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Almost complex structures on hyperbolic manifolds

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Abstract
We discuss the existence of almost complex structures on closed hyperbolic manifolds of even dimension at least four. We prove that for $n = 2$ and for all odd n every hyperbolic $2n$-manifold has a finite covering admitting an almost complex structure. Conjecturally this should be true for all n. For $n = 4$ we prove it for arithmetic manifolds.

Keywords Almost complex structure · Hyperbolic manifold · Characteristic number

Mathematics Subject Classification Primary 32Q60 · 22E40 · 57R15; Secondary 53C20 · 55R50 · 57R20

1 Introduction

The purpose of this paper is to discuss the following:

Conjecture 1 *Every $2n$-dimensional closed hyperbolic manifold has a finite-sheeted covering space which admits an almost complex structure.*

In other words, closed even-dimensional hyperbolic manifolds are virtually almost complex. As usual, a hyperbolic manifold is one that admits a metric of constant negative sectional curvature. If it has dimension $2n$ it is therefore of the form $\mathbb{H}^{2n}/\Gamma$ for a torsion-free uniform lattice Γ in the isometry group of hyperbolic space. For $n = 1$ it is of course classical that oriented hyperbolic structures and complex structures are the same, but in higher dimensions the two geometries diverge. To avoid this exceptional case, we assume throughout that $n > 1$, so the dimensions of our manifolds are ≥ 4.

The Davis [10] hyperbolic 4-manifold is orientable but not almost complex, cf. Example 14 below, showing that the passage to a finite-sheeted covering cannot be avoided, even if one considers only orientable manifolds.

One reason for believing the conjecture is that hyperbolic manifolds are virtually stably parallelizable by a result of Deligne and Sullivan [11, 23]. This implies that they are virtually stably almost complex. Therefore, to prove the conjecture one only has to bridge the gap from stably almost complex to genuinely almost complex structures. In dimension 6, it is

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known from work of Wall [26] that in fact there is no such gap. Another reason to believe the conjecture is the following result.

Theorem 2 *Conjecture 1 is true for hyperbolic manifolds of dimension* $2n$ *if* $n = 2$ *and if* n *is odd. For* $n = 4$ *it is true at least for all arithmetic manifolds.*

There are also good reasons for being skeptical about Conjecture 1. First of all, no hyperbolic manifold can be Kähler, and, even stronger, no such lattice can be a Kähler group, by a result of Carlson and Toledo [7], see [2, Ch. 6]. Second of all, without any Kähler assumption, I proved in [19], that no hyperbolic 4-manifold can be homotopy equivalent to a compact complex surface. This was later generalised to show that no lattice in hyperbolic space can be the fundamental group of a compact complex surface, cf. [2, Ch. 1, Sect. 3] and [8]. See [17, 18] for further generalizations.

In dimension 4, i.e. for $n = 2$, we know from the result in [19] that none of the almost complex structures provided by Theorem 2 can be integrable. In this dimension there are other examples which are almost complex without being complex. However, in higher dimensions no such examples are known at the time of writing, and Yau has speculated that they cannot exist. So either the almost complex structures produced by Theorem 2 are non-integrable, providing the first countexamples to Yau's suggestion in dimensions > 4, or they are integrable to non-Kähler complex structures that have little connection to hyperbolic geometry since Gauduchon [12] proved that an almost complex structure which is orthogonal for the hyperbolic metric cannot be integrable, cf. also [14].

2 Characteristic numbers of hyperbolic manifolds

The following is well known. For the convenience of the reader we provide a quick proof.

Lemma 3 *Let* $M = \mathbb{H}^{2n}/\Gamma$ *be a closed orientable hyperbolic manifold. Then the following two statements hold.*

(1) All real Pontryagin classes of M *vanish. In particular, its signature* $\sigma(M)$ *vanishes.*
(2) The Euler characteristic $\chi(M)$ *satisfies* $\chi(M) \equiv 0 \pmod 2$ *and* $(-1)^n\chi(M) > 0$.

Proof The hyperbolic metric is conformally flat, and therefore has vanishing Weyl tensor. Since the Chern–Weil representatives for Pontryagin classes depend only on the Weyl tensor, they all vanish. The vanishing of the signature follows by the Hirzebruch signature theorem.

Recall that the signature is zero by definition whenever n is odd. Poincaré duality gives $\chi(M) \equiv b_n(M) \equiv \sigma(M) \equiv 0 \pmod 2$. Finally, the sign of the Euler characteristic follows from the Chern–Gauss–Bonnet theorem. □

3 Odd complex dimensions

In this section we prove Theorem 2 in odd complex dimensions.

So let M be a closed hyperbolic $2n$-manifold, with n odd. After passing to a finite covering if necessary, we may assume that M is stably parallelizable by the result of Deligne and Sullivan [11, 23]. Therefore M is stably almost complex, and we may choose the stable almost complex structure to have trivial Chern classes.

We now need to modify this stable almost complex structure to make it genuinely almost complex. The criterion for this modification to work is provided by the following well known result, see for example [24, Theorem 1.1] or [25, Theorem 1.7].

Proposition 4 *A connected stably almost complex manifold M of real dimension $2n$ has an almost complex structure if and only if it has a stable almost complex structure with $\langle c_n, [M]\rangle$ equal to its Euler characteristic $\chi(M)$.*

This tells us that we need to modify the top Chern number of the stable almost complex structure to arrange it to equal the Euler characteristic of M. We make this modification via the connected sum with S^{2n}. This leaves the manifold unchanged, but changes the stable almost complex structure.

The next result is also straightforward; see for example [16, Section 5.2] or [13, Lemma 2.1].

Proposition 5 *Let $M\#N$ be a connected sum with collapsing maps $p_M\colon M\#N \longrightarrow M$ and $p_N\colon M\#N \longrightarrow N$ to the summands. Then $p_M^*TM \oplus p_N^*TN$ is stably isomorphic to $T(M\#N)$.*

As remarked by Kahn [16, Lemma 3], this means that stable almost complex structures on M and N give rise to stable almost complex structures on their connected sum, and that the Chern numbers are additive in this construction.

Consider first the case $n = 3$. The standard almost complex structure on S^6 has $\langle c_3, [S^6]\rangle = 2$, which is the Euler characteristic of the sphere. However, we can take its complex conjugate, which changes the sign of the third Chern class. Evaluating this on the fundamental class of S^6 for the orientation induced by the standard almost complex structure, rather than by its conjugate, we get the value -2. Connect summing this stable almost complex structure to the given one on M, which has trivial c_3, we lower the value of c_3 on M by 2. Since the Euler characteristic of M is a negative even integer by Lemma 3, if we sum M with $-\chi(M)/2$ many copies of this conjugate structure on S^6, we obtain a stable almost complex structure on M which, by Proposition 4, induces an almost complex structure on M. This completes the proof for $n = 3$.

Remark 6 The argument we have given proves Wall's result [26, Theorem 9] that every stably almost complex closed oriented 6-manifold is genuinely almost complex, because such a manifold automatically has even Euler characteristic by Poincaré duality.

For n odd and ≥ 5, the sphere S^{2n} is not almost complex, but it is stably almost complex.

Proposition 7 *For every odd $n \geq 5$, the sphere S^{2n} admits stable almost complex structures with Chern number $c_n \neq 0$.*

Proof The proof is a standard application of Bott periodicity [5].

Via the clutching construction, vector bundles over S^{2n} with structure group G are classified by $\pi_{2n-1}(G)$. An oriented real bundle of rank $2n$ admits a complex structure if and only if the homotopy class of its clutching function is in the image of the inclusion-induced map $f\colon \pi_{2n-1}(U(n)) \longrightarrow \pi_{2n-1}(SO(2n))$. We think of these maps as forgetful maps as they forget the complex structure of a vector bundle. They fit into the following commutative diagram, in which s denotes the stabilisation maps:

$$\begin{array}{ccccccc}
 & & \pi_{2n-1}(U(n)) & \xrightarrow{f} & \pi_{2n-1}(SO(2n)) & & \\
 & & \downarrow s & & \downarrow s & & \\
\pi_{2n}(SO/U) & \xrightarrow{\partial} & \pi_{2n-1}(U) & \xrightarrow{f} & \pi_{2n-1}(SO) & \longrightarrow & \pi_{2n-1}(SO/U) \\
\| & & \| & & \| & & \| \\
\mathbb{Z} & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{f} & \pi_{2n-1}(SO) & \longrightarrow & 0
\end{array}$$

The second line of the diagram is a piece of the homotopy exact sequence of the fibration $U \longrightarrow SO \longrightarrow SO/U$. The information given about the homotopy groups in the third line is from [5]. Since the group $\pi_{2n-1}(SO)$ is finite—either trivial or of order 2, depending on n, see again [5]—it is clear that its neutral element, corresponding to the stable tangent bundle of S^{2n} because the sphere is stably parallelisable, is hit by non-zero elements of $\pi_{2n-1}(U)$ under the stabilised forgetful map $f: \pi_{2n-1}(U) \longrightarrow \pi_{2n-1}(SO)$ at the bottom. This gives stable almost complex structures on S^{2n} with $c_n \neq 0$ since the isomorphism between $\pi_{2n-1}(U)$ and $\mathbb{Z}$ is given by c_n, suitably normalised. □

Now let $c(n)$ be the smallest positive Chern number for a stable almost complex structure on S^{2n}. The exact value of $c(n)$ is known, cf. [6], but is not important for the argument that follows, in which we use a theorem of Lubotzky to pass to a suitable covering of a given hyperbolic manifold. Lubotzky's result [20, Theorem A], about arbitrary finitely generated linear groups, is more general, we only state what we need here.

Theorem 8 *For every natural number d, the fundamental group of a hyperbolic manifold has a subgroup of index divisible by d.*

Given a closed hyperbolic $2n$-manifold M with a stable almost complex structure with trivial Chern classes, the theorem tells us that M has a finite covering whose degree is a multiple of $c(n)$. Thus, replacing M by this covering, we may assume that $\chi(M)$ is a negative multiple of $c(n)$. Therefore, the connected sum with the appropriate number of copies of S^{2n} equipped with the stable almost complex structure with top Chern number $-c(n)$ provides a stable almost complex structure on M which induces a genuine almost complex structure because of Proposition 4.

This completes the proof of Theorem 2 for all odd n.

4 Even complex dimensions

In this section we discuss the case when n is even. In particular we will prove the cases $n = 2$ and $n = 4$ in Theorem 2.

We begin with the following observations.

Lemma 9 *Let M be a closed oriented hyperbolic $2n$-manifold with n even.*

(1) If M is almost complex, then $\chi(M) \equiv 0 \pmod 4$.
(2) There is no almost complex structure on M with $c_i = 0$ for $i < n$.

Proof If M is almost complex of even complex dimension n, then

$$\chi(M) \equiv (-1)^{n/2}\sigma(M) \pmod 4$$

by the Hodge index theorem, cf. Hirzebruch [15, p. 777]. Since the signature vanishes by Lemma 3, the first statement follows.

For the second statement recall that if $c_i = 0$ for $i < n$, then $(-1)^{n/2}p_{n/2} = 2c_n$. Now in the hyperbolic case the Pontryagin numbers vanish, but the Euler characteristic, which is the evaluation of c_n, does not by Lemma 3, so this is impossible. □

Remark 10 Either part of Lemma 9 can be applied to S^{2n}, the compact symmetric space dual to $\mathbb{H}^{2n}$. This gives two different proofs—both well known—for the fact that S^{2n} is not almost complex for even n.

The second statement in Lemma 9 shows that the stable almost complex structures found on hyperbolic manifolds via the result of Deligne and Sullivan [11, 23] cannot be modified easily to make them genuinely almost complex. Any modification cannot be restricted to the top cell, and has to start in dimension $2n-2$, or earlier. Therefore, for even n, unlike for odd n, we do not use the result of [11, 23].

In dimension four the first part of Lemma 9 has a converse.

Proposition 11 *A closed oriented hyperbolic* 4*-manifold M is almost complex if and only if its Euler characteristic is divisible by* 4*.*

Proof If M is almost complex, then its Euler characteristic is a multiple of 4 by Lemma 9. For the converse note that $\chi(M) > 0$ by Lemma 3, so once it is a multiple of 4, it is at least 4. This means that $b_2(M) \geq 2$. By the vanishing of the signature, we know that the intersection form is indefinite. Therefore, by the Hasse–Minkowski classification, it is either diagonal or a sum of hyperbolic pairs. In both cases one can explicitly find characteristic elements that have the correct square to satisfy the Wu formula. Therefore these characteristic elements are first Chern classes of almost complex structures. □

The Euler characteristics of closed orientable hyperbolic 4-manifolds are even, again by Lemma 3. Therefore the multiplicativity of the Euler characteristic in coverings yields:

Corollary 12 *Every connected even-degree covering space of a closed oriented hyperbolic* 4*-manifold is almost complex.*

Since the fundamental groups are residually finite, we know that there are plenty of finite coverings. More precisely, Theorem 8 shows that there are even degree coverings. This proves the case $n = 2$ in Theorem 2.

Remark 13 In response to a question from D. Toledo, I had proved Proposition 11 and Corollary 12 in the 1990s. This was not published at the time, but has since been mentioned occasionally in the literature, for example in [14, p. 206]. Beware however, that in the parenthetical remark before this statement, in [14, p. 205/6], the Wu criterion seems to be misstated.

Example 14 The Davis manifold [10] has Euler characteristic 26, and is therefore not almost complex. However, all its even-degree covering spaces are almost complex.

Example 15 Conder and Machlachlan [9] constructed a closed orientable hyperbolic 4-manifold of Euler characteristic 16. This is almost complex, as are all its covering spaces.

Example 16 Agol and Lin [1] have constructed hyperbolic 4-manifolds with vanishing Seiberg–Witten invariants. This means in particular that they cannot support symplectic structures. The construction starts with a certain 3-dimensional L-space Y, which is embedded in an orientable hyperbolic 4-manifold M as a non-separating hypersurface. This means that $b_1(M)$ is positive. They then take a double cover $\overline{M} \longrightarrow M$ to find a separating L-space in $\overline{M}$. By construction, $\chi(\overline{M})$ is divisible by 4. So $\overline{M}$ and all its covering spaces are almost complex.

Example 17 Martelli, Riolo and Slavich [21] have constructed an orientable hyperbolic 4-manifold with odd intersection form. In particular, it is not spin, although by [11, 23] it does have spin coverings. This manifold has Euler characteristic divisible by 4, and the same will be true for any manifold constructed by the colouring method of [21]. (Private communication from B. Martelli). Therefore, this manifold is almost complex.

Example 18 Battista and Martelli [3] recently constructed an orientable hyperbolic 4-manifold with a perfect circle-valued Morse function. It has Euler characteristic 272, and so it too is almost complex.

We now move on to dimension eight. Let M be a closed oriented hyperbolic 8-manifold. Like in dimension 4, Lemma 9 shows that having Euler characteristic divisible by 4 is a necessary condition for the existence of an almost complex structure on M. Unlike in dimension 4, this divisibility does not imply positivity of the second Betti number via Lemma 3. We now get only

$$2b_2(M) + b_4(M) \geq 2$$

so it is conceivable that M would have Euler characteristic a multiple of 4 but $b_2(M) = 0$, and that this would persist on all finite covering spaces. The existence of such an M would disprove Conjecture 1 in dimension 8:

Lemma 19 *A closed orientable hyperbolic* 8*-manifold* M *with* $b_2(M) = 0$ *is not almost complex.*

Proof Since by assumption there is no second cohomology, any almost complex structure would have vanishing first Chern class. The vanishing of the first Pontryagin class from Lemma 3 then shows that the second Chern class would also vanishs. By Poincaré duality there is no cohomology in degree 6, so c_3 would also vanish. Therefore the conclusion follows from the second part of Lemma 9. □

With the added assumption of positive second Betti number, Proposition 11 generalizes.

Proposition 20 *A closed oriented hyperbolic* 8*-manifold* M *with* $b_2(M) > 0$ *is almost complex if and only if its Euler characteristic is divisible by* 4.

Proof The necessity of the divisibility condition was proved in the first part of Lemma 9. The sufficiency follows from the analog of the Wu criterion in dimension 8 proved by Müller and Geiges [22, Theorem 4]. □

Corollary 21 *Every connected even-degree covering space of a closed oriented hyperbolic* 8*-manifold with positive second Betti number is almost complex.*

Again even-degree covering spaces exist in all cases by Theorem 8.

Let us now assume that M is an arithmetic hyperbolic 8-manifold. Then a deep result of Bergeron and Clozel [4, Corollaire 1.8] shows that M has a finite connected covering $\overline{M}$ with positive first and second Betti numbers. In case that $\chi(\overline{M})$ is not divisible by 4, we take another covering, of even degree. This exists, either because $b_1(\overline{M}) > 0$, or by invoking Theorem 8 again. This further covering is almost complex by Proposition 20. This completes the proof of the statement for $n = 4$ in Theorem 2.

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