possible choices on which a certain indecomposable Pontryagin number is unbounded. Moreover, with the same dimension assumption, these generators admit metrics of nonnegative sectional curvature and are in the kernel of the elliptic genus. In terms of dimensions, this is the best one can hope for, since in dimensions up to 8 all Pontryagin numbers factor through the elliptic genus. In dimension 4, because of the Lichnerowicz theorem [6], it is not possible to choose a spin generator with nonnegative curvature for the bordism group. This makes the proof of Theorem 1 much more complicated than the proof in [5], since one has to work around the lack of a suitable 4-dimensional generator. This is achieved by uncovering certain polynomial relations between different families of generators for the bordism ring, see Proposition 12 in Section 4.

2 | A FAMILY OF PROJECTIVE BUNDLES OVER COMPLEX PROJECTIVE SPACES

For integers c denote by H^c the tensor powers of the hyperplane bundle $H \longrightarrow \mathbb{C}P^n$. (For negative c these are the $|c|$-fold tensor products of the dual line bundle.) Consider the complex vector bundle $E_c \longrightarrow \mathbb{C}P^n$ of rank $k+1$ of the form $E_c = H^c \oplus \underline{\mathbb{C}}^k$, and let $X_n^k(c) = P(E_c)$ denote its projectivization. This is a complex manifold of real dimension $2(n+k)$. To have non-trivial Pontryagin numbers, we need manifolds of real dimension divisible by 4, and so we will take n and k of the same parity.

The total Chern class of E_c is $c(E_c) = 1 + c \cdot x$, where $x \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is the positive generator. Therefore, the Leray–Hirsch theorem immediately gives the following statement.

Lemma 2. *The cohomology ring of $X_n^k(c)$ is generated by two classes $x, y \in H^2(X_n^k(c))$ subject to the relations*

$$x^{n+1} = 0, \qquad y^{k+1} + cxy^k = 0. \tag{1}$$

Here x is the generator coming from the base $\mathbb{C}P^n$. Next we compute the characteristic classes of the $X_n^k(c)$.

Lemma 3. *The total Chern class of $X_n^k(c)$ is given by*

$$c(X_n^k(c)) = (1+x)^{n+1}(1+y)^k(1+y+cx). \tag{2}$$

Therefore, its total Pontryagin class is

$$p(X_n^k(c)) = (1+x^2)^{n+1}(1+y^2)^k(1+(y+cx)^2). \tag{3}$$

Proof. We have the decomposition $TX_n^k(c) = T\pi \oplus \pi^* T\mathbb{C}P^n$, where $T\pi$ is the tangent bundle along the fibers, and so

$$c(X_n^k(c)) = c(T\pi)\cdot \pi^* c(T\mathbb{C}P^n)$$

with the latter factor being $\pi^* c(\mathbb{C}P^n) = (1+x)^{n+1}$. It remains to compute $c(T\pi)$. We have the relative Euler sequence

$$0 \to L^{-1} \to \pi^* E_c \to L^{-1} \otimes T\pi \to 0,$$

where L is the fiberwise hyperplane bundle with $c(L) = 1+y$. Tensoring by L yields an isomorphism of bundles $L \otimes \pi^* E_c \cong T\pi \oplus \underline{\mathbb{C}}$, and this shows

$$c(T\pi) = c(L\otimes \pi^* E_c) = (1+y)^k(1+y+cx).$$

This proves the formula for the total Chern class, which in turn gives the one for the total Pontryagin class. □

The following lemma tells us under which conditions $X_n^k(c)$ is spin.

Lemma 4. *Assume that n and k have the same parity. Then $X_n^k(c)$ is spin if and only if k and n are odd and c is even.*

Proof. Recall that a complex manifold is spin if and only if its first Chern class is divisible by 2 in integral cohomology. For $X_n^k(c)$, the formula (2) gives us

$$c_1(X_n^k(c)) = (n+1)x + ky + (y+cx) = (k+1)y + (n+1+c)x.$$

This is divisible by 2 if and only if both $k+1$ and $n+1+c$ are even. Since we assumed that k and n have the same parity this happens if and only if k and n are odd and c is even. □

In later sections, we will always assume that the conditions in this lemma are satisfied, so that we are dealing with spin manifolds of real dimension $4m$, where $2m = k+n$. To end this section, we note that the manifolds $X_n^k(c)$ are nonnegatively curved.

Lemma 5. *Every $X_n^k(c)$ admits a Riemannian metric of nonnegative sectional curvature.*

Proof. Since $X_n^k(0)$ is a product of complex projective spaces, we may assume $c \neq 0$. The vector bundle $E_c = H^c \oplus \underline{\mathbb{C}}^k$ over $\mathbb{C}P^n$ has structure group $U(1) = S^1$, and is therefore associated to an S^1-bundle $P_c \longrightarrow \mathbb{C}P^n$ with Euler class $c \cdot x$. The total space of this circle bundle is the lens space $S^{2n+1}/\mathbb{Z}_{|c|}$, carrying an S^1-invariant metric of constant positive curvature. Now $X_n^k(c)$ has the form $(P_c \times \mathbb{C}P^k)/S^1$, where S^1 acts freely by isometries of the product metric formed by the positively curved metric on the lens space and the Fubini–Study metric on $\mathbb{C}P^k$, since S^1 acts on $\mathbb{C}P^k$ via the appropriate inclusion $S^1 \hookrightarrow U(k+1)$. The non-decreasing property of curvature in submersions implies that the induced metric on $X_n^k(c)$ is nonnegatively curved. □

3 | COMPUTATIONS OF PONTRYAGIN NUMBERS

We now prove some results about the Pontryagin numbers of $X_n^k(c)$ in the case where k and n are both odd. Here by Pontryagin number we mean not just the evaluation of monomials in the Pontryagin classes, but any rational linear combination of such evaluations. The general shape of these numbers is as follows.

Proposition 6. *As a function of c, every Pontryagin number of $X_n^k(c)$ is an odd polynomial of degree at most n.*

Recall that a polynomial is odd if it involves only odd powers of the variable c. We will see in Proposition 7 that the maximal degree n does occur.

Proof. Recall from (3) that the total Pontryagin class is given by

$$p(X_n^k(c)) = (1+y^2)^k(1+x^2)^{n+1}(1+(y+cx)^2).$$

Therefore, every Pontryagin number is given by the evaluation of a homogeneous polynomial of degree $m = \frac{1}{2}(k+n)$ in the variables y^2, x^2 and $(y+cx)^2$. In other words, we are looking into evaluating linear combinations of monomials of the form

$$y^{2a} \cdot x^{2b} \cdot (y+cx)^{2(m-a-b)}$$

with nonnegative exponents. Expanding the third factor with the binomial theorem we find a linear combination of the monomials

$$y^{2a+i} \cdot x^{2m-2a-i} \cdot c^{2(m-a-b)-i} \quad \text{with} \quad 0 \leqslant i \leqslant 2(m-a-b).$$

Note that x and y are cohomology classes of degree 2, and c is an integer. Setting $j = 2m-2a-i$, these monomials become

$$y^{2m-j} \cdot x^j \cdot c^{j-2b} \quad \text{with} \quad 0 \leqslant j \leqslant n.$$

Using the cohomology relation from Lemma 2 repeatedly, we find

$$y^{2m-j} \cdot x^j \cdot c^{j-2b} = (-1)^{n-j} y^k \cdot x^n \cdot c^{n-2b}.$$

Here $y^k \cdot x^n$ is the generator of the top-degree cohomology of $X_n^k(c)$. Since n is odd, the exponent $n-2b$ of c is always odd, and this finally shows that any Pontryagin number of $X_n^k(c)$ is a rational linear combination of terms which contain only odd powers of c. Moreover, the exponent $n-2b$ of c is bounded above by n since b is nonnegative. □

For any closed oriented manifold M of dimension $4m$ with total Pontryagin class

$$p(TM) = \prod_i (1 + y_i^2)$$

the Milnor–Thom number $s_n(M)$ is defined by

$$s_m(M) = \sum_i \langle y_i^{2m}, [M] \rangle.$$

The splitting principle implies that this is a Pontryagin number. The significance of s_m is that its non-vanishing on M is equivalent to M being a generator of the rational bordism ring, see Section 4.

Proposition 7. *Assume that k and n are odd, and let $m = \frac{1}{2}(k+n)$. Then the Milnor–Thom number of $X_n^k(c)$ is given by*

$$s_m(X_n^k(c)) = c^n \left[\binom{k+n-1}{n} - k \right]. \tag{4}$$

In particular, $s_m(X_n^k(c)) \neq 0$ whenever $k \geqslant 3$, $n \geqslant 3$ and $c \neq 0$.

Proof. This formula is a special case of a calculation of Schreieder [8], who considered arbitrary projectivizations. Nevertheless, we include a direct proof, which is no more complicated than the explanation of how to extract what we need from [8, Lemma 2.4].

Using the formula (3), the definition of the Milnor–Thom number gives

$$s_m(X_n^k(c)) = \langle ky^{n+k} + (n+1)x^{n+k} + (y+cx)^{n+k}, [X_n^k(c)] \rangle.$$

The term $(n+1)x^{n+k}$ vanishes since $x^{n+1} = 0$. We expand $(y+cx)^{n+k}$ using the binomial formula, and drop all terms where the exponent of x is $> n$. Finally, we trade all terms with an exponent of y that is larger than k using the relation $y^{k+1} = -cxy^k$ from Lemma 2 repeatedly. This leads to

$$\begin{aligned} ky^{n+k} + (y+cx)^{n+k} &= ky^{n+k} + \sum_{i=0}^{n} \binom{n+k}{i} c^i x^i y^{n+k-i} \\ &= (-1)^n k c^n x^n y^k + \sum_{i=0}^{n} (-1)^{n-i} \binom{n+k}{i} c^n x^n y^k \\ &= (-1)^n c^n \left[k + \sum_{i=0}^{n} (-1)^i \binom{n+k}{i} \right] x^n y^k. \end{aligned}$$

Since n is odd, the term $(-1)^n = -1$. Further, $x^n y^k$ evaluates as 1 on the fundamental class, and therefore

$$s_m(X_n^k(c)) = -c^n \left[k + \sum_{i=0}^{n} (-1)^i \binom{n+k}{i} \right].$$

Now replace each binomial coefficient with $i > 0$ in the sum using the recursion

$$\binom{n+k}{i} = \binom{n+k-1}{i-1} + \binom{n+k-1}{i}. \tag{5}$$

Since in the resulting sum almost all binomial coefficients appear twice with opposite signs, they cancel in pairs, except for the last summand $\binom{n+k-1}{n}$. This finally gives

$$s_m(X_n^k(c)) = -c^n\left[k - \binom{n+k-1}{n}\right]$$

as claimed. □

We will need to consider another special Pontryagin number, defined as follows.

Definition 8. For a closed oriented manifold M of dimension $4m$ the q-number is defined by

$$q_m(M) = \sum_i \sum_{j\neq i} \langle y_i^2 y_j^{2m-2}, [M]\rangle,$$

where the y_i^2 are the Pontryagin roots of M.

By the splitting principle, this is indeed a Pontryagin number of M. From its definition it is clear that q_m vanishes on product manifolds which have no factor of dimension at least $4m - 4$. (To avoid trivialities, we may assume that $m > 1$.)

Proposition 9. *Assume that k and n are odd and* $\geqslant 3$. *Setting* $m = \frac{1}{2}(k+n)$, *the q-number of* $X_n^k(c)$ *is given by*

$$q_m(X_n^k(c)) = k\left[\binom{n+k-3}{n} - (k-1)\right]c^n + (n+1)\left[\binom{n+k-3}{n-2} - k\right]c^{n-2}. \tag{6}$$

Proof. We compute $q_m(X_n^k(c))$ from the definition using the formula (3):

$$\begin{aligned} q_m(X_n^k(c)) = \Big\langle &(n+1)x^2\left[ky^{n+k-2} + nx^{n+k-2} + (y+cx)^{n+k-2}\right] \\ &+ ky^2\left[(k-1)y^{n+k-2} + (n+1)x^{n+k-2} + (y+cx)^{n+k-2}\right] \\ &+ (y+cx)^2\left[ky^{n+k-2} + nx^{n+k-2}\right], [X_n^k(c)]\Big\rangle. \end{aligned}$$

The cohomology relations from Lemma 2 give $x^{n+k-2} = 0$ and $(y+cx)\cdot y^k = 0$, therefore the above sum reduces to

$$\begin{aligned} q_m(X_n^k(c)) = &\Big\langle (n+1)x^2\left[ky^{n+k-2} + (y+cx)^{n+k-2}\right], [X_n^k(c)]\Big\rangle \\ &+ \Big\langle ky^2\left[(k-1)y^{n+k-2} + (y+cx)^{n+k-2}\right], [X_n^k(c)]\Big\rangle. \end{aligned}$$

We expand the powers of $y + cx$ using the binomial theorem, and, as in the proof of Proposition 7, we drop terms where the exponent of x is $> n$, and we trade powers of y with exponent $> k$ using the relation $y^{k+1} = -cxy^k$ from Lemma 2. Finally, evaluating on the fundamental class using $\langle y^k x^n, [X^k_n(c)]\rangle = 1$ we find

$$q_m(X^k_n(c)) = (n+1)\left[-k + \sum_{i=0}^{n-2}(-1)^{i+1}\binom{n+k-2}{i}\right]c^{n-2} \\ +k\left[-(k-1) + \sum_{i=0}^{n}(-1)^{i+1}\binom{n+k-2}{i}\right]c^n.$$

The first sum is simplified using the recursion (5) to get

$$\sum_{i=0}^{n-2}(-1)^{i+1}\binom{n+k-2}{i} = \binom{n+k-3}{n-2},$$

and similarly for the second sum. This completes the proof. □

Remark 10. When $n = 3$, then the shape of formula (6) says that $q_m(X^k_3(c))$ is an integral linear combination of c^3 and c. However, an inspection of the linear term reveals that it actually vanishes in this case, so $q_m(X^k_3(c))$ is simply a multiple of c^3.

4 | CALCULATIONS IN SPIN BORDISM

In this section, we work in Ω_*, the rational spin bordism ring, that is, the usual spin bordism ring tensored with $\mathbb{Q}$. As is well-known, at the rational level there is no difference between oriented bordism and spin bordism, see [9, chapter XI], and so Ω_* is a polynomial algebra over $\mathbb{Q}$ with one ring generator α_i in each dimension of the form $4i$. Moreover, by a result of Thom [10], a $4i$-dimensional closed oriented manifold M may be taken as a representative for α_i if and only if its Milnor–Thom number $s_i(M)$ does not vanish; cf. [4, 9]. We will write simply $\alpha_i = M \in \Omega_{4i}$, identifying M with its bordism class, without using brackets or other notation to specify that we are taking the equivalence class of M in the $\mathbb{Q}$-vector space Ω_{4i}, the degree $4i$ part of Ω_*.

Let $K3$ be the smooth closed oriented 4-manifold underlying a complex $K3$ surface, and $\mathbb{H}P^2$ the quaternionic projective plane.

Proposition 11. *Let $\alpha_1 = K3$, $\alpha_2 = \mathbb{H}P^2$. For $i \geqslant 3$ consider any decomposition $2i = n + k$ with both n and k odd and $\geqslant 3$, and let $\alpha_i(c) = X^k_n(c)$. Then as long as c is even and nonzero, the α_i form a basis sequence for Ω_*.*

Proof. The $K3$ surface has trivial first Chern class, and so is spin. Moreover, its signature is not zero, which means $s_1(K3) \neq 0$. The quaternionic projective plane is 2-connected, and therefore spin. It is well-known that $s_2(\mathbb{H}P^2) \neq 0$. Finally, if k and n are odd and c is even, then $X^k_n(c)$ is spin by Lemma 4, and we have $s_i(X^k_n(c)) \neq 0$ by Proposition 7 as soon as $c \neq 0$ and $n, k \geqslant 3$. □

The universal elliptic genus is a surjective homomorphism of graded rings

$$\varphi : \Omega_* \longrightarrow \mathbb{Q}[\delta, \epsilon],$$

where δ and ϵ have degrees 4 and 8 respectively; cf. [4, 7]. Ochanine's theorem [7] characterizes $\ker(\varphi)$ as the ideal generated by all $\mathbb{C}P^{\mathrm{odd}}$-bundles with compact Lie groups as structure groups. Since the manifolds $X_n^k(c)$ are $\mathbb{C}P^{\mathrm{odd}}$-bundles with structure group S^1, they are in the kernel of the elliptic genus. Moreover, in terms of the above basis sequences we can think of φ as being the projection

$$\Omega_* = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3(c), \ldots] \longrightarrow \mathbb{Q}[\alpha_1, \alpha_2]$$

which kills all the α_i with $i \geqslant 3$.

Proposition 12. *Let p and q be odd and $\geqslant 3$ and c be even and nonzero. Then in Ω_* the product $X_p^q(c) \times K3$ equals a polynomial in $\mathbb{H}P^2$ and the various $X_n^k(c)$, but not involving $K3$.*

Proof. The proof is by induction on the dimension $2(p+q)$. The base case is where this dimension is 12 and $p = q = 3$. Then we look at $X_3^3(c) \times K3$, of dimension 16. In dimension 16, we have two choices of indecomposable generators, namely $X_3^5(c)$ and $X_5^3(c)$ whose Milnor–Thom numbers according to Proposition 7 are

$$s_4(X_3^5(c)) = 30c^3, \quad s_4(X_5^3(c)) = 18c^5.$$

This implies that the element $Y(c) = 3c^2 X_3^5(c) - 5X_5^3(c)$ is in the kernel of s_4, and is therefore expressible as a polynomial in the generators of dimension $\leqslant 12$. However, $Y(c)$ is also in the kernel of the elliptic genus, and in dimension 16 this kernel is spanned by $\alpha_3(c) \times K3 = X_3^3(c) \times K3$ and $\alpha_4(c)$. Thus $Y(c)$ is a rational multiple of $X_3^3(c) \times K3$, and we only have to show that it is not the zero multiple. We can do this conveniently by using the calculations of the q-number in Proposition 9:

$$\begin{aligned} q_4(Y(c)) &= 3c^2 q(X_3^5(c)) - 5q(X_5^3(c)) \\ &= 3c^2 \cdot 30c^3 - 5 \cdot (-3c^5 + 42c^3) \\ &= 105c^3(c^2 - 2). \end{aligned}$$

This is nonzero since c is a nonzero integer. Thus, we have shown that $X_3^3(c) \times K3$ is a rational multiple of $Y(c) = 3c^2 X_3^5(c) - 5X_5^3(c)$.

For the inductive step, consider some $X_p^q(c)$ of dimension $4m = 2(p+q) \geqslant 16$. We fix a basis sequence α_i as in Proposition 11, taking $X_p^q(c)$ in its dimension: $\alpha_m(c) = X_p^q(c)$ for $m = \frac{1}{2}(p + q)$. Now in dimension $4m+4$ we may consider the two manifolds $X_3^{2m-1}(c)$ and $X_{2m-1}^3(c)$. By Proposition 7 their Milnor–Thom numbers are

$$\begin{aligned} s_{m+1}(X_3^{2m-1}(c)) &= c^3 \left[\binom{2m+1}{3} - (2m-1) \right] = \lambda c^3, \\ s_{m+1}(X_{2m-1}^3(c)) &= c^{2m-1} \left[\binom{2m+1}{2m-1} - 3 \right] \qquad = \mu c^{2m-1}. \end{aligned}$$

This implies that the element $Z(c) = \mu c^{2m-4} X_3^{2m-1}(c) - \lambda X_{2m-1}^3(c)$ is in the kernel of s_{m+1}, and is therefore expressible as a polynomial in the generators α_i with $i \leqslant m$.

We want to check that $q_{m+1}(Z(c)) \neq 0$. Using the formula from Proposition 9 we see that $q_{m+1}(Z(c))$ is an integral linear combination of c^{2m-1} and c^{2m-3}. However, in light of Remark 10, $c^{2m-4} X_3^{2m-1}(c)$ does not contribute to the term of degree $2m-3$, and so we can simply read off the coefficient of c^{2m-3} in $q_{m+1}(Z(c))$ to be

$$-\lambda \cdot 2m \cdot \left[\binom{2m-1}{2m-3} - 3\right] = -\left[\binom{2m+1}{3} - (2m-1)\right] \cdot 2m \cdot \left[\binom{2m-1}{2} - 3\right].$$

As this is clearly nonzero, we do indeed have $q_{m+1}(Z(c)) \neq 0$ for nonzero c.

Now expressing $Z(c)$ as a polynomial in the generators α_i with $i \leqslant m$, the non-vanishing of the q-number tells us that the monomial $\alpha_1 \cdot \alpha_m(c)$ must appear with a nonzero coefficient, since q_{m+1} vanishes on all monomials consisting only of α_i with $i < m$. We can then solve the resulting equation for $\alpha_1 \cdot \alpha_m(c)$ and this expresses $K3 \times X_p^q(c)$ as a rational linear combination of $X_3^{2m-1}(c)$, $X_{2m-1}^3(c)$ and monomials in the α_i with $i < m$. As $K3 \times X_p^q(c)$ is in the kernel of the elliptic genus, each monomial appearing in the linear combination must contain an α_i with $3 \leqslant i < m$ represented by some $X_n^k(c)$. By the inductive hypothesis, whenever such a monomial also contains $\alpha_1 = K3$ we can replace $K3 \times X_n^k(c)$ by an expression not involving $K3$. This completes the inductive step. □

After these preparations, we can now prove the theorem.

Proof of Theorem 1. We fix a basis sequence $\alpha_1 = K3$, $\alpha_2 = \mathbb{H}P^2$, $\alpha_i(c) = X_n^k(c)$ for $i \geqslant 3$ for Ω_* as in Proposition 11. As discussed above, the elliptic genus φ is the projection to the quotient by the ideal generated by the α_i with $i \geqslant 3$.

Consider spin manifolds of dimension $4m$, and a non-trivial linear combination f of their Pontryagin numbers. If the linear map

$$f : \Omega_{4m} \longrightarrow \mathbb{Q}$$

does not factor through $\varphi|\Omega_{4m}$, then it does not vanish identically on $\ker(\varphi)$. This means that $m \geqslant 3$ and that there is an element in $\ker(\varphi)$ on which f does not vanish. However, a vector space basis for $\ker(\varphi)$ is given by the monomials in the α_i containing at least one index $i \geqslant 3$. Thus, there is a Cartesian product of $K3$, $\mathbb{H}P^2$ and the $X_n^k(c)$ on which f does not vanish, and this product definitely contains at least one factor of the form $X_n^k(c)$. If this product does not contain $K3$, then it is a product of nonnegatively curved manifolds, and therefore nonnegatively curved. The value of f on this Cartesian product can be thought of as a linear combination of Pontryagin numbers of one of the factors of the form $X_n^k(c)$, with coefficients that depend on the Pontryagin numbers of the other factors. Moreover, the non-vanishing of f means that a non-zero Pontryagin number of $X_n^k(c)$ does appear. Since the Pontryagin numbers of $X_n^k(c)$ are odd polynomials in c by Proposition 6, they are certainly non-constant. Varying c, we see that f is unbounded on this family of nonnegatively curved manifolds.

Finally, if the monomial in the generators α_i on which f does not vanish contains a $K3$ factor, then, because it also contains at least one $X_n^k(c)$, we can replace $K3 \times X_n^k(c)$ by a linear combination of terms not involving $K3$ using Proposition 12. This replaces the monomial in the generators by a linear combination of terms, but since f does not vanish on the monomial, it does not vanish on at least one of the summands of the linear combination. Repeating this procedure until there are no $K3$ factors left we find a product of factors all of which are $\mathbb{H}P^2$ or of the form $X_n^k(c)$, and on which f does not vanish. Since there is at least one $X_n^k(c)$-factor, as above we conclude that f is unbounded on these nonnegatively curved manifolds as we vary c. This completes the proof. □

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