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The Logic of Lexical Connectives

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Abstract
Natural language does not express all connectives definable in classical logic as simple lexical items. Coordination in English is expressed by conjunction *and*, disjunction *or*, and negated disjunction *nor*. Other languages pattern similarly. Non-lexicalized connectives are typically expressed compositionally: in English, negated conjunction is typically expressed by combining negation and conjunction (*not both*). This is surprising: if $\wedge$ and $\vee$ are duals, and the negation of the latter can be expressed lexically (*nor*), why not the negation of the former? I present a two-tiered model of the semantics of the binary connectives. The first tier captures the expressive power of the lexicon: it is a bilateral state-based semantics that, under a restriction, can express all and only the distinctions that can be expressed by the lexicon of natural language (*and, or, nor*). This first tier is characterized by rejection as non-assertion and a Neglect Zero assumption. The second tier is obtained by dropping the Neglect Zero assumption and enforcing a stronger notion of rejection, thereby recovering classical logic and thus definitions for all Boolean connectives. On the two-tiered model, we distinguish the limited expressive resources of the lexicon and the greater combinatorial expressive power of the language as a whole. This gives us a logic-based account of compositionality for the Boolean fragment of the language.

Keywords Lexicalization · Horn's Puzzle · Semantics · Natural language semantics · Bilateralism · Assertion · Rejection · Propositional logic · Connectives · NAND · Neglect Zero effects

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1 Introduction

Languages across the world have simple (monomorphemic) words for some but not all of the binary connectives of classical logic [21, 22, 31]. Coordination in English is expressed by conjunction *and*, disjunction *or*, and negated disjunction *nor*. No language appears to include coordinators that express more or other connectives [21, 46]. Connectives that are definable in classical logic but are not lexicalized are usually expressed compositionally: for example, negated conjunction by *not both*, and exclusive disjunction by *either but not both*.

This surprising distribution is known as Horn's puzzle [21]. The surprise is due in part to standard semantics. A common assumption in standard semantics is that conjunction and disjunction in natural language express Boolean operators, which are duals under negation, and so if the negation of one can be expressed by a simple word (*nor*), one would expect that the negation of the other could be so expressed (by a hypothetical **nand*). But this is not so.

More generally, if the logic of semantic representations is Boolean, we face an overgeneration problem: there are many binary Boolean operators but only *and, or*, and *nor* are ever lexicalized.[1]

There is a Gricean approach to the overgeneration problem, according to which pragmatic principles of communication explain the lexical gaps, in particular scalar implicatures [4, 13, 21, 30, 46]. Implementations of this approach differ in various respects, but two main concerns with this strategy have arisen.

(a) Most Gricean approaches are limited in scope. The arguments of [13, 21, 30], aim to show how the set {*and, or, nor, *nand*} can be thinned to the target set *and, or, nor*, but do not apply to the full set of classically definable binary Boolean connectives. Additional assumptions are needed to rule out non-trivial but unattested operators.

(b) No Gricean approach explains or predicts that compositionality is the fundamental linguistic tool to overcome the expressive limits of the lexicon [25]: on Gricean approaches, the combinatorial power of logical operators is not part of the account.

Gricean approaches maintain a background Boolean logic and invoke principles of communication to account for Horn's puzzle. The perspective of this paper is to change

[1] Some preliminary clarifications. (i) The focus of this paper are coordinating expressions: this is to exclude binary operators such as *because, until*, or *if*, that are not merely coordination devices (on *if*, see [33, 39], among others). (ii) As emphasized below, *nor* coordinates rejections, unlike *and* and *or* that coordinate assertions. In many natural languages, including English, rejections must be morphologically marked, hence occurrences of *nor* are accompanied by rejection-markers on the first argument (in English, *not* and *neither*):

(1) a. She is not American nor Canadian.
b. Neither the President nor the Prime Minister was found guilty.

In other languages (the analog of) *nor* need not come with a different partner. Italian *né*, for example, behaves as conjunction (*sia*) and disjunction (*o*) in constructions akin to (1b).

(2) {Sia/O/Né} questo {sia/o/né} quello
{Both/Either/Neither} this {and/or/nor} that

(iii) A separate issue is raised by the view that *nor* is not syntactically simple [41, 51]. This seems tangential to me. The puzzle is to explain why *nor* may occur as a monosyllabic item (syntactically complex or not), unlike other possible operators such as **nand*.

the background logic and make the overgeneration problem disappear. In the logic of the lexicon, issues (a) and (b) do not arise: no additional principles are needed to exclude non-trivial unattested operators, and compositionality is a core component of the explanation.

A logic of the lexicon is a formal system that captures all and only the semantic distinctions that natural language expresses lexically, relative to a fragment of the language, and nothing else. Regarding the coordinators, there are two desiderata: to describe the semantics of *and, or,* and *nor*, and to show that no other connective is definable. In this paper I will introduce two logics. The two logics share the same language and, in a sense, the same semantics for the operators $\neg$, $\wedge$ and $\vee$ that belong to the common language. The semantics is bilateral, and stated in terms of assertion- and rejection-conditions of sentences at information states (sets of possible worlds). The first logic, LLC, is characterized by (i) a definition of rejection as non-assertion, and (ii) the possibility of avoiding vacuous information in the performance of speech acts. LLC is the logic of the lexical connectives: under a restriction, only *and, or*, and *nor* can be expressed in LLC. The second logic, CL, is characterized by (i) a stronger notion of rejection, and (ii) a requirement to always allow for vacuous speech. CL is classical logic.

The two systems can be understood as two specifications (by different parameter settings) of the same underlying semantic structure, which can capture, on the one hand, the expressive power of the lexicon (LLC), and on the other, the expressive power of natural language as a whole (CL). A two-tiered account of lexicalization emerges. On the first tier, LLC, we define what the lexicon can express. Classically definable non-lexical connectives are introduced at the second tier, CL, which is obtained from the first by requiring that non-vacuous reasoning is always permitted (a staple of classical logic) and by strengthening rejection into the classical contradictory of assertion. If we suppose, which seems plausible, that the overt negation operator *not* often expresses strong rejection [26], then the two tiers closely correspond to how natural language expresses the connectives: some at lexical level, and some by combining the lexical connectives with overt negation (e.g., *not both*).

1.1 Linguistic Motivations

There is a widely accepted explanation of the lexicalization pattern of color terms. We start with a space of possible lexical meanings, some of which are selected in any given natural language. The space of color terms' possible lexical meanings is, plausibly, any partition of the visible light spectrum that satisfies certain universal constraints (such as connectedness [10]). Partitions of the color spectrum that satisfy such universal constraints are "up" for lexicalization, so to speak, and languages differ lexically in patterns that have been explained by informativity and cost pressures on communication [32, 35, 36].

The case of binary connectives may be analogous. The first question is: what is the space of possible lexical meanings for the connectives? Gricean approaches to Horn's puzzle assume a "worst case" hypothesis [24], namely that the space of possible lexical meanings is determined by Boolean logic. However, there is no linguistic evidence that any coordinating connective is lexicalizable besides *and, or,* and *nor*, and we know that further constraints are in place (such as monotonicity [5, 45]), that rule out some Boolean operators. In the logic introduced below, natural assumptions about assertion and rejection constrain the space of possible meanings for the connectives so as to include *and, or, nor*, and nothing else. On the model provided by this logic, only *and, or,* and *nor* are possible lexical meanings for the connectives. Languages may then differ by adopting different strategies concerning what to lexicalize from this space: some languages lack a word for *nor*, others lack disjunction. This part of the explanation may well depend on communicative trade-offs. For an account of cross-linguistic differences along these lines, compatible with the perspective of the present paper, see [9].

The project of a logic of the lexicon is continuous with the project of standard semantics. Standard semantics, as many understand it [49], is a model of semantic competence as revealed by native speakers' intuitive judgements. A logic of the lexicon builds on this by adding an account of lexicalizability—another source of evidence about semantic competence. There are methodological differences: while in standard semantics valid entailments are understood to be introspectively accessible to native speakers, in a logic of the lexicon some valid entailments are not meant to "ring" valid (knowledge of one's language does not imply knowledge of cross-linguistic regularities): these entailments are meant to show that the expressive power of the logic of the lexicon matches the expressive power of the lexicon of natural language. The more general perspective is that of explaining some aspects of lexicalizability, such as the lack of **nand*, by appealing to (un)definability in the proper logic.

For illustration, consider a language L closed under negation $\neg$, conjunction $\wedge$, and disjunction $\vee$. Let's assume a standard semantics and a classical metalanguage.

$$\phi \wedge \psi \text{ is true iff } \phi \text{ is true and } \psi \text{ is true}$$
$$\phi \vee \psi \text{ is true iff } \phi \text{ is true or } \psi \text{ is true}$$
$$\neg\phi \text{ is true iff } \phi \text{ is not true}$$

Why are $\wedge$ and $\vee$ the primitive binary connectives of L? Apparently by accident. For if the lexicon of L has the expressive resources invoked on the right-hand side of the "iffs", then it could in principle include all connectives definable in Boolean logic. Negated conjunction **nand*, for example, could be a primitive of L by stipulating that ϕ *nand* ψ is true iff ϕ is not true or ψ is not true. The same applies to the material biconditional, exclusive disjunction, and all other Boolean operators. The absence of simple words for all these operators would be unexplained in L as well as in natural language insofar as the former is a model of the latter.

Standard semantics is often understood to model compositionality in the following sense: the semantics shows *that* compositionality in natural language is possible, by means of a formal language that implements compositionality and interprets natural language. The clauses above show that the truth-conditions of *not both* ϕ *and* ψ can be obtained by combining the semantic contributions of $\neg$, $\wedge$, ϕ, and ψ. This approach does not show *how* compositionality in natural language works: it does not explain how natural language expresses *not both* ϕ *and* ψ. For the possibility remains open that such truth-conditions are expressed directly at the lexical level by a simple word such as **nand*. However, this is not a genuine possibility: it is an apparent one that arises because the expressive resources of the lexicon of L, as characterized by a standard semantics and a classical metalanguage, outstrip those of the lexicon of natural language.

Semantic clauses such as the ones above are rather rudimentary, of course, but the point is general. A semantic metalanguage should be expressive enough to capture the truth-conditions of all expressions in the intended fragment of the object language. But this assumption may overgenerate. A further constraint is thus desirable: that the semantic metalanguage does not express more than what is expressed by the lexicon of the intended fragment of natural language. A logic of the lexicon is thus continuous with standard truth-conditional semantics, but builds in an account of why we don't have more logic at the lexical level than we do. (For recent work oriented in a similar direction, see [24].)

1.2 Overview

Binary sentential coordinators are natural language expressions such as *and, or, nor*. The Logic of Lexical Connectives (LLC) may be studied by means of a formal language with standard syntax.

$$\mathscr{L}_{\text{LLC}} := p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

I assume two basic distinctions: between assertion and rejection, and between conjunction and disjunction. The lexicon of natural language plausibly keeps track of both distinctions, which justify the syntactic primitives of $\mathscr{L}_{\text{LLC}}$. The distinction between conjunction $\wedge$ and disjunction $\vee$ is attested in the lexicon of many natural languages [34]. The distinction between assertion and rejection is also well-documented [22, 26], and justifies a form of negation $\neg$ as a switch between the two speech acts.

Perhaps the most conspicuous detail of LLC is that the classical duality of conjunction and disjunction is broken: while assertions of conjunctions and assertions of disjunctions behave classically, their rejections coincide if vacuous information is avoided, and express the classical meaning of negated disjunction, which is the semantic value of *nor*. In other words, under certain conditions LLC fails to express the contradictory of conjunction. This is what we want, because a connective such as **nand* is not lexicalized in natural language, and LLC is designed to model the expressive resources of the lexicon of natural language. The same consideration applies to other non-lexical connectives.

LLC may be depicted by the Triangle of Oppositions of Fig. 1. The dashed vertical line represents the assertion/rejection distinction, and the horizontal the conjunction/disjunction distinction. These two lines divide the plane in four quadrants, one of which is empty (down-right). The ghost of the Square of Oppositions for classical propositional logic, that goes back to the Aristotelian tradition [7, p. 14], is indicated by a dotted line.

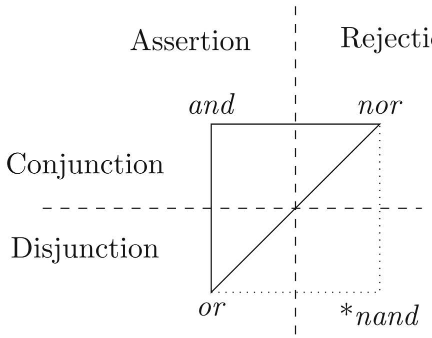

Fig. 1 The triangle of oppositions

Evidence for placing *nor* up-right is that (3a) implies both (3b) and (3c) and is jointly implied by them. This is the same inferential relation as that between a conjunction (up-left) and its arguments.

(3) a. Neither Maria nor Robert tried the new restaurant.
b. Robert did not try the new restaurant.
c. Maria did not try the new restaurant.

The Rejection side of Fig. 1 is characterized in classical logic by the De Morgan equivalences. The up-right quadrant is characterized by the equivalence between $\neg(\phi \vee \psi)$ and $\neg\phi \wedge \neg\psi$, and the empty down-right quadrant by the equivalence between $\neg(\phi \wedge \psi)$ and $\neg\phi \vee \neg\psi$. A sentence of any of these four forms will be called a *De Morgan sentence*. The strategy I will follow to triangulate the Square will be to (a) collapse the De Morgan sentences on the same meaning, and (b) show that it is the classical meaning of *nor*. The entailments that account for part (a) are summarized below: on the two left columns, the classically valid entailments among De Morgan sentences. All of these are unrestrictedly valid in LLC. On the right column, the classically invalid entailments, to which I will refer collectively as Negative Collapse. These will also come out valid in LLC, under a restriction, thereby causing all De Morgan sentences to collapse on the same meaning. Moreover, and this is part (b), this is the classical meaning of *nor*, for LLC captures the inferential pattern in (3) under the same restriction.[2]

Classically Valid		Negative Collapse
$\neg\phi \vee \neg\psi \vDash \neg(\phi \wedge \psi)$	$\neg(\phi \vee \psi) \vDash \neg\phi \vee \neg\psi$	$\neg\phi \vee \neg\psi \vDash \neg(\phi \vee \psi)$
$\neg(\phi \wedge \psi) \vDash \neg\phi \vee \neg\psi$	$\neg(\phi \vee \psi) \vDash \neg(\phi \wedge \psi)$	$\neg(\phi \wedge \psi) \vDash \neg(\phi \vee \psi)$
$\neg\phi \wedge \neg\psi \vDash \neg(\phi \vee \psi)$	$\neg\phi \wedge \neg\psi \vDash \neg(\phi \wedge \psi)$	$\neg(\phi \wedge \psi) \vDash \neg\phi \wedge \neg\psi$
$\neg(\phi \vee \psi) \vDash \neg\phi \wedge \neg\psi$	$\neg\phi \wedge \neg\psi \vDash \neg\phi \vee \neg\psi$	$\neg\phi \vee \neg\psi \vDash \neg\phi \wedge \neg\psi$

2 Horn [pp. 216ff] [22] mentions a 'three-cornered Square of Oppositions', which illustrates a proposal about lexicalization originally due to Jespersen [28] (see [23] for a negative assessment). On the three-cornered Square, *or* is interpreted as exclusive disjunction. The Square is thus made into a triangle, since *or* expresses two vertices at once (disjunction and the contradictory of conjunction), but this comes at the price of a non-standard disjunction. The triangulation of the Square that I am proposing is different: I propose not to revise standard semantics but to complement it.

Because of Negative Collapse, triviality looms. Suppose that $\neg\phi$, therefore $\neg\phi \vee \neg\psi$ by Disjunction Introduction, therefore $\neg\phi \wedge \neg\psi$ by Negative Collapse, therefore $\neg\psi$ by Conjunction Elimination. It appears that from any $\neg\phi$ we can conclude any $\neg\psi$. This is the Triviality Argument.

(i)	$\neg\phi$	Assumption
(ii)	$\neg\phi \vee \neg\psi$	From (i) by Disjunction Introduction
(iii)	$\neg\phi \wedge \neg\psi$	From (ii) by Negative Collapse
(iv)	$\neg\psi$	From (iii) by Conjunction Elimination

The step from (i) to (ii) arguably captures something fundamental about disjunction, and is valid in classical logic along with the step from (iii) to (iv), which arguably captures something fundamental about conjunction. Negative Collapse supports the inference from (ii) to (iii), which is classically aberrant but necessary to claim that $\neg\phi \vee \neg\psi$ and $\neg\phi \wedge \neg\psi$ express the same meaning in LLC. Thus, each of the steps from (i) to (iv) seems irresistible. There are similar triviality arguments from any of the other validities in Negative Collapse, but they will be dealt with in the same way as this.

The inference from (ii) to (iii) has the superficial form of a (wide scope) FREE CHOICE inference, and it is labelled accordingly.

FC	$\neg\phi \vee \neg\psi \vDash \neg\phi \wedge \neg\psi$
FREE CHOICE	$\Diamond\phi \vee \Diamond\psi \vDash \Diamond\phi \wedge \Diamond\psi$

FREE CHOICE is classically invalid but perceived to be valid in ordinary uses of epistemic or deontic modals. For example, (4b) seems to follow from (4a).

(4) a. Paul might be Dutch or might be Danish.
 b. Paul might be Dutch and might be Danish.

The analogy between FC and FREE CHOICE is perhaps only superficial, but it may offer some technical insight to block the Triviality Argument. Consider the analogous argument with epistemic modals below [29].

(i')	Paul might be Dutch	Assumption
(ii')	Paul might be Dutch or might be Danish	From (i') by Disjunction Introduction
(iii')	Paul might be Dutch and might be Danish	From (ii') by FREE CHOICE
(iv')	Paul might be Danish	From (iii') by Conjunction Elimination

Someone who is in a position to assert (i') seems to be led to (iv') by inexorable steps of logical reasoning. A possible diagnosis of the argument from (i') to (iv') is to reject not any single inferential step, but rather the way the steps are chained together. Intuitively, one's ground for asserting (i') is also, somewhat loosely, a ground for asserting (ii').

However, asserting (ii') as the premise of an inference to (iii') requires more than such loose ground. One must not equivocate between the epistemic state of someone who asserts a disjunction, and the state of someone who loosely asserts a disjunction because they can assert a disjunct. For a diagnosis of FREE CHOICE roughly along these lines, see [2]. While I do not commit to an account of FREE CHOICE here, I do endorse an analogous equivocation diagnosis of the Triviality Argument via Negative Collapse. In addition, I will look at recent work on FREE CHOICE to design models that validate the FC inference [2, 20].

1.3 The Semantic Core

As I anticipated, I will describe two logical systems: the Logic of the Lexical Connectives LLC and Classical Logic CL. Perhaps surprisingly, the two systems are closely related. In this section I will introduce their common core. The language $\mathscr{L}_{\text{LLC}}$, already introduced in the previous section, is also shared.

A model is a structure $M = (W, V)$ of a non-empty set of possible worlds and a valuation function. Worlds are classical indices. Valuations are functions from worlds to the Boolean truth values.

Definition 1 Let $M = (W, V)$ be a model, with $W \neq \varnothing$. For all atomic sentences p in $\mathscr{L}_{\text{LLC}}$ and all worlds $w \in W$, a *valuation* V in M is a function from p and w to a truth value 0 or 1.

The syntactic primitives of $\mathscr{L}_{\text{LLC}}$, $\{\neg, \vee, \wedge\}$, have bilateral truth-conditions, following a rich logical tradition [26, 37, 40, 44]. Truth-conditions of sentences will be specified relative to information states, that is, subsets of W, following [12]. We can think of the statement '$M, s \vDash \phi$' as saying that ϕ is asserted in state s relative to a model M, while '$M, s \dashv \phi$' says that ϕ is rejected in s relative to M. I will also gloss this notation, occasionally, by saying that s asserts (resp., rejects) ϕ relative to a model. For simplicity, reference to M is omitted whenever no confusion arises.

A form of negation is available "for free", as it were, in a bilateral setting, as a switch between assertion and rejection. A sentence $\neg\phi$ is asserted if and only if ϕ is rejected, and *vice versa*.

$$s \vDash \neg\phi \text{ iff } s \dashv \phi$$
$$s \dashv \neg\phi \text{ iff } s \vDash \phi$$

Classical disjunction ($s \vDash \phi \vee \psi$ iff $s \vDash \phi$ or $s \vDash \psi$) would quickly lead to triviality via the Triviality Argument: by classical disjunction, Disjunction Introduction holds unrestrictedly in the metalanguage. Instead of classical disjunction, I adopt a rule of so-called split disjunction. A disjunction is asserted in an information state if and only if the state can be "split" into two substates, each asserting a disjunct, while a state rejects a disjunction if and only if both disjuncts are rejected.

$$s \vDash \phi \vee \psi \text{ iff for some } t \text{ and } t' : s = t \cup t' \text{ and } t \vDash \phi \text{ and } t' \vDash \psi$$
$$s \dashv \phi \vee \psi \text{ iff } s \dashv \phi \text{ and } s \dashv \psi$$

Split disjunction has been studied in state-based models of intuitionistic and classical logic [12], team logic [50], and for the semantics of FREE CHOICE [2, 20]. Split disjunction gets us half way to avoiding trivialization, as we shall see.

A similar issue arises with the conditions for rejecting a conjunction. Classical rejected conjunction ($s \dashv \phi \wedge \psi$ iff $s \dashv \phi$ or $s \dashv \psi$) would allow for unrestricted Disjunction Introduction in the metalanguage, which threatens to trivialize the account. The truth-conditions of $\wedge$ are the mirror image of those for $\vee$. Indeed, the former can be derived from the latter by setting $\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi)$.

$$s \vDash \phi \wedge \psi \text{ iff } s \vDash \phi \text{ and } s \vDash \psi$$
$$s \dashv \phi \wedge \psi \text{ iff for some } t \text{ and } t' : s = t \cup t' \text{ and } t \dashv \phi \text{ and } t' \dashv \psi$$

Summarizing, I assume the following truth-conditions for the operators in $\mathscr{L}_{\text{LLC}}$. These truth-conditions are common for LLC and CL. In this sense, conjunction, disjunction, and negation, have the same meaning in the two systems.[3]

Definition 2 Bilateral truth-conditions of the operators in $\mathscr{L}_{\text{LLC}}$.

$$s \vDash \neg\phi \text{ iff } s \dashv \phi$$
$$s \dashv \neg\phi \text{ iff } s \vDash \phi$$
$$s \vDash \phi \vee \psi \text{ iff for some } t \text{ and } t' : s = t \cup t' \text{ and } t \vDash \phi \text{ and } t' \vDash \psi$$
$$s \dashv \phi \vee \psi \text{ iff } s \dashv \phi \text{ and } s \dashv \psi$$
$$s \vDash \phi \wedge \psi \text{ iff } s \vDash \phi \text{ and } s \vDash \psi$$
$$s \dashv \phi \wedge \psi \text{ iff for some } t \text{ and } t' : s = t \cup t' \text{ and } t \dashv \phi \text{ and } t' \dashv \psi$$

Logical consequence for LLC and CL is the familiar assertion-based notion: a sentence ψ follows from some sentences Γ if and only if ψ is asserted in all models and states in which all sentences in Γ are asserted.

Definition 3 Logical consequence.

$$\Gamma \vDash_{\text{LLC/CL}} \psi \text{ iff } M, s \vDash \psi \text{ for all models M and states s such that } M, s \vDash \gamma \text{ for all } \gamma \in \Gamma$$

Therefore, ϕ is a tautology in LLC or CL, written '$\vDash_{\text{LLC/CL}} \phi$', if and only if ϕ is asserted relative to all models and states. In the following, the subscript on the consequence relation will often be omitted. There is thus a triple ambiguity, for '$\vDash$' may stand for assertion (a relation between a model, a state, and a sentence), consequence in LLC, and consequence in CL (which are relations between some sentences and a sentence). The context will always clarify which sense is intended: if assertion, then a variable

[3] Insofar as meaning is captured by assertion- and rejection-conditions. There is a further dimension of meaning (a "structural" one that concerns which entailments are valid) on which the logical operators have different meanings in LLC and CL, since the two logics have different sets of validities. See Section 4.3 below.

for an information state will always occur to the left of '$\vDash$',[4] and the two logics LLC and CL are the focus of different sections of the paper.

The logical scaffolding I have described thus far is not a logic because assertion- and rejection-conditions for atomic sentences are yet to be specified. The definition of rejection for atoms is one point where LLC and CL diverge.

2 The Logic of Lexical Connectives

A logic is obtained from the semantic core described in the previous section by adding conditions for the assertion and rejection of atomic sentences, and a policy for the management of vacuous speech. The conditions for assertion and rejection of an atomic sentence in LLC reflect the intuitive idea that assertion and rejection are polar opposites. A state asserts p if and only if all worlds in the state verify p. A state rejects p either vacuously, if the state is empty, or if it fails to assert p. Since the (non-vacuous) condition for rejection is non-assertion, I will call this condition polar rejection.

Definition 4 Assertion and Polar Rejection.

$$s \vDash p \text{ iff for all } w \in s : V(w, p) = 1$$
$$s \dashv p \text{ iff } s = \varnothing \text{ or } s \nvDash p$$

Equivalently, polar rejection can be stated directly in terms of possible worlds rather than states, as for the assertion clause.

$$s \dashv p \text{ iff } s = \varnothing \text{ or for some } w \in s : V(w, p) = 0$$

Assertion and polar rejection are opposite but unequal. There is a demanding rule of unanimity for assertion: all worlds in a state have to agree that p is true in order for the state to assert p. However one dissenting world is enough for polar rejection. Intuitively, the epistemic standard for assertion is higher.

An immediate consequence of Definition 4 is that the empty state—the "zero"—asserts and rejects any atom. A simple induction through the clauses of Definition 2 extends this property to all sentences. Following [2], I take $\varnothing$ to be a state of logical absurdity.

Observation 5 $\varnothing \vDash \phi$ and $\varnothing \dashv \phi$ for any ϕ in $\mathscr{L}_{\text{LLC}}$

By Observation 5, vacuous information is a ground for asserting anything: *ex absurdo quodlibet*. However, there is an important difference between asserting something on the basis of specific information, and asserting something because one is in a vacuous information state and could assert anything. Likewise for rejection. The next assumption I will introduce, which completes the description of LLC, is a policy on

[4] In particular, '$\varnothing \vDash \phi$' indicates that ϕ is asserted in the empty state, not that it follows from the empty set of premises, for which I will write '$\vDash_{\text{LLC/CL}} \phi$' (see Observation 27).

the management of the zero: in LLC, the empty ground can be avoided. In certain contexts, vacuous information is not an acceptable ground for assertion and rejection. This assumption is called 'Neglect Zero', following [2].

To model Neglect Zero I introduce the star operator $\star$: a speech act modifier whose function is to refine the grounds for assertion and rejection in order to avoid the vacuous support provided by the zero. Formally, the star is a function from a speech act, $s \vDash \phi$, to its non-vacuous counterpart, $s \vDash \phi^\star$, which says that ϕ is non-vacuously asserted in s.[5] Likewise for rejection. For atomic sentences, $s \vDash p^\star$ ($s \Dashv p^\star$) if and only if s is non-empty and asserts (resp., rejects) p. Non-vacuous assertion and rejection of complex sentences are defined in terms of the non-vacuous assertion and rejection of their constituents.[6]

Definition 6 The star.

$$s \vDash p^\star \text{ iff } s \vDash p \text{ and } s \neq \varnothing$$
$$s \Dashv p^\star \text{ iff } s \Dashv p \text{ and } s \neq \varnothing$$
$$s \vDash [\neg\phi]^\star \text{ iff } s \vDash \neg\phi^\star$$
$$s \Dashv [\neg\phi]^\star \text{ iff } s \Dashv \neg\phi^\star$$
$$s \vDash [\phi \vee \psi]^\star \text{ iff } s \vDash \phi^\star \vee \psi^\star$$
$$s \Dashv [\phi \vee \psi]^\star \text{ iff } s \Dashv \phi^\star \vee \psi^\star$$
$$s \vDash [\phi \wedge \psi]^\star \text{ iff } s \vDash \phi^\star \wedge \psi^\star$$
$$s \Dashv [\phi \wedge \psi]^\star \text{ iff } s \Dashv \phi^\star \wedge \psi^\star$$

Neglect Zero is an optional environment for the performance of speech acts in LLC. The possibility of neglecting the zero enjoys some independent motivation. Disregarding the role of vacuous configurations in reasoning can be plausibly considered a natural

[5] Although the star appears as a superscript on a sentence, it is not a sentential operator: its scope is the whole $(s \vDash \phi)^\star$. The notation has the benefit of allowing us to write '$\phi^\star \vDash \psi^\star$' to mean that, for all M and s, if $M, s \vDash \phi^\star$ then $M, s \vDash \psi^\star$, namely, if ϕ is non-vacuously asserted, then ψ is non-vacuously asserted: a claim of entailment in a context in which the speaker is forbidden from vacuous speech. Intuitively, the job of the star operator is to separate two dimensions of speech acts: obligatorily non-vacuous, when the Zero is neglected, and possibly vacuous, as in classical logic.

[6] It might be objected that calling the star a speech act operator does not sit well with Definition 6, on which the star can take narrow scope under the logical constants. However, logical scope need not be understood as an environment utterly inaccessible to illocutionary force, even if it is on some theories of logic, particularly Frege's. A simple observation is the following. Consider a function ? that turns an assertion into a question. No doubt, ? is a speech act operator. In many logics of questions [11], a distinction is made between $?(Fx \vee Gx)$ and $(?Fx) \vee (?Gx)$, corresponding to the intuitive difference between (5a) and (5b).

(5) a. Is he going to Portugal or Spain? (I am also traveling to the Iberian peninsula) $?(Fx \vee Gx)$
b. Is he going to Portugal or is he going to Spain? (I forgot which.) $(?Fx) \vee (?Gx)$

In context, it is natural to understand the speaker of (5b) as asserting the disjunction but asking about the disjuncts. In contrast, the speaker of (5a) is not asserting the disjunction but asking about it. This example illustrates that speech act operators can interact with logical scope in subtle ways. There are several further questions about the interaction between force and logic: in particular, the Frege-Geach problem and its various manifestations [15, 17, 27, 42, 47], about which I will defer to the relevant literature. Thanks to an anonymous reviewer for asking me to clarify this point.

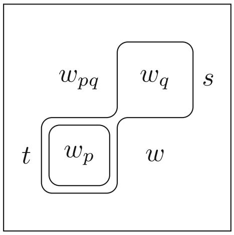

Fig. 2 States $s = \{w_p, w_q\}$ and $t = \{w_p\}$

cognitive tendency of speakers [8]. Moreover, despite some differences, the star is inspired by Maria Aloni's 'pragmatic intrusion' operator for the treatment of FREE CHOICE in Bilateral State-based Modal Logic [2], and it is similarly motivated. The option to Neglect Zero is also available in BSML.[7]

An immediate consequence of the enforcement of Neglect Zero is the dual of Observation 5 (the proof is omitted but straightforward): no sentence is non-vacuously asserted or rejected in the empty state.

Observation 7 $\varnothing \nvDash \phi^\star$ and $\varnothing \not\dashv \phi^\star$ for any ϕ in $\mathscr{L}_{\text{LLC}}$

Moreover, if a state non-vacuously asserts ϕ, then it asserts ϕ, likewise for rejection. The converse, of course, fails because of the empty state.

Observation 8 If $s \vDash \phi^\star$ then $s \vDash \phi$ and if $s \dashv \phi^\star$ then $s \dashv \phi$

For illustration of what the star does, consider a state $s = \{w_p, w_q\}$, as in Fig. 2, with subscripts indicating which atomic sentences are true at a world. By the conditions on the atoms (Definition 4), $s \nvDash p$ and $s \nvDash q$. However, $s \vDash p \vee q$ because s can be split into two substates, $\{w_p\}$ and $\{w_q\}$, which assert p and q respectively. Consider now state $t = \{w_p\}$. Despite $t \vDash p$ and $t \dashv q$, still $t \vDash p \vee q$ since $t = t \cup \varnothing$ and the empty state asserts anything, hence t can be split into two substates that assert p and q respectively. However, there is a difference in the assertability of disjunction in s and t, which is brought up by the star: $s \vDash [p \vee q]^\star$ and $t \nvDash [p \vee q]^\star$, since s is made out of non-empty substates that assert the disjuncts, unlike t. So s is a non-vacuous ground for asserting $p \vee q$, but t is not.

Another interesting remark concerning Fig. 2 is about polar rejection, and does not depend on the star. As we just saw, $s \vDash p \vee q$. Moreover, since $s \dashv p$ and $s \dashv q$, we have $s \dashv p \vee q$ as well. Hence $s \vDash \neg(p \vee q)$. Intuitively, $p \vee q$ is asserted in s because, from s, both a state that asserts p and a state that asserts q are "accessible"—interpreting set-theoretic inclusion as accessibility. However s is just as good a ground for rejecting $p \vee q$, since the state rejects both disjuncts. This remains so under the star: relative to Fig. 2, $s \vDash [p \vee q]^\star$ and $s \vDash [\neg(p \vee q)]^\star$. Thus the empty state is not

[7] In BSML, Neglect Zero is encoded in the syntax of the object language by means of an optional atom NE. Since such atom does not correspond to a lexicalized expression of natural language, following Aloni's notation would be somewhat at odds with the goals of this paper.

the only source of inconsistency in LLC: the zero is not the only "glutty" information state. It follows that negation in LLC does not satisfy the standard truth-table.[8]

2.1 Classical Validities

Despite the conclusion of the previous section, a number of classically valid results hold in LLC. As for the case of Observations 5 and 7, it is interesting to check whether the validity of an entailment depends on the zero. All of the entailments discussed in this section are valid in classical logic as well as in LLC, and remain valid whether the zero is allowed or not—so the topic of this section is the part of LLC that does not depend on the Neglect Zero assumption. I will write '$\phi \Leftrightarrow \psi$' if and only if $\phi \vDash \psi$ and $\psi \vDash \phi$.

Double negation is valid. Suppose $s \vDash \neg\neg\phi$. Then $s \dashv \neg\phi$, hence $s \vDash \phi$. In the opposite direction we reason in a similar fashion.

Observation 9 $\neg\neg\phi \Leftrightarrow \phi$

Conjunction implies disjunction but disjunction does not imply conjunction. Let $s \vDash \phi \wedge \psi$, then both $s \vDash \phi$ and $s \vDash \psi$. Since $s = s \cup s$, it follows that $s \vDash \phi \vee \psi$. Conversely, a countermodel to the disjunction-to-conjunction inference is given by a state s made out of two substates, each verifying one disjunct but not the other, so that the disjunction but not the conjunction is asserted in s. An example is in Fig. 2: $\{w_p, w_q\} \vDash p \vee q$, but since $\{w_p, w_q\} \nvDash p$ and $\{w_p, w_q\} \nvDash q$, $\{w_p, w_q\} \nvDash p \wedge q$.

Observation 10 $\phi \wedge \psi \vDash \phi \vee \psi$
$\phi \vee \psi \nvDash \phi \wedge \psi$

Observation 10 establishes the familiar relation between the top-left and down-left corners of the Triangle of Oppositions (Fig. 1) for LLC. I will now turn to the classically valid De Morgan entailments, all of which are valid in LLC.

$$\neg(\phi \wedge \psi) \vDash \neg\phi \vee \neg\psi \qquad \neg\phi \vee \neg\psi \vDash \neg(\phi \wedge \psi)$$
$$\neg(\phi \vee \psi) \vDash \neg\phi \wedge \neg\psi \qquad \neg\phi \wedge \neg\psi \vDash \neg(\phi \vee \psi)$$
$$\neg\phi \wedge \neg\psi \vDash \neg\phi \vee \neg\psi \qquad \neg(\phi \vee \psi) \vDash \neg(\phi \wedge \psi)$$
$$\neg\phi \wedge \neg\psi \vDash \neg(\phi \wedge \psi) \qquad \neg(\phi \vee \psi) \vDash \neg\phi \vee \neg\psi$$

Let $s \vDash \neg(\phi \wedge \psi)$. Then $s \dashv \phi \wedge \psi$. Then there are t, t' such that $s = t \cup t'$, $t \dashv \phi$ and $t' \dashv \psi$. Hence $t \vDash \neg\phi$ and $t' \vDash \neg\psi$. Hence $s \vDash \neg\phi \vee \neg\psi$. The opposite direction is similar.

Observation 11 $\neg\phi \vee \neg\psi \Leftrightarrow \neg(\phi \wedge \psi)$

Next, let $s \vDash \neg(\phi \vee \psi)$. Hence $s \dashv \phi$ and $s \dashv \psi$. Hence $s \vDash \neg\phi$ and $s \vDash \neg\psi$. Hence $s \vDash \neg\phi \wedge \neg\psi$. The converse is similar.

[8] Consequently, the down-left to up-right diagonal of the Triangle of Oppositions in Fig. 1 is not the relation of classical incompatibility. The notion of incompatibility in LLC is the topic of Section 4.2. Classical exclusionary negation depends on the stronger notion of rejection that will be introduced in Section 4.3.

Observation 12 $\neg\phi \wedge \neg\psi \Leftrightarrow \neg(\phi \vee \psi)$

The next one is simply an instance of Observation 10, and for the same reason its opposite direction fails.

Observation 13 $\neg\phi \wedge \neg\psi \vDash \neg\phi \vee \neg\psi$

Next, assume that $s \vDash \neg\phi \wedge \neg\psi$. Then $s \vDash \neg\phi$ and $s \vDash \neg\psi$. Thus $s \dashv \phi$ and $s \dashv \psi$. Hence $s \dashv \phi$ and $s \dashv \psi$. Since $s = s \cup s$ then $s \dashv \phi \wedge \psi$ hence $s \vDash \neg(\phi \wedge \psi)$.

Observation 14 $\neg\phi \wedge \neg\psi \vDash \neg(\phi \wedge \psi)$

Next, let $s \vDash \neg(\phi \vee \psi)$, and so $s \dashv \phi \vee \psi$. Hence $s \dashv \phi$ and $s \dashv \psi$. Since $s = s \cup s$ then $s \dashv \phi \wedge \psi$, and so $s \vDash \neg(\phi \wedge \psi)$.

Observation 15 $\neg(\phi \vee \psi) \vDash \neg(\phi \wedge \psi)$

Next, we reason from $s \vDash \neg(\phi \vee \psi)$ to $s \dashv \phi$ and $s \dashv \psi$. Thus $s \vDash \neg\phi$ and $s \vDash \neg\psi$, and so $s \vDash \neg\phi \vee \neg\psi$ by taking $s = s \cup s$.

Observation 16 $\neg(\phi \vee \psi) \vDash \neg\phi \vee \neg\psi$

Inspection of these entailments shows that they hold regardless of the empty state. Thus, they remain valid under the star. The star, however, is not an idle wheel in the system, as it will be apparent in the next section, when we turn to Negative Collapse. All entailments of Negative Collapse, which are not valid in CL, are valid in LLC (under a restriction) if assertion and rejection are non-vacuous.

2.2 Non-Standard Validities

In this section I will discuss some important consequences of Neglect Zero. The basic observation is the following: in LLC, speech acts performed non-vacuously are "stable" with respect to the order on the states. Let's first distinguish positive and negative sentences.

Definition 17 A sentence ϕ of $\mathscr{L}_{\text{LLC}}$ is *positive* iff every atom in ϕ is in the scope of an even number of $\neg$s. A sentence ϕ of $\mathscr{L}_{\text{LLC}}$ is *negative* iff every atom in ϕ is in the scope of an odd number of $\neg$s.

For example, $p \vee q$ is positive, $\neg p$ and $\neg(p \vee q)$ are negative, and $p \wedge \neg q$ is neither positive nor negative. Positive sentences have the following property: if non-vacuously rejected in a state, they are non-vacuously rejected in all superstates. No matter how much information is added by expanding an information state, illocutionary force does not flip. Negative sentences have the opposite property: if they are non-vacuously asserted, they are non-vacuously asserted in all superstates.

Lemma 1 LEXICAL REJECTION. *Let ϕ be positive. If $t \dashv \phi^\star$ then for all $s \supseteq t : s \dashv \phi^\star$*

In other words, non-vacuous rejection is upward monotonic for positive sentences in LLC. Intuitively, if a singleton state $s = \{w\}$ rejects p, that's because $V(w, p) = 0$. Then, w is a member of all superstates of s, and so all superstates of s non-vacuously reject p. A proof of LEXICAL REJECTION is in the Appendix. The positive counterpart of the Lemma also holds: non-vacuous assertion is upward monotonic for negative sentences (as shown in the Appendix).

LEXICAL REJECTION is necessary to prove all four classically invalid entailments of Negative Collapse, repeated below. Since LEXICAL REJECTION holds for positive sentences, Negative Collapse is valid under a restriction to negative sentences and assuming Neglect Zero. I will write '$\phi^\star \vDash \psi^\star$' to indicate a valid argument whose single premise and conclusion are non-vacuously asserted.

$$\text{(FC) } [\neg\phi \vee \neg\psi]^\star \vDash [\neg\phi \wedge \neg\psi]^\star \qquad [\neg(\phi \wedge \psi)]^\star \vDash [\neg(\phi \vee \psi)]^\star$$
$$[\neg(\phi \wedge \psi)]^\star \vDash [\neg\phi \wedge \neg\psi]^\star \qquad [\neg\phi \vee \neg\psi]^\star \vDash [\neg(\phi \vee \psi)]^\star$$

FC is a disjunction to conjunction inference. The countermodel described in Observation 10 is blocked in LLC in case assertion is non-vacuous and the disjunction is a negative sentence. Suppose that $s \vDash [\neg\phi \vee \neg\psi]^\star$ and assume that the disjunction is a negative sentence. Then $s \vDash [\neg\phi]^\star \vee [\neg\psi]^\star$. Then there are t, t' such that $s = t \cup t'$ and $t \vDash [\neg\phi]^\star$ and $t' \vDash [\neg\psi]^\star$. Then $t \vDash \neg\phi^\star$ and $t' \vDash \neg\psi^\star$. Then $t \dashv \phi^\star$ and $t' \dashv \psi^\star$ and both ϕ and ψ are positive. By LEXICAL REJECTION, $s \dashv \phi^\star$ and $s \dashv \psi^\star$. Hence $s \vDash [\neg\phi]^\star$ and $s \vDash [\neg\psi]^\star$. Hence $s \vDash [\neg\phi \wedge \neg\psi]^\star$.

Observation 18 (Restricted) FC: $[\neg\phi \vee \neg\psi]^\star \vDash [\neg\phi \wedge \neg\psi]^\star$

The restriction to negative sentences is, in a way, to be expected, since in LLC double negation holds (Observation 9). Hence, just as $p \vee q \nvDash p \wedge q$ (Observation 10), so too $\neg\neg p \vee q \nvDash \neg\neg p \wedge q$. But, by Definition 17, $\neg\neg p \vee q$ is not a negative sentence, despite the occurrence of the negation symbol.

Next, a conjunction of negatives follows from negated conjunction under the restrictions indicated. Let $s \vDash [\neg(\phi \wedge \psi)]^\star$ and assume that the sentence is negative. Then $s \dashv \phi^\star \wedge \psi^\star$. Then there are t, t' such that $s = t \cup t'$ and $t \dashv \phi^\star$ and $t' \dashv \psi^\star$, and both ϕ and ψ are positive. Since s is a superstate of both t and t' then $s \dashv \phi^\star$ and $s \dashv \psi^\star$ by LEXICAL REJECTION. Hence $s \vDash \neg\phi^\star$ and $s \vDash \neg\psi^\star$ and the conclusion follows.[9]

Observation 19 (Restricted) $[\neg(\phi \wedge \psi)]^\star \vDash [\neg\phi \wedge \neg\psi]^\star$

Consider next the inference from negated conjunction to negated disjunction. Let $s \vDash [\neg(\phi \wedge \psi)]^\star$ and assume that the sentence is negative. Hence $s \dashv \phi^\star \wedge \psi^\star$. Hence $s = t \cup t'$, $t \dashv \phi^\star$, and $t' \dashv \psi^\star$, and both ϕ and ψ are positive. Since s is a superstate of both t and t', $s \dashv \phi^\star$ and $s \dashv \psi^\star$ by LEXICAL REJECTION. Hence $s \dashv \phi^\star \vee \psi^\star$. Hence $s \vDash [\neg(\phi \vee \psi)]^\star$.

Observation 20 (Restricted) $[\neg(\phi \wedge \psi)]^\star \vDash [\neg(\phi \vee \psi)]^\star$

The last inference of Negative Collapse is from a disjunction of negatives to negation of disjunction. Assume that $s \vDash [\neg\phi \vee \neg\psi]^\star$ and that the sentence is negative. Hence there are t, t' such that $s = t \cup t'$ and $t \vDash \neg\phi^\star$ and $t' \vDash \neg\psi^\star$. Hence $t \dashv \phi^\star$ and

[9] Observation 19 shows that in LLC non-vacuously asserted conjunction behaves like informational conjunction in bilattice algebras [3, 14, 18]. Informational conjunction coincides with Boolean conjunction in its assertion-conditions, but its rejection-conditions require both conjuncts to be rejected.

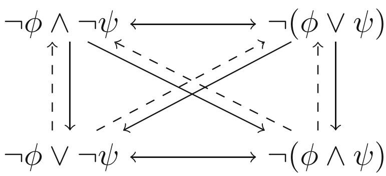

Fig. 3 Lemma 3. Full arrows are classically valid entailments, also valid in LLC whether or not Neglect Zero is assumed. Dashed arrows are entailments that are valid in LLC, relative to negative sentences, if Neglect Zero is assumed

$t' \Dashv \psi^\star$ and both sentences are positive. By LEXICAL REJECTION, $s \Dashv \phi^\star$ and $s \Dashv \psi^\star$. Hence $s \Dashv \phi^\star \vee \psi^\star$. Therefore $s \vDash [\neg(\phi \vee \psi)]^\star$.

Observation 21 (Restricted) $[\neg\phi \vee \neg\psi]^\star \vDash [\neg(\phi \vee \psi)]^\star$

Observations 18, 19, 20, and 21, together with the last four observations of the previous section (Observations 13, 14, 15, 16, all of which hold unrestrictedly), establish that the following Lemma is valid under the restriction to negative sentences. I use '$\phi^\star \Leftrightarrow \psi^\star$' to abbreviate $\phi^\star \vDash \psi^\star$ and $\psi^\star \vDash \phi^\star$.

Lemma 2 (Restricted) *The following equivalences hold in* LLC.

$$[\neg\phi \vee \neg\psi]^\star \Leftrightarrow [\neg\phi \wedge \neg\psi]^\star \qquad [\neg(\phi \wedge \psi)]^\star \Leftrightarrow [\neg(\phi \vee \psi)]^\star$$
$$[\neg(\phi \wedge \psi)]^\star \Leftrightarrow [\neg\phi \wedge \neg\psi]^\star \qquad [\neg\phi \vee \neg\psi]^\star \Leftrightarrow [\neg(\phi \vee \psi)]^\star$$

Together with the two classically valid De Morgan equivalences (Observations 11 and 12), which are valid in LLC with no restrictions, Lemma 2 entails the following result.

Lemma 3 (Restricted) *If Neglect Zero is assumed, all De Morgan sentences are equivalent in* LLC.

That is, if assertion and rejection are non-vacuous, and relative to negative sentences, distinctions among De Morgan sentences collapse in LLC. Hence, under certain conditions, LLC expresses but one binary operator on the negative side of Fig. 1. Figure 3 summarizes this result.

3 The Binary Connectives

LLC is the logic of lexical connectives because, under Neglect Zero, the only binary connectives that can be expressed are *and, or*, and *nor* (setting aside trivial connectives, see Section 3.2 below).

Let's begin with conjunction and disjunction. As I noted above (Observation 10), assertion of conjunction implies assertion of disjunction but not *vice versa*, and this relation is preserved under the star. The relation between $\wedge$ and $\vee$ is thus the familiar one. Moreover, the behavior of conjunction is entirely ordinary. A conjunct does not imply a conjunction but both do. The second claim is obvious from the assertion-conditions of conjunction. For the first claim, suppose that $s \vDash p$. Let $s = \{w\}$ and suppose that $V(w, p) = 1$ and $V(w, q) = 0$. Then $s \nvDash q$. Hence $s \nvDash p \wedge q$. Finally, conjunction implies its conjuncts (obviously, given its assertion-conditions). It is easy to verify that all of these inferences are valid under the star.

Observation 22 $\phi, \psi \vDash \phi \wedge \psi$
$\phi \nvDash \phi \wedge \psi$
$\phi \wedge \psi \vDash \phi$

A disjunct suffices to support a disjunction, but only if the zero is allowed. Let $s \vDash \phi$. Since $\varnothing \vDash \psi$ for any ψ (Observation 5) and $s = s \cup \varnothing$, then $s \vDash \phi \vee \psi$. Thus $\phi \vDash \phi \vee \psi$. However, this inference fails under Neglect Zero. If both disjuncts are asserted in a state s, then of course s asserts the disjunction, since one can always take $s = s \cup s$. Finally, and as expected, a disjunction does not imply a disjunct (regardless of the zero; proof omitted but obvious).

Observation 23 $\phi \vDash \phi \vee \psi$
$\phi, \psi \vDash \phi \vee \psi$
$\phi \vee \psi \nvDash \phi$

Unlike conjunction, whose inferential behavior is not affected by Neglect Zero, Disjunction Introduction is valid only when we consider all states in a model, including in particular the empty state. Thus, one cannot generally go from $s \vDash \phi$ or $s \vDash \phi^\star$ to $s \vDash [\phi \vee \psi]^\star$. Nevertheless, there are at least three reasons to suggest that natural language *or* is expressed in LLC. The first reason is that the truth-conditions for $\vee$, given in Definition 2, are unchanged whether or not the Neglect Zero assumption is in place. Some inferences might fail if the context requires high standards for the performance of speech acts, but this does not imply a change of meaning.

The second reason is that it would be unwise to assume that Disjunction Introduction in generally valid in natural language. Disjunction Introduction might fail in LLC but this is how it should be. Depending on context, the inference from (6a) to (6b) may be a pragmatic norm violation [19, 43], and in combination with modals, imperatives, and performatives, Disjunction Introduction fails: this is the case of the inference from (7a) to (7b), of the inference from (8a) to (8b), also known as Ross's paradox, and of the inference from (9a) to (9b) [1, 38, 48, 52].

(6) a. Russell was a great philosopher.
b. Therefore, Russell was a great philosopher or Hegel is overestimated.
(7) a. She may go to the beach.
b. Therefore, she may go to the beach or she may go to the cinema.
(8) a. Clean your room.
b. Therefore, clean your room or burn the house down.
(9) a. You are fired.
b. Therefore, you are fired or you are our new CEO.

Finally, failure of Disjunction Introduction may also explain why, in contexts that support FREE CHOICE, it is not the case that (10a) implies (10b) (see [2, 29], and Section 1 above).

(10) a. Paul might be Dutch.

b. Paul might be Dutch and might be Danish.

Hence, LLC expresses the meanings of *and* and *or*, even under Neglect Zero, as meaning is revealed by assertion- and rejection-conditions and inferential role.

3.1 The Meaning of *nor*

By Lemma 3, LLC expresses but one negative relation between atoms p and q if the zero is out. We may indicate such relation equivalently by '$s \vDash [\neg p \wedge \neg q]^\star$' or '$s \vDash [\neg p \vee \neg q]^\star$'. This relation is the meaning of p *nor* q: the following inferences hold, and characterize the inferential profile of *nor* (cf. example (3) above).

$$\begin{array}{rlrl}
\neg\phi & \nvDash [\neg\phi \wedge \neg\psi]^\star & \neg\phi & \nvDash [\neg(\phi \wedge \psi)]^\star \\
\neg\phi, \neg\psi & \vDash [\neg\phi \wedge \neg\psi]^\star & \neg\phi, \neg\psi & \vDash [\neg(\phi \wedge \psi)]^\star \\
\neg\phi & \nvDash [\neg\phi \vee \neg\psi]^\star & \neg\phi & \nvDash [\neg(\phi \vee \psi)]^\star \\
\neg\phi, \neg\psi & \vDash [\neg\phi \vee \neg\psi]^\star & \neg\phi, \neg\psi & \vDash [\neg(\phi \vee \psi)]^\star \\
[\neg\phi \wedge \neg\psi]^\star & \vDash \neg\phi & [\neg(\phi \wedge \psi)]^\star & \vDash \neg\phi \\
[\neg\phi \vee \neg\psi]^\star & \vDash \neg\phi & [\neg(\phi \vee \psi)]^\star & \vDash \neg\phi
\end{array}$$

That is, $\neg\phi$ alone does not suffice to support the one negative meaning expressed in LLC under Neglect Zero, for $s \vDash \neg\phi$ fails to entail $s \vDash \neg\phi \wedge \neg\psi$ (Observation 22), hence it fails to entail $s \vDash [\neg\phi \wedge \neg\psi]^\star$ (Observation 8), and it fails to entail $s \vDash [\neg\phi \vee \neg\psi]^\star$ (Observation 23). However, $\neg\phi$ and $\neg\psi$ together support the relevant inferences. Conversely, this one negative meaning suffices to support $\neg\phi$ (under the restrictions we are considering), for $[\neg\phi \wedge \neg\psi]^\star \vDash \neg\phi \wedge \neg\psi$ by Observation 8 and $\neg\phi \wedge \neg\psi \vDash \neg\phi$ by the truth-conditions of conjunction, and $[\neg\phi \vee \neg\psi]^\star \vDash \neg\phi$ by FC (Observation 18). Thus the one negative meaning of LLC, under Neglect Zero, exhibits the inferential pattern of *nor* seen in example (3): the truth of *neither* ϕ nor ψ follows from the truth of both $\neg\phi$ and $\neg\psi$ jointly but not individually, and implies each. The same holds if $\neg$ takes wide scope (see the right column above). It also follows that this one negative meaning expressed in LLC under Neglect Zero is not **nand*.

3.2 Other Connectives

So, under a restriction, LLC expresses *and, or, nor* and does not express *nand*. Similarly, the lexicon of natural language does not express any of the other binary Boolean operators. Setting aside trivial operators, LLC does not express them either. The trivial connectives, the Falsum $\bot$ and the Verum $\top$, are expressible in LLC but are not attested in the lexicon of natural language. However, there are good independent reasons why these are not lexicalized. A common and plausible assumption is that trivial connectives have not evolved for reasons of cognitive economy [16].

The interesting candidates are the material conditional, the material biconditional, and exclusive disjunction. However, none of these are expressed in LLC under the restrictions that define the lexical connectives. The material conditional is expressed in classical logic as '$\neg\phi \vee \psi$'. However, suppose that $s \vDash [\neg p \vee q]^\star$. Then there are substates t and t' such that $s = t \cup t'$ and $t \vDash \neg p^\star$ and $t \vDash q^\star$. Hence $t \dashv p^\star$ and so, since p is positive, $s \dashv p^\star$ by LEXICAL REJECTION. Hence $s \vDash \neg p$. Therefore, if assertion is non-vacuous the inferential behavior of $\neg p \vee q$ is not that of the material conditional.

Observation 24 $[\neg p \vee q]^\star \vDash \neg p$

Exclusive disjunction is expressed in a classical setting by '$(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$' but we know that this sentence does not capture Boolean exclusive disjunction since its second conjunct does not express the classical complement of conjunction under Neglect Zero. Therefore, the material biconditional isn't expressed in LLC either. Thus, if Neglect Zero is assumed, LLC expresses only *and, or,* and *nor* (the latter, under a restriction to negative sentences), and no other non-trivial connective. I conclude with the following result, which follows from Lemma 3 and the remarks in this section.

Theorem 1 *If Neglect Zero is assumed,* LLC *characterizes the expressive power of the lexicon of natural language with regards to the binary connectives.*

In the next section I will focus on the relationship between LLC and classical logic CL.

4 Relation to Classical Logic

This section has three parts: on the Triviality Argument, on the notion of incompatibility, and on "recapturing" classical logic.

4.1 The Triviality Argument

As we have seen, $\neg\phi \vDash \neg\phi \vee \neg\psi$ (Observation 23), and $[\neg\phi \vee \neg\psi]^\star \vDash \neg\psi$ (by Observation 18). Doesn't this imply that any $\neg\psi$ follows from any $\neg\phi$? This is a version of the Triviality Argument. But the Triviality Argument is invalid, due to an equivocation between disjunction loosely supported by a state that asserts one disjunct only, and disjunction under the star, in which such a configuration is ruled out. In other words, in LLC we have:

$$s \vDash \neg\phi \Rightarrow s \vDash \neg\phi \vee \neg\psi \qquad \not\Rightarrow s \vDash \neg\psi$$
$$s \vDash \neg\phi \not\Rightarrow s \vDash [\neg\phi \vee \neg\psi]^\star \quad \Rightarrow s \vDash \neg\psi$$

So there is no path from $\neg\phi$ to $\neg\psi$. The first path is blocked because, although Disjunction Introduction is valid if the zero is allowed, disjunction asserted while allowing for the zero does not entail a disjunct. Intuitively, if one somewhat vacuously asserts that Lisa is not at school or not at home (having information that Lisa is not at school), it doesn't follow that Lisa is not at home. The second path is blocked because, although non-vacuous assertion of disjunction may entail a disjunct, the star blocks Disjunction Introduction. Intuitively, it is not the case that any non-vacuous ground to assert 'Lisa is not at the movies' is a non-vacuous ground to assert 'Lisa is not at the movies nor at the supermarket', since a non-vacuous ground to assert the former but not the latter is given by information that Lisa is at the supermarket. Other apparent routes to triviality via the entailments in Neglect Zero are analyzed in a similar fashion.

4.2 Negation and Incompatibility

Incompatibility is regarded as the hallmark of negation [6]. However, as observed above with respect to Fig. 2, the same state s may support both $p \vee q$ and $\neg(p \vee q)$. However, it may be that $s \nvDash p$ and therefore contradiction does not entail everything. Indeed, EFQ holds in LLC among atoms but not in general.

Observation 25 $p \wedge \neg p \vDash \psi$

This is because if $s \vDash p$ and $s \vDash \neg p$, then $s \Dashv p$, hence $s = \varnothing$. Interestingly, if Neglect Zero is assumed it is still the case that EFQ holds only among atoms, but for a different reason: no state (not even the empty state) is such that $s \vDash [p \wedge \neg p]^{\star}$. Consequently, contradictory atomic sentences entail anything under Neglect Zero because logical consequence is void (Definition 3).

Failures of EFQ might raise a general worry about the relationship between negation and the notion of incompatibility in LLC. One way to express this relationship is as empty intersection between states. $s \vDash \neg\phi$ if and only if, for all t, $t \vDash \phi$ only if $s \cap t = \varnothing$. Both directions fail in LLC. A counterexample to the LtR direction is of the same sort as that just seen: it may be that $s \vDash \neg(p \vee q)$ and $s \vDash p \vee q$, and yet $s \neq \varnothing$. The other direction fails because of the empty state. Since $s \cap \varnothing = \varnothing$ for any s, it follows that if $\varnothing \vDash p$ then $s \cap \varnothing = \varnothing$. But it is not the case that $s \vDash \neg p$ for any s. Both directions fail under Neglect Zero as well. The above mentioned counterexample to the LtR direction also shows that $s \vDash [\neg\phi]^{\star}$ does not imply that $t \vDash \phi^{\star}$ only if $s \cap t = \varnothing$ for any t. For the opposite direction, notice that assuming that $s \cap t = \varnothing$ for any t such that $t \vDash \phi^{\star}$ is compatible with $s = \varnothing$. In this case, $s \nvDash [\neg\phi]^{\star}$.

However, a more limited characterization of incompatibility holds—but only if we are considering all states in a model, that is, including the empty state in particular. In bilateral systems, besides the standard unilateral *reductio* rules, there are also *Smileian* or bilateral *reductio* rules [26, 44]. Intuitively, if the assertion of ϕ leads to contradiction, then ϕ must be rejected, and *vice versa*. Smileian *reductio*s hold in LLC.

Lemma 4 *Smileian reductio: If* $s \vDash \phi$ *then* $s = \varnothing \Rightarrow s \Dashv \phi$
If $s \Dashv \phi$ *then* $s = \varnothing \Rightarrow s \vDash \phi$

Consider the first implication. Suppose that either $s \nvDash \phi$ or $s = \varnothing$. If $s = \varnothing$, the conclusion is obvious by Observation 5. Otherwise, the conclusion follows from the following lemma: failure of assertion is rejection, and *vice versa*. The second implication is proved in a similar way.

Lemma 5 *If* $s \nvDash \phi$ *then* $s \Dashv \phi$ *and if* $s \not\Dashv \phi$ *then* $s \vDash \phi$

For the proof of this Lemma, see the Appendix. The converse implications fail. One example is given by Fig. 2 (in which $s \Dashv p \vee q$, which does not entail that if $s \vDash p \vee q$

then $s = \varnothing$), other examples are due to the empty set. Moreover, the proof strategy for Smileian *reductio* relies on the empty state, and it fails if Neglect Zero is assumed.

A straightforward consequence of Lemma 5 is that the unilateral *reductio* rules are valid—again, provided we are considering "full" models that include the zero. For the intuitionistic version of *reductio*, let '$\bot$' be short for '$q \wedge \neg q$' and suppose that $\Gamma, \phi \vDash \bot$. From the assumption, for all models and states, if $s \vDash \phi$ (and s supports all of Γ) then $s \vDash q \wedge \neg q$. Hence if $s \vDash \phi$ then $s \vDash q$ and $s \vDash \neg q$, and so $s = \varnothing$ by Definition 2. Therefore if $s \vDash \phi$ (and s supports all of Γ) then $s = \varnothing$. Therefore $s \dashv \phi$ by Smileian *reductio* (Lemma 4). The classical version of unilateral *reductio* can be established in a similar way.

Observation 26 If $\Gamma, \phi \vDash \bot$ then $\Gamma \vDash \neg\phi$
If $\Gamma, \neg\phi \vDash \bot$ then $\Gamma \vDash \phi$

The previous remarks about Smileian and unilateral *reductio* crucially rely on the role of the empty state. Indeed, all *reductio* rules fail in LLC under Neglect Zero. This is not surprising: after all, appeal to absurdity is crucial to *reductio* reasoning, and vacuous grounds are ruled out by Neglect Zero. A consequence of this failure is that some classical inference patterns that are valid in LLC when one considers all states within a model have countermodels under Neglect Zero. A case in point is the Law of Excluded Middle. Suppose for *reductio* that for some model M and state s, $s \nvDash \phi \vee \neg\phi$. Thus for any t and t' such that $s = t \cup t'$ it is not the case that both $t \vDash \phi$ and $t' \vDash \neg\phi$. However, either t or t' could be empty. Let $t = \varnothing$. Then $t \vDash \phi$ by Observation 5 and therefore the assumption entails that no state $t' = s$ is such that $t' \vDash \neg\phi$. This is false. Thus there could be no such s.

Observation 27 LEM. $\vDash \phi \vee \neg\phi$

So LEM is a tautology of LLC. Parallel reasoning establishes the Law of Non-Contradiction. However, the proof in both cases fails if we are not allowed to assume that one of the substates is empty: indeed, it is easy to find counterexamples to $s \vDash [\phi \vee \neg\phi]^{\star}$. There are other failures of classical reasoning under the star.

In sum, if the zero is allowed, ϕ and $\neg\phi$ are exhaustive in LLC but not mutually exclusive, except on atoms. Nevertheless, negation expresses incompatibility in LLC in the sense captured by the *reductio* rules, provided all states are considered. However, this is not full-blown classical incompatibility, for assertion is not in general equivalent to failure of rejection, except on atoms.

4.3 Classical Logic

A simple and well-motivated adjustment leads from LLC to the classical propositional calculus CL. The adjustment is in two steps. First, it must always be possible to assert and reject on the basis of the empty state. That is, there is no longer an option to neglect the zero. Second, as seen in the previous Section 4.2, the negation operator in LLC is not classical negation. As I mentioned above, the language $\mathscr{L}_{\text{LLC}}$ is the same for LLC and CL, and so are the assertion- and rejection-conditions for $\neg$, $\vee$, $\wedge$.

Negation is definable in a bilateral setting by the opposition of assertion and rejection. This, in turn, means that the behavior of $\neg$ rests on the properties of assertion and rejection. Polar rejection is insufficient for classical negation. Classical rejection is a stronger requirement on states: s classically rejects p if and only if all non-empty substates of s fail to assert p.

$$\text{Polar Rejection: } s \dashv p \text{ iff either } s = \varnothing \text{ or } s \nvDash p$$
$$\text{Classical Rejection: } s \dashv p \text{ iff for all } t \subseteq s: \text{ either } t = \varnothing \text{ or } t \nvDash p$$

Of course, polar rejection is implied by classical rejection, but not *vice versa*. Classical rejection entails that s rejects p if and only if every singleton substate $\{w\}$ of s fails to assert p. Therefore, every $w \in s$ must falsify p. This means that assertion and classical rejection can both be stated as universal conditions on states.

Definition 28 Classical bilateral truth-conditions for atoms.

$$s \vDash p \text{ iff for all } w \in s : V(w, p) = 1$$
$$s \dashv p \text{ iff for all } w \in s : V(w, p) = 0$$

As I noted above, assertion has a demanding standard: all worlds in the state must agree that p is true. Polar rejection is comparatively undemanding: one world in which p is false is enough for polar rejection. Classical rejection restores the balance: both assertion and classical rejection require unanimity amongst worlds within a state.

The logic obtained by the classical assertion- and rejection-conditions for atoms of Definition 28, together with the semantic core of the Introduction (Section 1.3: in particular, Definitions 2 and 3), always allowing for the empty state, is classical logic CL [12]. In other words, classical logic is the bilateral system obtained from LLC with classical rejection, and in which no speech acts are starred and thus models always include the empty state. This latter condition is unsurprising: after all, a staple of classical reasoning is its reliance on reasoning from absurdity. It is worth noting that Neglect Zero may still have a role to play for the reasons noted in Section 3 about the behavior of disjunction in natural language, and that may suggest a departure from classical logic. Indeed, working with a classical definition of rejection, Aloni [2] assumes Neglect Zero to account for FREE CHOICE and related phenomena.

There is a sense in which the meaning of any of the operators in $\mathscr{L}_{\text{LLC}}$ does not change between CL and LLC, at least insofar as meaning is expressed by the relation between speech acts, information states, and logical form. After all, $\neg$, $\wedge$, and $\vee$ have the same assertion- and rejection-conditions in CL and in LLC. In particular, negation just expresses the opposition between assertion and rejection. With classical rejection, however, assertion of $\neg\phi$ is thoroughly incompatible with assertion of ϕ: if two states support a formula and its negation, respectively, then their intersection must be empty.

On what grounds is it legitimate to introduce classical rejection? Natural language negation *not* allows us to express all Boolean connectives compositionally. For example, ϕ *nand* ψ is typically expressed in English by *not both* ϕ *and* ψ. It is clear that the negation of LLC based on polar rejection does not allow us to express all Boolean connectives. And so it is plausible to assume that natural language negation *not* can and often does express classical rejection [26]. Thanks to classical rejection, it is possible to re-capture all connectives of classical logic. For example, ϕ *and* ψ may be expressed as $\neg(\phi \wedge \psi)$ once the clause for $\neg$ is defined in terms of classical rejection.

The semantic model thus described has two tiers. We begin with polar rejection, characterized as non-assertion. Assertion and polar rejection determine the expressive limits of the lexicon, which are defined in LLC assuming Neglect Zero. The binary connectives that may be so introduced at this first tier are only *and, or,* and *nor*, and these are the only coordinating connectives attested in the lexicon of natural language. But of course the expressive power of natural language as a whole is vastly greater. The additional expressive power is typically achieved by combining the lexicalized binary connectives, *and, or, nor*, with negation *not*. This step can be modeled by assuming that *not* can contribute a stronger notion of rejection, classical rejection, and by imposing no restrictions on the zero. It is at this second tier that, as in natural language, we may express all distinctions of classical logic. The two-tiered model accounts for the differences in expressive power between the lexicon and natural language as a whole. On this picture, compositionality arises, in the way it does, by reason of logic.

5 Conclusion

The lexical coordinators *and, or, nor*, are usually analyzed as Boolean operators. This analysis captures the truth-conditions of sentences in which the coordinators occur, but it leaves something out. Conjunction and disjunction are treated as duals under negation, but natural language does not treat them as such in all respects: the negation of disjunction may be expressed lexically but not the negation of conjunction. Existing accounts of lexicalization assume a background Boolean logic and leave to Gricean pragmatic principles the task to explain failures of lexicalization, but such accounts are open to criticism. I indicated two main issues: (a) most Gricean accounts need to be supplemented with additional assumptions to rule out non-trivial unattested material, (b) Gricean accounts don't shed light on the relation between compositionality and the expressive limitations of the lexicon [9, 25].

The main contribution of this article is the Logic of Lexical Connectives: a two-tiered model of the expressive power of the lexicon, as apparent from the semantics of *and, or,* and *nor*, together with cross-linguistic generalizations about lexicalizability, as well as of the expressive power of natural language as a whole. I have shown how to link the two levels in terms of natural assumptions about the speech acts of assertion and rejection, and the possibility of reasoning from vacuous information. The overt negation operator *not* of natural language can express a strong notion of rejection, that permits the recovery of all classical meanings if vacuous reasoning is always allowed. On this model we have a single explanation of the limits of lexicalization for the binary coordinators, and an indication of the role of compositionality, improving on both issues (a) and (b).

An important component of the present account is the Neglect Zero assumption, implemented as the star operator. The assumption is independently motivated and independently plausible as a cognitive constraint on the performance of speech acts. Like Gricean accounts of lexicalization, the present account rests on assumptions about language use and cognitive resources, but unlike Gricean accounts, the present account does not include a model of communication and does not rely on scalar reasoning.

The resulting two-tiered model is a complete and compositional account of the semantic space of the lexical coordinators, which accounts for why we don't have more logic at the lexical level than we do, and explains in logical terms why the expressive power of natural language greatly outstrips the expressive power of the lexicon.

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Declarations

Ethics Approval Not applicable.

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Appendix

Proof of Lemma 1. LEXICAL REJECTION. Let ϕ be positive. If $t \dashv \phi^\star$ then for all $s \supseteq t : s \dashv \phi^\star$.

Recall Definition 17: a sentence ϕ of $\mathscr{L}_{\text{LLC}}$ is positive iff every atom in ϕ is in the scope of an even number of $\neg$s; a sentence ϕ of $\mathscr{L}_{\text{LLC}}$ is negative iff every atom in ϕ is in the scope of an odd number of $\neg$s. In order to establish the Lemma we prove something stronger, namely the following generalization alongside the Lemma, for the purposes of induction. Let ϕ be negative: If $t \vDash \phi^\star$ then for all $s \supseteq t : s \vDash \phi^\star$. We run two inductions in parallel for positive and negative sentences. The basis for the positive case are positive literals: these sentences are in the scope of the least even number of $\neg$s, namely 0. Suppose that $t \dashv p^\star$. Then $t \dashv p$ and $t \neq \varnothing$ by Definition 6. Then there is a world $w \in t : V(w, p) = 0$. Let $s \supseteq t$. Then $w \in s$, hence $s \dashv p$ and $s \neq \varnothing$. Therefore $s \dashv p^\star$. The basis for the negative case are negative literals. Suppose that $t \vDash [\neg p]^\star$. Then $t \vDash \neg p^\star$ and so $t \dashv p^\star$ hence $t \dashv p$ and $t \neq \varnothing$. Therefore $s \dashv p^\star$ by the previous reasoning. Hence $s \vDash \neg p^\star$ and so $s \vDash [\neg p]^\star$.

Suppose that $\neg\phi$ is positive and $t \dashv [\neg\phi]^\star$. Then $t \dashv \neg\phi^\star$, hence $t \vDash \phi^\star$ and ϕ is negative since it has one wide-scoping negation less. Assume the IH about ϕ. Then for all $s \supseteq t : s \vDash \phi^\star$. Hence $s \dashv \neg\phi^\star$ and so $s \dashv [\neg\phi]^\star$. Suppose that $\neg\phi$ is negative and $t \vDash [\neg\phi]^\star$. Then $t \vDash \neg\phi^\star$. Hence $t \dashv \phi^\star$ and ϕ is positive. Assume the IH about ϕ. Then for all $s \supset t$: $s \dashv \phi^\star$. Hence $s \vDash \neg\phi^\star$ and so $s \vDash [\neg\phi]^\star$.

Suppose that $\phi \vee \psi$ is positive and $t \dashv [\phi \vee \psi]^\star$. Then both ϕ and ψ are positive and $t \dashv \phi^\star$ and $t \dashv \psi^\star$. Assume the IH about ϕ and ψ. Then for all $s \supseteq t : s \dashv \phi^\star$ and $s \dashv \psi^\star$. Hence $s \dashv \phi^\star \vee \psi^\star$ and so $s \dashv [\phi \vee \psi]^\star$. Suppose that $\phi \vee \psi$ is negative and $t \vDash [\phi \vee \psi]^\star$. Then both ϕ and ψ are negative and there are t' and t'' such that $t = t' \cup t''$ and $t' \vDash \phi^\star$ and $t'' \vDash \psi^\star$. Assume the IH about ϕ and ψ. Then for all $s \supseteq t' : s \vDash \phi^\star$ and for all $s \supseteq t'' : s \vDash \psi^\star$. Since all superstates of t are superstates of both t' and t'', for all $s \supseteq t$ both $s \vDash \phi^\star$ and $s \vDash \psi^\star$. Since $s = s \cup s$ then $s \vDash \phi^\star \vee \psi^\star$ for all $s \supseteq t$. Hence $s \vDash [\phi \vee \psi]^\star$ for all such states.

Suppose that $\phi \wedge \psi$ is positive and $t \dashv [\phi \wedge \psi]^\star$. Then both ϕ and ψ are positive and there are t' and t'' such that $t = t' \cup t''$ and $t' \dashv \phi^\star$ and $t'' \dashv \psi^\star$. Assume the IH about ϕ and ψ. Then for all $s \supseteq t' : s \dashv \phi^\star$ and for all $s \supseteq t'' : s \dashv \psi^\star$. Since all superstates of t are superstates of both t' and t'', for all $s \supseteq t$ both $s \dashv \phi^\star$ and $s \dashv \psi^\star$. Since $s = s \cup s$ then $s \dashv \phi^\star \wedge \psi^\star$ for all $s \supseteq t$. Hence $s \dashv [\phi \wedge \psi]^\star$ for all such states. Suppose that $\phi \wedge \psi$ is negative and $t \vDash [\phi \wedge \psi]^\star$. Then both ϕ and ψ are negative and $t \vDash \phi^\star$ and $t \vDash \psi^\star$. Assume the IH about ϕ and ψ. Then for all $s \supseteq t : s \vDash \phi^\star$ and $s \vDash \psi^\star$. Hence $s \vDash \phi^\star \wedge \psi^\star$ and so $s \vDash [\phi \wedge \psi]^\star$.

Proof of Lemma 5. If $s \nvDash \phi$ then $s \dashv \phi$ and if $s \not\dashv \phi$ then $s \vDash \phi$.
This is shown by induction, generalizing from the atomic case. For the atomic case, (i) suppose that $s \nvDash p$. Then $s \dashv p$ by Definition 2. (ii) Suppose that $s \not\dashv p$. Then $s \neq \varnothing$ and $s \vDash p$ by the same definition.

Assume the Induction Hypothesis about ϕ. (i) Suppose that $s \nvDash \neg\phi$. Then $s \not\dashv \phi$. Then $s \vDash \phi$ by the IH. Then $s \dashv \neg\phi$ (ii) Suppose that $s \not\dashv \neg\phi$. Then $s \nvDash \phi$. Then $s \dashv \phi$ by the IH. Then $s \vDash \neg\phi$.

Assume the IH about ϕ and ψ. (i) Suppose that $s \nvDash \phi \vee \psi$. Then it is not the case that there are t and t' such that $s = t \cup t'$ and $t \vDash \phi$ and $t' \vDash \psi$. Let $s = t$ and $t' = \varnothing$. Since $\varnothing \vDash \psi$, the assumption implies that $s \nvDash \phi$. Then $s \dashv \phi$ by the IH. Suppose for contradiction that $s \not\dashv \psi$. Then $s \vDash \psi$ by IH and since $s = s \cup \varnothing$ and $\varnothing \vDash \phi$, we would have $s \vDash \phi \vee \psi$, contrary to assumption. Hence $s \dashv \psi$. Hence $s \dashv \phi \vee \psi$. (ii) Suppose that $s \not\dashv \phi \vee \psi$. Then either $s \not\dashv \phi$ or $s \not\dashv \psi$. Suppose the former. Then $s \vDash \phi$ by the IH. Since $s = s \cup \varnothing$ and $\varnothing \vDash \psi$, then $s \vDash \phi \vee \psi$. Likewise if you suppose the latter.
(i) Suppose that $s \nvDash \phi \wedge \psi$. Then either $s \nvDash \phi$ or $s \nvDash \psi$. Suppose the former. Then $s \dashv \phi$ by the IH. Since $s = s \cup \varnothing$ and $\varnothing \dashv \psi$, then $s \dashv \phi \wedge \psi$. Likewise if you suppose the latter. (ii) Suppose that $s \not\dashv \phi \wedge \psi$. Then it is not the case that there are t and t' such that $s = t \cup t'$ and $t \dashv \phi$ and $t' \dashv \psi$. Let $s = t$ and $t' = \varnothing$. Since $\varnothing \dashv \psi$, the assumption implies that $s \not\dashv \phi$. Then $s \vDash \phi$ by the IH. Moreover, suppose for contradiction that $s \nvDash \psi$. Then $s \dashv \psi$ by IH and since $s = s \cup \varnothing$ and $\varnothing \dashv \phi$, we would have $s \dashv \phi \wedge \psi$, contrary to assumption. Hence $s \vDash \psi$. Hence $s \vDash \phi \wedge \psi$.

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