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The Gell-Mann–Brueckner Formula for the Correlation Energy of the Electron Gas: A Rigorous Upper Bound in the Mean-Field Regime

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Abstract: We prove a rigorous upper bound on the correlation energy of interacting fermions in the mean-field regime for a wide class of interaction potentials. Our result covers the Coulomb potential, and in this case we obtain the analogue of the Gell-Mann–Brueckner formula $c_1 \rho \log(\rho) + c_2 \rho$ in the high density limit. We do this by refining the analysis of our bosonization method to deal with singular potentials, and to capture the exchange contribution which is absent in the purely bosonic picture.

Contents

1. Introduction

Although interacting Fermi gases have been studied extensively from the beginning of quantum mechanics, their rigorous understanding remains one of the major issues of condensed matter physics. From first principles, a system of N fermions in $\mathbb{R}^3$ can be described by a Schrödinger equation in $\mathbb{R}^{3N}$, subject to the anti-symmetry condition between the variables due to Pauli's exclusion principle. However, this fundamental theory becomes very complex when $N \to \infty$, leading to the need of various approximations. Justifying these approximations is an important task of mathematical physics.

One of the most basic approximations for fermions is the Hartree–Fock (HF) theory. In HF theory, the particles are assumed to be independent, namely the HF energy is computed by restricting the consideration to Slater determinants. In spite of its simplicity, the HF theory is used very successfully in computational physics and chemistry to compute the ground state energy of atoms and molecules. The accuracy of the HF energy (in comparison to the full quantum energy) for large Coulomb systems was investigated in the 1990 s by Fefferman and Seco [12], Bach [1], and Graf and Solovej [15].

On the other hand, for the *electron gas* (e.g. jellium, a homogeneous electron gas moving in a background of uniform positive charge), the HF theory is essentially trivial in the high density limit since the HF energy only contains an exponentially small correction to the energy of the Fermi state, the ground state of the non-interacting gas [14]. Therefore, computing the *correlation energy*,[1] namely the correction to the HF energy, is a crucial task to understand the effect of the interaction. It was already noticed by Wigner in 1934 [23] and confirmed by Heisenberg in 1947 [17] that it would be very challenging to accomplish this task within perturbation theory due to the long-range property of the Coulomb potential. Nevertheless, a remarkable attempt in this direction was done by Macke in 1950 [18] when he used a partial resummation of the divergent series to predict the leading order contribution $c_1 \rho \log(\rho)$ of the correlation energy (with density $\rho \to \infty$).

A cornerstone in the correlation analysis of the electron gas is the *random phase approximation* (RPA) which was proposed by Bohm and Pines in the 1950 s [7–9,19]. As an important consequence of the Bohm-Pines RPA theory, the electron gas could be decoupled into collective plasmon excitations and quasi-electrons that interacted via a screened Coulomb interaction. The latter fact justified the independent particle approach commonly used for many-body fermion systems. The justification of the RPA was a major question in condensed matter and nuclear physics in the late 1950 s and 1960 s. An important justification was given by Gell-Mann and Brueckner in 1957 [13] when they formally derived the RPA from a resummation of Feynman diagrams where each term separately diverges but the sum is convergent. More precisely, by considering the diagrams corresponding to the interaction of pairs of fermions, one from inside and one from outside the Fermi state, Gell-Mann and Brueckner were able to produce the leading order contribution $c_1 \rho \log(\rho) + c_2 \rho$ of the correlation energy.

Soon after the achievement of Gell-Mann and Brueckner, Sawada [21] and Sawada–Brueckner–Fukuda–Brout [22] proposed an alternative approach to the RPA where the pairs of electrons are interpreted as bosons, leading to an effective Hamiltonian which is quadratic in terms of the bosonic creation and annihilation operators. Note that within the purely bosonic picture, quadratic Hamiltonians can be diagonalized by Bogolubov transformations [6], and hence their spectra can be computed explicitly. Therefore, the

[1] This name comes from the fact that Slater determinants are the least correlated wave functions under Pauli's exclusion principle.

Hamiltonian approach in [21,22] is conceptually more transparent than the resummation method in [13]. Unfortunately the analysis in [21,22] only gives the contribution $c_1 \rho \log(\rho)$ of the correlation energy because the *exchange contribution* of order ρ is missed in the purely bosonic picture.

Recently, the bosonization argument in [21,22] has been revisited and made rigorous in the mean-field regime with smooth interaction potentials [2–5,10,11,16]. In principle, if the interaction is sufficiently weak, then the non-bosonizable terms of the interaction energy are negligible, and the quasi-bosonic Hamiltonian can be analyzed with great precision. In particular, the correlation energy has been successfully computed to the leading order [2,3,5,10]. However, the boundedness of interaction potentials is crucial for all of these works, and extending the analysis to the electron gas remains a very interesting open question.

In the present paper, we will give the first rigorous upper bound to the correlation energy of the electron gas in the mean-field regime. Our bound is consistent with the Gell-Mann–Brueckner formula $c_1 \rho \log(\rho) + c_2 \rho$ for jellium in the high density limit [13]. Although our trial state argument is inspired by the bosonization method in [21,22], we are able to capture correctly the exchange contribution by carefully distinguishing the purely bosonic picture and the quasi-bosonic one. On the mathematical side, we will use the general method in our recent work [10], but several new estimates are needed to deal with the singularity of the potential. The matching lower bound in the mean-field regime, as well as the corresponding result in the thermodynamic limit, remain open, and we hope to be able to come back to these issues in the future.

On the technical side, the key idea of [10] is that while the bosonic property of fermionic pairs holds only in an average sense, this weak bosonic property is sufficient to extract correctly the correlation energy by implementing a quasi-bosonic Bogolubov transformation. The main contribution of the present paper is to show that this approach is also sufficient to extract the exchange correction to the purely bosonic computation. On the other hand, another bosonization method has been proposed in [2], where the bosonic property of fermionic pairs is strengthened by using suitable patches on the Fermi sphere for the quasi-bosonic creation and annihilation operators, making the comparison with the purely bosonic computation significantly easier. In fact, as explained in [5], the approach in [2] can be extended to give the leading order of the correlation energy upper bound for potentials satisfying $\sum V_k^2 |k| < \infty$. Although this condition only barely fails for the Coulomb potential, there is a huge difference to the Coulomb case. While for $\sum V_k^2 |k| < \infty$ the bosonic correlation contribution is of order k_F and the exchange correlation is of lower order $o(k_F)$, for the Coulomb potential the exchange contribution raises to the order k_F, whereas the bosonic correlation behaves as $k_F \log(k_F)$, which makes the Coulomb case much more challenging (here k_F is the radius of the Fermi ball). In particular, the method in [2,5] does not seem to capture the exchange contribution which is indeed important for the Coulomb potential.

1.1. Main result. Let $\mathbb{T}^3 = [0, 2\pi]^3$ with periodic boundary conditions. Let $V : \mathbb{T}^3 \to \mathbb{R}$ be defined by

$$V(x) = \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k e^{ik\cdot x}, \quad \mathbb{Z}^3_* = \mathbb{Z}^3 \setminus \{0\}, \tag{1.1}$$

with Fourier coefficients satisfying

$$\hat{V}_k \geq 0, \quad \hat{V}_k = \hat{V}_{-k}, \quad \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 < \infty. \tag{1.2}$$

We implicitly assume that $\hat{V}_0 = 0$, or equivalently that the "background" has been subtracted.

For $k_F > 0$, let $N = |B_F|$ be the number of integer points in the Fermi ball $B_F = \overline{B}(0, k_F) \cap \mathbb{Z}^3$ and consider the mean-field Hamiltonian

$$H_N = -\sum_{i=1}^{N} \Delta_i + k_F^{-1} \sum_{1 \leq i < j \leq N} V\left(x_i - x_j\right) \tag{1.3}$$

on the fermionic space $\mathcal{H}_N = \bigwedge^N \mathfrak{h}$ with $\mathfrak{h} = L^2\left(\mathbb{T}^3\right)$.[2] The leading order of the ground state energy of H_N is given by the Fermi state

$$\psi_{\mathrm{FS}} = \bigwedge_{p \in B_F} u_p, \quad u_p(x) = (2\pi)^{-\frac{3}{2}} e^{ip \cdot x}. \tag{1.4}$$

It is straightforward to find (see e.g. [10, Eqs. (1.10) and (1.20)])

$$E_{\mathrm{FS}} = \langle \psi_{\mathrm{FS}}, H_N \psi_{\mathrm{FS}} \rangle = \sum_{p \in B_F} |p|^2 + \frac{1}{2(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}(k)\left(|L_k| - N\right) \tag{1.5}$$

where for every $k \in \mathbb{Z}^3_*$, we denoted the *lune* associated to k by

$$L_k = (B_F + k) \setminus B_F = \left\{ p \in \mathbb{Z}^3 \mid |p - k| \leq k_F < |p| \right\}. \tag{1.6}$$

Our main result concerns the corrections to the ground state energy. For every $k \in \mathbb{Z}^3_*$, define

$$\lambda_{k,p} = \frac{1}{2}\left(|p|^2 - |p-k|^2\right), \quad \forall p \in L_k. \tag{1.7}$$

We will prove the following:

Theorem 1.1. *As $k_F \to \infty$ it holds that*

$$\inf \sigma\left(H_N\right) \leq E_{\mathrm{FS}} + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\left\{|k|, k_F\right\}}$$

where

$$E_{\mathrm{corr,bos}} = \frac{1}{\pi} \sum_{k \in \mathbb{Z}^3_*} \int_0^\infty F\left(\frac{\hat{V}_k k_F^{-1}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right) dt, \quad F(x) = \log(1+x) - x,$$

[2] We consider spinless particles for simplicity. Including the spin only requires slight modifications of the analysis.

is the bosonic contribution and

$$E_{\mathrm{corr,ex}} = \frac{k_F^{-2}}{4\,(2\pi)^6} \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in L_k} \frac{\hat{V}_k \hat{V}_{p+q-k}}{\lambda_{k,p} + \lambda_{k,q}}$$

is the exchange contribution, for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2$.

Some remarks on our result:

1. Consider the Coulomb potential, $\hat{V}_k = g\,|k|^{-2}$ for a constant $g > 0$. Following the analysis of [15], we find that

$$\inf \sigma\,(H_N) = E_{\mathrm{FS}} + o\left(k_F^3\right) \tag{1.8}$$

where E_{FS} contains the kinetic energy of order k_F^5, the direct interaction energy of order k_F^5 and the exchange interaction energy of order k_F^3. Furthermore, it is straightforward to adapt the proof in [14] to see that the difference between E_{FS} and the HF energy is exponentially small as $k_F \to \infty$. Therefore our result really concerns the correlation energy, which we bound from above by

$$E_{\mathrm{corr,bos}} \sim -k_F \log\,(k_F) \quad \text{and} \quad E_{\mathrm{corr,ex}} \sim k_F \tag{1.9}$$

plus the error term of order

$$\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|\,, k_F\}} \sim \sqrt{\log\,(k_F)}. \tag{1.10}$$

In fact, it is easy to verify (1.10) using $\sum_{|k|\le k_F} \hat{V}_k^2 |k| \sim \log(k_F)$ and $\sum_{|k|\ge k_F} \hat{V}_k^2 \sim k_F^{-1}$. To see the leading order behavior $E_{\mathrm{corr,ex}} \sim k_F$ in (1.9), one may use that $\lambda_{k,p} \sim |k| \max\{|k|, k_F\}$ (in an average sense) and that $|L_k| \sim k_F^2 \min\{|k|, k_F\}$. Moreover, from the expansion

$$\log(1+x) - x \approx -x^2/2 + o(x^3)_{x\to 0} \tag{1.11}$$

we have

$$\begin{aligned} E_{\mathrm{corr,bos}} &\approx -\frac{1}{4(2\pi)^6} \sum_{k\in\mathbb{Z}^3_*} (\hat{V}_k k_F^{-1})^2 \frac{2}{\pi} \int_0^\infty \left(\sum_{p\in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \right)^2 dt \\ &= -\frac{1}{4(2\pi)^6} \sum_{k\in\mathbb{Z}^3_*} (\hat{V}_k k_F^{-1})^2 \sum_{p,q\in L_k} \frac{1}{\lambda_{k,p} + \lambda_{k,q}}, \end{aligned} \tag{1.12}$$

and hence the asymptotic behavior $E_{\mathrm{corr,bos}} \sim -k_F \log(k_F)$ in (1.9) follows similarly.

Note that the correlation energy $E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}}$ in Theorem 1.1 is exactly the mean-field analogue of the Gell-Mann–Brueckner formula $c_1\rho\log\,(\rho) + c_2\rho$ for jellium in the thermodynamic limit [13]. Indeed, substituting $k_F^{-1}\hat{V}_k \to 4\pi e^2\,|k|^{-2}$ and $(2\pi)^3 \to$ the volume Ω, $E_{\mathrm{corr,bos}}$ agrees with [22, Eq. (34)] which is equivalent with [13, Eq. (19)] (accounting also for spin). In the thermodynamic limit, the right-hand side of (1.12) always diverges, no matter if we have the mean-field scaling or not, but the full expression on the left-hand side converges in either case.

Furthermore, we also obtain the exchange contribution $E_{\text{corr,ex}}$, which is the analogue of [13, Eq. (9)], which is completely absent from the bosonic model of [22]. With the same substitutions as above, the exchange contribution takes the form

$$E_{\text{corr,ex}} = 2 \cdot \frac{1}{4\Omega^2} \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in L_k} \frac{4\pi e^2}{|k|^2} \frac{4\pi e^2}{|p+q-k|^2} \frac{1}{\frac{1}{2}\left(|p|^2+|p-k|^2\right)+\frac{1}{2}\left(|q|^2+|q-k|^2\right)}$$
$$= \frac{8\pi^2 e^4}{\Omega^2} \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in L_k} \frac{1}{|k|^2\,|p+q-k|^2\, k\cdot(p+q-k)} \tag{1.13}$$

which agrees with [20, Eq. (9.14)] (noting that we take $m = 1/2$).

2. If the potential satisfies $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2\,|k| < \infty$, and so is less singular than the Coulomb potential, then the bosonic contribution $E_{\text{corr,bos}}$ is of order k_F, while the exchange contribution is $o(k_F)$. In this case, the upper bound

$$\inf \sigma(H_N) \le E_{\text{FS}} + E_{\text{corr,bos}} + o(k_F) \tag{1.14}$$

is already known; see [10, Remark 1 after Theorem 1.3] and [5, Appendix A]. Under the stronger condition $\sum \hat{V}_k\,|k| < \infty$ the matching lower bound was established in [5,10] (see also [2] and [3] for previous results on the upper and lower bounds, respectively, when $\hat{V}_k$ is finitely supported). In comparison, the Coulomb potential is much more challenging to analyze, since it leads to an additional logarithmic factor in the bosonic contribution, and lifts the exchange contribution to the order k_F. On the mathematical side, working with the Coulomb potential thus requires a substantial refinement of the bosonization method compared to the existing works.

3. Although the case of the greatest physical interest is the Coulomb potential, our result covers a far greater class of singular potentials: Under the condition $\sum_{k\mathbb{Z}^3_*} \hat{V}_k^2 < \infty$, the error term $\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}}$ is of order at most $O\left(\sqrt{k_F}\right)$, and so Theorem 1.1 is always a meaningful result.

1.2. Overview of the proof. We will construct a trial state by applying a quasi-bosonic Bogolubov transformation to the Fermi state ψ_{FS}. We will follow the general formulation of the bosonization method in [10]. We quickly recall this here for the reader's convenience, after which we explain the new components of the proof and the structure of the rest of the paper.

Rewriting the Hamiltonian We will use the second quantization formalism in which we associate to every plane wave state u_p of equation (1.4) the creation and annihilation operators $c_p^* = a^*(u_p)$ and $c_p = a(u_p)$ on the fermionic Fock space. They obey the canonical anti-commutation relations (CAR)

$$\{c_p, c_q\} = \left\{c_p^*, c_q^*\right\} = 0, \quad \left\{c_p, c_q^*\right\} = \delta_{p,q}, \quad p, q \in \mathbb{Z}^3. \tag{1.15}$$

The Hamiltonian H_N of equation (1.3) can then be written as $H_N = H_{\text{kin}} + k_F^{-1} H_{\text{int}}$ where

$$H_{\text{kin}} = \sum_{p\in\mathbb{Z}^3} |p|^2\, c_p^* c_p, \quad H_{\text{int}} = \frac{1}{2(2\pi)^3} \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in\mathbb{Z}^3} \hat{V}_k c_{p+k}^* c_{q-k}^* c_q c_p. \tag{1.16}$$

Note that the Fermi state ψ_{FS} obeys (B_F^c denoting the complement of B_F with respect to $\mathbb{Z}^3$)

$$c_p \psi_{\mathrm{FS}} = 0 = c_q^* \psi_{\mathrm{FS}}, \quad p \in B_F^c,\ q \in B_F, \tag{1.17}$$

and so it follows by the CAR that the kinetic energy of the Fermi state is

$$\langle \psi_{\mathrm{FS}}, H_{\mathrm{kin}} \psi_{\mathrm{FS}} \rangle = \sum_{p \in B_F} |p|^2 . \tag{1.18}$$

We define the *localized kinetic operator* H'_{kin} by

$$\begin{aligned} H'_{\mathrm{kin}} = H_{\mathrm{kin}} - \langle \psi_{\mathrm{FS}}, H_{\mathrm{kin}} \psi_{\mathrm{FS}} \rangle &= \sum_{p \in B_F^c} |p|^2 c_p^* c_p - \sum_{p \in B_F} |p|^2 c_p c_p^* \\ &= \sum_{p \in B_F^c} \left(|p|^2 - k_F^2 \right) c_p^* c_p + \sum_{p \in B_F} \left(k_F^2 - |p|^2 \right) c_p c_p^*, \end{aligned} \tag{1.19}$$

where we for the last identity used the "particle-hole symmetry"

$$\mathcal{N}_E := \sum_{p \in B_F^c} c_p^* c_p = \sum_{p \in B_F} c_p c_p^* \quad \text{on } \mathcal{H}_N. \tag{1.20}$$

From the last identity of equation (1.19) it is clear that H'_{kin} is non-negative.

We normal-order H_{int} with respect to ψ_{FS}: Using the CAR and the fact that $\sum_{p \in \mathbb{Z}^3} c_p^* c_p = \mathcal{N} = N$ on $\mathcal{H}_N$, it factorizes as

$$H_{\mathrm{int}} = \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k \left(\left(\sum_{p \in \mathbb{Z}^3} c_p^* c_{p+k} \right)^* \left(\sum_{q \in \mathbb{Z}^3} c_{q-k}^* c_q \right) - N \right). \tag{1.21}$$

Decomposing for every $k \in \mathbb{Z}_*^3$

$$\sum_{p \in \mathbb{Z}^3} c_{p-k}^* c_p = B_k + B_{-k}^* + D_k, \quad B_k = \sum_{p \in L_k} c_{p-k}^* c_p, \tag{1.22}$$

we can write

$$\begin{aligned} H_{\mathrm{int}} &= \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k \left(\left(B_k + B_{-k}^* \right)^* \left(B_k + B_{-k}^* \right) - N \right) \\ &\quad + \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k \left(2\,\mathrm{Re} \left(\left(B_k + B_{-k}^* \right)^* D_k \right) + D_k^* D_k \right). \end{aligned} \tag{1.23}$$

Using the CAR again it is easy to compute that

$$\left[B_k, B_k^* \right] = |L_k| - \sum_{p \in L_k} \left(c_p^* c_p + c_{p-k} c_{p-k}^* \right) \tag{1.24}$$

whence (using also that $\hat{V}_k = \hat{V}_{-k}$)

$$H_{\text{int}} = -\frac{1}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(N-|L_k|\right) + \frac{1}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(2B_k^*B_k + B_kB_{-k} + B_{-k}^*B_k^*\right) + \frac{1}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(2\,\text{Re}\left(\left(B_k + B_{-k}^*\right)^* D_k\right) + D_k^*D_k - \sum_{p\in L_k}\left(c_p^*c_p + c_{p-k}c_{p-k}^*\right)\right). \tag{1.25}$$

Note that the first sum is finite as $|L_k| = N$ for $|k| > 2k_F$. It is easily verified that $D_k\psi_{\text{FS}} = D_k^*\psi_{\text{FS}} = B_k\psi_{\text{FS}} = 0$, so we deduce from this identity that

$$\langle\psi_{\text{FS}}, H_{\text{int}}\psi_{\text{FS}}\rangle = -\frac{1}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(N-|L_k|\right) \tag{1.26}$$

and we summarize the calculations above in the following:

Proposition 1.2. *It holds that*

$$H_N = E_{\text{FS}} + H'_{\text{kin}} + \sum_{k\in\mathbb{Z}^3_*}\frac{\hat{V}_k k_F^{-1}}{2(2\pi)^3}\left(2B_k^*B_k + B_kB_{-k} + B_{-k}^*B_k^*\right) + \mathcal{C} + \mathcal{Q}$$

where $E_{\text{FS}} = \langle\psi_{\text{FS}}, H_N\psi_{\text{FS}}\rangle$ *and the* cubic *and* quartic *terms,* $\mathcal{C}$ *and* $\mathcal{Q}$*, are defined by*

$$\mathcal{C} = \frac{k_F^{-1}}{(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\,\text{Re}\left(\left(B_k + B_{-k}^*\right)^* D_k\right),$$

$$\mathcal{Q} = \frac{k_F^{-1}}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(D_k^*D_k - \sum_{p\in L_k}\left(c_p^*c_p + c_{p-k}c_{p-k}^*\right)\right).$$

We will prove that the cubic and quartic terms are negligible, and so the main contribution to the correlation energy comes from the *bosonizable terms*

$$H_{\text{eff}} = H'_{\text{kin}} + \sum_{k\in\mathbb{Z}^3_*}\frac{\hat{V}_k k_F^{-1}}{2(2\pi)^3}\left(2B_k^*B_k + B_kB_{-k} + B_{-k}^*B_k^*\right). \tag{1.27}$$

We will write these in terms of quasi-bosonic operators, which will lead us to define a quasi-bosonic Bogolubov transformation that serves to effectively diagonalize them.

The quasi-bosonic quadratic Hamiltonian We define the *excitation operators* $b_{k,p}^*, b_{k,p}$, for $k \in \mathbb{Z}^3_*$ and $p \in L_k$, by

$$b_{k,p} = c_{p-k}^*c_p, \quad b_{k,p}^* = c_p^*c_{p-k}. \tag{1.28}$$

The name is due to the fact that $b_{k,p}^*$ acts by annihilating a state with momentum $p-k \in B_F$ and creating a state with momentum $p \in B_F^c$, i.e. it excites the state $p-k$ to the state p.

For the purpose of computations it is convenient to also introduce a basis-independent notation for the quasi-bosonic operators. Consider for $k \in \mathbb{Z}^3_*$ the auxilliary space $\ell^2(L_k)$, which we will consider only as a real vector space, with standard orthonormal basis $\left(e_p\right)_{p\in L_k}$. For any $k \in \mathbb{Z}^3_*$ and $\varphi \in \ell^2(L_k)$ we define the *generalized excitation operators* $b_k(\varphi)$ and $b^*_k(\varphi)$ by

$$b_k(\varphi) = \sum_{p\in L_k} \left\langle \varphi, e_p\right\rangle b_{k,p}, \quad b^*_k(\varphi) = \sum_{p\in L_k} \left\langle e_p, \varphi\right\rangle b^*_{k,p}. \tag{1.29}$$

Note that the assignments $\varphi \mapsto b_k(\varphi), b^*_k(\varphi)$ are both linear (as we only consider $\ell^2(L_k)$ as a real vector space). In this notation we simply have that $b_k\left(e_p\right) = b_{k,p}$. A short calculation using the CAR shows that these operators are quasi-bosonic in the following sense:

Lemma 1.3. *For any $k, l \in \mathbb{Z}^3_*$, $\varphi \in \ell^2(L_k)$ and $\psi \in \ell^2(L_l)$ it holds that*

$$[b_k(\varphi), b_l(\psi)] = \left[b^*_k(\varphi), b^*_l(\psi)\right] = 0, \quad \left[b_k(\varphi), b^*_l(\psi)\right] = \delta_{k,l}\langle \varphi, \psi\rangle + \varepsilon_{k,l}(\varphi; \psi),$$

where the exchange correction $\varepsilon_{k,l}(\varphi; \psi)$ *is given by*

$$\varepsilon_{k,l}(\varphi; \psi) = -\sum_{p\in L_k}\sum_{q\in L_l} \left\langle \varphi, e_p\right\rangle\left\langle e_q, \psi\right\rangle \left(\delta_{p,q} c_{q-l} c^*_{p-k} + \delta_{p-k,q-l} c^*_q c_p\right).$$

Note that in the purely bosonic picture the exchange correction is absent. In our quasi-bosonic case, these corrections are small but non-zero; it will be important to keep careful track of them as it is these that gives rise to the exchange contribution $E_{\mathrm{corr,ex}}$.

For any operators A, B on $\ell^2(L_k)$, we define the *associated quadratic operators* $Q^k_1(A)$, $Q^k_2(B)$ on $\mathcal{H}_N$ by[3]

$$Q^k_1(A) = \sum_{p,q\in L_k} \left\langle e_p, Ae_q\right\rangle b^*_{k,p} b_{k,q} = \sum_{p\in L_k} b^*_k(Ae_p) b_{k,p} \tag{1.30}$$

and

$$\begin{aligned} Q^k_2(B) &= \sum_{p,q\in L_k} \left\langle e_p, Be_q\right\rangle \left(b_{k,p} b_{-k,-q} + b^*_{-k,-q} b^*_{k,p}\right) \\ &= \sum_{p\in L_k} \left(b_k(Be_p) b_{-k,-p} + b^*_{-k,-p} b^*_k(Be_p)\right). \end{aligned} \tag{1.31}$$

Defining the operator P_k on $\ell^2(L_k)$ by

$$P_k = |v_k\rangle\langle v_k|, \quad v_k = \sqrt{\frac{\hat{V}_k k_F^{-1}}{2(2\pi)^3}} \sum_{p\in L_k} e_p \in \ell^2(L_k), \quad \text{so that} \quad \left\langle e_p, P_k e_q\right\rangle = \frac{\hat{V}_k k_F^{-1}}{2(2\pi)^3}, \tag{1.32}$$

[3] Note that these definitions differ slightly from those of [10]. The main change is the definition of $Q^k_1(A)$; this operator is what was denoted $\tilde{Q}^k_1(A)$ in that paper.

we can express the interaction part of the bosonizable terms as

$$\sum_{k\in\mathbb{Z}^3_*}\frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\left(2B_k^*B_k+B_kB_{-k}+B_{-k}^*B_k^*\right)=\sum_{k\in\mathbb{Z}^3_*}\left(2\,Q_1^k(P_k)+Q_2^k(P_k)\right)$$
$$=\sum_{k\in\mathbb{Z}^3_*}\left(2\sum_{p,q\in L_k}\frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}b_{k,p}^*b_{k,q}+\sum_{p,q\in L_k}\frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\left(b_{k,p}b_{-k,-q}+b_{-k,-q}^*b_{k,p}^*\right)\right) \tag{1.33}$$

The localized kinetic operator H'_{kin} cannot be written exactly in a quadratic quasi-bosonic form, but due to the commutation relation

$$\left[H'_{\text{kin}},b_{k,p}^*\right]=\left(|p|^2-|p-k|^2\right)b_{k,p}^*=2\lambda_{k,p}b_{k,p}^* \tag{1.34}$$

(see [10, Eq. (1.76)]) and the quasi-bosonicity of the $b_{k,p}^*$ operators, it is sensible to consider it analogous to a quadratic operator of the form

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}2\lambda_{k,p}b_{k,p}^*b_{k,p}=\sum_{k\in\mathbb{Z}^3_*}2\,Q_1^k(h_k) \tag{1.35}$$

where the operators h_k on $\ell^2(L_k)$ are simply defined by $h_k e_p=\lambda_{k,p}e_p$. In all we thus consider the bosonizable terms as being analogous to a quasi-bosonic quadratic operator as

$$H_{\text{eff}}\approx\sum_{k\in\mathbb{Z}^3_*}\left(2\,Q_1^k(h_k+P_k)+Q_2^k(P_k)\right). \tag{1.36}$$

The quasi-bosonic Bogolubov transformation If the quadratic Hamiltonian on the right-hand side of equation (1.36) was exactly bosonic, it could be diagonalized by a Bogolubov transformation. Motivated by this we define such a transformation in the quasi-bosonic setting, while keeping careful track of the additional terms arising from the exchange correction.

Let $K_k:\ell^2(L_k)\to\ell^2(L_k), k\in\mathbb{Z}^3_*$, be a collection of symmetric operators satisfying

$$\left\langle e_p,K_ke_q\right\rangle=\left\langle e_{-p},K_{-k}e_{-q}\right\rangle,\quad k\in\mathbb{Z}^3_*,\ p,q\in L_k. \tag{1.37}$$

Then we define the *associated quasi-bosonic Bogolubov kernel* $\mathcal{K}$ on $\mathcal{H}_N$ by

$$\mathcal{K}=\frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\left\langle e_p,K_le_q\right\rangle\left(b_{l,p}b_{-l,-q}-b_{-l,-q}^*b_{l,p}^*\right)$$
$$=\frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left(b_l(K_le_q)b_{-l,-q}-b_{-l,-q}^*b_l^*(K_le_q)\right). \tag{1.38}$$

It is obvious from the second equation that $\mathcal{K}$ is skew-symmetric; $\mathcal{K}$ thus generates a unitary transformation $e^{\mathcal{K}}:\mathcal{H}_N\to\mathcal{H}_N$ - the quasi-bosonic Bogolubov transformation.

We consider the case $\sum_{k\in\mathbb{Z}^3_*}\|K_k\|_{\text{HS}}^2<\infty$, in which case $\mathcal{K}$ is not only well-defined but even bounded as an operator on $\mathcal{H}_N$, as we will prove in the next section.

We choose the operators (K_k) such that $e^{\mathcal{K}}$ would diagonalize the right-hand side of equation (1.36) if it was exactly bosonic. As explained in [10, Sect. 3] the diagonalizing kernel is

$$K_k = -\frac{1}{2}\log\left(h_k^{-\frac{1}{2}}\left(h_k^{\frac{1}{2}}\left(h_k + 2P_k\right)h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}h_k^{-\frac{1}{2}}\right). \tag{1.39}$$

Keeping careful track of the quasi-bosonic corrections, the action of $e^{\mathcal{K}}$ on the bosonizable terms are as follows:

Theorem 1.4. *Let H_{eff} be as in (1.27). Assume $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 < \infty$. Then $e^{\mathcal{K}}$ is well-defined and*

$$e^{\mathcal{K}}H_{\text{eff}}e^{-\mathcal{K}} = E_{\text{corr,bos}} + H'_{\text{kin}} + 2\sum_{k\in\mathbb{Z}^3_*}Q_1^k(e^{-K_k}h_ke^{-K_k} - h_k)$$
$$+\sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k(\{K_k, B_k(t)\})+2\,\mathrm{Re}\left(\mathcal{E}_k^1(A_k(t))\right)+2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\right)e^{-(1-t)\mathcal{K}}dt$$

where for any symmetric operators $A_k, B_k : \ell^2(L_k) \to \ell^2(L_k)$ we define

$$\varepsilon_k(A_k) = -\sum_{p\in L_k}\langle e_p, A_ke_p\rangle\left(c_p^*c_p + c_{p-k}c_{p-k}^*\right),$$
$$\mathcal{E}_k^1(A_k) = \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l}b_k^*(A_ke_p)\left\{\varepsilon_{k,l}(e_p; e_q), b_{-l}^*(K_{-l}e_{-q})\right\},$$
$$\mathcal{E}_k^2(B_k) = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l}\left\{b_k(B_ke_p), \left\{\varepsilon_{-k,-l}(e_{-p}; e_{-q}), b_l^*(K_le_q)\right\}\right\},$$

and for $t\in[0,1]$ the operators $A_k(t), B_k(t) : \ell^2(L_k)\to\ell^2(L_k)$ are given by

$$A_k(t) = \frac{1}{2}\left(e^{tK_k}\left(h_k + 2P_k\right)e^{tK_k} + e^{-tK_k}h_ke^{-tK_k}\right) - h_k,$$
$$B_k(t) = \frac{1}{2}\left(e^{tK_k}\left(h_k + 2P_k\right)e^{tK_k} - e^{-tK_k}h_ke^{-tK_k}\right).$$

This result is essentially the same as [10, Proposition 5.7], except that we now do not introduce a momentum cut-off and assume only that $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 < \infty$. For the readers convenience, we include in Appendix A the proof of the identity of Theorem 1.4 - that the condition $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 < \infty$ is sufficient to define $e^{\mathcal{K}}$ is proved in the next section.

Outline of the paper Now we come to the main part of the paper. We will choose as our trial state $\Psi = e^{-\mathcal{K}}\psi_{\text{FS}}$. As mentioned the cubic and quartic terms are negligible, so the energy of our trial state energy is by Theorem 1.4, to leading order,

$$\langle\Psi, H_N\Psi\rangle \approx E_{\text{FS}} + E_{\text{corr,bos}}$$
$$+\sum_{k\in\mathbb{Z}^3_*}\int_0^1\left\langle\psi_{\text{FS}}, e^{(1-t)\mathcal{K}}\left(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\left(\mathcal{E}_k^1(A_k(t))\right)\right.\right.$$

$$+2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k(t))\right)e^{-(1-t)\mathcal{K}}\psi_{\mathrm{FS}}\Big\rangle\,dt. \tag{1.40}$$

The main task will thus be to extract the exchange contribution $E_{\mathrm{corr,ex}}$ from this last term. The outline of the paper is as follows:

In Sect. 2 we show that $e^{\mathcal{K}}$ is well-defined by proving that $\mathcal{K}$ is bounded under the condition $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2<\infty$. We do this by employing a type of higher-order fermionic estimate, resulting in a bound of the form

$$\pm\,\mathcal{K}\le C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\,\mathcal{N}_E \tag{1.41}$$

which will also be crucial in allowing us to control $\mathcal{N}_E$ later.

In Sect. 3 we establish various bounds on the one-body operators K_k, $A_k(t)$ and $B_k(t)$. This is conceptually similar to the one-body analysis in our previous paper [10], but we must refine several estimates in order to establish control using only the assumption that $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2<\infty$.

In Sect. 4 comes the main new work: We engage in a detailed study of the exchange terms $\mathcal{E}_k^1(A_k)$ and $\mathcal{E}_k^2(B_k)$ so that we can extract $E_{\mathrm{corr,ex}}$ from the last term of equation (1.40), first in the form

$$\sum_{k\in\mathbb{Z}_*^3}\int_0^1\Big\langle\psi_{\mathrm{FS}},2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\psi_{\mathrm{FS}}\Big\rangle\,dt, \tag{1.42}$$

and then analyze this expression further to obtain the leading order of this, which is precisely $E_{\mathrm{corr,ex}}$ as given in Theorem 1.1.

Finally in Sect. 5 we control the non-bosonizable cubic and quartic terms, and bound the number operator $\mathcal{N}_E$ and its powers by a Gronwall argument. We end the paper by concluding Theorem 1.1.

2. The Bogolubov Kernel

We consider the kernel $\mathcal{K}$ defined by (1.38). We prove the following:

Proposition 2.1. *Let $K_l:\ell^2(L_l)\to\ell^2(L_l)$, $l\in\mathbb{Z}_*^3$, be a collection of symmetric operators. Then provided $\sum_{l\in\mathbb{Z}_*^3}\|K_l\|_{\mathrm{HS}}^2<\infty$, the expression*

$$\mathcal{K}=\frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p,q\in L_l}\langle e_p,K_le_q\rangle\left(b_{l,p}b_{-l,-q}-b_{-l,-q}^*b_{l,p}^*\right)$$

defines a bounded operator $\mathcal{K}:\mathcal{H}_N\to\mathcal{H}_N$, and for any $\Psi,\Phi\in\mathcal{H}_N$ we have

$$|\langle\Psi,\mathcal{K}\Phi\rangle|\le\sqrt{5}\sqrt{\sum_{l\in\mathbb{Z}_*^3}\|K_l\|_{\mathrm{HS}}^2}\sqrt{\langle\Psi,(\mathcal{N}_E+1)\,\Psi\rangle\,\langle\Phi,(\mathcal{N}_E+1)\,\Phi\rangle}.$$

Note that $\mathcal{N}_E = \sum_{p\in B_F^c} c_p^* c_p = \sum_{p\in B_F} c_p c_p^* \le |B_F| = N$ on $\mathcal{H}_N$. Moreover, it was shown in [10] (see also Theorem 3.1) that the kernels in (1.39) satisfy $\|K_k\|_{\mathrm{HS}} \le C\hat{V}_k$, and hence the boundedness of $\mathcal{K}$ follows from the assumption $\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 < \infty$. Let us write

$$\mathcal{K} = \tilde{\mathcal{K}} - \tilde{\mathcal{K}}^*, \quad \tilde{\mathcal{K}} = \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p,q\in L_l} \langle e_p, K_l e_q\rangle b_{l,p} b_{-l,-q}, \tag{2.1}$$

and focus on the boundedness of $\tilde{\mathcal{K}}$. Since

$$\begin{aligned} 2\tilde{\mathcal{K}} &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p,q\in L_l} \langle e_p, K_l e_q\rangle b_{l,p} c_{-q+l}^* c_{-q} \\ &= \sum_{q\in B_F^c}\left(\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} 1_{L_l}(q)\langle e_p, K_l e_q\rangle b_{l,p} c_{-q+l}^*\right) c_{-q}, \end{aligned} \tag{2.2}$$

for any $\Psi, \Phi \in \mathcal{H}_N$ we may estimate by the Cauchy–Schwarz inequality

$$\begin{aligned} |\langle \Psi, \tilde{\mathcal{K}}\Phi\rangle| &\le \frac{1}{2}\sqrt{\sum_{q\in B_F^c}\left\|\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} 1_{L_l}(q)\langle K_l e_q, e_p\rangle c_{-q+l} b_{l,p}^*\Psi\right\|^2}\sqrt{\sum_{q\in B_F^c}\left\|c_{-q}\Phi\right\|^2} \\ &= \frac{1}{2}\sqrt{\sum_{q\in B_F^c}\left\|\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} 1_{L_l}(q)\langle K_l e_q, e_p\rangle c_{-q+l} b_{l,p}^*\Psi\right\|^2}\sqrt{\langle \Phi, \mathcal{N}_E\Phi\rangle}. \end{aligned} \tag{2.3}$$

The operator appearing under the root can be written as

$$\begin{aligned} \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} 1_{L_l}(q)\langle K_l e_q, e_p\rangle c_{-q+l} b_{l,p}^* &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} 1_{L_l}(q)\langle K_l e_q, e_p\rangle c_p^* c_{p-l} c_{-q+l}, \\ = \sum_{p'\in B_F^c}\sum_{q',r'\in B_F}&\left(\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l}\delta_{p',p}\delta_{q',p-l}\delta_{r',-q+l} 1_{L_l}(q)\langle K_l e_q, e_p\rangle\right) c_{p'}^* c_{q'} c_{r'}. \end{aligned} \tag{2.4}$$

Let us estimate the following general expression, with some coefficients $A_{p,q,r}$,

$$\sum_{p\in B_F^c}\sum_{q,r\in B_F} A_{p,q,r} c_p^* c_q c_r. \tag{2.5}$$

A higher order fermionic estimate. Note that the Cauchy–Schwarz inequality trivially implies that

$$\left\|\sum A_p c_p\Psi\right\| \le \sum |A_p|\|c_p\Psi\| \le \sqrt{\sum |A_p|^2}\sqrt{\sum \|c_p\Psi\|^2}, \tag{2.6}$$

but this is non-optimal for fermionic states. The "standard fermionic estimate" states that

$$\left\|\sum A_p c_p\Psi\right\|, \left\|\sum A_p c_p^*\Psi\right\| \le \sqrt{\sum \left|A_p\right|^2}\,\|\Psi\|, \tag{2.7}$$

which can be proved by appealing to the CAR as follows:

$$\left(\sum A_p c_p\right)^* \left(\sum A_q c_q\right) \le \left\{\left(\sum A_p c_p\right)^*, \left(\sum A_q c_q\right)\right\} = \sum \overline{A_p} A_q \left\{c_p^*, c_q\right\}$$
$$= \sum \left|A_p\right|^2. \tag{2.8}$$

One can imagine generalizing this to quadratic expressions of the form $\sum_{p,q} A_{p,q} c_p c_q$, but this fails since the CAR only yields a commutation relation for such expressions, and not an anticommutation relation. However, for cubic expressions, such as $\sum_{p,q,r} A_{p,q,r} c_p^* c_q c_r$, the CAR does yield an anticommutation relation, allowing the trick to be applied. The anticommutator is of course not constant, but rather a combination of quadratic, linear and constant expressions, but this still yields a reduction in "number operator order", which will be crucial for our estimation of $e^{\mathcal{K}} \mathcal{N}_E^m e^{-\mathcal{K}}$ later on. We will need the following basic anticommutator:

Lemma 2.2. *For any $p, p' \in B_F^c$ and $q, q', r, r' \in B_F$ it holds that*

$$\begin{aligned}\left\{\left(c_p^* c_q c_r\right)^*, c_{p'}^* c_{q'} c_{r'}\right\} = {} & \delta_{p,p'} c_{q'} c_{r'} c_r^* c_q^* + \delta_{q,q'} c_{p'}^* c_{r'} c_r^* c_p + \delta_{r,r'} c_{p'}^* c_{q'} c_q^* c_p \\ & - \delta_{r,q'} c_{p'}^* c_{r'} c_q^* c_p - \delta_{r,q'} c_{p'}^* c_{r'} c_q^* c_p \\ & - \delta_{q,q'} \delta_{r,r'} c_{p'}^* c_p - \delta_{p,p'} \delta_{r,r'} c_{q'} c_q^* - \delta_{p,p'} \delta_{q,q'} c_{r'} c_r^* \\ & + \delta_{q,r'} \delta_{r,q'} c_{p'}^* c_p + \delta_{p,p'} \delta_{r,q'} c_{r'} c_q^* + \delta_{p,p'} \delta_{q,r'} c_{q'} c_r^* \\ & + \delta_{p,p'} \delta_{q,q'} \delta_{r,r'} - \delta_{p,p'} \delta_{q,r'} \delta_{r,q'}.\end{aligned}$$

We can now conclude the desired bound:

Proposition 2.3. *Let $A_{p,q,r} \in \mathbb{C}$ for $p \in B_F^c$ and $q, r \in B_F$ with $\sum_{p \in B_F^c} \sum_{q,r \in B_F} \left|A_{p,q,r}\right|^2 < \infty$ be given. Then for any $\Psi \in \mathcal{H}_N$*

$$\left\|\sum_{p \in B_F^c} \sum_{q,r \in B_F} A_{p,q,r} c_p^* c_q c_r \Psi\right\|^2 \le 5 \sum_{p \in B_F^c} \sum_{q,r \in B_F} \left|A_{p,q,r}\right|^2 \langle \Psi, (\mathcal{N}_E + 1) \Psi \rangle.$$

Proof. As in the proof of the standard fermionic estimate (2.8), we have

$$\begin{aligned}&\left\|\sum_{p \in B_F^c} \sum_{q,r \in B_F} A_{p,q,r} c_p^* c_q c_r \Psi\right\|^2 \\ &\quad \le \sum_{p,p' \in B_F^c} \sum_{q,q',r,r' \in B_F} \overline{A_{p,q,r}} A_{p',q',r'} \left\langle \Psi, \left\{\left(c_p^* c_q c_r\right)^*, c_{p'}^* c_{q'} c_{r'}\right\} \Psi \right\rangle.\end{aligned}$$

Hence, by the identity of Lemma 2.2, we bound the left-hand side by

$$\begin{aligned}&\sum_{p,p' \in B_F^c} \sum_{q,q',r,r' \in B_F} \overline{A_{p,q,r}} A_{p',q',r'} \left\langle \Psi, \left(\delta_{p,p'} c_{q'} c_{r'} c_r^* c_q^* + \delta_{q,q'} c_{p'}^* c_{r'} c_r^* c_p + \delta_{r,r'} c_{p'}^* c_{q'} c_q^* c_p\right) \Psi \right\rangle \\ &- \sum_{p,p' \in B_F^c} \sum_{q,q',r,r' \in B_F} \overline{A_{p,q,r}} A_{p',q',r'} \left\langle \Psi, \left(\delta_{r,q'} c_{p'}^* c_{r'} c_q^* c_p + \delta_{r,q'} c_{p'}^* c_{r'} c_q^* c_p\right) \Psi \right\rangle\end{aligned}$$

$$
\begin{aligned}
&-\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{q,q'}\delta_{r,r'}c_{p'}^*c_p+\delta_{p,p'}\delta_{r,r'}c_{q'}c_q^*+\delta_{p,p'}\delta_{q,q'}c_{r'}c_r^*\right)\Psi\right\rangle\\
&+\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{q,r'}\delta_{r,q'}c_{p'}^*c_p+\delta_{p,p'}\delta_{r,q'}c_{r'}c_q^*+\delta_{p,p'}\delta_{q,r'}c_{q'}c_r^*\right)\Psi\right\rangle\\
&+\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{p,p'}\delta_{q,q'}\delta_{r,r'}-\delta_{p,p'}\delta_{q,r'}\delta_{r,q'}\right)\Psi\right\rangle. \qquad (2.9)
\end{aligned}
$$

We estimate the different types of expressions appearing above. Firstly, by the standard fermionic estimate (2.8),

$$
\begin{aligned}
&\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{p,p'}c_{q'}c_{r'}c_r^*c_q^*\right)\Psi\right\rangle=\sum_{p\in B_F^c}\left\|\sum_{q,r\in B_F}\overline{A_{p,q,r}}c_r^*c_q^*\Psi\right\|^2\\
&\leq\sum_{p\in B_F^c}\left(\sum_{q\in B_F}\left\|\left(\sum_{r\in B_F}\overline{A_{p,q,r}}c_r^*\right)c_q^*\Psi\right\|\right)^2\leq\sum_{p\in B_F^c}\left(\sum_{q\in B_F}\sqrt{\sum_{r\in B_F}\left|A_{p,q,r}\right|^2}\left\|c_q^*\Psi\right\|\right)^2\\
&\leq\sum_{p\in B_F^c}\sum_{q,r\in B_F}\left|A_{p,q,r}\right|^2\left(\sum_{q\in B_F}\left\|c_q^*\Psi\right\|^2\right)=\sum_{p\in B_F^c}\sum_{q,r\in B_F}\left|A_{p,q,r}\right|^2\langle\Psi,\mathcal{N}_E\Psi\rangle \qquad (2.10)
\end{aligned}
$$

and likewise for the other two terms on the first line of equation (2.9). For the terms on the second line we similarly estimate

$$
\begin{aligned}
&\left|\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{r,q'}c_{p'}^*c_{r'}c_q^*c_p\right)\Psi\right\rangle\right|\\
&\leq\sum_{r\in B_F}\left\|\sum_{p'\in B_F^c}\sum_{r'\in B_F}A_{p',r,r'}c_{p'}c_{r'}^*\Psi\right\|\left\|\sum_{p\in B_F^c}\sum_{q\in B_F}\overline{A_{p,q,r}}c_q^*c_p\Psi\right\|\\
&\leq\sum_{p\in B_F^c}\sum_{r,r'\in B_F}\sqrt{\sum_{p'\in B_F^c}\left|A_{p',r,r'}\right|^2}\left\|c_{r'}^*\Psi\right\|\sqrt{\sum_{q\in B_F}\left|A_{p,q,r}\right|^2}\left\|c_p\Psi\right\|\\
&\leq\sum_{r\in B_F}\sqrt{\sum_{p\in B_F^c}\sum_{r'\in B_F}\left|A_{p,r,r'}\right|^2}\sqrt{\sum_{p\in B_F^c}\sum_{q\in B_F}\left|A_{p,q,r}\right|^2}\sqrt{\sum_{r'\in B_F}\left\|c_{r'}^*\Psi\right\|^2}\sqrt{\sum_{p\in B_F^c}\left\|c_p\Psi\right\|^2}\\
&\leq\sum_{p\in B_F^c}\sum_{q\in B_F}\left|A_{p,q,r}\right|^2\langle\Psi,\mathcal{N}_E\Psi\rangle. \qquad (2.11)
\end{aligned}
$$

The terms on the third line of equation (2.9) all factorize in a manifestly non-positive fashion, and so can be dropped, while for the fourth line

$$
\begin{aligned}
&\left|\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{q,r'}\delta_{r,q'}c_{p'}^*c_p\right)\Psi\right\rangle\right|\\
&=\left|\sum_{q,r\in B_F}\left\langle\sum_{p'\in B_F^c}A_{p',r,q}c_{p'}\Psi,\sum_{p\in B_F^c}\overline{A_{p,q,r}}c_p\Psi\right\rangle\right|
\end{aligned}
$$

$$\leq \sum_{q,r\in B_F}\left\|\sum_{p'\in B_F^c} A_{p',r,q}c_{p'}\Psi\right\|\left\|\sum_{p\in B_F^c}\overline{A_{p,q,r}}c_p\Psi\right\|$$
$$\leq \sum_{q,r\in B_F}\sqrt{\sum_{p'\in B_F^c}\left|A_{p',r,q}\right|^2}\sqrt{\sum_{p\in B_F^c}\left|A_{p,q,r}\right|^2}\,\|\Psi\|^2 \leq \sum_{p\in B_F^c}\sum_{q,r\in B_F}\left|A_{p,q,r}\right|^2\|\Psi\|^2. \tag{2.12}$$

Lastly, the terms on the fifth line are seen to simply be constant and easily bounded by $\sum_{p\in B_F^c}\sum_{q,r\in B_F}\left|A_{p,q,r}\right|^2$, whence the proposition follows. □

We can now conclude the following bound for $\tilde{\mathcal{K}}$, which in turn implies Proposition 2.1.

Proposition 2.4. *For any* $\Psi, \Phi \in \mathcal{H}_N$ *it holds that*

$$|\langle\Psi, \tilde{\mathcal{K}}\Phi\rangle| \leq \frac{\sqrt{5}}{2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\sqrt{\langle\Psi, (\mathcal{N}_E+1)\,\Psi\rangle\,\langle\Phi, \mathcal{N}_E\Phi\rangle}.$$

Proof. By (2.3) and (2.4), combined with the estimate of Proposition 2.3, we can bound

$$|\langle\Psi, \tilde{\mathcal{K}}\Phi\rangle| \leq \frac{\sqrt{5}}{2}\sqrt{\sum_{q\in B_F^c}\sum_{p'\in B_F^c}\sum_{q',r'\in B_F}\left|\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l}\delta_{p',p}\delta_{q',p-l}\delta_{r',-q+l}1_{L_l}(q)\left\langle K_l e_q, e_p\right\rangle\right|^2}$$
$$\cdot\sqrt{\langle\Psi, (\mathcal{N}_E+1)\,\Psi\rangle\,\langle\Phi, \mathcal{N}_E\Phi\rangle}. \tag{2.13}$$

The sum inside the first square root is exactly equal to $\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}$.

3. Analysis of the One-Body Operators

In this section we analyze the operators K_k, $A_k(t)$ and $B_k(t)$ which appear in Theorem 1.4, obtaining the following:

Theorem 3.1. *For any* $k \in \mathbb{Z}^3_*$ *it holds that*

$$\|K_k\|_{\mathrm{HS}} \leq C\hat{V}_k\min\{1, k_F^2\,|k|^{-2}\}.$$

Moreover, for all $p, q \in L_k$ *and* $t \in [0,1]$,

$$\left|\left\langle e_p, K_k e_q\right\rangle\right| \leq C\frac{\hat{V}_k k_F^{-1}}{\lambda_{k,p}+\lambda_{k,q}},$$
$$\left|\left\langle e_p, (-K_k)\,e_q\right\rangle - \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\right| \leq C\frac{\hat{V}_k^2 k_F^{-1}}{\lambda_{k,p}+\lambda_{k,q}},$$
$$\left|\left\langle e_p, A_k(t)e_q\right\rangle\right|, \left|\left\langle e_p, B_k(t)e_q\right\rangle\right| \leq C\left(1+\hat{V}_k^2\right)\hat{V}_k k_F^{-1},$$
$$\left|\left\langle e_p, \{K_k, B_k(t)\}\,e_q\right\rangle\right| \leq C\left(1+\hat{V}_k^2\right)\hat{V}_k^2 k_F^{-1},$$

$$\left|\left\langle e_p, \left(\int_0^1 B_k(t)dt\right) e_q\right\rangle - \frac{\hat{V}_k k_F^{-1}}{4(2\pi)^3}\right| \le C\left(1+\hat{V}_k\right)\hat{V}_k^2 k_F^{-1},$$

for a constant $C > 0$ independent of all relevant quantities.

The analysis of this section is similar to that of [10, Sect. 7], but compared to that section, the estimates of this section are considerably more precise: We quantify the error of the upper bound on $\langle e_p, (-K_k)\, e_q\rangle$, obtain elementwise estimates for $A_k(t)$ and $B_k(t)$ (rather than only estimates for the norm $\|\cdot\|_{\infty,2}$ as in [10]), and determine the leading term of the operator $\int_0^1 B_k(t)dt$ which will be needed to extract the exchange contribution in the next section.

3.1. Matrix element estimates for K-quantities. To ease the notation we will abstract the problem slightly: Instead of $\ell^2(L_k)$ we consider a general n-dimensional Hilbert space $(V, \langle\cdot,\cdot\rangle)$, let $h : V \to V$ be a positive self-adjoint operator on V with eigenbasis $(x_i)_{i=1}^n$ and eigenvalues $(\lambda_i)_{i=1}^n$, and let $v \in V$ be any vector such that $\langle x_i, v\rangle \ge 0$ for all $1 \le i \le n$, and let $P_w(\cdot) = \langle w, \cdot\rangle\, w$ be the projection onto $w \in V$. Theorem 3.1 will then be obtained at the end by insertion of the particular operators h_k and P_k.

We define $K : V \to V$ by

$$K = -\frac{1}{2}\log\left(h^{-\frac{1}{2}}\left(h^{\frac{1}{2}}(h+2P_v)\,h^{\frac{1}{2}}\right)^{\frac{1}{2}} h^{-\frac{1}{2}}\right) = -\frac{1}{2}\log\left(h^{-\frac{1}{2}}\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}} h^{-\frac{1}{2}}\right). \tag{3.1}$$

As $\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}} \ge h$ we see that $K \le 0$. In [10, Sect. 7.2] we proved the following result.

Proposition 3.2. *For all $1 \le i, j \le n$ it holds that*

$$\frac{2}{1+2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j} \le \left\langle x_i, \left(e^{-2K}-1\right)x_j\right\rangle, \left\langle x_i, \left(1-e^{2K}\right)x_j\right\rangle \le 2\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}.$$

Below it will be more convenient to consider the hyperbolic functions $\sinh(-2K)$ and $\cosh(-2K)$ rather than e^{-2K} and e^{2K}. The previous proposition implies the following for these operators:

Corollary 3.3. *For any $1 \le i, j \le n$ it holds that*

$$\left\langle x_i, \sinh(-2K)\, x_j\right\rangle \le 2\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j},$$
$$\left\langle x_i, (\cosh(-2K)-1)\, x_j\right\rangle \le \frac{2\langle v, h^{-1}v\rangle}{1+2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}.$$

Proof. These bounds follow from Proposition 3.2 and the identities

$$\sinh(-2K) = \frac{1}{2}\left(\left(e^{-2K}-1\right)+\left(1-e^{2K}\right)\right),$$
$$\cosh(-2K)-1 = \frac{1}{2}\left(\left(e^{-2K}-1\right)-\left(1-e^{2K}\right)\right). \tag{3.2}$$

□

Now we extend our elementwise estimates to more general operators. These estimates are similar to those of Proposition 7.10 of [10], but more precise. First we consider K itself:

Proposition 3.4. *For any* $1 \leq i, j \leq n$ *it holds that*

$$\frac{1}{1+2\left\langle v, h^{-1} v\right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} \leq \left\langle x_i, (-K)\, x_j\right\rangle \leq \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j}.$$

Proof. From the identity

$$-x = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left(1 - e^{2x}\right)^m, \quad x \leq 0, \tag{3.3}$$

which follows by the Mercator series, we thus have that $-K = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left(1 - e^{2K}\right)^m$. Noting that Proposition 3.2 in particular implies that $\left\langle x_i, \left(1 - e^{2K}\right) x_j\right\rangle \geq 0$ for all $1 \leq i, j \leq n$, whence also $\left\langle x_i, \left(1 - e^{2K}\right)^m x_j\right\rangle \geq 0$ for any $m \in \mathbb{N}$, we may estimate

$$\begin{aligned} \left\langle x_i, (-K)\, x_j\right\rangle &= \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left\langle x_i, \left(1 - e^{2K}\right)^m x_j\right\rangle \geq \frac{1}{2} \left\langle x_i, \left(1 - e^{2K}\right) x_j\right\rangle \\ &\geq \frac{1}{1+2\left\langle v, h^{-1} v\right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} \end{aligned} \tag{3.4}$$

which is the lower bound. This similarly implies that $\left\langle x_i, (-K)^m x_j\right\rangle \geq 0$ for all $1 \leq i, j \leq n$, $m \in \mathbb{N}$, so the upper bound now also follows from Proposition 3.2 by noting that

$$\frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} \geq \frac{1}{2} \left\langle x_i, \left(e^{-2K} - 1\right) x_j\right\rangle = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle x_i, (-2K)^m x_j\right\rangle \geq \left\langle x_i, (-K)\, x_j\right\rangle. \tag{3.5}$$

The proof of Proposition 3.4 is complete. □

The fact that $\left\langle x_i, (-K)^m x_j\right\rangle \geq 0$ for all $1 \leq i, j \leq n$, $m \in \mathbb{N}$, has the important consequence that for any such i and j, the functions

$$t \mapsto \left\langle x_i, \sinh(-tK)\, x_j\right\rangle, \left\langle x_i, (\sinh(-tK) + tK)\, x_j\right\rangle, \left\langle x_i, (\cosh(-tK) - 1)\, x_j\right\rangle \tag{3.6}$$

are non-negative and convex for $t \in [0, \infty)$, as follows by considering the Taylor expansions of the operators involved. This allows us to extend the bounds of Corollary 3.3 to arbitrary $t \in [0, 1]$:

Proposition 3.5. *For all* $1 \leq i, j \leq n$ *and* $t \in [0, 1]$ *it holds that*

$$\frac{1}{1+2\left\langle v, h^{-1} v\right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} t \leq \left\langle x_i, \sinh(-tK)\, x_j\right\rangle \leq \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} t,$$

$$0 \leq \left\langle x_i, (\cosh(-tK) - 1)\, x_j \right\rangle \leq \frac{\left\langle v, h^{-1} v \right\rangle}{1 + 2\left\langle v, h^{-1} v \right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j \right\rangle}{\lambda_i + \lambda_j},$$

$$\left| \left\langle x_i, \left(e^{tK} - 1\right) x_j \right\rangle \right| \leq \frac{\langle x_i, v\rangle \left\langle v, x_j \right\rangle}{\lambda_i + \lambda_j}.$$

Proof. By the noted convexity we immediately conclude the upper bounds

$$\left\langle x_i, \sinh(-tK)\, x_j \right\rangle \leq \frac{t}{2} \left\langle x_i, \sinh(-2K)\, x_j \right\rangle \leq \frac{\langle x_i, v\rangle \left\langle v, x_j \right\rangle}{\lambda_i + \lambda_j} t$$

$$\left\langle x_i, (\cosh(-tK) - 1)\, x_j \right\rangle \leq \frac{t}{2} \left\langle x_i, (\cosh(-2K) - 1)\, x_j \right\rangle \leq \frac{\left\langle v, h^{-1} v \right\rangle}{1 + 2\left\langle v, h^{-1} v \right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j \right\rangle}{\lambda_i + \lambda_j} \tag{3.7}$$

and by non-negativity of $\left\langle x_i, (\sinh(-tK) + tK)\, x_j \right\rangle$ and Proposition 3.4, the lower bound

$$\left\langle x_i, \sinh(-tK)\, x_j \right\rangle \geq \left\langle x_i, (-tK)\, x_j \right\rangle \geq \frac{1}{1 + 2\left\langle v, h^{-1} v \right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j \right\rangle}{\lambda_i + \lambda_j} t. \tag{3.8}$$

Lastly we can apply the non-negativity of the hyperbolic operators to conclude the bound for $e^{tK} - 1$ as

$$\begin{aligned} \left| \left\langle x_i, \left(e^{tK} - 1\right) x_j \right\rangle \right| &= \left| \left\langle x_i, ((\cosh(-tK) - 1) - \sinh(-tK))\, x_j \right\rangle \right| \\ &\leq \max \left\{ \left\langle x_i, (\cosh(-tK) - 1)\, x_j \right\rangle, \left\langle x_i, \sinh(-tK)\, x_j \right\rangle \right\} \leq \frac{\langle x_i, v\rangle \left\langle v, x_j \right\rangle}{\lambda_i + \lambda_j}. \end{aligned} \tag{3.9}$$

□

3.2. Matrix element estimates for $A(t)$ and $B(t)$. We now consider operators $A(t), B(t) : V \to V$ defined by

$$\begin{aligned} A(t) &= \frac{1}{2} \left(e^{tK} (h + 2P_v)\, e^{tK} + e^{-tK} h e^{-tK} \right) - h, \\ B(t) &= \frac{1}{2} \left(e^{tK} (h + 2P_v)\, e^{tK} - e^{-tK} h e^{-tK} \right), \end{aligned} \tag{3.10}$$

for $t \in [0, 1]$. We decompose these as

$$A(t) = A_h(t) + e^{tK} P_v e^{tK}, \quad B(t) = (1 - t)\, P_v + B_h(t) + e^{tK} P_v e^{tK} - P_v \tag{3.11}$$

with

$$\begin{aligned} C_K(t) &= \cosh(-tK) - 1, \quad S_K(t) = \sinh(-tK), \\ A_h(t) &= \cosh(-tK)\, h \cosh(-tK) + \sinh(-tK)\, h \sinh(-tK) - h \\ &= \{h, C_K(t)\} + S_K(t) h\, S_K(t) + C_K(t) h\, C_K(t), \\ B_h(t) &= -\sinh(-tK)\, h \cosh(-tK) - \cosh(-tK)\, h \sinh(-tK) + t P_v \\ &= t P_v - \{h, S_K(t)\} - S_K(t) h\, C_K(t) - C_K(t) h\, S_K(t). \end{aligned} \tag{3.12}$$

We begin by estimating the $e^{tK} P_v e^{tK}$ terms:

Proposition 3.6. *For all* $1 \leq i, j \leq n$ *and* $t \in [0, 1]$ *it holds that*

$$\left|\left\langle x_i, \left(e^{tK} P_v e^{tK} - P_v\right) x_j\right\rangle\right| \leq \left(2 + \left\langle v, h^{-1} v\right\rangle\right) \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle.$$

Proof. Writing

$$e^{tK} P_v e^{tK} - P_v = \left\{P_v, e^{tK} - 1\right\} + \left(e^{tK} - 1\right) P_v \left(e^{tK} - 1\right) \tag{3.13}$$

we see that

$$\begin{aligned}\left\langle x_i, \left(e^{tK} P_v e^{tK} - P_v\right) x_j\right\rangle &= \langle x_i, v\rangle \left\langle \left(e^{tK} - 1\right) v, x_j\right\rangle + \left\langle x_i, \left(e^{tK} - 1\right) v\right\rangle \left\langle v, x_j\right\rangle \\ &\quad + \left\langle x_i, \left(e^{tK} - 1\right) v\right\rangle \left\langle \left(e^{tK} - 1\right) v, x_j\right\rangle. \end{aligned} \tag{3.14}$$

Now, by Proposition 3.5 we can for any $1 \leq i \leq n$ estimate

$$\begin{aligned}\left|\left\langle x_i, \left(e^{tK} - 1\right) v\right\rangle\right| &= \left|\sum_{j=1}^n \left\langle x_i, \left(e^{tK} - 1\right) x_j\right\rangle \left\langle x_j, v\right\rangle\right| \leq \sum_{j=1}^n \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} \left\langle x_j, v\right\rangle \\ &\leq \langle x_i, v\rangle \sum_{j=1}^n \frac{\left|\left\langle x_j, v\right\rangle\right|^2}{\lambda_j} = \langle x_i, v\rangle \left\langle v, h^{-1} v\right\rangle \end{aligned} \tag{3.15}$$

whence the claim follows. □

Note that for $\left\langle x_i, e^{tK} P_v e^{tK} x_j\right\rangle$ this in particular implies the bound

$$\left|\left\langle x_i, e^{tK} P_v e^{tK} x_j\right\rangle\right| \leq \left(1 + \left\langle v, h^{-1} v\right\rangle\right)^2 \langle x_i, v\rangle \left\langle v, x_j\right\rangle. \tag{3.16}$$

We now consider $A_h(t)$ and $B_h(t)$:

Proposition 3.7. *For all* $1 \leq i, j \leq n$ *and* $t \in [0, 1]$ *it holds that*

$$\left|\left\langle x_i, A_h(t) x_j\right\rangle\right|, \left|\left\langle x_i, B_h(t) x_j\right\rangle\right| \leq 4 \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle.$$

Proof. The estimates of Proposition 3.5 imply that

$$\begin{aligned}\left|\left\langle x_i, \{h, C_K(t)\} x_j\right\rangle\right| &= \left(\lambda_i + \lambda_j\right) \left|\left\langle x_i, C_K(t) x_j\right\rangle\right| \\ &\leq \left(\lambda_i + \lambda_j\right) \frac{\left\langle v, h^{-1} v\right\rangle}{1 + 2\left\langle v, h^{-1} v\right\rangle} \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j} \leq \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\left|\left\langle x_i, S_K(t) h\, S_K(t) x_j\right\rangle\right| &= \left|\sum_{k=1}^n \lambda_k \langle x_i, S_K(t) x_k\rangle \left\langle x_k, S_K(t) x_j\right\rangle\right| \\ &\leq \sum_{k=1}^n \lambda_k \frac{\langle x_i, v\rangle \langle v, x_k\rangle}{\lambda_i + \lambda_k} \frac{\langle x_k, v\rangle \left\langle v, x_j\right\rangle}{\lambda_k + \lambda_j} \\ &\leq \langle x_i, v\rangle \left\langle v, x_j\right\rangle \sum_{k=1}^n \frac{|\langle x_k, v\rangle|^2}{\lambda_k} = \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle. \end{aligned} \tag{3.18}$$

The latter estimate only relied on the inequality

$$\left|\left\langle x_i, S_K(t)x_j\right\rangle\right| \le \frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{\lambda_i + \lambda_j}, \tag{3.19}$$

which is also true for $C_K(t)$, so the terms $C_K(t)h\,C_K(t)$, $C_K(t)h\,S_K(t)$ and $S_K(t)h\,C_K(t)$ also obey this estimate. It thus only remains to bound $tP_v - \{h, S_K(t)\}$. From Proposition 3.5 we see that

$$\frac{\langle x_i, v\rangle \left\langle v, x_j\right\rangle}{1 + 2\left\langle v, h^{-1}v\right\rangle} t \le \left\langle x_i, \{h, S_K(t)\}\, x_j\right\rangle \le \langle x_i, v\rangle \left\langle v, x_j\right\rangle t \tag{3.20}$$

whence

$$\begin{aligned} \left|\left\langle x_i, (tP_v - \{h, S_K(t)\})\, x_j\right\rangle\right| &= \left\langle x_i, P_v x_j\right\rangle t - \left\langle x_i, \{h, S_K(t)\}\, x_j\right\rangle \\ &\le \left(1 - \frac{1}{1 + 2\left\langle v, h^{-1}v\right\rangle}\right) \langle x_i, v\rangle \left\langle v, x_j\right\rangle t \le 2\left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle. \end{aligned} \tag{3.21}$$

□

Combining equation (3.16) and Proposition 3.7 we conclude the following:

Proposition 3.8. *For all* $1 \le i, j \le n$ *and* $t \in [0, 1]$ *it holds that*

$$\left|\left\langle x_i, A(t)x_j\right\rangle\right|, \ \left|\left\langle x_i, B(t)x_j\right\rangle\right| \le 3\left(1 + \left\langle v, h^{-1}v\right\rangle\right)^2 \langle x_i, v\rangle \left\langle v, x_j\right\rangle.$$

Analysis of $\{K, B(t)\}$ *and* $\int_0^1 B(t)dt$ We end by estimating $\{K, B(t)\}$ and $\int_0^1 B(t)dt$, the latter of which will be needed for the analysis of the exchange contribution in the next section.

Proposition 3.9. *For all* $1 \le i, j \le n$ *and* $t \in [0, 1]$ *it holds that*

$$\left|\left\langle x_i, \{K, B(t)\}\, x_j\right\rangle\right| \le 6\left(1 + \left\langle v, h^{-1}v\right\rangle\right)^2 \left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle.$$

Proof. Using the Propositions 3.4 and 3.8 we see that

$$\begin{aligned} \left|\left\langle x_i, KB(t)x_j\right\rangle\right| &= \left|\sum_{k=1}^n \langle x_i, Kx_k\rangle \left\langle x_k, B(t)x_j\right\rangle\right| \\ &\le 3\left(1 + \left\langle v, h^{-1}v\right\rangle\right)^2 \sum_{k=1}^n \frac{\langle x_i, v\rangle \langle v, x_k\rangle}{\lambda_i + \lambda_k} \langle x_k, v\rangle \left\langle v, x_j\right\rangle \\ &\le 3\left(1 + \left\langle v, h^{-1}v\right\rangle\right)^2 \sum_{k=1}^n \frac{|\langle x_k, v\rangle|^2}{\lambda_k} \langle x_i, v\rangle \left\langle v, x_j\right\rangle \\ &= 3\left(1 + \left\langle v, h^{-1}v\right\rangle\right)^2 \left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle. \end{aligned} \tag{3.22}$$

This estimate is also valid for $\left|\left\langle x_i, B(t)Kx_j\right\rangle\right|$ whence the claim follows. □

Proposition 3.10. *For all* $1 \le i, j \le n$ *it holds that*

$$\left|\left\langle x_i, \left(\int_0^1 B(t)dt\right) x_j\right\rangle - \frac{1}{2}\langle x_i, v\rangle \left\langle v, x_j\right\rangle\right| \le \left(6 + \left\langle v, h^{-1}v\right\rangle\right)\left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle.$$

Proof. Noting that $\frac{1}{2}\langle x_i, v\rangle \left\langle v, x_j\right\rangle = \frac{1}{2}\left\langle x_i, P_v x_j\right\rangle$ and that

$$\begin{aligned}\int_0^1 B(t)dt - \frac{1}{2}P_v &= \int_0^1 \left((1-t)\,P_v + B_h(t) + e^{tK}P_v e^{tK} - P_v\right)dt - \frac{1}{2}P_v \\ &= \int_0^1 \left(B_h(t) + e^{tK}P_v e^{tK} - P_v\right)dt \end{aligned} \tag{3.23}$$

we can estimate using the Propositions 3.6 and 3.7 that

$$\begin{aligned}\left|\left\langle x_i, \left(\int_0^1 B(t)dt - \frac{1}{2}P_v\right) x_j\right\rangle\right| &\le \int_0^1 \left(\left|\left\langle x_i, B_h(t)x_j\right\rangle\right| + \left|\left\langle x_i, \left(e^{tK}P_v e^{tK} - P_v\right)x_j\right\rangle\right|\right)dt \\ &\le \left(6 + \left\langle v, h^{-1}v\right\rangle\right)\left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \left\langle v, x_j\right\rangle. \end{aligned} \tag{3.24}$$

□

Insertion of the particular operators h_k and P_k Recall that the particular operators we must consider are $h_k, P_k : \ell^2(L_k) \to \ell^2(L_k)$ defined by

$$\begin{aligned} h_k e_p &= \lambda_{k,p} e_p, \quad \lambda_{k,p} = \tfrac{1}{2}\left(|p|^2 - |p-k|^2\right), \\ P_k(\cdot) &= \langle v_k, \cdot\rangle\, v_k, \quad v_k = \sqrt{\tfrac{\hat{V}_k k_F^{-1}}{2(2\pi)^3}} \textstyle\sum_{p \in L_k} e_p. \end{aligned} \tag{3.25}$$

For these we have that

$$\left\langle v_k, h_k^{-1} v_k\right\rangle = \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \sum_{p \in L_k} \frac{1}{\lambda_{k,p}}. \tag{3.26}$$

In [10] the following estimates for sums of the form $\sum_{p \in L_k} \lambda_{k,p}^{\beta}$ were proved:

Proposition 3.11. *For any* $k \in \mathbb{Z}^3_*$ *and* $\beta \in [-1, 0]$ *it holds that*

$$\sum_{p \in L_k} \lambda_{k,p}^{\beta} \le C_\beta \begin{cases} k_F^{2+\beta}\,|k|^{1+\beta} & |k| \le 2k_F \\ k_F^3\,|k|^{2\beta} & |k| > 2k_F \end{cases}$$

for a constant $C_\beta > 0$ *independent of* k *and* k_F.

In particular, it holds that

$$\sum_{p \in L_k} \lambda_{k,p}^{-1} \le C k_F \min\{1, k_F^2\,|k|^{-2}\}, \tag{3.27}$$

so $\left\langle v_k, h_k^{-1} v_k\right\rangle \le C\hat{V}_k$. Additionally,

$$\left\langle e_p, v_k\right\rangle\left\langle v_k, e_q\right\rangle = \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}. \tag{3.28}$$

Inserting these quantities into the statements of the Propositions 3.4, 3.8 and 3.9 yields Theorem 3.1, noting also that by Proposition 3.4

$$\|K_k\|_{\mathrm{HS}} = \sqrt{\sum_{p,q\in L_k} \left|\left\langle e_p, K_k e_q\right\rangle\right|^2} \le \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \sqrt{\sum_{p,q\in L_k} \frac{1}{\left(\lambda_{k,p}+\lambda_{k,q}\right)^2}} \le \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \sum_{p\in L_k} \frac{1}{\lambda_{k,p}}$$
$$\le C\hat{V}_k \min\{1, k_F^2\,|k|^{-2}\}. \tag{3.29}$$

4. Analysis of the Exchange Terms

In this section we analyze the *exchange terms*, by which we mean the quantities of the expression

$$\sum_{k\in\mathbb{Z}^3_*} \int_0^1 e^{(1-t)\mathcal{K}} \left(\varepsilon_k(\{K_k, B_k(t)\})+2\,\mathrm{Re}\left(\mathcal{E}^1_k(A_k(t))\right)+2\,\mathrm{Re}\left(\mathcal{E}^2_k(B_k(t))\right)\right) e^{-(1-t)\mathcal{K}}dt \tag{4.1}$$

which appears in Theorem 1.4. The name is apt as these enter our calculations due to the presence of the exchange correction $\varepsilon_{k,l}(p;q)$ of the quasi-bosonic commutation relations (see Lemma 1.3). To be precise, we will consider in this section the operators $\varepsilon_k(\{K_k, B_k(t)\})$, $\mathcal{E}^1_k(A_k(t))$ and $\mathcal{E}^2_k(B_k(t))$, and the effect of the integration will be handled in the next section. The main result of this section is the following estimates for them:

Theorem 4.1. *For any $\Psi \in \mathcal{H}_N$ and $t \in [0,1]$ it holds that*

$$\left|\sum_{k\in\mathbb{Z}^3_*} \langle \Psi, \varepsilon_k(\{K_k, B_k(t)\})\Psi\rangle\right| \le Ck_F^{-1}\langle \Psi, \mathcal{N}_E\Psi\rangle,$$
$$\sum_{k\in\mathbb{Z}^3_*} \left|\left\langle \Psi, \mathcal{E}^1_k(A_k(t))\Psi\right\rangle\right| \le C\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}}\left\langle \Psi, \left(\mathcal{N}_E^3+1\right)\Psi\right\rangle,$$
$$\sum_{k\in\mathbb{Z}^3_*} \left|\left\langle \Psi, \left(\mathcal{E}^2_k(B_k(t)) - \left\langle \psi_{\mathrm{FS}}, \mathcal{E}^2_k(B_k(t))\psi_{\mathrm{FS}}\right\rangle\right)\Psi\right\rangle\right|$$
$$\le C\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}}\left\langle \Psi, \mathcal{N}_E^3\Psi\right\rangle$$

for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2$.

The constant terms in the final estimate of the theorem give the *exchange contribution*

$$\sum_{k\in\mathbb{Z}^3_*} \int_0^1 \left\langle \psi_{\mathrm{FS}}, 2\,\mathrm{Re}\left(\mathcal{E}^2_k(B_k(t))\right)\psi_{\mathrm{FS}}\right\rangle dt. \tag{4.2}$$

It is not generally negligible for singular potentials V, and the leading behavior is given by by

Proposition 4.2. *It holds that*

$$\left|\sum_{k\in\mathbb{Z}^3_*}\int_0^1\left\langle\psi_{\mathrm{FS}},2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\psi_{\mathrm{FS}}\right\rangle dt-E_{\mathrm{corr,ex}}\right|\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}$$

for a constant $C>0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$*, where*

$$E_{\mathrm{corr,ex}}=\frac{k_F^{-2}}{4\,(2\pi)^6}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\frac{\hat{V}_k\hat{V}_{p+q-k}}{\lambda_{k,p}+\lambda_{k,q}}.$$

Analysis of ε_k *terms.* Let us first consider terms of the form $\sum_{k\in\mathbb{Z}^3_*}\varepsilon_k(A_k)$, where we recall that

$$\varepsilon_k(A_k)=-\sum_{p\in L_k}\langle e_p,A_ke_p\rangle\left(c_p^*c_p+c_{p-k}c_{p-k}^*\right).\tag{4.3}$$

When summing over $k\in\mathbb{Z}^3_*$, we can split the sum into two parts and interchange the summations as follows:

$$\begin{aligned}-\sum_{k\in\mathbb{Z}^3_*}\varepsilon_k(A_k)&=\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\langle e_p,A_ke_p\rangle c_p^*c_p+\sum_{k\in\mathbb{Z}^3_*}\sum_{q\in(L_k-k)}\langle e_{q+k},A_ke_{q+k}\rangle c_qc_q^*\\&=\sum_{p\in B_F^c}\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\langle e_p,A_ke_p\rangle\right)c_p^*c_p+\sum_{q\in B_F}\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(q+k)\langle e_{q+k},A_ke_{q+k}\rangle\right)c_qc_q^*.\end{aligned}\tag{4.4}$$

Recalling that $\mathcal{N}_E=\sum_{p\in B_F^c}c_p^*c_p=\sum_{q\in B_F}c_qc_q^*$ on $\mathcal{H}_N$, we can then immediately conclude that

$$\begin{aligned}\pm\sum_{k\in\mathbb{Z}^3_*}\varepsilon_k(A_k)&\leq\left(\sup_{p\in B_F^c}\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\left|\langle e_p,A_ke_p\rangle\right|+\sup_{q\in B_F}\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(q+k)\left|\langle e_{q+k},A_ke_{q+k}\rangle\right|\right)\mathcal{N}_E\\&\leq 2\left(\sum_{k\in\mathbb{Z}^3_*}\sup_{p\in L_k}\left|\langle e_p,A_ke_p\rangle\right|\right)\mathcal{N}_E.\end{aligned}\tag{4.5}$$

By the estimates of the previous section we thus obtain the first estimate of Theorem 4.1:

Proposition 4.3. *For any* $\Psi\in\mathcal{H}_N$ *and* $t\in[0,1]$ *it holds that*

$$\left|\sum_{k\in\mathbb{Z}^3_*}\langle\Psi,\varepsilon_k(\{K_k,B_k(t)\})\Psi\rangle\right|\leq Ck_F^{-1}\langle\Psi,\mathcal{N}_E\Psi\rangle$$

for a constant $C>0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$.

Proof. By Theorem 3.1 we have that

$$\left|\left\langle e_p, \{K_k, B_k(t)\} e_q\right\rangle\right| \le C\left(1+\hat{V}_k^2\right)\hat{V}_k^2 k_F^{-1}, \quad k\in\mathbb{Z}_*^3,\ p,q\in L_k, \tag{4.6}$$

for a constant $C>0$ independent of all quantities, so

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}_*^3}\langle\Psi,\varepsilon_k(\{K_k,B_k(t)\})\Psi\rangle\right| &\le 2\left(\sum_{k\in\mathbb{Z}_*^3}\sup_{p\in L_k}\left|\left\langle e_p,\{K_k,B_k(t)\}e_p\right\rangle\right|\right)\langle\Psi,\mathcal{N}_E\Psi\rangle\\ &\le Ck_F^{-1}\sum_{k\in\mathbb{Z}_*^3}\left(1+\hat{V}_k^2\right)\hat{V}_k^2\langle\Psi,\mathcal{N}_E\Psi\rangle \le Ck_F^{-1}\left(1+\|\hat{V}\|_\infty^2\right)\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\langle\Psi,\mathcal{N}_E\Psi\rangle.\end{aligned} \tag{4.7}$$

As $\|\hat{V}\|_\infty^2\le\|\hat{V}\|_2^2=\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2$ the claim follows. □

4.1. Analysis of $\mathcal{E}_k^1$ terms. We consider terms of the form

$$\mathcal{E}_k^1(A_k)=\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}b_k^*(A_ke_p)\left\{\varepsilon_{k,l}(e_p;e_q),b_{-l}^*(K_{-l}e_{-q})\right\}. \tag{4.8}$$

Recalling that

$$\varepsilon_{k,l}(e_p;e_q)=-\left(\delta_{p,q}c_{q-l}c_{p-k}^*+\delta_{p-k,q-l}c_q^*c_p\right) \tag{4.9}$$

we see that $\mathcal{E}_k^1(A_k)$ splits into two sums as

$$\begin{aligned}-\mathcal{E}_k^1(A_k)&=\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}b_k^*(A_ke_p)\left\{\delta_{p,q}c_{q-l}c_{p-k}^*,b_{-l}^*(K_{-l}e_{-q})\right\}\\ &\quad+\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)}\sum_{q\in(L_l-l)}b_k^*(A_ke_{p+k})\left\{\delta_{p,q}c_{q+l}^*c_{p+k},b_{-l}^*(K_{-l}e_{-q-l})\right\}\\ &=\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}b_k^*(A_ke_p)\left\{c_{p-l}c_{p-k}^*,b_{-l}^*(K_{-l}e_{-p})\right\}\\ &\quad+\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)\cap(L_l-l)}b_k^*(A_ke_{p+k})\left\{c_{p+l}^*c_{p+k},b_{-l}^*\left(K_{-l}e_{-p-l}\right)\right\}.\end{aligned} \tag{4.10}$$

The two sums on the right-hand side have the same "schematic form": They can be written as

$$\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in S_k\cap S_l}b_k^*\left(A_ke_{p_1}\right)\left\{\tilde{c}_{p_2}^*\tilde{c}_{p_3},b_{-l}^*\left(K_{-l}e_{p_4}\right)\right\},\quad \tilde{c}_p=\begin{cases}c_p & p\in B_F^c\\ c_p^* & p\in B_F\end{cases}, \tag{4.11}$$

where the index set is either the lune $S_k=L_k$ or the corresponding hole states $S_k=L_k-k$, and depending on this index set the variables p_1,p_2,p_3,p_4 are given by

$$(p_1,p_2,p_3,p_4)=\begin{cases}(p,p-l,p-k,-p) & S_k=L_k\\ (p+k,p+l,p+k,-p-l) & S_k=L_k-k\end{cases}. \tag{4.12}$$

Note that in either case p_1, p_3 only depend on p and k, while p_2, p_4 depend only on p and l. Additionally, p_1 is always an element of L_k and p_4 is always an element of L_{-l}.

Since $b_{k,p} = c^*_{p-k}c_p = \tilde{c}_{p-k}\tilde{c}_p$ it is easily seen that $\left[b, \tilde{c}\right] = 0$, so in normal-ordering (with respect to ψ_{FS}) the summand of equation (4.11) we find

$$\begin{aligned}
&b^*_k\left(A_k e_{p_1}\right)\left\{\tilde{c}^*_{p_2}\tilde{c}_{p_3}, b^*_{-l}\left(K_{-l}e_{p_4}\right)\right\}\\
&\quad = b^*_k\left(A_k e_{p_1}\right)\tilde{c}^*_{p_2}\tilde{c}_{p_3}b^*_{-l}\left(K_{-l}e_{p_4}\right) + b^*_k\left(A_k e_{p_1}\right)b^*_{-l}\left(K_{-l}e_{p_4}\right)\tilde{c}^*_{p_2}\tilde{c}_{p_3}\\
&\quad = 2\,\tilde{c}^*_{p_2}b^*_k\left(A_k e_{p_1}\right)b^*_{-l}\left(K_{-l}e_{p_4}\right)\tilde{c}_{p_3} + \tilde{c}^*_{p_2}b^*_k\left(A_k e_{p_1}\right)\left[\tilde{c}_{p_3}, b^*_{-l}\left(K_{-l}e_{p_4}\right)\right]. \quad (4.13)
\end{aligned}$$

To bound a sum of the form $\sum_{k\in\mathbb{Z}^3_*}\mathcal{E}^k_1(A_k)$ it thus suffices to estimate the two schematic forms

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}b^*_k\left(A_k e_{p_1}\right)b^*_{-l}\left(K_{-l}e_{p_4}\right)\tilde{c}_{p_3},\\
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}b^*_k\left(A_k e_{p_1}\right)\left[b_{-l}\left(K_{-l}e_{p_4}\right), \tilde{c}^*_{p_3}\right]^*. \quad (4.14)
\end{aligned}$$

Preliminary estimates We prepare for the estimation of these schematic forms by deriving some auxilliary bounds for the operators involved. Recall that for any $k \in \mathbb{Z}^3_*$ and $\varphi \in \ell^2(L_k)$,

$$b_k(\varphi) = \sum_{p\in L_k}\langle\varphi, e_p\rangle b_{k,p} = \sum_{p\in L_k}\langle\varphi, e_p\rangle c^*_{p-k}c_p. \quad (4.15)$$

Denote $\mathcal{N}_k = \sum_{p\in L_k} b^*_{k,p}b_{k,p}$. We can bound both $b_k(\varphi)$ and $b^*_k(\varphi)$ as follows:

Proposition 4.4. *For any $k \in \mathbb{Z}^3_*$, $\varphi \in \ell^2(L_k)$ and $\Psi \in \mathcal{H}_N$ it holds that*

$$\|b_k(\varphi)\Psi\| \le \|\varphi\|\,\|\mathcal{N}_k^{\frac{1}{2}}\Psi\|, \quad \left\|b^*_k(\varphi)\Psi\right\| \le \|\varphi\|\,\|\left(\mathcal{N}_k + 1\right)^{\frac{1}{2}}\Psi\|.$$

Proof. By the triangle and Cauchy-Schwarz inequalities we immediately obtain

$$\|b_k(\varphi)\Psi\| \le \sum_{p\in L_k}\left|\langle\varphi, e_p\rangle\right|\left\|b_{k,p}\Psi\right\| \le \|\varphi\|\sqrt{\sum_{p\in L_k}\left\|b_{k,p}\Psi\right\|^2} = \|\varphi\|\,\|\mathcal{N}_k^{\frac{1}{2}}\Psi\| \quad (4.16)$$

and the bound for $\left\|b^*_k(\varphi)\Psi\right\|$ now follows from (4.16) and the fact that

$$\varepsilon_{k,k}\left(\varphi;\varphi\right) = \left[b_k(\varphi), b^*_k(\varphi)\right] - \|\varphi\|^2 = -\sum_{p\in L_k}\left|\langle e_p, \varphi\rangle\right|^2\left(c_{p-k}c^*_{p-k} + c^*_p c_p\right) \le 0. \quad (4.17)$$

□

It is straightforward to see that $\mathcal{N}_k \leq \mathcal{N}_E$. Moreover, by rearranging the summations,

$$\sum_{k\in\mathbb{Z}^3_*} \mathcal{N}_k = \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in L_k} c_p^* c_{p-k} c_{p-k}^* c_p = \sum_{p\in B_F^c} c_p^* c_p \sum_{k\in(B_F+p)} c_{p-k} c_{p-k}^* = \mathcal{N}_E^2 \quad (4.18)$$

on $\mathcal{H}_N$. We also note that for any $\Psi \in \mathcal{H}_N$ and $p \in \mathbb{Z}^3$

$$\begin{aligned} \|\mathcal{N}_k^{\frac{1}{2}} \tilde{c}_p \Psi\| &\leq \|\tilde{c}_p \mathcal{N}_k^{\frac{1}{2}} \Psi\| \leq \|\tilde{c}_p \mathcal{N}_E^{\frac{1}{2}} \Psi\| \\ \| (\mathcal{N}_k + 1)^{\frac{1}{2}} \tilde{c}_p \Psi\| &\leq \|\tilde{c}_p (\mathcal{N}_k + 1)^{\frac{1}{2}} \Psi\| \leq \|\tilde{c}_p (\mathcal{N}_E + 1)^{\frac{1}{2}} \Psi\|, \end{aligned} \quad (4.19)$$

as follows by the inequality (considering $p \in B_F^c$ for definiteness)

$$\begin{aligned} \tilde{c}_p^* \mathcal{N}_k \tilde{c}_p &= \sum_{q\in L_k} c_p^* c_q^* c_{q-k} c_{q-k}^* c_q c_p = \sum_{q\in L_k} c_q^* c_{q-k} c_{q-k}^* \left(c_q c_p^* - \delta_{p,q} \right) c_p \\ &= \mathcal{N}_k c_p^* c_p - 1_{L_k}(p) c_p^* c_{p-k} c_{p-k}^* c_p \leq \mathcal{N}_k c_p^* c_p \end{aligned} \quad (4.20)$$

and the fact that $\left[\tilde{c}_p^* c_p, \mathcal{N}_k\right] = 0 = \left[\tilde{c}_p^* c_p, \mathcal{N}_E\right]$. Similarly

$$\|\mathcal{N}_E^{\frac{1}{2}} \tilde{c}_p \Psi\| \leq \|\tilde{c}_p \mathcal{N}_E^{\frac{1}{2}} \Psi\|, \quad \| (\mathcal{N}_E + 1)^{\frac{1}{2}} \tilde{c}_p \Psi\| \leq \|\tilde{c}_p (\mathcal{N}_E + 1)^{\frac{1}{2}} \Psi\|. \quad (4.21)$$

To analyze the commutator term $\left[b_{-l}\left(K_{-l} e_{p_4}\right), \tilde{c}_{p_3}^*\right]$ we calculate a general identity: For any $l \in \mathbb{Z}^3_*$, $\psi \in \ell^2(L_l)$ and $p \in \mathbb{Z}^3$

$$\left[b_l(\psi), \tilde{c}_p^*\right] = \begin{cases} -1_{L_l}(p+l) \langle \psi, e_{p+l} \rangle \tilde{c}_{p+l} & p \in B_F \\ 1_{L_l}(p) \langle \psi, e_p \rangle \tilde{c}_{p-l} & p \in B_F^c \end{cases}, \quad (4.22)$$

so for our particular commutator we obtain

$$\left[b_{-l}\left(K_{-l} e_{p_4}\right), \tilde{c}_{p_3}^*\right] = \begin{cases} -1_{L_{-l}}(p_3 - l) \langle K_{-l} e_{p_4}, e_{p_3-l} \rangle \tilde{c}_{p_3-l} & S_k = L_k \\ 1_{L_{-l}}(p_3) \langle K_{-l} e_{p_4}, e_{p_3} \rangle \tilde{c}_{p_3+l} & S_k = L_k - k \end{cases}. \quad (4.23)$$

It will be crucial to our estimates that the prefactors obey the following:

Proposition 4.5. *For any $k, l \in \mathbb{Z}^3_*$ and $p \in S_k \cap S_l$ it holds that*

$$\begin{aligned} \left|1_{L_{-l}}(p_3 - l) \langle K_{-l} e_{p_4}, e_{p_3-l} \rangle\right| &\leq C \hat{V}_{-l} k_F^{-1} \frac{1_{L_{-k}}(p_2 - k) 1_{L_{-l}}(p_3 - l)}{\sqrt{\lambda_{k,p_1} + \lambda_{-k,p_2-k}} \sqrt{\lambda_{-l,p_3-l} + \lambda_{-l,p_4}}}, \quad S_k = L_k, \\ \left|1_{L_{-l}}(p_3) \langle K_{-l} e_{p_4}, e_{p_3} \rangle\right| &\leq C \hat{V}_{-l} k_F^{-1} \frac{1_{L_{-k}}(p_2) 1_{L_{-l}}(p_3)}{\sqrt{\lambda_{k,p_1} + \lambda_{-k,p_2}} \sqrt{\lambda_{-l,p_3} + \lambda_{-l,p_4}}}, \quad S_k = L_k - k. \end{aligned}$$

Proof. Recall that p_1, p_2, p_3, p_4 are given by

$$(p_1, p_2, p_3, p_4) = \begin{cases} (p, p - l, p - k, -p) & S_k = L_k \\ (p + k, p + l, p + k, -p - l) & S_k = L_k - k \end{cases}. \quad (4.24)$$

From this we see that for any $p \in S_k \cap S_l$

$$\begin{cases} 1_{L_{-l}}(p_3 - l) & S_k = L_k \\ 1_{L_{-l}}(p_3) & S_k = L_k - k \end{cases} = \begin{cases} 1_{L_{-k}}(p_2 - k) & S_k = L_k \\ 1_{L_{-k}}(p_2) & S_k = L_k - k \end{cases} \tag{4.25}$$

where the assumption that $p \in S_k \cap S_l$ enters to ensure that $1_{B_F}(p-k) = 1 = 1_{B_F}(p-l)$ or $1_{B_F^c}(p+k) = 1 = 1_{B_F^c}(p+l)$, respectively. Importantly this also implies that, when combined with such an indicator function, we also have the identity

$$\begin{cases} \lambda_{-l,p_3-l} + \lambda_{-l,p_4} & S_k = L_k \\ \lambda_{-l,p_3} + \lambda_{-l,p_4} & S_k = L_k - k \end{cases} = \begin{cases} \lambda_{k,p_1} + \lambda_{-k,p_2-k} & S_k = L_k \\ \lambda_{k,p_1} + \lambda_{-k,p_2} & S_k = L_k - k \end{cases}. \tag{4.26}$$

The claim now follows by applying these identities to the estimates

$$\begin{aligned} \left|1_{L_{-l}}(p_3 - l)\left\langle K_{-l}e_{p_4}, e_{p_3-l}\right\rangle\right| &\leq C\frac{1_{L_{-l}}(p_3 - l)\hat{V}_{-l}k_F^{-1}}{\lambda_{-l,p_3-l} + \lambda_{-l,p_4}}, \qquad S_k = L_k, \\ \left|1_{L_{-l}}(p_3)\left\langle K_{-l}e_{p_4}, e_{p_3}\right\rangle\right| &\leq C\frac{1_{L_{-l}}(p_3)\hat{V}_{-l}k_F^{-1}}{\lambda_{-l,p_3} + \lambda_{-l,p_4}}, \qquad S_k = L_k - k, \end{aligned} \tag{4.27}$$

which are given by Theorem 3.1. □

Below we will only use the simpler bound

$$\begin{cases} \left|1_{L_{-l}}(p_3 - l)\left\langle K_{-l}e_{p_4}, e_{p_3-l}\right\rangle\right| & S_k = L_k \\ \left|1_{L_{-l}}(p_3)\left\langle K_{-l}e_{p_4}, e_{p_3}\right\rangle\right| & S_k = L_k - k \end{cases} \leq C\frac{\hat{V}_{-l}k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{-l,p_4}}} \tag{4.28}$$

but for the $\mathcal{E}_k^2$ terms the more general ones will be needed.

Estimation of $\sum_{k\in\mathbb{Z}_*^3} \mathcal{E}_k^1(A_k(t))$ Now the main estimate of this subsection:

Proposition 4.6. *For any collection of symmetric operators* (A_k) *and* $\Psi \in \mathcal{H}_N$ *it holds that*

$$\begin{aligned} &\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l} \left|\left\langle \Psi, \tilde{c}_{p_2}^* b_k^*\left(A_k e_{p_1}\right) b_{-l}^*\left(K_{-l}e_{p_4}\right)\tilde{c}_{p_3}\Psi\right\rangle\right| \\ &\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \max_{p\in L_k} \left\|A_k e_p\right\|^2} \|\left(\mathcal{N}_E + 1\right)^{\frac{3}{2}}\Psi\|^2 \\ &\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l} \left|\left\langle \Psi, \tilde{c}_{p_2}^* b_k^*\left(A_k e_{p_1}\right)\left[b_{-l}\left(K_{-l}e_{p_4}\right), \tilde{c}_{p_3}^*\right]^* \tilde{c}_{p_3}\Psi\right\rangle\right| \\ &\leq Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}_*^3} \|A_k h_k^{-\frac{1}{2}}\|_{\mathrm{HS}}^2} \|\left(\mathcal{N}_E + 1\right)\Psi\|^2. \end{aligned}$$

Proof. Using the triangle and Cauchy-Schwarz inequalities and Proposition 4.4 we estimate

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left|\left\langle\Psi,\tilde{c}^*_{p_2}b^*_k\left(A_ke_{p_1}\right)b^*_{-l}\left(K_{-l}e_{p_4}\right)\tilde{c}_{p_3}\Psi\right\rangle\right|\\
&\leq\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left\|b_k\left(A_ke_{p_1}\right)\tilde{c}_{p_2}\Psi\right\|\left\|b^*_{-l}\left(K_{-l}e_{p_4}\right)\tilde{c}_{p_3}\Psi\right\|\\
&\leq\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|A_ke_{p_1}\right\|\left\|K_{-l}e_{p_4}\right\|\|\mathcal{N}_k^{\frac{1}{2}}\tilde{c}_{p_2}\Psi\|\|\left(\mathcal{N}_{-l}+1\right)^{\frac{1}{2}}\tilde{c}_{p_3}\Psi\|\\
&\leq\sum_{k\in\mathbb{Z}^3_*}\left(\max_{p\in L_k}\left\|A_ke_p\right\|\right)\sum_{p\in S_k}\|\tilde{c}_{p_3}\left(\mathcal{N}_E+1\right)^{\frac{1}{2}}\Psi\|\\
&\quad\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|K_{-l}e_{p_4}\right\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\|\tilde{c}_{p_2}\mathcal{N}_k^{\frac{1}{2}}\Psi\|^2}\\
&\leq\sum_{k\in\mathbb{Z}^3_*}\left(\max_{p\in L_k}\left\|A_ke_p\right\|\right)\|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\|\sqrt{\sum_{p\in S_k}\|\tilde{c}_{p_3}\left(\mathcal{N}_E+1\right)^{\frac{1}{2}}\Psi\|^2}\\
&\quad\sqrt{\sum_{p\in S_k}\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|K_{-l}e_{p_4}\right\|^2}\\
&\leq\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\left\|A_ke_p\right\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\,\|\left(\mathcal{N}_E+1\right)\Psi\|\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\|^2}\\
&=\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\left\|A_ke_p\right\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\,\|\left(\mathcal{N}_E+1\right)\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\| \qquad (4.29)
\end{aligned}$$

and the first bound now follows by recalling that $\|K_l\|^2_{\mathrm{HS}}\leq C\hat{V}_l$. For the second we have by the equations (4.23) and (4.28) that

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left|\left\langle\Psi,\tilde{c}^*_{p_2}b^*_k\left(A_ke_{p_1}\right)\left[b_{-l}\left(K_{-l}e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\Psi\right\rangle\right|\\
&\leq\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left\|\left[b_{-l}\left(K_{-l}e_{p_4}\right),\tilde{c}^*_{p_3}\right]\tilde{c}_{p_2}\Psi\right\|\left\|b^*_k\left(A_ke_{p_1}\right)\Psi\right\|\\
&\leq C\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_l}\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\left\|A_ke_{p_1}\right\|\frac{\hat{V}_{-l}k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{-l,p_4}}}\left\|\tilde{c}_{p_3\mp l}\tilde{c}_{p_2}\Psi\right\|\|\left(\mathcal{N}_k+1\right)^{\frac{1}{2}}\Psi\|\\
&\leq Ck_F^{-1}\|\left(\mathcal{N}_E+1\right)^{\frac{1}{2}}\Psi\|\sum_p\sum_{l\in\mathbb{Z}^3_*}\frac{1_{S_l}(p)\hat{V}_{-l}}{\sqrt{\lambda_{-l,p_4}}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|^2}\\
&\quad\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\left\|\tilde{c}_{p_3\mp l}\tilde{c}_{p_2}\Psi\right\|^2}
\end{aligned}$$

$$\le Ck_F^{-1}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\sum_p\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\frac{\hat{V}^2_{-l}}{\lambda_{-l,p_4}}}$$
$$\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\|\tilde{c}_{p_2}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2}$$
$$\le Ck_F^{-1}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}^2_{-l}\sum_{p\in S_l}\frac{1}{\lambda_{-l,p_4}}}$$
$$\le Ck_F^{-1}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|A_kh_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}}}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}^2_l\sum_{p\in L_l}\frac{1}{\lambda_{l,p}}}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\| \tag{4.30}$$

where we used $\left\|A_ke_{p_1}\right\|\lambda_{k,p_1}^{-\frac{1}{2}}=\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|$. The claim follows by $\sum_{p\in L_l}\lambda_{l,p}^{-1}\le Ck_F$. □

The bound on $\sum_{k\in\mathbb{Z}^3_*}\mathcal{E}^1_k(A_k(t))$ of Theorem 4.1 now follows by our matrix element estimates:

Proposition 4.7. *For any $\Psi\in\mathcal{H}_N$ and $t\in[0,1]$ it holds that*

$$\sum_{k\in\mathbb{Z}^3_*}\left|\left\langle\Psi,\mathcal{E}^1_k(A_k(t))\Psi\right\rangle\right|\le C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}^2_k\min\{|k|,k_F\}}\left\langle\Psi,\left(\mathcal{N}_E^3+1\right)\Psi\right\rangle$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}^2_k$.

Proof. By Theorem 3.1 we have

$$\left|\left\langle e_p,A_k(t)e_q\right\rangle\right|\le C\left(1+\hat{V}^2_k\right)\hat{V}_kk_F^{-1},\quad k\in\mathbb{Z}^3_*,\ p,q\in L_k. \tag{4.31}$$

Combining with $|L_k|\le C\min\left\{k_F^2|k|,k_F^3\right\}$ since $\sum_{q\in L_k}\lambda_{k,q}^{-1}\le Ck_F$, we get

$$\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\left\|A_k(t)e_p\right\|^2\le\frac{C}{k_F^2}\sum_{k\in\mathbb{Z}^3_*}\left(1+\hat{V}^2_k\right)^2\hat{V}^2_k|L_k|$$
$$=C\left(1+\|\hat{V}\|^4_\infty\right)\sum_{k\in\mathbb{Z}^3_*}\hat{V}^2_k\min\{|k|,k_F\}$$

$$\sum_{k\in\mathbb{Z}^3_*}\|A_k(t)h_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}}=\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\left|\left\langle e_p,A_k(t)h_k^{-\frac{1}{2}}e_q\right\rangle\right|^2$$
$$\le Ck_F^{-2}\sum_{k\in\mathbb{Z}^3_*}\left(1+\hat{V}^2_k\right)^2\hat{V}^2_k|L_k|\sum_{q\in L_k}\frac{1}{\lambda_{k,q}}$$
$$\le Ck_F\left(1+\|\hat{V}\|^4_\infty\right)\sum_{k\in\mathbb{Z}^3_*}\hat{V}^2_k\min\{|k|,k_F\}. \tag{4.32}$$

Inserting these estimates into Proposition 4.6 yields the claim. □

4.2. Analysis of $\mathcal{E}_k^2$ terms. Now we come to the terms

$$\mathcal{E}_k^2(B_k) = \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\left\{b_k(B_k e_p), \left\{\varepsilon_{-k,-l}(e_{-p}; e_{-q}), b_l^*(K_l e_q)\right\}\right\}. \quad (4.33)$$

We will analyze these similarly to the $\mathcal{E}_k^1(A_k)$ terms. Noting that

$$\varepsilon_{-k,-l}(e_{-p}; e_{-q}) = -\left(\delta_{p,q} c_{-q+l} c_{-p+k}^* + \delta_{p-k,q-l} c_{-q}^* c_{-p}\right) \quad (4.34)$$

we find that $\mathcal{E}_k^2(B_k)$ splits into two sums as

$$\begin{aligned} -2\,\mathcal{E}_k^2(B_k) &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\left\{b_k(B_k e_p), \left\{\delta_{p,q} c_{-q+l} c_{-p+k}^*, b_l^*(K_l e_q)\right\}\right\} \\ &\quad + \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)}\sum_{q\in(L_l-l)}\left\{b_k\left(B_k e_{p+k}\right), \left\{\delta_{p,q} c_{-q-l}^* c_{-p-k}, b_l^*\left(K_l e_{q+l}\right)\right\}\right\} \\ &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}\left\{b_k(B_k e_p), \left\{c_{-p+l} c_{-p+k}^*, b_l^*\left(K_l e_p\right)\right\}\right\} \\ &\quad + \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)\cap(L_l-l)}\left\{b_k\left(B_k e_{p+k}\right), \left\{c_{-p-l}^* c_{-p-k}, b_l^*\left(K_l e_{p+l}\right)\right\}\right\} \end{aligned} \quad (4.35)$$

and again these share a common schematic form, namely

$$\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in S_k\cap S_l}\left\{b_k\left(B_k e_{p_1}\right), \left\{\tilde{c}_{p_2}^*\tilde{c}_{p_3}, b_l^*\left(K_l e_{p_4}\right)\right\}\right\} \quad (4.36)$$

where the momenta are now

$$(p_1, p_2, p_3, p_4) = \begin{cases} (p, -p+l, -p+k, p) & S_k = L_k \\ (p+k, -p-l, -p-k, p+l) & S_k = L_k - k \end{cases}. \quad (4.37)$$

Again p_1, p_3 only depend on p and k while p_2, p_4 only depend on p and l.

We normal order the summand: As

$$\begin{aligned} & b_k\left(B_k e_{p_1}\right)\left\{\tilde{c}_{p_2}^*\tilde{c}_{p_3}, b_l^*\left(K_l e_{p_4}\right)\right\} \\ &= \tilde{c}_{p_2}^* b_k\left(B_k e_{p_1}\right)\left\{\tilde{c}_{p_3}, b_l^*\left(K_l e_{p_4}\right)\right\} + \left[b_k\left(B_k e_{p_1}\right), \tilde{c}_{p_2}^*\right]\left\{\tilde{c}_{p_3}, b_l^*\left(K_l e_{p_4}\right)\right\} \\ &= 2\,\tilde{c}_{p_2}^* b_k\left(B_k e_{p_1}\right) b_l^*\left(K_l e_{p_4}\right)\tilde{c}_{p_3} + \tilde{c}_{p_2}^* b_k\left(B_k e_{p_1}\right)\left[b_l\left(K_l e_{p_4}\right), \tilde{c}_{p_3}^*\right]^* \\ &\quad + 2\left[b_k\left(B_k e_{p_1}\right), \tilde{c}_{p_2}^*\right] b_l^*\left(K_l e_{p_4}\right)\tilde{c}_{p_3} + \left[b_k\left(B_k e_{p_1}\right), \tilde{c}_{p_2}^*\right]\left[b_l\left(K_l e_{p_4}\right), \tilde{c}_{p_3}^*\right]^* \\ &= 2\,\tilde{c}_{p_2}^* b_l^*\left(K_l e_{p_4}\right) b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3} + 2\,\tilde{c}_{p_2}^*\left[b_k\left(B_k e_{p_1}\right), b_l^*\left(K_l e_{p_4}\right)\right]\tilde{c}_{p_3} \\ &\quad + \tilde{c}_{p_2}^*\left[b_l\left(K_l e_{p_4}\right), \tilde{c}_{p_3}^*\right]^* b_k\left(B_k e_{p_1}\right) + \tilde{c}_{p_2}^*\left[b_k\left(B_k e_{p_1}\right), \left[b_l\left(K_l e_{p_4}\right), \tilde{c}_{p_3}^*\right]^*\right] \\ &\quad + 2\,b_l^*\left(K_l e_{p_4}\right)\left[b_k\left(B_k e_{p_1}\right), \tilde{c}_{p_2}^*\right]\tilde{c}_{p_3} + 2\left[b_l\left(K_l e_{p_4}\right), \left[b_k\left(B_k e_{p_1}\right), \tilde{c}_{p_2}^*\right]^*\right]^*\tilde{c}_{p_3} \end{aligned}$$

$$-\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\left[b_k\left(B_k e_{p_1}\right),\tilde{c}^*_{p_2}\right]+\left\{\left[b_k\left(B_k e_{p_1}\right),\tilde{c}^*_{p_2}\right],\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\right\} \tag{4.38}$$

and simply

$$\begin{aligned}\left\{\tilde{c}^*_{p_2}\tilde{c}_{p_3},b^*_l\left(K_l e_{p_4}\right)\right\}b_k\left(B_k e_{p_1}\right)&=\tilde{c}^*_{p_2}\left\{\tilde{c}_{p_3},b^*_l\left(K_l e_{p_4}\right)\right\}b_k\left(B_k e_{p_1}\right)\\&=2\,\tilde{c}^*_{p_2}b^*_l\left(K_l e_{p_4}\right)b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3}+\tilde{c}^*_{p_2}\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*b_k\left(B_k e_{p_1}\right)\end{aligned} \tag{4.39}$$

the summand decomposes into 8 schematic forms as

$$\begin{aligned}&\left\{b_k\left(B_k e_{p_1}\right),\left\{\tilde{c}^*_{p_2}\tilde{c}_{p_3},b^*_l\left(K_l e_{p_4}\right)\right\}\right\}\\&\quad=4\,\tilde{c}^*_{p_2}b^*_l\left(K_l e_{p_4}\right)b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3}+2\,\tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right),b^*_l\left(K_l e_{p_4}\right)\right]\tilde{c}_{p_3}\\&\qquad+2\,\tilde{c}^*_{p_2}\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*b_k\left(B_k e_{p_1}\right)+2\,b^*_l\left(K_l e_{p_4}\right)\left[b_k\left(B_k e_{p_1}\right),\tilde{c}^*_{p_2}\right]\tilde{c}_{p_3}\\&\qquad+\tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right),\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\right]+2\left[b_l\left(K_l e_{p_4}\right),\left[b_k\left(B_k e_{p_1}\right),\tilde{c}^*_{p_2}\right]^*\right]^*\tilde{c}_{p_3}\\&\qquad-\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\left[b_k\left(B_k e_{p_1}\right),\tilde{c}^*_{p_2}\right]+\left\{\left[b_k\left(B_k e_{p_1}\right),\tilde{c}^*_{p_2}\right],\left[b_l\left(K_l e_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\right\}.\end{aligned} \tag{4.40}$$

Of these it should be noted that only the last one is proportional to a constant (i.e. does not contain any creation or annihilation operators). As the rest annihilate ψ_{FS}, it follows that (when summed) the constant term yields precisely $\langle\psi_{\mathrm{FS}},\mathcal{E}^2_k(B_k)\psi_{\mathrm{FS}}\rangle$, whence bounding the other terms amounts to estimating the operator

$$\mathcal{E}^2_k(B_k)-\left\langle\psi_{\mathrm{FS}},\mathcal{E}^2_k(B_k)\psi_{\mathrm{FS}}\right\rangle \tag{4.41}$$

as in the statement of Theorem 4.1.

Estimation of the top terms We begin by bounding the "top" terms

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}b^*_l\left(K_l e_{p_4}\right)b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3}\quad\text{and}$$

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right),b^*_l\left(K_l e_{p_4}\right)\right]\tilde{c}_{p_3}.$$

By the quasi-bosonic commutation relations, the commutator term reduces to

$$\begin{aligned}&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right),b^*_l\left(K_l e_{p_4}\right)\right]\tilde{c}_{p_3}\\&\quad=\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\langle B_k e_{p_1},K_k e_{p_1}\rangle\tilde{c}^*_{p_3}\tilde{c}_{p_3}+\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}\varepsilon_{k,l}\left(B_k e_{p_1};K_l e_{p_4}\right)\tilde{c}_{p_3}\end{aligned} \tag{4.42}$$

where we used that $p_1=p_4$ and $p_2=p_3$ when $k=l$. Now, the exchange correction of the second sum splits as

$$-\varepsilon_{k,l}\left(B_k e_{p_1};K_l e_{p_4}\right)=\sum_{q\in L_k}\sum_{q'\in L_l}\langle B_k e_{p_1},e_q\rangle\langle e_{q'},K_l e_{p_4}\rangle\left(\delta_{q,q'}c_{q'-l}c^*_{q-k}+\delta_{q-k,q'-l}c^*_{q'}c_q\right)$$

$$= \sum_{q\in L_k\cap L_l} \left\langle B_k e_{p_1}, e_q\right\rangle \left\langle e_q, K_l e_{p_4}\right\rangle \tilde{c}^*_{q-l}\tilde{c}_{q-k} + \sum_{q\in(L_k-k)\cap(L_l-l)} \left\langle B_k e_{p_1}, e_{q+k}\right\rangle \left\langle e_{q+l}, K_l e_{p_4}\right\rangle \tilde{c}^*_{q+l}\tilde{c}_{q+k} \tag{4.43}$$

which are both of the schematic form $\sum_{q\in S'_k\cap S'_l} \left\langle B_k e_{p_1}, e_{q_1}\right\rangle \left\langle e_{q_4}, K_l e_{p_4}\right\rangle \tilde{c}^*_{q_2}\tilde{c}_{q_3}$.

To estimate $\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l} \tilde{c}^*_{p_2}\varepsilon_{k,l}\left(B_k e_{p_1}; K_l e_{p_4}\right)\tilde{c}_{p_3}$ it thus suffices to consider

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\sum_{q\in S'_k\cap S'_l} \left\langle B_k e_{p_1}, e_{q_1}\right\rangle \left\langle e_{q_4}, K_l e_{p_4}\right\rangle \tilde{c}^*_{p_2}\tilde{c}^*_{q_2}\tilde{c}_{q_3}\tilde{c}_{p_3}. \tag{4.44}$$

The estimates for the top terms are as follows:

Proposition 4.8. *For any collection of symmetric operators (B_k) and $\Psi\in\mathcal{H}_N$ it holds that*

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l} \left|\left\langle \Psi, \tilde{c}^*_{p_2} b^*_l\left(K_l e_{p_4}\right) b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3}\Psi\right\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\left\|B_k e_p\right\|^2}\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|^2$$

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l} \left|\left\langle \Psi, \tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right), b^*_l\left(K_l e_{p_4}\right)\right]\tilde{c}_{p_3}\Psi\right\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}\left|\left\langle e_p, B_k e_q\right\rangle\right|^2}\,\|\mathcal{N}_E\Psi\|^2$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$.

Proof. The first term we can estimate as in Proposition 4.6 by

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l} \left|\left\langle \Psi, \tilde{c}^*_{p_2} b^*_l\left(K_l e_{p_4}\right) b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3}\Psi\right\rangle\right| \\
&\leq \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l} \left\|b_l\left(K_l e_{p_4}\right)\tilde{c}_{p_2}\Psi\right\| \left\|b_k\left(B_k e_{p_1}\right)\tilde{c}_{p_3}\Psi\right\| \\
&\leq \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\sum_{l\in\mathbb{Z}^3_*} 1_{S_l}(p)\left\|B_k e_{p_1}\right\| \left\|K_l e_{p_4}\right\| \|\mathcal{N}_l^{\frac{1}{2}}\tilde{c}_{p_2}\Psi\| \|\mathcal{N}_k^{\frac{1}{2}}\tilde{c}_{p_3}\Psi\| \\
&\leq \sum_{k\in\mathbb{Z}^3_*}\left(\max_{p\in L_k}\left\|B_k e_p\right\|\right)\sum_{p\in S_k}\|\tilde{c}_{p_3}\mathcal{N}_k^{\frac{1}{2}}\Psi\|\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|K_l e_{p_4}\right\|^2} \\
&\quad\times\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\|\tilde{c}_{p_2}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2} \\
&\leq \|\mathcal{N}_E\Psi\|\sum_{k\in\mathbb{Z}^3_*}\left(\max_{p\in L_k}\left\|B_k e_p\right\|\right)\sqrt{\sum_{p\in S_k}\|\tilde{c}_{p_3}\mathcal{N}_k^{\frac{1}{2}}\Psi\|^2}\sqrt{\sum_{p\in S_k}\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|K_l e_{p_4}\right\|^2}
\end{aligned}$$

$$\leq \sqrt{\sum_{l\in\mathbb{Z}^3_*} \|K_l\|^2_{\mathrm{HS}}}\, \|\mathcal{N}_E\Psi\| \sum_{k\in\mathbb{Z}^3_*} \left(\max_{p\in L_k} \|B_k e_p\|\right) \|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\| \tag{4.45}$$

and obviously $\|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\| \leq \|\mathcal{N}_E\Psi\|\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|$. For the commutator term we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k} \left|\langle B_k e_{p_1}, K_k e_{p_1}\rangle \left\langle \Psi, \tilde{c}^*_{p_3}\tilde{c}_{p_3}\Psi\right\rangle\right| &\leq \sum_{k\in\mathbb{Z}^3_*} \max_{p\in L_k} |\langle B_k e_p, K_k e_p\rangle| \sum_{p\in S_k} \left\langle \Psi, \tilde{c}^*_{p_3}\tilde{c}_{p_3}\Psi\right\rangle \\ &\leq \sum_{k\in\mathbb{Z}^3_*} \max_{p\in L_k} |\langle e_p, B_k K_k e_p\rangle| \,\langle\Psi, \mathcal{N}_E\Psi\rangle\,. \end{aligned} \tag{4.46}$$

By the matrix element estimate for K_k of Theorem 3.1 we have for any $p \in L_k$ that

$$\begin{aligned}|\langle B_k e_p, K_k e_p\rangle| &\leq \sum_{q\in L_k} |\langle B_k e_p, e_q\rangle|\, |\langle e_q, K_k e_p\rangle| \leq C \sum_{q\in L_k} |\langle e_p, B_k e_q\rangle| \frac{\hat{V}_k k_F^{-1}}{\lambda_{k,q}+\lambda_{k,p}} \\ &\leq C\hat{V}_k k_F^{-1}\left(\max_{q\in L_k} |\langle e_p, B_k e_q\rangle|\right) \sum_{q\in L_k} \frac{1}{\lambda_{k,q}} \leq C\hat{V}_k \max_{q\in L_k} |\langle e_p, B_k e_q\rangle| \end{aligned} \tag{4.47}$$

since $\sum_{q\in L_k} \lambda_{k,q}^{-1} \leq C k_F$. Consequently

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k} \left|\langle B_k e_{p_1}, K_k e_{p_1}\rangle \left\langle \Psi, \tilde{c}^*_{p_3}\tilde{c}_{p_3}\Psi\right\rangle\right| \leq C \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \left(\max_{p,q\in L_k} |\langle e_p, B_k e_q\rangle|\right) \langle\Psi, \mathcal{N}_E\Psi\rangle \\ &\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p,q\in L_k} |\langle e_p, B_k e_q\rangle|^2}\, \langle\Psi, \mathcal{N}_E\Psi\rangle \end{aligned} \tag{4.48}$$

and clearly

$$\max_{p,q\in L_k} |\langle e_p, B_k e_q\rangle|^2 \leq \sum_{p\in L_k}\max_{q\in L_k} |\langle e_p, B_k e_q\rangle|^2\,. \tag{4.49}$$

Finally

$$\begin{aligned}&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\sum_{q\in S'_k\cap S'_l} \left|\langle B_k e_{p_1}, e_{q_1}\rangle\langle e_{q_4}, K_l e_{p_4}\rangle \left\langle\Psi, \tilde{c}^*_{p_2}\tilde{c}^*_{q_2}\tilde{c}_{q_3}\tilde{c}_{p_3}\Psi\right\rangle\right| \\ &\leq \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\sum_{q\in S'_k\cap S'_l} |\langle B_k e_{p_1}, e_{q_1}\rangle|\, |\langle e_{q_4}, K_l e_{p_4}\rangle|\, \|\tilde{c}_{q_2}\tilde{c}_{p_2}\Psi\|\, \|\tilde{c}_{q_3}\tilde{c}_{p_3}\Psi\| \\ &\leq \sqrt{\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\sum_{q\in S'_k\cap S'_l} |\langle B_k e_{p_1}, e_{q_1}\rangle|^2 \|\tilde{c}_{q_2}\tilde{c}_{p_2}\Psi\|^2} \\ &\quad\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\sum_{q\in S'_k\cap S'_l} |\langle e_{q_4}, K_l e_{p_4}\rangle|^2 \|\tilde{c}_{q_3}\tilde{c}_{p_3}\Psi\|^2} \end{aligned}$$

$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\max_{q\in L_k}\left|\langle e_{p_1}, B_k e_q\rangle\right|^2 \sum_{l\in\mathbb{Z}^3_*} 1_{S_l}(p)\|\tilde{c}_{p_2}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2}$$
$$\sqrt{\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_l}\left\|K_l e_{p_4}\right\|^2 \sum_{k\in\mathbb{Z}^3_*} 1_{S_k}(p)\left\|\tilde{c}_{p_3}\Psi\right\|^2}$$
$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}\left|\langle e_p, B_k e_q\rangle\right|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\| \tag{4.50}$$

whence the claim follows as $\|K_l\|_{\mathrm{HS}} \leq C\hat{V}_l$. □

Estimation of the single commutator terms For the single commutator terms

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}\left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]^* b_k\left(B_k e_{p_1}\right) \quad \text{and}$$
$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l} b^*_l\left(K_l e_{p_4}\right)\left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right]\tilde{c}_{p_3}$$

we note that by equation (4.22), the commutator $\left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]$ is given by

$$\left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right] = \begin{cases} -1_{L_l}(p_3+l)\langle K_l e_{p_4}, e_{p_3+l}\rangle\tilde{c}_{p_3+l} & S_k = L_k \\ 1_{L_l}(p_3)\langle K_l e_{p_4}, e_{p_3}\rangle\tilde{c}_{p_3-l} & S_k = L_k - k \end{cases}. \tag{4.51}$$

The prefactors again obey an estimate as in Proposition 4.5:

Proposition 4.9. *For any $k, l \in \mathbb{Z}^3_*$ and $p \in S_k \cap S_l$ it holds that*

$$\left|1_{L_l}(p_3+l)\langle K_l e_{p_4}, e_{p_3+l}\rangle\right| \leq C\hat{V}_l k_F^{-1}\frac{1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}, \quad S_k = L_k,$$
$$\left|1_{L_l}(p_3)\langle K_l e_{p_4}, e_{p_3}\rangle\right| \leq C\hat{V}_l k_F^{-1}\frac{1_{L_k}(p_2)1_{L_l}(p_3)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2}}\sqrt{\lambda_{l,p_3}+\lambda_{l,p_4}}}, \quad S_k = L_k - k.$$

The proof is essentially the same as that of Proposition 4.5 (indeed, this proposition can be obtained directly from the former by appropriate substition, but some care must be used since the p_i's differ in their definition).

For the single commutator terms we again only need the simpler bound

$$\begin{cases} \left|1_{L_l}(p_3+l)\langle K_l e_{p_4}, e_{p_3+l}\rangle\right| & S_k = L_k \\ \left|1_{L_l}(p_3)\langle K_l e_{p_4}, e_{p_3}\rangle\right| & S_k = L_k - k \end{cases} \leq C\frac{\hat{V}_l k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{l,p_4}}} \tag{4.52}$$

but the full one will be needed for the double commutator terms below. Now the estimate:

Proposition 4.10. *For any collection of symmetric operators* (B_k) *and* $\Psi \in \mathcal{H}_N$ *it holds that*

$$\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left| \left\langle \Psi, \tilde{c}^*_{p_2} \left[b_l \left(K_l e_{p_4} \right), \tilde{c}^*_{p_3} \right]^* b_k \left(B_k e_{p_1} \right) \Psi \right\rangle \right| \leq C k_F^{-\frac{1}{2}} \sqrt{\sum_{k \in \mathbb{Z}^3_*} \| B_k h_k^{-\frac{1}{2}} \|^2_{\mathrm{HS}}} \, \| \mathcal{N}_E \Psi \|^2 ,$$

$$\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left| \left\langle \Psi, b^*_l \left(K_l e_{p_4} \right) \left[b_k \left(B_k e_{p_1} \right), \tilde{c}^*_{p_2} \right] \tilde{c}_{p_3} \Psi \right\rangle \right| \leq C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} \max_{q \in L_k} \left| \langle e_p, B_k e_q \rangle \right|^2} \, \| \mathcal{N}_E \Psi \|^2$$

for a constant $C > 0$ *depending only on* $\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2$.

Proof. As in the second estimate of Proposition 4.6 we have

$$\begin{aligned}
&\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left| \left\langle \Psi, \tilde{c}^*_{p_2} \left[b_l \left(K_l e_{p_4} \right), \tilde{c}^*_{p_3} \right]^* b_k \left(B_k e_{p_1} \right) \Psi \right\rangle \right| \\
&\leq \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left\| \left[b_l \left(K_l e_{p_4} \right), \tilde{c}^*_{p_3} \right] \tilde{c}_{p_2} \Psi \right\| \left\| b_k \left(B_k e_{p_1} \right) \Psi \right\| \\
&\leq C \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in S_l} \sum_{k \in \mathbb{Z}^3_*} 1_{S_k}(p) \left\| B_k e_{p_1} \right\| \frac{\hat{V}_l k_F^{-1}}{\sqrt{\lambda_{k,p_1} \lambda_{l,p_4}}} \left\| \tilde{c}_{p_3 \pm l} \tilde{c}_{p_2} \Psi \right\| \| \mathcal{N}_k^{\frac{1}{2}} \Psi \| \\
&\leq C k_F^{-1} \| \mathcal{N}_E^{\frac{1}{2}} \Psi \| \sum_p \sum_{l \in \mathbb{Z}^3_*} \frac{1_{S_l}(p) \hat{V}_l}{\sqrt{\lambda_{l,p_4}}} \sqrt{\sum_{k \in \mathbb{Z}^3_*} 1_{S_k}(p) \| B_k h_k^{-\frac{1}{2}} e_{p_1} \|^2} \\
&\quad \times \sqrt{\sum_{k \in \mathbb{Z}^3_*} 1_{S_k}(p) \left\| \tilde{c}_{p_3 \pm l} \tilde{c}_{p_2} \Psi \right\|^2} \\
&\leq C k_F^{-1} \| \mathcal{N}_E^{\frac{1}{2}} \Psi \| \sum_p \sqrt{\sum_{k \in \mathbb{Z}^3_*} 1_{S_k}(p) \| B_k h_k^{-\frac{1}{2}} e_{p_1} \|^2} \sqrt{\sum_{l \in \mathbb{Z}^3_*} 1_{S_l}(p) \frac{\hat{V}_l^2}{\lambda_{l,p_4}}} \\
&\quad \times \sqrt{\sum_{l \in \mathbb{Z}^3_*} 1_{S_l}(p) \| \tilde{c}_{p_2} \mathcal{N}_E^{\frac{1}{2}} \Psi \|^2} \\
&\leq C k_F^{-1} \| \mathcal{N}_E^{\frac{1}{2}} \Psi \| \, \| \mathcal{N}_E \Psi \| \sqrt{\sum_{k \in \mathbb{Z}^3_*} \sum_{p \in S_k} \| B_k h_k^{-\frac{1}{2}} e_{p_1} \|^2} \sqrt{\sum_{l \in \mathbb{Z}^3_*} \hat{V}_l^2 \sum_{p \in S_l} \frac{1}{\lambda_{l,p_4}}} \\
&\leq C k_F^{-\frac{1}{2}} \sqrt{\sum_{k \in \mathbb{Z}^3_*} \| B_k h_k^{-\frac{1}{2}} \|^2_{\mathrm{HS}}} \sqrt{\sum_{l \in \mathbb{Z}^3_*} \hat{V}_l^2} \| \mathcal{N}_E^{\frac{1}{2}} \Psi \| \, \| \mathcal{N}_E \Psi \| .
\end{aligned} \tag{4.53}$$

By equation (4.22) it holds that

$$\left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right] = \begin{cases} -1_{L_k}(p_2+k)\left\langle B_k e_{p_1}, e_{p_2+k}\right\rangle \tilde{c}_{p_2+k} & p \in B_F \\ 1_{L_k}(p_2)\left\langle B_k e_{p_1}, e_{p_2}\right\rangle \tilde{c}_{p_2-k} & p \in B_F^c \end{cases} \tag{4.54}$$

so the second term can be bounded as

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left|\left\langle \Psi, b^*_l\left(K_l e_{p_4}\right)\left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right]\tilde{c}_{p_3}\Psi\right\rangle\right| \\
&\quad\leq \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left\|b_l\left(K_l e_{p_4}\right)\Psi\right\|\left\|\left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right]\tilde{c}_{p_3}\Psi\right\| \\
&\quad\leq \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left(\max_{q\in L_k}\left|\left\langle e_{p_1}, B_k e_q\right\rangle\right|\right)\left\|K_l e_{p_4}\right\|\|\mathcal{N}_l^{\frac{1}{2}}\Psi\|\left\|\tilde{c}_{p_2\pm k}\tilde{c}_{p_3}\Psi\right\| \\
&\quad\leq \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\sum_{p}\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\left(\max_{q\in L_k}\left|\left\langle e_{p_1}, B_k e_q\right\rangle\right|\right)\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|K_l e_{p_4}\right\|^2} \\
&\qquad\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|\tilde{c}_{p_2\pm k}\tilde{c}_{p_3}\Psi\right\|^2} \\
&\quad\leq \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\sum_{p}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\left\|K_l e_{p_4}\right\|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\left(\max_{q\in L_k}\left|\left\langle e_{p_1}, B_k e_q\right\rangle\right|^2\right)} \\
&\qquad\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\|\tilde{c}_{p_3}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2} \\
&\quad\leq \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\sqrt{\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_l}\left\|K_l e_{p_4}\right\|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\max_{q\in L_k}\left|\left\langle e_{p_1}, B_k e_q\right\rangle\right|^2} \\
&\quad\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}\left|\left\langle e_p, B_k e_q\right\rangle\right|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\,\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|.
\end{aligned} \tag{4.55}$$

□

Estimation of the double commutator terms Finally we have the double commutator terms

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right), \left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]^*\right], \\
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left[b_l\left(K_l e_{p_4}\right), \left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right]^*\right]^*\tilde{c}_{p_3}, \\
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]^*\left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right].
\end{aligned} \tag{4.56}$$

An identity for the iterated commutators is obtained by applying the identity of equation (4.22) to itself: For any $k, l \in \mathbb{Z}^3_*$, $\varphi \in \ell^2(L_k)$, $\psi \in \ell^2(L_l)$ and $p \in \mathbb{Z}^3_*$

$$\left[b_k(\varphi), \left[b_l(\psi), \tilde{c}^*_p\right]^*\right] = \begin{cases} -1_{L_l}(p+l)\left\langle e_{p+l}, \psi\right\rangle \left[b_k(\varphi), \tilde{c}^*_{p+l}\right] & p \in B_F \\ 1_{L_l}(p)\left\langle e_p, \psi\right\rangle \left[b_k(\varphi), \tilde{c}^*_{p-l}\right] & p \in B_F^c \end{cases}$$
$$= \begin{cases} -1_{L_k}(p+l)1_{L_l}(p+l)\left\langle \varphi, e_{p+l}\right\rangle\left\langle e_{p+l}, \psi\right\rangle \tilde{c}_{p+l-k} & p \in B_F \\ -1_{L_k}(p-l+k)\,1_{L_l}(p)\left\langle \varphi, e_{p-l+k}\right\rangle\left\langle e_p, \psi\right\rangle \tilde{c}_{p-l+k} & p \in B_F^c \end{cases}. \tag{4.57}$$

The estimates are the following:

Proposition 4.11. *For any collection of symmetric operators (B_k) and $\Psi \in \mathcal{H}_N$ it holds that*

$$\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left|\left\langle \Psi, \tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right), \left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]^*\right]\Psi\right\rangle\right|,$$
$$\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left|\left\langle \Psi, \left[b_l\left(K_l e_{p_4}\right), \left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right]^*\right]^* \tilde{c}_{p_3}\Psi\right\rangle\right|,$$
$$\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in S_k \cap S_l} \left|\left\langle \Psi, \left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]^* \left[b_k\left(B_k e_{p_1}\right), \tilde{c}^*_{p_2}\right]\Psi\right\rangle\right|,$$

are all bounded by

$$C k_F^{-\frac{1}{2}} \sqrt{\sum_{k \in \mathbb{Z}^3_*} \max_{p \in L_k} \|h_k^{-\frac{1}{2}} B_k e_p\|^2}\, \langle \Psi, \mathcal{N}_E \Psi\rangle$$

for a constant $C > 0$ depending only on $\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2$.

Proof. For these estimates we consider only the case $S_k = L_k$ for the sake of clarity, i.e. we let

$$(p_1, p_2, p_3, p_4) = (p, -p+l, -p+k, p); \tag{4.58}$$

the case $S_k = L_k - k$ can be handled by similar manipulations.

Using the identity of equation (4.57) we start by estimating (by the bound of Proposition 4.9)

$$\sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in L_k \cap L_l} \left|\left\langle \Psi, \tilde{c}^*_{p_2}\left[b_k\left(B_k e_{p_1}\right), \left[b_l\left(K_l e_{p_4}\right), \tilde{c}^*_{p_3}\right]^*\right]\Psi\right\rangle\right|$$
$$= \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in L_k \cap L_l} \left|1_{L_k}(p_3+l)1_{L_l}(p_3+l)\left\langle B_k e_{p_1}, e_{p_3+l}\right\rangle\left\langle e_{p_3+l}, K_l e_{p_4}\right\rangle\left\langle \Psi, \tilde{c}^*_{p_2}\tilde{c}_{p_3+l-k}\Psi\right\rangle\right|$$
$$\leq C \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p \in L_k \cap L_l} 1_{L_k}(p_3+l)\left|\left\langle B_k e_{p_1}, e_{p_3+l}\right\rangle\right| \frac{\hat{V}_l k_F^{-1} 1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}\left\langle \Psi, \tilde{c}^*_{p_2}\tilde{c}_{p_2}\Psi\right\rangle$$
$$\leq C k_F^{-1} \sum_{l \in \mathbb{Z}^3_*} \hat{V}_l \sum_{p \in L_l} \sqrt{\sum_{k \in \mathbb{Z}^3_*} 1_{L_k}(p)1_{L_k}(p_3+l)\left|\left\langle e_p, h_k^{-\frac{1}{2}} B_k e_{p_3+l}\right\rangle\right|^2}$$

$$\cdot\sqrt{\sum_{k\in\mathbb{Z}^3_*}\frac{1_{L_l}(p_3+l)}{\lambda_{l,p_3+l}}\left\langle\Psi,\tilde{c}^*_{-p+l}\tilde{c}_{-p+l}\Psi\right\rangle}$$

$$\le Ck_F^{-\frac{1}{2}}\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l\sum_{p\in(L_l-l)}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p+l)1_{L_k}(p_3)\left|\left\langle e_{p+l},h_k^{-\frac{1}{2}}B_ke_{p_3}\right\rangle\right|^2}\left\langle\Psi,\tilde{c}^*_{-p}\tilde{c}_{-p}\Psi\right\rangle$$

$$\le Ck_F^{-\frac{1}{2}}\sum_{p\in B_F}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k}(p+l)1_{L_k}(p_3)\left|\left\langle e_{p+l},h_k^{-\frac{1}{2}}B_ke_{p_3}\right\rangle\right|^2}\left\langle\Psi,\tilde{c}^*_{-p}\tilde{c}_{-p}\Psi\right\rangle$$

$$\le Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle \tag{4.59}$$

where we used $\sum_{k\in\mathbb{Z}^3_*}1_{L_l}(p_3+l)\lambda_{l,p_3+l}^{-1}\le\sum_{q\in L_l}\lambda_{l,q}^{-1}\le Ck_F$. From (4.57) we have

$$\begin{aligned}&\left[b_l\left(K_le_{p_4}\right),\left[b_k\left(B_ke_{p_1}\right),\tilde{c}^*_p\right]^*\right]\\&\quad=-1_{L_l}(p_2+k)1_{L_k}(p_2+k)\left\langle K_le_{p_4},e_{p_2+k}\right\rangle\left\langle e_{p_2+k},B_ke_{p_1}\right\rangle\tilde{c}_{p_2+k-l}\\&\quad=-1_{L_k}(p_2+k)1_{L_l}(p_3+l)\left\langle K_le_{p_4},e_{p_3+l}\right\rangle\left\langle e_{p_2+k},B_ke_{p_1}\right\rangle\tilde{c}_{p_3}\end{aligned}\tag{4.60}$$

so the second term can be similarly estimated as

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left|\left\langle\Psi,\left[b_l\left(K_le_{p_4}\right),\left[b_k\left(B_ke_{p_1}\right),\tilde{c}^*_{p_2}\right]^*\right]^*\tilde{c}_{p_3}\Psi\right\rangle\right|$$

$$\le C\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\frac{\hat{V}_lk_F^{-1}1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}\left|\left\langle e_{p_2+k},B_ke_{p_1}\right\rangle\right|\left\langle\Psi,\tilde{c}^*_{p_3}\tilde{c}_{p_3}\Psi\right\rangle$$

$$\le Ck_F^{-1}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{L_l}(p)\frac{\hat{V}_l^2}{\lambda_{l,p_4}}}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{L_k}(p_2+k)\left|\left\langle e_{p_2+k},h_k^{-\frac{1}{2}}B_ke_{p_1}\right\rangle\right|^2}$$

$$\left\langle\Psi,\tilde{c}^*_{-p+k}\tilde{c}_{-p+k}\Psi\right\rangle$$

$$\le Ck_F^{-1}\sum_{p\in B_F}\sum_{k\in\mathbb{Z}^3_*}1_{L_k-k}(p)\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2\frac{1_{L_l}(p+k)}{\lambda_{l,p+k}}}\|h_k^{-\frac{1}{2}}B_ke_{p+k}\|\left\langle\Psi,\tilde{c}^*_{-p}\tilde{c}_{-p}\Psi\right\rangle$$

$$\le Ck_F^{-1}\sum_{p\in B_F}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2\sum_{k\in\mathbb{Z}^3_*}\frac{1_{L_l}(p+k)}{\lambda_{l,p+k}}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|h_k^{-\frac{1}{2}}B_ke_{p+k}\|^2}\left\langle\Psi,\tilde{c}^*_{-p}\tilde{c}_{-p}\Psi\right\rangle$$

$$\le Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle. \tag{4.61}$$

Finally, from (4.51) and (4.54) we see that $\left[b_l\left(K_le_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\left[b_k\left(B_ke_{p_1}\right),\tilde{c}^*_{p_2}\right]$ is equal to

$$1_{L_k}(p_2+k)1_{L_l}(p_3+l)\left\langle B_ke_{p_1},e_{p_2+k}\right\rangle\left\langle e_{p_3+l},K_le_{p_4}\right\rangle\tilde{c}^*_{p_3+l}\tilde{c}_{p_2+k}, \tag{4.62}$$

so we estimate

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}\left|\left\langle\Psi,\left[b_l\left(K_le_{p_4}\right),\tilde{c}^*_{p_3}\right]^*\left[b_k\left(B_ke_{p_1}\right),\tilde{c}^*_{p_2}\right]\Psi\right\rangle\right|\\
&\leq C\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\frac{\hat{V}_lk_F^{-1}1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}\left|\left\langle B_ke_{p_1},e_{p_2+k}\right\rangle\right|\left\langle\Psi,\tilde{c}^*_{p_3+l}\tilde{c}_{p_2+k}\Psi\right\rangle\\
&\leq Ck_F^{-1}\sum_{p\in B_F^c}\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k\cap L_l}(p)1_{L_k\cap L_l}(-p+k+l)\frac{\hat{V}_l}{\sqrt{\lambda_{l,p}}}\left|\left\langle e_p,h_k^{-\frac{1}{2}}B_ke_{-p+k+l}\right\rangle\right|\\
&\quad\cdot\left\langle\Psi,\tilde{c}^*_{-p+k+l}\tilde{c}_{-p+k+l}\Psi\right\rangle\\
&=Ck_F^{-1}\sum_{p\in B_F^c}\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k\cap L_l}(p+k+l)1_{L_k\cap L_l}(-p)\frac{\hat{V}_l}{\sqrt{\lambda_{l,p+k+l}}}\\
&\quad\left|\left\langle e_{p+k+l},h_k^{-\frac{1}{2}}B_ke_{-p}\right\rangle\right|\left\langle\Psi,\tilde{c}^*_{-p}\tilde{c}_{-p}\Psi\right\rangle\\
&\leq Ck_F^{-1}\sum_{p\in B_F^c}\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k}(p+k+l)1_{L_k}(-p)\left|\left\langle e_{p+k+l},h_k^{-\frac{1}{2}}B_ke_{-p}\right\rangle\right|^2}\\
&\quad\cdot\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_l^2\frac{1_{L_l}(p+k+l)}{\lambda_{l,p+k+l}}}\left\langle\Psi,\tilde{c}^*_{-p}\tilde{c}_{-p}\Psi\right\rangle\\
&\leq Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle.
\end{aligned}\tag{4.63}$$

□

The $\mathcal{E}_k^2$ bound of Theorem 4.1 now follows:

Proposition 4.12. *For any* $\Psi\in\mathcal{H}_N$ *and* $t\in[0,1]$ *it holds that*

$$\sum_{k\in\mathbb{Z}^3_*}\left|\left\langle\Psi,\left(\mathcal{E}_k^2(B_k(t))-\left\langle\psi_{\mathrm{FS}},\mathcal{E}_k^2(B_k(t))\psi_{\mathrm{FS}}\right\rangle\right)\Psi\right\rangle\right|\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\left\langle\Psi,\mathcal{N}_E^3\Psi\right\rangle$$

for a constant $C>0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$.

Proof. Clearly

$$\max_{p\in L_k}\left\|B_ke_p\right\|^2\leq\sum_{p\in L_k}\max_{q\in L_k}\left|\left\langle e_p,B_ke_q\right\rangle\right|^2,\quad\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2\leq\|B_kh_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}},\tag{4.64}$$

for any B_k, and as our estimate for $B_k(t)$ in Theorem 3.1 is the same as that for $A_k(t)$, the bounds

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}\left|\left\langle e_p,B_ke_q\right\rangle\right|^2,\ k_F^{-1}\sum_{k\in\mathbb{Z}^3_*}\|B_kh_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}}\leq C\left(1+\|\hat{V}\|^4_\infty\right)\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}$$

follow exactly as those of Proposition 4.7. Insertion into the Propositions 4.8, 4.10 and 4.11 yields the claim. □

4.3. Analysis of the exchange contribution. Finally we determine the leading order of the exchange contribution. To begin we derive a general formula for a quantity of the form $\langle \psi_{\mathrm{FS}}, \mathcal{E}_k^2(B_k)\psi_{\mathrm{FS}}\rangle$: We can write

$$\begin{aligned}
&-2\left\langle \psi_{\mathrm{FS}}, \mathcal{E}_k^2(B_k)\psi_{\mathrm{FS}}\right\rangle \\
&\quad= -\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l}\langle \psi_{\mathrm{FS}}, b_k(B_k e_p)\varepsilon_{-k,-l}(e_{-p}; e_{-q}) b_l^*(K_l e_q)\psi_{\mathrm{FS}}\rangle \\
&\quad= \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left\langle \psi_{\mathrm{FS}}, b_k(B_k e_p)\tilde{c}^*_{-p+l}\tilde{c}_{-p+k} b_l^*\left(K_l e_p\right)\psi_{\mathrm{FS}}\right\rangle \\
&\qquad+\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)\cap(L_l-l)}\left\langle \psi_{\mathrm{FS}}, b_k\left(B_k e_{p+k}\right)\tilde{c}^*_{-p-l}\tilde{c}_{-p-k} b_l^*\left(K_l e_{p+l}\right)\psi_{\mathrm{FS}}\right\rangle \\
&\quad=: A + B
\end{aligned} \tag{4.65}$$

where, using equation (4.22) in the form

$$\left[b_l(\psi), \tilde{c}_p^*\right] = \begin{cases} -\sum_{q\in L_l}\delta_{p,q-l}\left\langle \psi, e_q\right\rangle \tilde{c}_q & p\in B_F \\ \sum_{q\in(L_l-l)}\delta_{p,q+l}\left\langle \psi, e_{q+l}\right\rangle \tilde{c}_q & p\in B_F^c \end{cases}, \tag{4.66}$$

the terms A and B are given by

$$\begin{aligned}
A &= \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left\langle \psi_{\mathrm{FS}}, \left[b_k(B_k e_p), \tilde{c}^*_{-p+l}\right]\left[b_l\left(K_l e_p\right), \tilde{c}^*_{-p+k}\right]^*\psi_{\mathrm{FS}}\right\rangle \\
&= \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left\langle \psi_{\mathrm{FS}}, \left(\sum_{q\in L_k}\delta_{-p+l,q-k}\left\langle B_k e_p, e_q\right\rangle \tilde{c}_q\right)\right. \\
&\qquad\left.\left(\sum_{q'\in L_l}\delta_{-p+k,q'-l}\left\langle e_{q'}, K_l e_p\right\rangle \tilde{c}^*_{q'}\right)\psi_{\mathrm{FS}}\right\rangle \\
&= \sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\left\langle e_p, B_k e_q\right\rangle\left\langle e_q, K_l e_p\right\rangle
\end{aligned} \tag{4.67}$$

and similarly

$$\begin{aligned}
B &= \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)\cap(L_l-l)}\left\langle \psi_{\mathrm{FS}}, \left[b_k\left(B_k e_{p+k}\right), \tilde{c}^*_{-p-l}\right]\left[b_l\left(K_l e_{p+l}\right), \tilde{c}^*_{-p-k}\right]^*\psi_{\mathrm{FS}}\right\rangle \\
&= \sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in(L_k-k)\cap(L_l-l)}\delta_{-p-q,k+l}\left\langle e_{p+k}, B_k e_{q+k}\right\rangle\left\langle e_{q+l}, K_l e_{p+l}\right\rangle.
\end{aligned} \tag{4.68}$$

Although non-obvious, there holds the identity $A = B$. To see this we rewrite both terms: First, for A, we note that the presence of the $\delta_{p+q,k+l}$ makes the L_l of the summation $p, q \in L_k \cap L_l$ redundant: For any $p, q \in B_F^c$ there holds the equivalence

$$p, q \in L_{p+q-k} \iff p, q \in L_k \tag{4.69}$$

by the trivial identities

$$|p-k| = |q-(p+q-k)|, \quad |q-k| = |p-(p+q-k)|, \tag{4.70}$$

so A can be written as

$$A = \sum_{p,q\in L_k}\sum_{l\in\mathbb{Z}^3_*}\delta_{p+q,k+l}\left\langle e_p, B_k e_q\right\rangle\left\langle e_q, K_l e_p\right\rangle = \sum_{p,q\in L_k}\left\langle e_p, B_k e_q\right\rangle\left\langle e_q, K_{p+q-k} e_p\right\rangle. \tag{4.71}$$

A similar observation applies to B: For any $p, q \in B_F$ we likewise have

$$p, q \in \left(L_{-p-q-k} + p + q + k\right) \iff p+k, q+k \in L_{p+q+k} \iff p, q \in (L_k - k) \tag{4.72}$$

so

$$\begin{aligned} B &= \sum_{p,q\in(L_k-k)}\sum_{l\in\mathbb{Z}^3_*}\delta_{-p-q,k+l}\left\langle e_{p+k}, B_k e_{q+k}\right\rangle\left\langle e_{q+l}, K_l e_{p+l}\right\rangle \\ &= \sum_{p,q\in(L_k-k)}\left\langle e_{p+k}, B_k e_{q+k}\right\rangle\left\langle e_{-p-k}, K_{-p-q-k} e_{-q-k}\right\rangle \\ &= \sum_{p,q\in L_k}\left\langle e_p, B_k e_q\right\rangle\left\langle e_q, K_{p+q-k} e_p\right\rangle \end{aligned} \tag{4.73}$$

where we lastly used that the kernels K_k obey

$$\left\langle e_{-p}, K_{-k} e_{-q}\right\rangle = \left\langle e_p, K_k e_q\right\rangle = \left\langle e_q, K_k e_p\right\rangle, \quad k \in \mathbb{Z}^3_*,\ p, q \in L_k. \tag{4.74}$$

In all we thus have the identity

$$\begin{aligned} \left\langle \psi_{\mathrm{FS}}, \mathcal{E}^2_k(B_k)\psi_{\mathrm{FS}}\right\rangle &= -\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\left\langle e_p, B_k e_q\right\rangle\left\langle e_q, K_l e_p\right\rangle \\ &= -\sum_{p,q\in L_k}\left\langle e_p, B_k e_q\right\rangle\left\langle e_q, K_{p+q-k} e_p\right\rangle. \end{aligned} \tag{4.75}$$

Our matrix element estimates of the last section now yield the following:

Proposition. (4.2) *It holds that*

$$\left|\sum_{k\in\mathbb{Z}^3_*}\int_0^1\left\langle \psi_{\mathrm{FS}}, 2\,\mathrm{Re}\left(\mathcal{E}^2_k(B_k(t))\right)\psi_{\mathrm{FS}}\right\rangle dt - E_{\mathrm{corr,ex}}\right| \le C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|, k_F\}}$$

for a constant $C > 0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$, *where*

$$E_{\mathrm{corr,ex}} = \frac{k_F^{-2}}{4(2\pi)^6}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\frac{\hat{V}_k\hat{V}_{p+q-k}}{\lambda_{k,p}+\lambda_{k,q}}.$$

Proof. Since all the one-body operators are real-valued we can drop the Re $(\cdot)$ and apply the above identity for

$$\sum_{k\in\mathbb{Z}^3_*}\int_0^1\left\langle\psi_{\mathrm{FS}},2\,\mathrm{Re}\left(\mathcal{E}^2_k(B_k(t))\right)\psi_{\mathrm{FS}}\right\rangle dt=\sum_{k\in\mathbb{Z}^3_*}2\left\langle\psi_{\mathrm{FS}},\mathcal{E}^2_k\left(\int_0^1 B_k(t)dt\right)\psi_{\mathrm{FS}}\right\rangle$$
$$=2\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\left\langle e_p,\left(\int_0^1 B_k(t)dt\right)e_q\right\rangle\left\langle e_q,(-K_l)\,e_p\right\rangle. \tag{4.76}$$

Now, note that $E_{\mathrm{corr,ex}}$ can be written as

$$E_{\mathrm{corr,ex}}=\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\frac{\hat{V}_l k_F^{-1}}{2\,(2\pi)^3}\frac{1}{\lambda_{l,p}+\lambda_{l,q}} \tag{4.77}$$

since, much as in Proposition 4.5, the $\delta_{p+q,k+l}$ implies the following identity for the denominators:

$$\lambda_{l,p}+\lambda_{l,q}=\frac{1}{2}\left(|p|^2-|p-l|^2\right)+\frac{1}{2}\left(|q|^2-|q-l|^2\right)$$
$$=\frac{1}{2}\left(|p|^2-|q-k|^2\right)+\frac{1}{2}\left(|q|^2-|p-k|^2\right)=\lambda_{k,p}+\lambda_{k,q}. \tag{4.78}$$

In conclusion we thus see that

$$\sum_{k\in\mathbb{Z}^3_*}\int_0^1\left\langle\psi_{\mathrm{FS}},2\,\mathrm{Re}\left(\mathcal{E}^2_k(B_k(t))\right)\psi_{\mathrm{FS}}\right\rangle dt-E_{\mathrm{corr,ex}}$$
$$=2\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\left(\left\langle e_p,\left(\int_0^1 B_k(t)dt\right)e_q\right\rangle-\frac{\hat{V}_k k_F^{-1}}{4\,(2\pi)^3}\right)\left\langle e_q,(-K_l)\,e_p\right\rangle$$
$$+\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\left(\left\langle e_q,(-K_l)\,e_p\right\rangle-\frac{\hat{V}_l k_F^{-1}}{2\,(2\pi)^3}\frac{1}{\lambda_{l,p}+\lambda_{l,q}}\right)=:A+B. \tag{4.79}$$

We estimate A and B. By the matrix element estimates of Theorem 3.1 we have that (using our freedom to replace $\lambda_{l,p}+\lambda_{l,q}$ by $\lambda_{k,p}+\lambda_{k,q}$)

$$|A|\le C\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\left(1+\hat{V}_k\right)\hat{V}_k^2 k_F^{-1}\frac{\hat{V}_l k_F^{-1}}{\lambda_{l,p}+\lambda_{l,q}}$$
$$\le Ck_F^{-2}\left(1+\|\hat{V}\|_\infty\right)\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\sum_{p\in L_k}\frac{1}{\sqrt{\lambda_{k,p}}}\sum_{q\in L_k}\frac{\hat{V}_{p+q-k}}{\sqrt{\lambda_{k,q}}}$$
$$\le Ck_F^{-\frac{3}{2}}\left(1+\|\hat{V}\|_\infty\right)\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\sum_{p\in L_k}\frac{1}{\sqrt{\lambda_{k,p}}}$$
$$\le C\left(1+\|\hat{V}\|_\infty\right)\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\,|k|^{\frac{1}{2}}\min\{1,k_F^{\frac{3}{2}}|k|^{-\frac{3}{2}}\} \tag{4.80}$$

where we applied the inequality $\sum_{q\in L_k}\lambda_{k,q}^{-1}\leq Ck_F$ and also used that Proposition 3.11 implies that

$$\sum_{p\in L_k}\frac{1}{\sqrt{\lambda_{k,p}}}\leq Ck_F^{\frac{3}{2}}|k|^{\frac{1}{2}}\min\{1,k_F^{\frac{3}{2}}|k|^{-\frac{3}{2}}\} \tag{4.81}$$

for a $C>0$ independent of all quantities. By Cauchy-Schwarz we can further estimate

$$\begin{aligned}\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2|k|^{\frac{1}{2}}\min\{1,k_F^{\frac{3}{2}}|k|^{-\frac{3}{2}}\}&\leq\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2|k|\min\{1,k_F^3|k|^{-3}\}}\\&\leq\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\end{aligned} \tag{4.82}$$

for the bound of the statement. By similar estimation also

$$\begin{aligned}|B|&\leq C\sum_{k,l\in\mathbb{Z}_*^3}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\hat{V}_kk_F^{-1}\frac{\hat{V}_l^2k_F^{-1}}{\lambda_{l,p}+\lambda_{l,q}}\\&\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\sum_{l\in\mathbb{Z}_*^3}\hat{V}_l^2|l|^{\frac{1}{2}}\min\{1,k_F^{\frac{3}{2}}|l|^{-\frac{3}{2}}\}\end{aligned}$$

and the claim follows likewise. □

5. Estimation of the Non-Bosonizable Terms and Gronwall Estimates

In this section we perform the final work which will allow us to conclude Theorem 1.1.

The main content of this section lies in the estimation of the *non-bosonizable terms*, by which we mean the cubic and quartic terms

$$\begin{aligned}\mathcal{C}&=\frac{k_F^{-1}}{(2\pi)^3}\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k\,\mathrm{Re}\left(\left(B_k+B_{-k}^*\right)^*D_k\right),\\\mathcal{Q}&=\frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k\left(D_k^*D_k-\sum_{p\in L_k}\left(c_p^*c_p+c_{p-k}c_{p-k}^*\right)\right).\end{aligned} \tag{5.1}$$

The cubic terms $\mathcal{C}$ will not present a big obstacle to us: As was first noted in [2] (in their formulation), the expectation value of these in fact vanish identically with respect to the type of trial state we will consider. The bulk of the work will thus be to estimate the quartic terms. We prove the following bounds:

Theorem 5.1. *It holds that* $\mathcal{Q}=G+\mathcal{Q}_{\mathrm{LR}}+\mathcal{Q}_{\mathrm{SR}}$ *where for any* $\Psi\in\mathcal{H}_N$

$$|\langle\Psi,G\Psi\rangle|\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\,\langle\Psi,\mathcal{N}_E\Psi\rangle$$

$$|\langle\Psi,\mathcal{Q}_{\mathrm{LR}}\Psi\rangle|\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\left\langle\Psi,\mathcal{N}_E^2\Psi\right\rangle$$

and $e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}} + \int_0^1 e^{t\mathcal{K}}\left(2\,\mathrm{Re}\left(\mathcal{G}\right)\right)e^{-t\mathcal{K}}dt$ *for an operator* $\mathcal{G}$ *obeying*

$$|\langle\Psi, \mathcal{G}\Psi\rangle| \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\left\{|k|, k_F\right\}}\left\langle\Psi, \left(\mathcal{N}_E^3+1\right)\Psi\right\rangle,$$

$C > 0$ *being a constant independent of all quantities.*

With these all the general bounds are established. As all our error estimates are with respect to $\mathcal{N}_E$ and powers thereof, it then only remains to control the effect which the transformation $e^{\mathcal{K}}$ has on these. By a standard Gronwall-type argument this control will follow from the estimate of Proposition 2.4, and we then end the paper by concluding Theorem 1.1.

Analysis of the cubic terms. Expanding the $\mathrm{Re}\left(\cdot\right)$, the cubic terms are

$$\mathcal{C} = \frac{k_F^{-1}}{2\left(2\pi\right)^3}\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k\left(\left(B_k^*+B_{-k}\right)D_k + D_k^*\left(B_k+B_{-k}^*\right)\right). \tag{5.2}$$

The operators B_k can be written simply as $B_k = \sum_{p\in L_k} b_{k,p}$ in terms of the excitation operators $b_{k,p} = c_{p-k}^* c_p$, whence it is easily seen that

$$\left[\mathcal{N}_E, B_k\right] = -B_k, \quad \left[\mathcal{N}_E, B_k^*\right] = B_k^*. \tag{5.3}$$

As a consequence, B_k maps the eigenspace $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M-1\}$ and B_k^* maps $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M+1\}$. Meanwhile, the operators D_k preserve the eigenspaces: Writing $D_k = D_{1,k} + D_{2,k}$ for

$$\begin{aligned} D_{1,k} &= \mathrm{d}\Gamma\left(P_{B_F}e^{-ik\cdot x}P_{B_F}\right) = \sum_{p,q\in B_F}\delta_{p,q-k}c_p^*c_q = -\sum_{q\in B_F\cap(B_F+k)}\tilde{c}_q^*\tilde{c}_{q-k} \\ D_{2,k} &= \mathrm{d}\Gamma\left(P_{B_F^c}e^{-ik\cdot x}P_{B_F^c}\right) = \sum_{p,q\in B_F^c}\delta_{p,q-k}c_p^*c_q = \sum_{p\in B_F^c\cap\left(B_F^c-k\right)}\tilde{c}_p^*\tilde{c}_{p+k} \end{aligned} \tag{5.4}$$

these annihilate and create one hole or excitation, respectively, whence $\left[\mathcal{N}_E, D_k\right] = 0 = \left[\mathcal{N}_E, D_k^*\right]$.

It follows that $\mathcal{C}$ maps the eigenspace $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M-1\}\oplus\{\mathcal{N}_E = M+1\}$. Decomposing $\mathcal{H}_N$ orthogonally as $\mathcal{H}_N = \mathcal{H}_N^{\mathrm{even}}\oplus\mathcal{H}_N^{\mathrm{odd}}$ for

$$\mathcal{H}_N^{\mathrm{even}} = \bigoplus_{m=0}^{\infty}\{\mathcal{N}_E = 2m\}, \quad \mathcal{H}_N^{\mathrm{odd}} = \bigoplus_{m=0}^{\infty}\{\mathcal{N}_E = 2m+1\}, \tag{5.5}$$

we thus see that $\mathcal{C}$ maps each subspace into the other. On the other hand, since our transformation kernel $\mathcal{K}$ is of the form

$$\mathcal{K} = \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p,q\in L_l}\left\langle e_p, K_l e_q\right\rangle\left(b_{l,p}b_{-l,-q} - b_{-l,-q}^*b_{l,p}^*\right) \tag{5.6}$$

we note that $\mathcal{K}$ maps each $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M-2\}\oplus\{\mathcal{N}_E = M+2\}$, hence $\mathcal{K}$ preserves $\mathcal{H}_N^{\mathrm{even}}$ and $\mathcal{H}_N^{\mathrm{odd}}$, and so too does the transformation $e^{-\mathcal{K}}$. As any eigenstate $\Psi \in \mathcal{H}_N$ of $\mathcal{N}_E$ is contained in either $\mathcal{H}_N^{\mathrm{even}}$ or $\mathcal{H}_N^{\mathrm{odd}}$, and these are orthogonal, we conclude the following:

Proposition 5.2. *For any eigenstate Ψ of $\mathcal{N}_E$ it holds that*

$$\left\langle e^{-\mathcal{K}}\Psi, \mathcal{C}e^{-\mathcal{K}}\Psi\right\rangle = 0.$$

5.1. Analysis of the quartic terms. Now we consider the quartic terms

$$\mathcal{Q} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\mathbb{Z}_*^3} \hat{V}_k \left(D_k^* D_k - \sum_{p\in L_k} \left(c_p^* c_p + c_{p-k} c_{p-k}^* \right) \right). \tag{5.7}$$

We begin by rewriting these: Recalling the decomposition $D_k = D_{1,k} + D_{2,k}$ above, we calculate

$$\begin{aligned} D_{1,k}^* D_{1,k} &= \sum_{p,q\in B_F\cap(B_F+k)} \tilde{c}_{p-k}^* \tilde{c}_p \tilde{c}_q^* \tilde{c}_{q-k} = \sum_{p,q\in B_F\cap(B_F+k)} \tilde{c}_{p-k}^* \tilde{c}_q^* \tilde{c}_{q-k} \tilde{c}_p \\ &\quad + \sum_{q\in B_F\cap(B_F+k)} \tilde{c}_{q-k}^* \tilde{c}_{q-k} \\ &= \sum_{p,q\in B_F\cap(B_F+k)} \tilde{c}_{p-k}^* \tilde{c}_q^* \tilde{c}_{q-k} \tilde{c}_p + \sum_{p\in B_F} 1_{B_F}(q+k) \tilde{c}_q^* \tilde{c}_q \end{aligned} \tag{5.8}$$

and similarly

$$\begin{aligned} D_{2,k}^* D_{2,k} &= \sum_{p,q\in B_F^c\cap(B_F^c-k)} \tilde{c}_{p+k}^* \tilde{c}_p \tilde{c}_q^* \tilde{c}_{q+k} = \sum_{p,q\in B_F^c\cap(B_F^c-k)} \tilde{c}_{p+k}^* \tilde{c}_q^* \tilde{c}_{q+k} \tilde{c}_p \\ &\quad + \sum_{p\in B_F^c} 1_{B_F^c}(p-k) \tilde{c}_p^* \tilde{c}_p \\ &= \sum_{p,q\in B_F^c\cap(B_F^c-k)} \tilde{c}_{p+k}^* \tilde{c}_q^* \tilde{c}_{q+k} \tilde{c}_p + \mathcal{N}_E - \sum_{p\in B_F^c} 1_{B_F}(p-k) \tilde{c}_p^* \tilde{c}_p. \end{aligned} \tag{5.9}$$

For any $k \in \mathbb{Z}_*^3$ we can likewise write $\sum_{p\in L_k} \left(c_p^* c_p + c_{p-k} c_{p-k}^* \right)$ in the form

$$\begin{aligned} \sum_{p\in L_k} \left(c_p^* c_p + c_{p-k} c_{p-k}^* \right) &= \sum_{p\in B_F^c} 1_{B_F}(p-k) \tilde{c}_p^* \tilde{c}_p + \sum_{q\in B_F} 1_{B_F^c}(q+k) \tilde{c}_q^* \tilde{c}_q \\ &= \sum_{p\in B_F^c} 1_{B_F}(p-k) \tilde{c}_p^* \tilde{c}_p + \mathcal{N}_E - \sum_{q\in B_F} 1_{B_F}(q+k) \tilde{c}_q^* \tilde{c}_q. \end{aligned} \tag{5.10}$$

Noting that $D_{1,k} = 0$ for $|k| > 2k_F$, as then $B_F \cap (B_F + k) = \emptyset$, we thus obtain the decomposition

$$\mathcal{Q} = G + \mathcal{Q}_{\mathrm{LR}} + \mathcal{Q}_{\mathrm{SR}} \tag{5.11}$$

where G is the one-body operator

$$G = \frac{k_F^{-1}}{(2\pi)^3} \sum_{k\in\mathbb{Z}_*^3} \hat{V}_k \left(\sum_{q\in B_F} 1_{B_F}(q+k) \tilde{c}_q^* \tilde{c}_q - \sum_{p\in B_F^c} 1_{B_F}(p-k) \tilde{c}_p^* \tilde{c}_p \right), \tag{5.12}$$

the *long-range terms* $\mathcal{Q}_{\mathrm{LR}}$ are given by

$$\mathcal{Q}_{\mathrm{LR}}=\frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\overline{B}(0,2k_F)\cap\mathbb{Z}^3_*}\hat{V}_k\left(\sum_{p,q\in B_F\cap(B_F+k)}\tilde{c}^*_{p-k}\tilde{c}^*_q\tilde{c}_{q-k}\tilde{c}_p+D^*_{1,k}D_{2,k}+D^*_{2,k}D_{1,k}\right) \tag{5.13}$$

and the *short-range terms* $\mathcal{Q}_{\mathrm{SR}}$ are

$$\mathcal{Q}_{\mathrm{SR}}=\frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap(B_F^c-k)}\tilde{c}^*_{p+k}\tilde{c}^*_q\tilde{c}_{q+k}\tilde{c}_p. \tag{5.14}$$

Estimation of G and $\mathcal{Q}_{\mathrm{LR}}$ G and the long-range terms are easily controlled: First, interchanging the summations we can write G as

$$G=\frac{k_F^{-1}}{(2\pi)^3}\sum_{q\in B_F}\left(\sum_{k\in(B_F-q)\cap\mathbb{Z}^3_*}\hat{V}_k\right)\tilde{c}^*_q\tilde{c}_q-\frac{k_F^{-1}}{(2\pi)^3}\sum_{p\in B_F^c}\left(\sum_{k\in(B_F+p)\cap\mathbb{Z}^3_*}\hat{V}_k\right)\tilde{c}^*_p\tilde{c}_p \tag{5.15}$$

from which it is obvious that G obeys

$$\pm G\le\max_{p\in\mathbb{Z}^3_*}\left(\frac{k_F^{-1}}{(2\pi)^3}\sum_{k\in(B_F+p)\cap\mathbb{Z}^3_*}\hat{V}_k\right)\mathcal{N}_E. \tag{5.16}$$

This implies the following:

Proposition 5.3. *For any* $\Psi\in\mathcal{H}_N$ *it holds that*

$$|\langle\Psi,G\Psi\rangle|\le C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\,\langle\Psi,\mathcal{N}_E\Psi\rangle$$

for a constant $C>0$ *independent of all quantities.*

Proof. For any $p\in\mathbb{Z}^3$ we estimate by Cauchy-Schwarz

$$\begin{aligned}\sum_{k\in(B_F+p)\cap\mathbb{Z}^3_*}\hat{V}_k&\le\sqrt{\sum_{k\in(B_F+p)\cap\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\sqrt{\sum_{k\in(B_F+p)\cap\mathbb{Z}^3_*}\min\{|k|,k_F\}^{-1}}\\&\le\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\sqrt{\sum_{k\in B_F\setminus\{0\}}|k|^{-1}+k_F^{-1}}\end{aligned} \tag{5.17}$$

where we lastly used that $k\mapsto\min\{|k|,k_F\}^{-1}$ is radially decreasing and that $(B_F+p)\cap\mathbb{Z}^3_*$ contains at most $|B_F|$ points. As it is well-known that $\sum_{k\in\overline{B}(0,R)\setminus\{0\}}|k|^{-1}\le CR^2$ as $R\to\infty$ the bound follows. □

$\mathcal{Q}_{\mathrm{LR}}$ can be handled in a similar manner:

Proposition 5.4. *For any* $\Psi \in \mathcal{H}_N$ *it holds that*

$$|\langle \Psi, \mathcal{Q}_{\mathrm{LR}} \Psi \rangle| \leq C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle$$

for a constant $C > 0$ *independent of all quantities.*

Proof. Consider the first term in the parenthesis of (5.13): For any $k \in \mathbb{Z}^3_*$ we can estimate

$$\sum_{p,q \in B_F \cap (B_F + k)} \left| \left\langle \Psi, \tilde{c}^*_{p-k} \tilde{c}^*_q \tilde{c}_{q-k} \tilde{c}_p \Psi \right\rangle \right| \leq \sum_{p,q \in B_F \cap (B_F + k)} \left\| \tilde{c}_q \tilde{c}_{p-k} \Psi \right\| \left\| \tilde{c}_{q-k} \tilde{c}_p \Psi \right\|$$
$$\leq \sqrt{\sum_{p,q \in B_F \cap (B_F + k)} \left\| \tilde{c}_q \tilde{c}_{p-k} \Psi \right\|^2} \sqrt{\sum_{p,q \in B_F \cap (B_F + k)} \left\| \tilde{c}_{q-k} \tilde{c}_p \Psi \right\|^2} \leq \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle. \quad (5.18)$$

As e.g.

$$D^*_{1,k} D_{2,k} = \sum_{p \in B_F^c \cap (B_F^c - k)} \sum_{q \in B_F \cap (B_F + k)} \tilde{c}^*_{q-k} \tilde{c}_q \tilde{c}^*_p \tilde{c}_{p+k}$$
$$= \sum_{p \in B_F^c \cap (B_F^c - k)} \sum_{q \in B_F \cap (B_F + k)} \tilde{c}^*_p \tilde{c}^*_{q-k} \tilde{c}_q \tilde{c}_{p+k}$$

the terms $D^*_{1,k} D_{2,k}$ and $D^*_{2,k} D_{1,k}$ can be handled similarly, whence

$$|\langle \Psi, \mathcal{Q}_{\mathrm{LR}} \Psi \rangle| \leq \frac{3 k_F^{-1}}{2 (2\pi)^3} \left(\sum_{k \in \overline{B}(0, 2k_F) \cap \mathbb{Z}^3_*} \hat{V}_k \right) \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle$$
$$\leq C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle$$

where $\sum_{k \in \overline{B}(0, 2k_F) \cap \mathbb{Z}^3_*} \hat{V}_k$ was bounded as in equation (5.17). □

Analysis of $\mathcal{Q}_{\mathrm{SR}}$ Lastly we come to

$$\mathcal{Q}_{\mathrm{SR}} = \frac{k_F^{-1}}{2 (2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \sum_{p,q \in B_F^c \cap (B_F^c - k)} \tilde{c}^*_{p+k} \tilde{c}^*_q \tilde{c}_{q+k} \tilde{c}_p. \quad (5.19)$$

Recall that the transformation $\mathcal{K}$ can be written as $\mathcal{K} = \tilde{\mathcal{K}} - \tilde{\mathcal{K}}^*$ for

$$\tilde{\mathcal{K}} = \frac{1}{2} \sum_{l \in \mathbb{Z}^3_*} \sum_{p,q \in L_l} \left\langle e_p, K_l e_q \right\rangle b_{l,p} b_{-l,-q} = \frac{1}{2} \sum_{l \in \mathbb{Z}^3_*} \sum_{q \in L_l} b_l(K_l e_q) b_{-l,-q}. \quad (5.20)$$

To determine $e^{\mathcal{K}} \mathcal{Q}_{\mathrm{SR}} e^{-\mathcal{K}}$ we will need the commutator $[\mathcal{K}, \mathcal{Q}_{\mathrm{SR}}] = 2\,\mathrm{Re}\left(\left[\tilde{\mathcal{K}}, \mathcal{Q}_{\mathrm{SR}} \right] \right)$. Noting that for any $p \in B_F^c$ and $l \in \mathbb{Z}^3_*$, $q \in L_l$, we have

$$\left[b_{l,q}, \tilde{c}^*_p \right] = \left[c^*_{q-l} c_q, c^*_p \right] = \delta_{p,q} c^*_{q-l} = \delta_{p,q} \tilde{c}_{q-l}, \quad (5.21)$$

we deduce (with the help of Lemma A.1) that

$$\begin{aligned}\left[\tilde{\mathcal{K}}, \tilde{c}_p^*\right] &= \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{q\in L_l}\left(b_l(K_l e_q)\left[b_{-l,-q}, \tilde{c}_p^*\right] + \left[b_l(K_l e_q), \tilde{c}_p^*\right] b_{-l,-q}\right)\\ &= \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{q\in L_l}\left(b_l(K_l e_q)\left[b_{-l,-q}, \tilde{c}_p^*\right] + \left[b_{l,q}, \tilde{c}_p^*\right] b_{-l}(K_{-l}e_{-q})\right)\\ &= \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{q\in L_l}\left(b_l(K_l e_q)\delta_{p,-q}\tilde{c}_{-q+l} + \delta_{p,q}\tilde{c}_{q-l} b_{-l}(K_{-l}e_{-q})\right)\\ &= \sum_{l\in\mathbb{Z}_*^3}\sum_{q\in L_l}\delta_{p,-q} b_l(K_l e_q)\tilde{c}_{-q+l} = \sum_{l\in\mathbb{Z}_*^3} 1_{L_l}(-p) b_l\left(K_l e_{-p}\right)\tilde{c}_{p+l}. \end{aligned} \tag{5.22}$$

Using this we conclude the following:

Proposition 5.5. *It holds that* $e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}} + \int_0^1 e^{t\mathcal{K}}\left(2\,\mathrm{Re}\,(\mathcal{G})\right)e^{-t\mathcal{K}}dt$ *for*

$$\begin{aligned}\mathcal{G} = &\frac{k_F^{-1}}{(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p,q\in B_F^c\cap(B_F^c+k)} 1_{L_l}(q)\tilde{c}_p^* b_l(K_l e_q)\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}\\ &+ \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p,q\in B_F^c\cap(B_F^c+k)} 1_{L_l}(p)1_{L_l}(q)\left\langle K_l e_q, e_p\right\rangle \tilde{c}_{p-l}\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}.\end{aligned}$$

Proof. By the fundamental theorem of calculus

$$e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}} + \int_0^1 e^{t\mathcal{K}}\left[\mathcal{K}, \mathcal{Q}_{\mathrm{SR}}\right]e^{-t\mathcal{K}}dt \tag{5.23}$$

and as noted $[\mathcal{K}, \mathcal{Q}_{\mathrm{SR}}] = 2\,\mathrm{Re}\left(\left[\tilde{\mathcal{K}}, \mathcal{Q}_{\mathrm{SR}}\right]\right)$. Using equation (5.22) we compute that $\mathcal{G} := \left[\tilde{\mathcal{K}}, \mathcal{Q}_{\mathrm{SR}}\right]$ is given by

$$\begin{aligned}\mathcal{G} &= \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p\in B_F^c\cap(B_F^c+k)}\sum_{q\in B_F^c\cap(B_F^c-k)}\left(\tilde{c}_p^*\left[\tilde{\mathcal{K}}, \tilde{c}_q^*\right] + \left[\tilde{\mathcal{K}}, \tilde{c}_p^*\right]\tilde{c}_q^*\right)\tilde{c}_{q+k}\tilde{c}_{p-k}\\ &= \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p\in B_F^c\cap(B_F^c+k)}\sum_{q\in B_F^c\cap(B_F^c-k)} 1_{L_l}(-q)\tilde{c}_p^* b_l\left(K_l e_{-q}\right)\tilde{c}_{q+l}\tilde{c}_{q+k}\tilde{c}_{p-k}\\ &\quad+ \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p\in B_F^c\cap(B_F^c+k)}\sum_{q\in B_F^c\cap(B_F^c-k)} 1_{L_l}(-p) b_l\left(K_l e_{-p}\right)\tilde{c}_{p+l}\tilde{c}_q^*\tilde{c}_{q+k}\tilde{c}_{p-k}\\ &= \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p\in B_F^c\cap(B_F^c+k)}\sum_{q\in B_F^c\cap(B_F^c-k)} 1_{L_l}(-q)\left\{b_l\left(K_l e_{-q}\right), \tilde{c}_p^*\right\}\tilde{c}_{q+l}\tilde{c}_{q+k}\tilde{c}_{p-k}\\ &= \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k \sum_{p,q\in B_F^c\cap(B_F^c+k)} 1_{L_l}(q)\left\{b_l(K_l e_q), \tilde{c}_p^*\right\}\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}, \end{aligned} \tag{5.24}$$

where we for the third inequality substituted $p \to q$ and $k \to -k$ in the second sum. By the identity of equation (4.22) the anti-commutator is given by

$$\left\{ b_l(K_l e_q), \tilde{c}_p^* \right\} = 2\, \tilde{c}_p^* b_l(K_l e_q) + 1_{L_l}(p) \left\langle K_l e_q, e_p \right\rangle \tilde{c}_{p-l} \tag{5.25}$$

which is inserted into the previous equation for the claim. □

We bound the $\mathcal{G}$ operator as follows:

Proposition 5.6. *For any $\Psi \in \mathcal{H}_N$ it holds that*

$$|\langle \Psi, \mathcal{G}\Psi \rangle| \le C \sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left\langle \Psi, \left(\mathcal{N}_E^{3} + 1 \right) \Psi \right\rangle$$

for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2$.

Proof. Using Proposition 4.4 we estimate the sum of the first term of $\mathcal{G}$ as

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap(B_F^c+k)} 1_{L_l}(q) \left| \left\langle \Psi, \tilde{c}_p^* b_l(K_l e_q) \tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}\Psi \right\rangle \right| \\
&\le \sum_{k,l\in\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap(B_F^c+k)} 1_{L_l}(q) \left\| b_l^*(K_l e_q)\tilde{c}_p \Psi \right\| \left\| \tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}\Psi \right\| \\
&\le \sum_{k,l\in\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap(B_F^c+k)} 1_{L_l}(q) \left\| K_l e_q \right\| \|\tilde{c}_p \left(\mathcal{N}_E + 1\right)^{\frac{1}{2}} \Psi\| \left\| \tilde{c}_{p-k}\tilde{c}_{-q+l}\tilde{c}_{-q+k}\Psi \right\| \\
&\le \|(\mathcal{N}_E + 1)\, \Psi\| \sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l} \left\| K_l e_q \right\| \sum_{k\in\mathbb{Z}^3_*} 1_{B_F^c+k}(q)\hat{V}_k \|\tilde{c}_{-q+k}\tilde{c}_{-q+l}\mathcal{N}_E^{\frac{1}{2}}\Psi\| \\
&\le \sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2}\, \|(\mathcal{N}_E + 1)\, \Psi\| \sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l} \left\| K_l e_q \right\| \left\| \tilde{c}_{-q+l}\mathcal{N}_E\Psi \right\| \\
&\le \sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2} \left(\sum_{l\in\mathbb{Z}^3_*} \|K_l\|_{\mathrm{HS}} \right) \|(\mathcal{N}_E + 1)\, \Psi\|\, \|\mathcal{N}_E^{\frac{3}{2}}\Psi\|.
\end{aligned} \tag{5.26}$$

Now, the $\|K_k\|_{\mathrm{HS}}$ estimate of Theorem 3.1 and Cauchy-Schwarz lets us estimate

$$\sum_{k\in\mathbb{Z}^3_*} \|K_k\|_{\mathrm{HS}} \le C \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \min\{1, k_F^2\, |k|^{-2}\} \le C \sqrt{\sum_{k\in\mathbb{Z}^3_*} \frac{\min\{1, k_F^4\, |k|^{-4}\}}{\min\{|k|, k_F\}}} \times \sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}},$$

and

$$\sum_{k\in\mathbb{Z}^3_*} \frac{\min\{1, k_F^4\, |k|^{-4}\}}{\min\{|k|, k_F\}} = \sum_{k\in B_F\setminus\{0\}} \frac{1}{|k|} + k_F^3 \sum_{k\in\mathbb{Z}^3_*\setminus B_F} \frac{1}{|k|^4} \le C k_F^2 \tag{5.27}$$

for a constant $C > 0$ independent of all quantities, so in all the first term of $\mathcal{G}$ obeys

$$\frac{k_F^{-1}}{2(2\pi)^3}\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k\sum_{p,q\in B_F^c\cap(B_F^c+k)}1_{L_l}(q)\left|\left\langle\Psi,\tilde{c}_p^*b_l(K_le_q)\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}\Psi\right\rangle\right|$$
$$\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\,\|(\mathcal{N}_E+1)\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|. \tag{5.28}$$

Similarly, for the second term (using simply that $\left\|\tilde{c}_{p-l}\right\|_{\mathrm{Op}}=1$ at the beginning)

$$\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k\sum_{p,q\in B_F^c\cap(B_F^c+k)}1_{L_l}(p)1_{L_l}(q)\left|\left\langle K_le_q,e_p\right\rangle\left\langle\Psi,\tilde{c}_{p-l}\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}\Psi\right\rangle\right|$$
$$\leq\|\Psi\|\sum_{k,l\in\mathbb{Z}_*^3}\hat{V}_k\sum_{p,q\in B_F^c\cap(B_F^c+k)}1_{L_l}(p)1_{L_l}(q)\left|\left\langle K_le_q,e_p\right\rangle\right|\left\|\tilde{c}_{p-k}\tilde{c}_{-q+l}\tilde{c}_{-q+k}\Psi\right\|$$
$$\leq\|\Psi\|\sum_{l\in\mathbb{Z}_*^3}\sum_{q\in L_l}\left\|K_le_q\right\|\sum_{k\in\mathbb{Z}_*^3}1_{B_F^c+k}(q)\hat{V}_k\|\tilde{c}_{-q+k}\tilde{c}_{-q+l}\mathcal{N}_E^{\frac{1}{2}}\Psi\|$$
$$\leq\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\left(\sum_{l\in\mathbb{Z}_*^3}\|K_l\|_{\mathrm{HS}}\right)\|\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|. \tag{5.29}$$

□

5.2. Gronwall estimates. We now establish control over the operators $e^{\mathcal{K}}\mathcal{N}_E^m e^{-\mathcal{K}}$ for $m=1,2,3$. Consider first the mapping $t\mapsto e^{t\mathcal{K}}\mathcal{N}_E e^{-t\mathcal{K}}$: Noting that for any $\Psi\in\mathcal{H}_N$

$$\frac{d}{dt}\left\langle\Psi,e^{t\mathcal{K}}(\mathcal{N}_E+1)e^{-t\mathcal{K}}\Psi\right\rangle=\left\langle\Psi,e^{-t\mathcal{K}}[\mathcal{K},\mathcal{N}_E]e^{-t\mathcal{K}}\Psi\right\rangle, \tag{5.30}$$

Gronwall's lemma implies that to bound $e^{t\mathcal{K}}(\mathcal{N}_E+1)e^{-t\mathcal{K}}$ it suffices to control $[\mathcal{K},\mathcal{N}_E]$ with respect to $\mathcal{N}_E+1$ itself. We determine the commutator: As $\mathcal{K}=\tilde{\mathcal{K}}-\tilde{\mathcal{K}}^*$ for

$$\tilde{\mathcal{K}}=\frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p,q\in L_l}\left\langle e_p,K_le_q\right\rangle b_{l,p}b_{-l,-q} \tag{5.31}$$

and $\left[b_{l,p},\mathcal{N}_E\right]=b_{l,p}$ it holds that $\left[\tilde{\mathcal{K}},\mathcal{N}_E\right]=2\,\tilde{\mathcal{K}}$, whence

$$[\mathcal{K},\mathcal{N}_E]=2\,\mathrm{Re}\left(\left[\tilde{\mathcal{K}},\mathcal{N}_E\right]\right)=2\,\tilde{\mathcal{K}}+2\,\tilde{\mathcal{K}}^*. \tag{5.32}$$

The estimate of Proposition 2.4 immediately yields that

$$\pm[\mathcal{K},\mathcal{N}_E]\leq C(\mathcal{N}_E+1) \tag{5.33}$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2$, whence by Gronwall's lemma

$$\left\langle\Psi,e^{t\mathcal{K}}(\mathcal{N}_E+1)e^{-t\mathcal{K}}\Psi\right\rangle\leq e^{C|t|}\left\langle\Psi,(\mathcal{N}_E+1)\Psi\right\rangle\leq C'\left\langle\Psi,(\mathcal{N}_E+1)\Psi\right\rangle,\quad|t|\leq1. \tag{5.34}$$

This proves the bound for $\mathcal{N}_E$; for $\mathcal{N}_E^2$ we will as in [10] apply the following lemma:

Lemma 5.7. *Let A, B, Z be given with $A > 0$, $Z \geq 0$ and $[A, Z] = 0$. Then if $\pm [A, [A, B]] \leq Z$ it holds that*

$$\pm[A^{\frac{1}{2}}, [A^{\frac{1}{2}}, B]] \leq \frac{1}{4} A^{-1} Z.$$

The estimates are as follows:

Proposition 5.8. *For any $\Psi \in \mathcal{H}_N$ and $|t| \leq 1$ it holds that*

$$\left\langle e^{-t\mathcal{K}} \Psi, \left(\mathcal{N}_E^m + 1\right) e^{-t\mathcal{K}} \Psi \right\rangle \leq C \left\langle \Psi, \left(\mathcal{N}_E^m + 1\right) \Psi \right\rangle, \quad m = 1, 2, 3,$$

for a constant $C > 0$ depending only on $\sum_{k \in \mathbb{Z}_*^3} \hat{V}_k^2$.

Proof. The case of $m = 1$ was proved above. For $m = 2$ it suffices to control $\left[\mathcal{K}, \mathcal{N}_E^2\right]$ in terms of $\mathcal{N}_E^2 + 1$; by the identity $\{A, B\} = A^{\frac{1}{2}} B A^{\frac{1}{2}} + [A^{\frac{1}{2}}, [A^{\frac{1}{2}}, B]]$ we can write

$$\begin{aligned} \left[\mathcal{K}, \mathcal{N}_E^2\right] &= \{\mathcal{N}_E, [\mathcal{K}, \mathcal{N}_E]\} = \{\mathcal{N}_E + 1, [\mathcal{K}, \mathcal{N}_E]\} - 2\,[\mathcal{K}, \mathcal{N}_E] \\ &= (\mathcal{N}_E + 1)^{\frac{1}{2}} [\mathcal{K}, \mathcal{N}_E] (\mathcal{N}_E + 1)^{\frac{1}{2}} + [(\mathcal{N}_E + 1)^{\frac{1}{2}}, [(\mathcal{N}_E + 1)^{\frac{1}{2}}, \\ &\quad \times [\mathcal{K}, \mathcal{N}_E]]] - 2\,[\mathcal{K}, \mathcal{N}_E] \end{aligned} \tag{5.35}$$

and note that the commutator $\left[\tilde{\mathcal{K}}, \mathcal{N}_E\right] = 2\,\tilde{\mathcal{K}}$ also implies that

$$[\mathcal{N}_E, [\mathcal{N}_E, [\mathcal{K}, \mathcal{N}_E]]] = 4\,[\mathcal{K}, \mathcal{N}_E], \tag{5.36}$$

so by Lemma 5.7 and equation (5.33)

$$\pm\left[\mathcal{K}, \mathcal{N}_E^2\right] \leq C\left((\mathcal{N}_E + 1)^2 + 1 + (\mathcal{N}_E + 1)\right) \leq C'\left(\mathcal{N}_E^2 + 1\right). \tag{5.37}$$

Similarly, for $\mathcal{N}_E^3$,

$$\left[\mathcal{K}, \mathcal{N}_E^3\right] = 3\,\mathcal{N}_E\,[\mathcal{K}, \mathcal{N}_E]\,\mathcal{N}_E + [\mathcal{N}_E, [\mathcal{N}_E, [\mathcal{K}, \mathcal{N}_E]]] = 3\,\mathcal{N}_E\,[\mathcal{K}, \mathcal{N}_E]\,\mathcal{N}_E + 4\,[\mathcal{K}, \mathcal{N}_E] \tag{5.38}$$

implies that

$$\pm\left[\mathcal{K}, \mathcal{N}_E^3\right] \leq C\left(\mathcal{N}_E\,(\mathcal{N}_E + 1)\,\mathcal{N}_E + (\mathcal{N}_E + 1)\right) \leq C'\left(\mathcal{N}_E^3 + 1\right) \tag{5.39}$$

hence the $m = 3$ bound. □

Conclusion of Theorem 1.1 We can now conclude:

Theorem. (1.1) *It holds that*

$$\inf \sigma\left(H_N\right) \le E_F + E_{\text{corr,bos}} + E_{\text{corr,ex}} + C\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\left\{|k|, k_F\right\}}, \quad k_F \to \infty,$$

for a constant $C > 0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2$.

Proof. By the variational principle applied to the trial state $e^{-\mathcal{K}}\psi_{\text{FS}}$ we have by Proposition 1.2 and the Theorems 1.4, 3.1 and 5.1 that

$$\begin{aligned}
\inf \sigma\left(H_N\right) &\le E_F + \left\langle \psi_{\text{FS}}, e^{\mathcal{K}}\left(H'_{\text{kin}} + \sum_{k\in\mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\left(2\pi\right)^3}\left(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^*\right)\right) e^{-\mathcal{K}}\psi_{\text{FS}}\right\rangle \\
&\quad + \left\langle \psi_{\text{FS}}, e^{\mathcal{K}}\mathcal{C}e^{-\mathcal{K}}\psi_{\text{FS}}\right\rangle + \left\langle \psi_{\text{FS}}, e^{\mathcal{K}}\mathcal{Q}e^{-\mathcal{K}}\psi_{\text{FS}}\right\rangle \\
&= E_F + E_{\text{corr,bos}} + \left\langle \psi_{\text{FS}}, H'_{\text{kin}}\psi_{\text{FS}}\right\rangle + 2\sum_{k\in\mathbb{Z}^3_*}\left\langle \psi_{\text{FS}}, Q_1^k\left(e^{-K_k} h_k e^{-K_k} - h_k\right)\psi_{\text{FS}}\right\rangle \\
&\quad + \sum_{k\in\mathbb{Z}^3_*}\int_0^1 \left\langle e^{-(1-t)\mathcal{K}}\psi_{\text{FS}}, \left(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\text{Re}\left(\mathcal{E}_k^1(A_k(t))\right)\right.\right. \\
&\quad \left.\left. + 2\,\text{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\right) e^{-(1-t)\mathcal{K}}\psi_{\text{FS}}\right\rangle dt \\
&\quad + \left\langle e^{\mathcal{K}}\psi_{\text{FS}}, \left(G + \mathcal{Q}_{\text{LR}}\right) e^{-\mathcal{K}}\psi_{\text{FS}}\right\rangle + \left\langle \psi_{\text{FS}}, \mathcal{Q}_{\text{SR}}\psi_{\text{FS}}\right\rangle \\
&\quad + \int_0^1 \left\langle e^{-t\mathcal{K}}\psi_{\text{FS}}, \left(2\,\text{Re}\left(\mathcal{G}\right)\right) e^{-t\mathcal{K}}\psi_{\text{FS}}\right\rangle dt \\
&= E_F + E_{\text{corr,bos}} + E_{\text{corr,ex}} + \epsilon_1 + \epsilon_2 + \epsilon_3,
\end{aligned} \tag{5.40}$$

where we also used that

$$H'_{\text{kin}}\psi_{\text{FS}} = Q_1^k(A)\psi_{\text{FS}} = \mathcal{Q}_{\text{SR}}\psi_{\text{FS}} = 0 \tag{5.41}$$

and that $\left\langle \psi_{\text{FS}}, e^{\mathcal{K}}\mathcal{C}e^{-\mathcal{K}}\psi_{\text{FS}}\right\rangle = 0$ by Proposition 5.2. The errors ϵ_1, ϵ_2 and ϵ_3 obey

$$\begin{aligned}
\epsilon_1 &= \sum_{k\in\mathbb{Z}^3_*}\int_0^1 \left\langle \psi_{\text{FS}}, 2\,\text{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\psi_{\text{FS}}\right\rangle dt - E_{\text{corr,ex}} \\
&\le C\sum_{k\in\mathbb{Z}^3_*}\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\left\{|k|, k_F\right\}}
\end{aligned} \tag{5.42}$$

by Proposition 4.2,

$$\begin{aligned}
\epsilon_2 &= \sum_{k\in\mathbb{Z}^3_*}\int_0^1 \left\langle e^{-(1-t)\mathcal{K}}\psi_{\text{FS}}, \left(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\text{Re}\left(\mathcal{E}_k^1(A_k(t))\right)\right) e^{-(1-t)\mathcal{K}}\psi_{\text{FS}}\right\rangle dt \\
&\quad + \sum_{k\in\mathbb{Z}^3_*}\int_0^1 \left\langle e^{-(1-t)\mathcal{K}}\psi_{\text{FS}}, \left(2\,\text{Re}\left(\mathcal{E}_k^2(B_k(t)) - \left\langle \psi_F, \mathcal{E}_k^2(B_k(t))\psi_F\right\rangle\right)\right) e^{-(1-t)\mathcal{K}}\psi_{\text{FS}}\right\rangle dt
\end{aligned}$$

$$\leq Ck_F^{-1} + C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\{|k|, k_F\}} \leq C'\sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\{|k|, k_F\}} \tag{5.43}$$

by Theorem 4.1, and

$$\begin{aligned}\epsilon_3 &= \left\langle e^{-\mathcal{K}}\psi_{\mathrm{FS}}, (G + \mathcal{Q}_{\mathrm{LR}})\, e^{-\mathcal{K}}\psi_{\mathrm{FS}}\right\rangle + \int_0^1 \left\langle e^{-t\mathcal{K}}\psi_{\mathrm{FS}}, (2\,\mathrm{Re}\,(\mathcal{G}))\, e^{-t\mathcal{K}}\psi_{\mathrm{FS}}\right\rangle dt \\ &\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\{|k|, k_F\}} \end{aligned} \tag{5.44}$$

by Theorem 5.1, where we for the last error terms also used that

$$\left\langle e^{-t\mathcal{K}}\psi_{\mathrm{FS}}, \left(\mathcal{N}_E^m + 1\right) e^{-t\mathcal{K}}\psi_{\mathrm{FS}}\right\rangle \leq C, \quad |t| \leq 1,\ m = 1, 2, 3, \tag{5.45}$$

as follows by Proposition 5.8. □

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A. Diagonalization of the Bosonizable Terms

In this section we derive the identity of Theorem 1.4. This is to a degree equivalent with the contents of Section 5 of [10], but for the reader's convenience, and since the notation used in the papers differ, we include a brief derivation in this appendix.

To determine the action of $e^{\mathcal{K}}$, we must first compute several commutators involving $\mathcal{K}$. To simplify the calculations we will make repeated use of the following result ([10, Lemma 3.2]).

Lemma A.1. *Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional Hilbert space and let $q : V \times V \to W$ be a sesquilinear mapping into a vector space W. Let $(e_i)_{i=1}^N$ be an orthonormal basis for V. Then for any linear operators $S, T : V \to V$ it holds that*

$$\sum_{i=1}^n q\,(Se_i, Te_i) = \sum_{i=1}^n q\left(ST^*e_i, e_i\right).$$

The lemma is easily proved by orthonormal expansion. In our case, where we regard $\ell^2(L_k)$ as real vector spaces, sesquilinearity is simply bilinearity. Moreover, the operators K_k satisfy

$$I_k K_k = K_{-k} I_k \tag{A.1}$$

where $I_k : \ell^2(L_k) \to \ell^2(L_{-k})$ denotes the unitary mapping defined by $I_k e_p = e_{-p}$, $p \in L_k$. Thus Lemma A.1 allows us to move operators from one argument to another (when summed), as e.g.

$$\begin{aligned}\sum_{q\in L_l} b_l(K_l e_q) b_{-l,-q} &= \sum_{q\in L_l} b_l(K_l e_q) b_{-l}\left(I_l e_q\right)\\ &= \sum_{q\in L_l} b_l\left(e_q\right) b_{-l}\left(I_l K_l^* e_q\right) = \sum_{q\in L_l} b_{l,q} b_{-l}(K_{-l} e_{-q}).\end{aligned} \tag{A.2}$$

We start by computing the commutator of $\mathcal{K}$ with an excitation operator:

Proposition A.2. *For any $k \in \mathbb{Z}^3_*$ and $\varphi \in \ell^2(L_k)$ it holds that*

$$[\mathcal{K}, b_k(\varphi)] = b^*_{-k}\left(I_k K_k \varphi\right) + \mathcal{E}_k(\varphi), \quad \left[\mathcal{K}, b^*_k(\varphi)\right] = b_{-k}\left(I_k K_k \varphi\right) + \mathcal{E}_k(\varphi)^*$$

where

$$\mathcal{E}_k(\varphi) = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left\{\varepsilon_{k,l}\left(\varphi; e_q\right), b^*_{-l}(K_{-l} e_{-q})\right\}.$$

Proof. It suffices to determine $[\mathcal{K}, b_k(\varphi)]$. Using Lemmas A.1 and 1.3 we calculate that

$$\begin{aligned}[\mathcal{K}, b_k(\varphi)] &= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left\{\left[b_k(\varphi), b^*_{-l}\left(e_{-q}\right)\right], b^*_l(K_l e_q)\right\}\\ &= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left\{\delta_{k,-l}\left\langle\varphi, e_{-q}\right\rangle + \varepsilon_{k,-l}\left(\varphi; e_{-q}\right), b^*_l(K_l e_q)\right\}\\ &= b^*_{-k}\left(I_k K_k \varphi\right) + \mathcal{E}_k(\varphi)\end{aligned} \tag{A.3}$$

where in the last identity we recognized $K_{-k}\sum_{q\in L_{-k}}\left\langle\varphi, e_{-q}\right\rangle e_q = K_{-k} I_k \varphi = I_k K_k \varphi$. □

Using this relation we can now determine the commutators with Q^k_1 terms:

Proposition A.3. *For any $k \in \mathbb{Z}^3_*$ and symmetric operators $A_{\pm k} : \ell^2(L_{\pm k}) \to \ell^2(L_{\pm k})$ such that $I_k A_k = A_{-k} I_k$, it holds that*

$$\left[\mathcal{K}, 2\,Q^k_1(A_k) + 2\,Q^{-k}_1(A_{-k})\right] = Q^k_2(\{K_k, A_k\}) + 2\,\mathrm{Re}\left(\mathcal{E}^1_k(A_k)\right) + (k \to -k)$$

where

$$\mathcal{E}^1_k(A_k) = \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l} b^*_k(A_k e_p)\left\{\varepsilon_{k,l}(e_p; e_q), b^*_{-l}(K_{-l} e_{-q})\right\}.$$

Proof. Using Proposition A.2 (and Lemma A.1 together with symmetry of A_k) we find that

$$\begin{aligned}\left[\mathcal{K}, Q_1^k(A_k)\right] &= \sum_{p\in L_k}\left(b_k^*(A_k e_p)\left[\mathcal{K}, b_k\left(e_p\right)\right] + \left[\mathcal{K}, b_k^*(A_k e_p)\right] b_k\left(e_p\right)\right) \\ &= \sum_{p\in L_k}\left(b_{-k,-p}^* b_k^*\left(A_k K_k e_p\right) + b_k\left(A_k K_k e_p\right) b_{-k,-p}\right) \\ &\quad + 2\,\mathrm{Re}\left(\sum_{p\in L_k} b_k^*(A_k e_p)\mathcal{E}_k\left(e_p\right)\right) \\ &= Q_2^k(A_k K_k) + 2\,\mathrm{Re}\left(\sum_{p\in L_k} b_k^*(A_k e_p)\mathcal{E}_k\left(e_p\right)\right). \end{aligned} \tag{A.4}$$

The assumption that $I_k A_k = A_{-k} I_k$ yields $Q_2^k(A_k K_k) = Q_2^{-k}(K_{-k} A_{-k})$. Summing over both k and $-k$, we obtain the desired identity. □

To state the commutator of $\mathcal{K}$ with Q_2^k terms we note the identity

$$\sum_{p\in L_k} b_k\left(e_p\right) b_k^*(A_k e_p) = Q_1^k(A_k) + \mathrm{tr}(A_k) + \varepsilon_k(A_k) \tag{A.5}$$

where we introduced the convenient notation

$$\varepsilon_k(A_k) = \sum_{p\in L_k} \varepsilon_{k,k}\left(e_p; A_k e_p\right) = -\sum_{p\in L_k}\left\langle e_p, A_k e_p\right\rangle\left(c_p^* c_p + c_{p-k} c_{p-k}^*\right). \tag{A.6}$$

The commutator is then given as follows:

Proposition A.4. *For any $k \in \mathbb{Z}_*^3$ and symmetric operators $B_{\pm k} : \ell^2(L_{\pm k}) \to \ell^2(L_{\pm k})$ such that $I_k B_k = B_{-k} I_k$, it holds that*

$$\begin{aligned}\left[\mathcal{K}, Q_2^k(B_k) + Q_2^{-k}(B_{-k})\right] &= 2\,Q_1^k(\{K_k, B_k\}) + \mathrm{tr}(\{K_k, B_k\}) + \varepsilon_k(\{K_k, B_k\}) \\ &\quad + 2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k)\right) + (k \to -k)\end{aligned}$$

where

$$\mathcal{E}_k^2(B_k) = \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\left\{b_k(B_k e_p), \left\{\varepsilon_{-k,-l}(e_{-p}; e_{-q}), b_l^*(K_l e_q)\right\}\right\}.$$

Proof. Writing $Q_2^k(B_k) = 2\,\mathrm{Re}\left(\sum_{p\in L_k} b_k(B_k e_p) b_{-k}\left(e_{-p}\right)\right)$ and using Proposition A.2 we get

$$\left[\mathcal{K}, Q_2^k(B_k)\right] = 2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_k(B_k e_p)\left[\mathcal{K}, b_{-k}\left(e_{-p}\right)\right] + \left[\mathcal{K}, b_k(B_k e_p)\right] b_{-k}\left(e_{-p}\right)\right)\right)$$

$$= 2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_{k,p}b_k^*\left(K_kB_ke_p\right)+b_{-k}^*\left(K_{-k}B_{-k}e_{-p}\right)b_{-k,-p}\right)\right)$$
$$+2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_k(B_ke_p)\mathcal{E}_{-k}\left(e_{-p}\right)+\mathcal{E}_k\left(e_p\right)b_{-k}\left(B_{-k}e_{-p}\right)\right)\right)=\text{(I)+(II)}.$$

For (I), the first term on the right-hand side, using (A.5) we find that

$$\text{(I)} = Q_1^k(\{K_k, B_k\}) + \mathrm{tr}(\{K_k, B_k\}) + \varepsilon_k(\{K_k, B_k\}) + Q_1^{-k}(\{K_{-k}, B_{-k}\}). \tag{A.7}$$

Summing over k and $-k$ and using $\mathcal{E}_k^2(B_k) = \sum_{p\in L_k}\left\{b_k(B_ke_p), \mathcal{E}_{-k}\left(e_{-p}\right)\right\}$ for (II), we obtain the desired identity. □

Finally we calculate the commutator with H'_{kin}:

Proposition A.5. *It holds that*

$$\left[\mathcal{K}, H'_{\mathrm{kin}}\right] = \sum_{k\in\mathbb{Z}_*^3} Q_2^k(\{K_k, h_k\}).$$

Proof. By equation (1.34) we have

$$\left[H'_{\mathrm{kin}}, b_k(\varphi)\right] = -2\,b_k\left(h_k\varphi\right), \quad \left[H'_{\mathrm{kin}}, b_k^*(\varphi)\right] = 2\,b_k^*\left(h_k\varphi\right), \tag{A.8}$$

so using that $I_kh_k = h_{-k}I_k$ we find

$$\left[\mathcal{K}, H'_{\mathrm{kin}}\right] = \frac{1}{2}\sum_{k\in\mathbb{Z}_*^3}\sum_{q\in L_k}\left(\left[b_k\left(K_ke_q\right)b_{-k}\left(e_{-q}\right), H'_{\mathrm{kin}}\right] - \left[b_{-k}^*\left(e_{-q}\right)b_k^*\left(K_ke_q\right), H'_{\mathrm{kin}}\right]\right)$$
$$= \sum_{k\in\mathbb{Z}_*^3}\sum_{q\in L_k}\left(b_k\left(\{K_k, h_k\}e_q\right)b_{-k}\left(e_{-q}\right) + b_{-k}^*\left(e_{-q}\right)b_k^*\left(\{K_k, h_k\}e_q\right)\right)$$
$$= \sum_{k\in\mathbb{Z}_*^3} Q_2^k(\{K_k, h_k\}). \tag{A.9}$$

□

Now we can now determine the action of $e^{\mathcal{K}}$ on quadratic operators:

Proposition A.6. *For any $k \in \mathbb{Z}_*^3$ and symmetric operators $T_{\pm k} : \ell^2(L_{\pm k}) \to \ell^2(L_{\pm k})$ such that $I_kT_k = T_{-k}I_k$ it holds that*

$$e^{\mathcal{K}}\left(2\,Q_1^k(T_k) + 2\,Q_1^{-k}(T_{-k})\right)e^{-\mathcal{K}} = \mathrm{tr}\left(T_k^1(1) - T_k\right) + 2\,Q_1^k(T_k^1(1)) + Q_2^k\left(T_k^2(1)\right)$$
$$+\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k\left(\left\{K_k, T_k^2(t)\right\}\right) + 2\,\mathrm{Re}\left(\mathcal{E}_k^1(T_k^1(t))\right)\right.$$
$$\left.+2\,\mathrm{Re}\left(\mathcal{E}_k^2\left(T_k^2(t)\right)\right)\right)e^{-(1-t)\mathcal{K}}dt + (k\to -k)$$

and

$$e^{\mathcal{K}}\left(Q_2^k(T_k) + Q_2^{-k}(T_{-k})\right)e^{-\mathcal{K}} = \mathrm{tr}\left(T_k^2(1)\right) + 2\,Q_1^k\left(T_k^2(1)\right) + Q_2^k(T_k^1(1))$$

$$+\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k(\left\{K_k, T_k^1(t)\right\}) + 2\,\mathrm{Re}\left(\mathcal{E}_k^1\left(T_k^2(t)\right)\right)\right.$$
$$\left.+2\,\mathrm{Re}\left(\mathcal{E}_k^2(T_k^1(t))\right)\right)e^{-(1-t)\mathcal{K}}dt + (k\to -k)$$

where for $t\in[0,1]$,

$$T_k^1(t) = \frac{1}{2}\left(e^{tK_k}T_k e^{tK_k} + e^{-tK_k}T_k e^{-tK_k}\right),\quad T_k^2(t) = \frac{1}{2}\left(e^{tK_k}T_k e^{tK_k} - e^{-tK_k}T_k e^{-tK_k}\right).$$

Proof. We prove the first identity, the second following by a similar argument. Note that the operators $A_k(t) = T_k^1(t)$, $B_k(t) = T_k^2(t)$ satisfy

$$A_k'(t) = \{K_k, B_k(t)\},\quad B_k'(t) = \{K_k, A_k(t)\},\quad A_k(0) = T_k,\quad B_k(0) = 0. \tag{A.10}$$

By Propositions A.3 and A.4 we get

$$\begin{aligned}
&\frac{d}{dt}e^{-t\mathcal{K}}\left(2\,Q_1^k(A_k(t)) + Q_2^k(B_k(t))\right)e^{t\mathcal{K}} + (k\to -k)\\
&= e^{-t\mathcal{K}}\left(2\,Q_1^k\left(A_k'(t)\right) + Q_2^k\left(B_k'(t)\right) - \left[\mathcal{K}, 2\,Q_1^k(A_k(t)) + Q_2^k(B_k(t))\right]\right)e^{t\mathcal{K}} + (k\to -k)\\
&= -\mathrm{tr}(\{K_k, B_k(t)\}) + e^{-t\mathcal{K}}\left(2\,Q_1^k\left(A_k'(t) - \{K_k, B_k(t)\}\right) + Q_2^k\left(B_k'(t) - \{K_k, A_k(t)\}\right)\right)e^{t\mathcal{K}}\\
&\quad - e^{-t\mathcal{K}}\left(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\left(\mathcal{E}_k^1(A_k(t))\right) + 2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\right)e^{t\mathcal{K}} + (k\to -k)\,.
\end{aligned}$$

The second term on the right-hand side vanishes due to (A.10). Specifying also the initial conditions in (A.10) we conclude by the fundamental theorem of calculus,

$$\begin{aligned}
&e^{\mathcal{K}}\left(2\,Q_1^k(T_k) + 2\,Q_1^{-k}(T_{-k})\right)e^{-\mathcal{K}} = \mathrm{tr}\left(A_k(1) - T_k\right) + 2\,Q_1^k(A_k(1)) + Q_2^k(B_k(1))\\
&+\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\left(\mathcal{E}_k^1(A_k(t))\right)\right.\\
&\left.+2\,\mathrm{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\right)e^{-(1-t)\mathcal{K}}dt + (k\to -k)
\end{aligned}\tag{A.11}$$

where we also used that by the assumptions on $A_k(t)$ and $B_k(t)$

$$\int_0^1 \mathrm{tr}(\{K_k, B_k(t)\})dt = \mathrm{tr}\left(\int_0^1 A_k'(t)dt\right) = \mathrm{tr}\left(A_k(1) - T_k\right). \tag{A.12}$$

The proof of Proposition A.6 is complete. □

From this we can also easily deduce the action of $e^{\mathcal{K}}$ on H_{kin}':

Proposition A.7. *It holds that*

$$\begin{aligned}
e^{\mathcal{K}}H_{\mathrm{kin}}'e^{-\mathcal{K}} &= \sum_{k\in\mathbb{Z}_*^3}\mathrm{tr}\left(h_k^1(1) - h_k\right) + H_{\mathrm{kin}}' + \sum_{k\in\mathbb{Z}_*^3}\left(2\,Q_1^k\left(h_k^1(1) - h_k\right) + Q_2^k\left(h_k^2(1)\right)\right)\\
&+\sum_{k\in\mathbb{Z}_*^3}\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k\left(\left\{K_k, h_k^2(t)\right\}\right) + \mathcal{E}_k^1\left(h_k^1(t) - h_k\right) + \mathcal{E}_k^2\left(h_k^2(t)\right)\right)e^{-(1-t)\mathcal{K}}dt
\end{aligned}$$

where for $t\in[0,1]$,

$$h_k^1(t) = \frac{1}{2}\left(e^{tK_k}h_k e^{tK_k} + e^{-tK_k}h_k e^{-tK_k}\right),\quad h_k^2(t) = \frac{1}{2}\left(e^{tK_k}h_k e^{tK_k} - e^{-tK_k}h_k e^{-tK_k}\right).$$

Proof. By the Propositions A.3 and A.5 we see that

$$\left[\mathcal{K}, H'_{\mathrm{kin}} - \sum_{k\in\mathbb{Z}^3_*} 2\,Q^k_1(h_k)\right] = -\sum_{k\in\mathbb{Z}^3_*} 2\,\mathrm{Re}\left(\mathcal{E}^1_k(h_k)\right) \tag{A.13}$$

whence by the fundamental theorem of calculus

$$\begin{aligned} &e^{\mathcal{K}}\left(H'_{\mathrm{kin}} - \sum_{k\in\mathbb{Z}^3_*} 2\,Q^k_1(h_k)\right)e^{-\mathcal{K}} \\ &= H'_{\mathrm{kin}} - \sum_{k\in\mathbb{Z}^3_*} 2\,Q^k_1(h_k) - \sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{t\mathcal{K}}\left(2\,\mathrm{Re}\left(\mathcal{E}^1_k(h_k)\right)\right)e^{-t\mathcal{K}}dt. \end{aligned} \tag{A.14}$$

Applying Proposition A.6 now yields the claim. □

We are now equipped to conclude Theorem 1.4. By the two previous propositions, we see that

$$\begin{aligned} e^{\mathcal{K}}H_{\mathrm{eff}}e^{-\mathcal{K}} &= e^{\mathcal{K}}\left(H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\left(2\,Q^k_1(P_k) + Q^k_2(P_k)\right)\right)e^{-\mathcal{K}} \\ &= \sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(A_k(1) - P_k) + H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\left(2\,Q^k_1(A_k(1)) + Q^k_2(B_k(1))\right) \\ &\quad + \sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k(\{K_k, B_k(t)\}) + \mathcal{E}^1_k(A_k(t)) + \mathcal{E}^2_k(B_k(t))\right)e^{-(1-t)\mathcal{K}}dt \end{aligned} \tag{A.15}$$

where the operators $A_k(t), B_k(t) : \ell^2(L_k) \to \ell^2(L_k)$ are given by

$$\begin{aligned} A_k(t) &= h^1_k(t) + P^1_k(t) + P^2_k(t) - h_k = \frac{1}{2}\left(e^{tK_k}\,(h_k + 2P_k)\,e^{tK_k} + e^{-tK_k}h_k e^{-tK_k}\right) - h_k \\ B_k(t) &= h^2_k(t) + P^1_k(t) + P^2_k(t) = \frac{1}{2}\left(e^{tK_k}\,(h_k + 2P_k)\,e^{tK_k} - e^{-tK_k}h_k e^{-tK_k}\right). \end{aligned} \tag{A.16}$$

Now we choose K_k such that $B_k(1) = 0$. This amounts to the diagonalization condition

$$e^{K_k}\,(h_k + 2P_k)\,e^{K_k} = e^{-K_k}h_k e^{-K_k}, \tag{A.17}$$

of which the solution is given in (1.39). Since (A.17) is fulfilled, it follows that also $A_k(1) = e^{-K_k}h_k e^{-K_k} - h_k$, and so the identity in Theorem 1.4 follows provided we can show that

$$\sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}\left(e^{-K_k}h_k e^{-K_k} - h_k - P_k\right) = E_{\mathrm{corr,bos}}. \tag{A.18}$$

To establish this final identity we will use the following integral representation of the square root of a one-dimensional perturbation, first used in [2]:

Lemma A.8. *Let $A : V \to V$ be a positive self-adjoint operator. Then for any $w \in V$ and $g \in \mathbb{R}$ such that $A + gP_w > 0$ it holds that*

$$(A + gP_w)^{\frac{1}{2}} = A^{\frac{1}{2}} + \frac{2g}{\pi} \int_0^\infty \frac{t^2}{1 + g\left\langle w, \left(A + t^2\right)^{-1} w\right\rangle} P_{(A+t^2)^{-1}w} dt,$$

$$\mathrm{tr}\left((A + gP_w)^{\frac{1}{2}}\right) = \mathrm{tr}\left(A^{\frac{1}{2}}\right) + \frac{1}{\pi} \int_0^\infty \log\left(1 + g\left\langle w, \left(A + t^2\right)^{-1} w\right\rangle\right) dt.$$

The trace identity (A.18) now follows (note that this is essentially Proposition 7.6 of [10]):

Proposition A.9. *Let $F(x) = \log(1 + x) - x$. For any $k \in \mathbb{Z}^3_*$ it holds that*

$$\mathrm{tr}\left(e^{-K_k} h_k e^{-K_k} - h_k - P_k\right) = \frac{1}{\pi} \int_0^\infty F\left(\frac{\hat{V}_k k_F^{-1}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right) dt.$$

Proof. By cyclicity of the trace and Lemma A.8, $\mathrm{tr}\left(e^{-K_k} h_k e^{-K_k} - h_k\right)$ is equal to

$$\mathrm{tr}\left(\left(h_k^2 + 2P_{h_k^{\frac{1}{2}} v_k}\right)^{\frac{1}{2}}\right) = \frac{1}{\pi} \int_0^\infty \log\left(1 + 2\left\langle v_k, h_k \left(h_k^2 + t^2\right)^{-1} v_k\right\rangle\right) dt. \quad \text{(A.19)}$$

The claim follows by inserting the definition of h_k and v_k, and noting also that

$$\frac{2}{\pi} \int_0^\infty \left\langle v_k, h_k \left(h_k^2 + t^2\right)^{-1} v_k\right\rangle dt = \frac{\hat{V}_k k_F^{-1}}{2(2\pi)^3} |L_k| = \|v_k\|^2 = \mathrm{tr}(P_k) \quad \text{(A.20)}$$

where we used the integral identity $\int_0^\infty a/(a^2 + t^2) dt = \pi/2$ for every $a > 0$. □

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