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Is, Ought, and Cut

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Abstract
In this paper we use proof-theoretic methods, specifically sequent calculi, admissibility of cut within them and the resultant subformula property, to examine a range of philosophically-motivated deontic logics. We show that for all of those logics it is a (meta)theorem that the Special Hume Thesis holds, namely that no purely normative conclusion follows non-trivially from purely descriptive premises (nor vice versa). In addition to its interest on its own, this also illustrates one way in which proof theory sheds light on philosophically substantial questions.

Keywords Special Hume thesis · Is-Ought argument · Sequent calculus · Subformula property · Deontic logics

1 Introduction

Philosophical orthodoxy holds it that David Hume [11] has been the first philosopher who (more or less) clearly saw the need to offer a justification for obtaining an 'ought' from an 'is'. This insight marks the beginning of a long history of the philosophical debate and investigation that is still ongoing today. For an extensive overview and a wealth of theorems (up to the publication in 1997) see [23], for an overview of more recent debates [10].

Our interest in this meta-ethical debate rests on the assumption that sentences can be divided into purely descriptive and purely normative ones (categories which are mutually exclusive, but not necessarily exhaustive). This distinction, as modest as it

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may seem, has not gone without criticism. For instance, the analysis given in [21] is driven by semantic considerations, and is called therein *barriers to implication*, see also [25]. That this semantic (intensional) approach is insufficient has been argued in [5]. The gist of this criticism is: one should, in the light of the equivalence between a descriptive contingent proposition P and a normative one Q, not be committed to a *gap* between descriptive and normative statements. For this and also further philosophical reasons [5] develops a *hyperintensional* account.

Our approach, however, is more modest still, and merely syntactic. It rests on the assumptions (1) that there is simply a *syntactic* difference between purely descriptive and purely normative sentences and (2) that this difference is suitably formalized in the realm of readily available deontic logics. It suffices for our study of this meta-ethical issue that there are deontic logics which are accepted as reasoning tools.

Of course, the sheer vastness of deontic logic is bewildering. It comprises anything from (much maligned and frequently criticized, as we will also see in this paper) standard deontic logic SDL, up to *reasons as defaults*, as well as many more, so it is fair to say that is a very diverse field of philosophical research. Somewhat unexpectedly, the Is-Ought gap does also emerge in the reasons-as-defaults approach of [9], as shown in [6].

We interpret the last result to the effect that the existence of an Is-Ought gap is not a *given* in a formal system, but a non-trivial, and highly desirable, property. Moreover, as we will show in this paper, this desirable property is not stipulated (thus making it a methodological dictum), but rather a consequence of the meaning of 'ought' in a range of deontic logics. This is one goal of the present paper. On a more general level, it is also a vivid illustration of the usefulness of formal methods in obtaining philosophically significant insights via the employment of proof theory.

Our method for establishing that there is an Is-Ought gap, or that there are no Is-Ought inferences as we phrase it here, applies to a class of deontic logics containing SDL and a range of refinements (motivated by deontic paradoxes we recap below). Furthermore it shows that the formalization of deontic reasoning encoded in this class satisfies the philosophical claims that the transition from Is to Ought is neither trivial nor a purely logical affair.

Having said this, more clarifications are in order. This will be done in Section 2 of the present paper where we present examples of Is-Ought inferences, and make precise what exactly our target is. Section 3 then is the core of the paper, it introduces deontic logics and their sequent calculi to be employed, as well as some standard puzzles that motivate, and differentiate between, the systems used in this paper. Moreover, it contains the important separation lemma. Section 4 continues the discussion with the addition of further modalities, and in particular the so-called *bridge principles*. Finally Section 5 offers a quick recap and then points the way forward for both technical and philosophical continuations of the research presented in this paper.

1.1 Is-Ought Arguments

A substantial philosophical debate centers around the question of what does count as a *genuine* Is-Ought argument. Below there are several concrete examples, inspired by [20] and mostly exaggerated. Naturally, the list here is not meant to be exhaustive, but rather hint at the variety of potential Is-Ought arguments.

Example 1.1 (Is-Ought arguments)

1. It ought to be the case that Bertram marries Anna or Munich is in Austria. Munich is not in Austria. Hence, it ought to be the case that Bertram marries Anna.
2. Anna knows that helping people in need is obligatory. If Anna knows that helping people in need is obligatory, then helping people in need is obligatory. Hence, helping people in need is obligatory.
3. It ought to be the case that Bertram marries Anna, and Munich is in Bavaria. Hence, it ought to be the case that Bertram marries Anna.
4. Munich is in Bavaria. Hence, it ought to be the case that Bertram marries Anna, or Munich is in Bavaria.
5. Abortion takes place. Hence, abortion is obligatory.

The logical form of these arguments can be presented as follows:

1. $\dfrac{\mathcal{O}p\vee q, \neg q}{\therefore \mathcal{O}p}$
2. $\dfrac{\mathsf{K}\mathcal{O}r \quad \mathsf{K}\mathcal{O}r\rightarrow\mathcal{O}r}{\therefore \mathcal{O}r}$
3. $\dfrac{\mathcal{O}s\wedge t}{\therefore \mathcal{O}s}$
4. $\dfrac{t}{\therefore \mathcal{O}s\vee t}$
5. $\dfrac{v}{\therefore \mathcal{O}v}$

Each of these examples can be considered as a candidate for an Is-Ought argument. However, this paper concentrates on those Is-Ought arguments for which the form (or generalizations thereof) 5 is viewed as paradigmatic, and which constitute a significant class of Is-Ought arguments. In the literature the claim that Is-Ought arguments of the form 5 are not logically valid is known as the *Special Hume Thesis*:

Definition 1.2 (Special Hume Thesis, SHT) No purely normative conclusion which is not already logically true is inferable from a consistent set of purely descriptive premises.[1]

Terms like 'purely normative', and 'purely descriptive' will be made precise in a subsequent part of this paper. As a first approximation let us stipulate that a sentence (of some natural language) which is governed by a normative phrase such as 'it is obligatory that' or just 'it is obligatory' (and likewise for 'it is permitted that' and 'it is forbidden that') are considered to be *normative*. Given the usual understanding, a *purely descriptive* sentence is a sentence which is constituted, if complex, only from descriptive sentences. Given this rather basic distinction a *mixed* sentence can be understood as a sentence that is constituted from normative as well as descriptive sentences.

[1] This formulation is due to [13, 23, 24].

Bearing those considerations in mind the arguments 1, 3, and 4 consist either of mixed premises or conclusions and hence are not subject to the Special Hume Thesis. The argument 2 is of a different kind altogether. Observe that it relates epistemic (descriptive) and normative phrases and contains a premise which touches on the connection between normative and descriptive sentences. The role of interaction of normative and descriptive phrases will be touched upon below in Section 4.1.

There is of course a significant number of investigations that address the more general class of Is-Ought arguments, e.g. [1, 21, 23, 24], and for a general overview [18]. These approaches rely heavily on model-theoretic tools and methods. For example, in [21] a more general theory of barriers to implication has been developed. That is to say, not only has (a version of) Hume's thesis about Is and Ought been studied but also e.g. Russell's claim that no universal generalizations can be concluded from particular sentences. The authors of [21] carefully develop relations between models and then prove that the implication to, say, an Ought from an Is can be barred. The emphasis here, in contrast to the strategies mainly followed in order to establish some version of Hume's thesis, is very much syntactical and makes use of proof-theoretical results.

Moreover the approach in this paper differs significantly from approaches taken in truthmaker semantics, e.g. [3, 4], and specifically [5]. Our approach makes full use of the syntactic nuances expressible in the language(s), whereas in [5] the differences are achieved on the deontic stati of the states underlying the propositions – entirely independent of how these might be expressed [6, 8, 9].

1.2 The Language of Obligations

In the remainder of this introductory section we define the language and the basic concepts that will form the core of our investigation in this paper. The formal language $\mathcal{L}$ we use consists of countably many propositional atoms, the usual Boolean connectives and a modal operator $\mathcal{O}$ (with the intended reading of 'it is obligatory that', or 'it ought to be the case that'), and is specified as:

Definition 1.3 (Language $\mathcal{L}$)

$$A ::= \ p \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \mathcal{O}A$$

This is a rather liberal definition; some authors prefer for mainly philosophical reasons to have a more restricted formulation where in $\mathcal{O}A$ the formula A does not contain $\mathcal{O}$ (nothing in this paper precludes this limitation). The usual rules on saving brackets are applied. Usually $\mathcal{P}A$ (with the intended reading of 'permitted') is used to abbreviate $\neg\mathcal{O}\neg A$, and $\mathcal{F}A$ (with the intended reading of 'forbidden') for $\mathcal{O}\neg A$, but we will not utilize these modalities in the present paper.

Moreover, a *sequent* then has the following form:

$$\Gamma \Rightarrow \Delta$$

where Γ, Δ are multisets of formulas of $\mathcal{L}$, the left side of the sequent-arrow is called the antecedent and the right the succedent; either side is allowed to be empty.

2 Descriptive and Normative

The discussion in this paper will rely on the dichotomy of two *polarities* (the definitions presented here are due to [13], while the term *polarity* itself we introduce here), those of purely descriptive and purely normative formulas - inductively as:

Definition 2.1 (Purely descriptive formula) The set of purely descriptive formulas is the smallest subset of $\mathcal{L}$ which satisfies the following properties:

i. Every propositional variable is purely descriptive.
ii. If A is a purely descriptive formula, then $\neg A$ is a purely descriptive formula.
iii. If A, B are purely descriptive formulas, then $A \rightarrow B$, $A \wedge B$, and $A \vee B$ are purely descriptive formulas.

We then can define *purely normative formula* in the same vein, with the base case a formula under the scope of $\mathcal{O}$ instead of an atom:

Definition 2.2 (Purely normative formula) The set of purely normative formulas is the smallest subset of $\mathcal{L}$ which satisfies the following properties:

i. If A is a formula, then $\mathcal{O}A$ is a purely normative formula.
ii. If A is a purely normative formula, then $\neg A$ is a purely normative formula.
iii. If A, B are purely normative formulas, then $A \rightarrow B$, $A \wedge B$, and $A \vee B$ are purely normative formulas.

We will sometimes refer to purely normative formulas as *norms* for short (even though only the formulas of type (i) strictly speaking express a norm, while others are better thought of as being *about* norms).

These definitions formulate two mutually exclusive classes of formulas which are not jointly exhaustive. We refer to a formula that is neither purely descriptive nor purely normative as a *mixed* formula (these will, however, not play a significant role going forward). The definitions of purely normative and purely descriptive formulas (as well as mixed formulas) extend in a natural way to sequences, and multisets, of formulas.

Definition 2.3 (Overriding) A modality $\Box$ is *overriding* if it determines the polarity of $\Box A$ independently of formula A.

We can notice that

Observation 2.4 $\mathcal{O}$ is an overriding modality.

It is of paramount importance to have a clear understanding of an *Is-Ought argument* and Is-Ought inferences (i.e. derivations of Is-Ought arguments), which we offer in the following definitions. They in effect allow for a *purely syntactic* explications of the Special Hume Thesis.

Definition 2.5 (Is-Ought argument)
In a formal system a sequent $\Gamma \Rightarrow \Delta$ is an Is-Ought argument iff

i. each A in Γ is purely descriptive,
ii. Γ is consistent, i.e. $\Gamma \not\Rightarrow$,
iii. each B in Δ is purely normative,
iv. Δ is not already derivable, i.e. $\not\Rightarrow \Delta$, and
v. $\Gamma \Rightarrow \Delta$ is derivable.

We offer a few examples:

Example 2.6 The following would be Is-Ought arguments when derivable:

i. $A \Rightarrow \mathcal{O}A$
ii. $A \Rightarrow \mathcal{O}(A \vee B)$
iii. $A, B \Rightarrow \mathcal{O}(A \wedge B)$

We offer a few illustrative non-examples as well:

Example 2.7 The following are not Is-Ought arguments:

i. $\mathcal{O}A \wedge \neg\mathcal{O}A \Rightarrow \mathcal{O}A$
ii. $B \Rightarrow A \vee \neg A$
iii. $A \Rightarrow \mathcal{O}(A \rightarrow B)$

In the first sequent the antecedent is neither purely descriptive nor consistent, in the second the succedent is not purely normative as well as already derivable, while the third is not derivable (in any respectable system).

In a similar fashion as above we can also define Ought-Is arguments:

Definition 2.8 (Ought-Is argument)
In a formal system a sequent $\Gamma \Rightarrow \Delta$ is an Ought-Is argument iff

i. each A in Γ is purely normative,
ii. Γ is consistent, i.e. $\Gamma \not\Rightarrow$[2],
iii. each B in Δ is purely descriptive,
iv. Δ is not already derivable, i.e. $\not\Rightarrow \Delta$, and
v. $\Gamma \Rightarrow \Delta$ is derivable.

We again offer several examples and non-examples:

Example 2.9 The following would be Ought-Is arguments when derivable:

i. $\mathcal{O}A \Rightarrow A$
ii. $\mathcal{O}(A \wedge B) \Rightarrow A$

[2]Notice that in SDL or RD if Γ is normatively inconsistent, i.e. contains a pair of formulas $\mathcal{O}A$ and $\mathcal{O}\neg A$, then it is inconsistent.

iii. $\mathcal{O}A, \mathcal{O}B \Rightarrow A \wedge B$

Example 2.10 The following inferences are not Ought-Is arguments:

i. $\mathcal{O}B \Rightarrow A \vee \neg A$
ii. $\mathcal{O}A \Rightarrow \mathcal{O}A$
iii. $\mathcal{O}A, \neg\mathcal{O}A \Rightarrow B$

In the first case the succedent is derivable, in the second it is not purely descriptive, and in the third the antecedent is inconsistent.

The point (v) in Definitions 2.5 and 2.8 is a space-saver, since we are in any case only ever concerned with the *derivable* arguments. Therefore, in the following we do not need to make a distinction between Is-Ought (nor Ought-Is) arguments and inferences.

3 Formal Systems of Deontic Logic

In this paper we focus on four common deontic logics, Standard Deontic Logic SDL, Chellas' minimal deontic logic CMD [2] (known to be sound and complete) and a weaker version of each (RD and ED respectively). We thus run the gamut from very strong to very weak deontic logics.

The propositional basis for both SDL and CMD is the same – fully classical propositional logic (including modus ponens).
The deontic part of SDL consists of two further axioms:
K: $\mathcal{O}(A \to B) \to (\mathcal{O}A \to \mathcal{O}B)$ and
Consistency: $\mathcal{O}A \to \neg\mathcal{O}\neg A$
and the rule of Necessitation, **Nec**:

$$\frac{\vdash A}{\vdash \mathcal{O}A}Nec$$

RD on the other hand leaves out the rule of Necessitation.
The deontic part of CMD consists of the axiom
Weak consistency: $\neg\mathcal{O}(A \wedge \neg A)$
and the rule **M**:

$$\frac{\vdash A \to B}{\vdash \mathcal{O}A \to \mathcal{O}B}M$$

ED on the other hand leaves out the rule M. It is the weakest deontic logic characterized by neighborhood frames [12, 17].

The approach we use in this paper is proof-theoretic, utilizing a G3 sequent calculus (due to the fact structural properties are easy to obtain and it is highly modular [15, 16]). We now first introduce the sequent calculi (Fig. 1) used in the rest of the paper, further analyze the distinctions between these systems, and then discuss the philosophical reasons for adopting each in more detail.

Initial sequents: $p, \Gamma \Rightarrow \Delta, p$ where p is atomic/prime.

Propositional rules:

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\, L\neg \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\, R\neg$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\, L\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\, R\rightarrow$$

Deontic rules:

$$\frac{A \Rightarrow B \qquad B \Rightarrow A}{\mathcal{O}A, \Gamma \Rightarrow \Delta, \mathcal{O}B}\, LR\text{-}E \qquad \frac{A, \Pi \Rightarrow B}{\mathcal{O}A, \mathcal{O}\Pi, \Gamma \Rightarrow \Delta, \mathcal{O}B}\, LR\text{-}R$$

$$\frac{A \Rightarrow B}{\mathcal{O}A, \Gamma \Rightarrow \Delta, \mathcal{O}B}\, LR\text{-}M \qquad \frac{\Pi \Rightarrow B}{\mathcal{O}\Pi, \Gamma \Rightarrow \Delta, \mathcal{O}B}\, LR\text{-}K$$

$$\frac{A \Rightarrow}{\mathcal{O}A, \Gamma \Rightarrow \Delta}\, LD \qquad \frac{\Pi \Rightarrow}{\mathcal{O}\Pi, \Gamma \Rightarrow \Delta}\, LD^*$$

Fig. 1 G3xD

3.1 Sequent Calculi for Deontic Logics

All the rules of sequent calculi consist of one sequent, written below the inference line, which is its *conclusion*, and one or more sequents above the line called its *premises*. All the formulas except Γ and Δ are called *active* formulas of the rule if they occur in the premise(s) and *principal* if they occur in the conclusion of the rule. Γ and Δ are called a *context* of the rule. A *branch* is a series of sequents, starting with the endsequent, in which every element is a conclusion of a rule that the following element is a premise of (two-premise rules thus split the branches). The *height* of a derivation is the length (number of consecutive applications of derivation rules) of its longest branch. The propositional base for out calculi is the calculus G3cp [15].

The four calculi considered here are each obtained by adding two deontic rules to that propositional base:

G3ed:= G3cp + LR-E + LD **G3rd**:= G3cp + LR-R + LD*
G3cmd:= G3cp + LR-M + LD **G3sdl**:= G3cp + LR-K + LD*

Fact 3.1 (Structural properties) It has been established in [17] that in all of the above systems axiom generalization holds, weakening and contraction are height-preserving admissible and cut is admissible. Moreover [17] establishes the subformula property and decidability, consistency and deductive equivalence with their respective Hilbert-style counterparts, as well as soundness and completeness of the systems (w.r.t. their neighborhood semantics).

We take these to be sufficiently evident and present them without recapping the proofs. Fact 3.1 enables us to likewise make judgments on *underivability*, for which the subformula property plays a significant part, and will be made use of going forward. We therefore next present its definition, adjusted for our present purposes by highlighting the role of an immediate subformula (though equivalent to the standard one).

Definition 3.2 (Immediate subformula, $\sqsubset$) Immediate subformula relation $\sqsubset$ is the smallest subset of $\mathcal{L}^2$ which satisfies the following properties (for any formulas A and B):

i. $A \sqsubset \neg A$,
ii. $A, B \sqsubset A \circ B$; $\circ \in \{\wedge, \vee, \rightarrow\}$,
iii. $A \sqsubset \mathcal{O}A$.

Definition 3.3 (Subformula, $\sqsubseteq$) Subformula relation $\sqsubseteq$ is the smallest subset of $\mathcal{L}^2$ which satisfies the following properties (for any formulas A, B and C):

i. $A \sqsubseteq A$,
ii. If $A \sqsubset B$ and $B \sqsubseteq C$ then $A \sqsubseteq C$.

Observation 3.4 Note that combining points (i) and (ii) of the latter definition shows that every immediate subformula is a subformula.

We now illustrate for the reader the systems in action via some easy derivations which will also be informative of the differences between the four systems under consideration.

Observation 3.5 Whereas (i) the sequent $\Rightarrow \neg\mathcal{O}(A \wedge \neg A)$ is derivable in all, (ii) the sequent $\Rightarrow \mathcal{O}(A \vee \neg A)$ is derivable in G3sdl only:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\neg A, A \Rightarrow}\,L\neg}{A \wedge \neg A \Rightarrow}\,L\wedge}{\mathcal{O}(A \wedge \neg A) \Rightarrow}\,LD/LD^*}{\Rightarrow \neg\mathcal{O}(A \wedge \neg A)}\,R\neg \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\,R\neg}{\Rightarrow A \vee \neg A}\,R\vee}{\Rightarrow \mathcal{O}(A \vee \neg A)}\,LR\text{-}K$$

The "only" part is obtained from the subformula property by a close inspection of the deontic rules. Namely, only LR-K allows for the antecedent of the premise to be empty, but the succedent non-empty. Similar for subsequent observations below.

Observation 3.6 The sequents (i) $\mathcal{O}A \Rightarrow \neg\mathcal{O}\neg A$, (ii) $\mathcal{O}(A \to B) \Rightarrow \mathcal{O}A \to \mathcal{O}B$ and (iii) $\mathcal{O}A, \mathcal{O}B \Rightarrow \mathcal{O}(A \wedge B)$ are only derivable in G3rd and G3sdl:

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A}{\neg A, A \Rightarrow}\, L\neg}{\mathcal{O}\neg A, \mathcal{O}A \Rightarrow}\, LD^*}{\mathcal{O}A \Rightarrow \neg\mathcal{O}\neg A}\, R\neg \qquad \cfrac{\cfrac{A, A \to B \Rightarrow B}{\mathcal{O}A, \mathcal{O}(A \to B) \Rightarrow \mathcal{O}B}\, LR\text{-}R/K}{\mathcal{O}(A \to B) \Rightarrow \mathcal{O}A \to \mathcal{O}B}\, R\to$$

$$\cfrac{\cfrac{A, B \Rightarrow A \qquad A, B \Rightarrow B}{A, B \Rightarrow A \wedge B}\, R\wedge}{\mathcal{O}A, \mathcal{O}B \Rightarrow \mathcal{O}(A \wedge B)}\, LR\text{-}R/K$$

Notice that given the last proposition, Weak consistency is sufficient for Consistency, but since (iii) is not available in CMD and ED, we have the former axiom but not the latter.

Observation 3.7 The rule of Inheritance, RM:

$$\cfrac{A \Rightarrow B}{\mathcal{O}A \Rightarrow \mathcal{O}B}\, RM$$

is only admissible in G3cmd, G3rd and G3sdl.

Consequently, the sequent $\mathcal{O}(A \wedge B) \Rightarrow \mathcal{O}A$ is only derivable in G3cmd, G3rd and G3sdl:

$$\cfrac{\cfrac{A, B \Rightarrow A}{A \wedge B \Rightarrow A}\, L\wedge}{\mathcal{O}(A \wedge B) \Rightarrow \mathcal{O}A}\, LR\text{-}R/M/K$$

3.2 Deontic Puzzles

These observations will be useful in distinguishing the four systems, and likewise understanding the philosophical reasons for introducing the common systems of deontic logic discussed here. Each of these arises out of a consideration of some feature of SDL.

Note first that while it is not entirely unassailable, weak consistency principle,

$$\neg\mathcal{O}(A \wedge \neg A)$$

it plausible and widely accepted in the discussion of deontic logic [12], as seen in Observation 3.5.i.

To start with features which are not accepted, consider first a world where there are no obligations, like perhaps a time when no rational agents existed [12]. This seems plausible, and it would hold in such a world that, for any B,

$$\neg\mathcal{O}B$$

Therefore, the following instance likewise holds:

$$\neg\mathcal{O}(A \vee \neg A)$$

But in SDL it is derivable that $\Rightarrow \mathcal{O}(A \vee \neg A)$ (Observation 3.5.ii) - a contradiction. Consequently, SDL entails that something is always obligatory. If one finds this unacceptable or counter-intuitive, one solution is to abandon LR-K in favor of one of the weaker rules LR-E/R/M, all of which require at least one formula to occur in the antecedent of the premise (unlike the offending LR-K, where the multiset can potentially be empty). It should be obvious that none of these will allow us to conclude all tautologies are obligatory. This corresponds to abandoning Necessitation.

Next we observe dilemmas, in which one is faced with conflicting obligations, each of which entails violating the other, and with no easy mechanism for deciding precedence. This last feature is not relevant for our discussion here, since the issue we are about to consider arises even when one easily overrides the other, but is nonetheless a common feature of philosophically interesting dilemmas.

So, consider a situation when we have obligations to contradictories (a strong form of conflict indeed), $\mathcal{O}A$ and $\mathcal{O}\neg A$, for example when one does not pay enough attention when making promises [12]. Given strong consistency principle (Observation 3.6.i), a contradiction easily follows. It is again easy to see that adopting a weaker LD rule in place of LD* blocks this inference while still allowing weak consistency to hold.

Another (major) difficulty for SDL and RD is Chisholm's paradox. Consider the following situation and its most straightforward formalization (other possible formalizations violate the apparent independence of these claims) [12]:

1. It ought to be that Jones goes to the assistance of her neighbors, $\mathcal{O}G$.
2. It ought to be that if Jones goes she tells them she is coming, $\mathcal{O}(G \rightarrow T)$.
3. If Jones doesn't go, she ought to not tell them she is coming. $\neg G \rightarrow \mathcal{O}(\neg T)$.
4. Jones doesn't go, $\neg G$.

This seems like a plausible scenario. But, by Observation 3.6.ii it follows from (1) and (2) that $\mathcal{O}T$, while it follows from (3) and (4) that $\mathcal{O}\neg T$, which leads to a contradiction given Observation 3.6.i.

The problem of conflicting obligations and Chisholm's paradox illustrate the difficulties SDL and RD, with strong consistency and normality. Subsequent deontic logics under discussion, CMD and its weaker version ED, abandon both of these principles. Another paradox is in store, however, to demonstrate the difference between these two, Forrester's paradox of the gentle murderer. Consider the following situation and its formalization (as before, due to [12]):

1. It is obligatory that Smith does not kill his mother, $\mathcal{O}\neg K$.
2. If Smith does kill his mother, then it is obligatory that he kills her gently, $K \rightarrow \mathcal{O}G$.
3. Smith does kill his mother, K.
4. If Smith kills his mother gently, then he kills his mother, $G \rightarrow K$.

Assuming the latter is a logical truth, which it certainly seems to be, it follows by M that $\mathcal{O}G \rightarrow \mathcal{O}K$. Since from (2) and (3) it follows that $\mathcal{O}G$, it now follows that $\mathcal{O}K$ as well. As we have seen in several previous examples, in SDL and RD this would lead to a contradiction with (1), but even putting that aside in CMD we are left with the strange, to say the least, claim that Smith ought to kill his mother (while not strictly paradoxical, this is quite puzzling). It is clear that ED avoids this, since, while

G implies K, the inverse does not hold (a counterexample is as obvious as it is in poor taste).

Another set of problem faces the related rule RM (Observation 3.7). Consider Ross' paradox, and the claim

1. It is obligatory that the letter is mailed, $\mathcal{O}L$.

Since that the letter is mailed entails that it is either mailed or burnt, (1) by RM entails

2. It is obligatory that the letter is either mailed or burnt, $\mathcal{O}(L \vee B)$.

Of course, (2) is an obligation that can be satisfied by simply burning the letter. It is strange indeed that an obligation to mail a letter would entail something that effectively prevents that obligation from being fulfilled. It should again be clear ED avoids this weird conclusion, since sending or burning a letter does not imply it is sent.

We will not stray any further to discuss answers and resolutions to these puzzles. Naturally, one is not always required to abandon a system in face of paradox (as we've been doing here), and a standard reaction is to rather explain them away, thus leading to a spiral of explanations and revenge paradoxes. But we will not go that deep down this particular rabbit hole, because those presented already serve to motivate, as well as illustrate the distinctions between, the systems we presented.

Of course, even with what we have we could lay our further systems with different configurations of the rules, though not all combinations need to be examined separately, since quite clearly rule admissibility relation $\preceq$ orders the rules LR-E $\preceq$ LR-M $\preceq$ LR-R $\preceq$ L-K on one axis and LD $\preceq$ LD* on another. We have here just presented some of the systems common in literature, and developed in reaction to famous (and hence named) paradoxes.

It is in any case a handy consequence of modularity of G3 systems that the central lemma of this paper in the next section will hold for any such configuration.

3.3 Separation Lemma

The main goal of this paper is to show that it is a property of all of these deontic systems (and consequently also corresponding ways of understanding obligation) that there are no Is-Ought arguments (nor the converse Ought-Is). To prove this in a uniform manner, the crucial step is the following lemma which instructs us that the purely descriptive and purely normative parts of the sequent can be kept strictly separate. Since it is the central lemma of this paper, we will examine it in some detail.

Lemma 3.8 (Separation) *If* $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ *is derivable in G3xD, where* Γ_1 *and* Δ_1 *are all purely descriptive, and* Γ_2 *and* Δ_2 *are all purely normative, then either* $\Gamma_1 \Rightarrow \Delta_1$ *or* $\Gamma_2 \Rightarrow \Delta_2$ *are derivable in G3xD.*

Proof By induction on the height of the derivation.

Basic case. If $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is initial, then $\Gamma_1 \Rightarrow \Delta_1$ likewise is, since atoms are purely descriptive.

Inductive case. We consider first the propositional base.

(L→) Assume the last step in derivation was L→, and $A \to B$ is purely normative. Then the last step has the form (where $A \to B, \Gamma_2'$ is Γ_2 and similar for other formulas below):

$$\frac{\Gamma_1, \Gamma_2' \Rightarrow \Delta_1, \Delta_2, A \qquad \Gamma_1, B, \Gamma_2' \Rightarrow \Delta_1, \Delta_2}{\Gamma_1, A \to B, \Gamma_2' \Rightarrow \Delta_1, \Delta_2}\, L{\to}$$

It follows by Definition that A and B are purely normative, and therefore by the inductive hypothesis from the left premise that either

i. $\Gamma_1 \Rightarrow \Delta_1$ or
ii. $\Gamma_2' \Rightarrow \Delta_2, A$
and from the right premise that either
iii. $\Gamma_1 \Rightarrow \Delta_1$ or
iv. $B, \Gamma_2' \Rightarrow \Delta_2$

In cases (i) or (iii) we are done and otherwise we can derive

$$\frac{\Gamma_2' \Rightarrow \Delta_2, A \qquad B, \Gamma_2' \Rightarrow \Delta_2}{A \to B, \Gamma_2' \Rightarrow \Delta_2}\, L{\to}$$

and we are done. The same if $A \to B$ is purely descriptive.

(R→) Assume the last step in derivation was R→, and $A \to B$ is purely normative. Then the last step has the form:

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow \Delta_1, \Delta_2', B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2', A \to B}\, R{\to}$$

It follows again by Definition and the inductive hypothesis that either

i. $\Gamma_1 \Rightarrow \Delta_1$ or
ii. $A, \Gamma_2 \Rightarrow \Delta_2', B$.

In the first case we are done and otherwise we can derive

$$\frac{A, \Gamma_2 \Rightarrow \Delta_2', B}{\Gamma_2 \Rightarrow \Delta_2', A \to B}\, R{\to}$$

Again the same if $A \to B$ is purely descriptive.

(L¬) Assume the last step in derivation was L¬, and $\neg A$ is purely normative. Then the last step has the form:

$$\frac{\Gamma_1, \Gamma_2' \Rightarrow \Delta_1, A, \Delta_2}{\Gamma_1, \neg A, \Gamma_2' \Rightarrow \Delta_1, \Delta_2}\, L\neg$$

It follows once more by Definition and the inductive hypothesis that either

i. $\Gamma_1 \Rightarrow \Delta_1$ or

ii. $\Gamma_2' \Rightarrow A, \Delta_2$

In the first case we are done and otherwise we can derive

$$\frac{\Gamma_2' \Rightarrow A, \Delta_2}{\neg A, \Gamma_2' \Rightarrow \Delta_2}\ L\neg$$

The same if $\neg A$ is purely descriptive.

(R¬) Assume the last step in derivation was R¬, and $\neg A$ is purely normative. Then the last step has the form:

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow \Delta_1, \Delta_2'}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \neg A, \Delta_2'}\ R\neg$$

It follows again by Definition and the inductive hypothesis that either

i. $\Gamma_1 \Rightarrow \Delta_1$ or
ii. $A, \Gamma_2 \Rightarrow \Delta_2'$

In the first case we are done and otherwise we can derive

$$\frac{A, \Gamma_2 \Rightarrow \Delta_2'}{\Gamma_2 \Rightarrow \neg A, \Delta_2'}\ R\neg$$

The same if $\neg A$ is purely descriptive.

(LR-E) If the last step in derivation was LR-E, it has the form, where $\mathcal{O}A, \Gamma_2'$ is Γ_2 and similar for Δ and $\mathcal{O}B$ (the same for the subsequent cases):

$$\frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma_1, \mathcal{O}A, \Gamma_2' \Rightarrow \Delta_1, \Delta_2', \mathcal{O}B}\ LR\text{-}E$$

Then we can likewise, as required, derive:

$$\frac{A \Rightarrow B \qquad B \Rightarrow A}{\mathcal{O}A, \Gamma_2' \Rightarrow \Delta_2', \mathcal{O}B}\ LR\text{-}E$$

(LR-R/LR-M/LR-K) If the last step in derivation was LR-R (LR-M and LR-K a special case), it has the form:

$$\frac{A, \Pi \Rightarrow B}{\Gamma_1, \mathcal{O}A, \mathcal{O}\Pi, \Gamma_2' \Rightarrow \Delta_1, \Delta_2', \mathcal{O}B}\ LR\text{-}R$$

Then we can derive:

$$\frac{A, \Pi \Rightarrow B}{\mathcal{O}A, \mathcal{O}\Pi, \Gamma_2' \Rightarrow \Delta_2', \mathcal{O}B}\ LR\text{-}R$$

(LD/LD*) If the last step in derivation was LD* (LD a special case), it has the form:

$$\frac{\Pi \Rightarrow}{\Gamma_1, \mathcal{O}\Pi, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ LD^*$$

Then we can derive:

$$\frac{\Pi \Rightarrow}{\mathcal{O}\Pi, \Gamma_2 \Rightarrow \Delta_2}\ LD^*$$ □

We immediately put this lemma to work in showing the central point of this paper, namely that there are no non-trivial Is-Ought or Ought-Is arguments. More precisely,

Theorem 3.9 *No formal system G3xD allows for a derivation of an Is-Ought argument (as defined above, Definition 2.5), or of an Ought-Is argument (Definition 2.8).*

Proof Assume $\Gamma \Rightarrow \Delta$ is an Is-Ought argument per Definition 2.5. Then all of Γ are purely descriptive, and all of Δ purely normative, and so by Lemma 3.8, either $\Gamma \Rightarrow$, contrary to point (ii) of the definition, or $\Rightarrow \Delta$, contrary to point (iv). Contradiction either way.

Likewise for Ought-Is arguments. □

In a nutshell, this result is obtained as a result of two properties. The first is the fact that the operator $\mathcal{O}$ is overriding (Observation 2.4) and binds all the principal formulas in the deontic rules. Therefore, simply omitting the context (i.e. having Γ, Δ empty in the conclusion of the rule) allows for the required separation. The second fact is that by Definitions 2.1 and 2.2, the connectives carry over the purely descriptive/normative status to their immediate subformulas (which are the active formulas of their respective rules), and thus make them available for the applications of inductive hypotheses. This latter fact illustrates the usefulness of the cut admissibility property, since the subformula property is its straightforward corollary, and with all the propositional rules generating (read bottom-up as usual) only subformulas (immediate subformulas in the first step), the proof of Lemma 3.8 goes through. This in turn allows us to show a very strong fact about *non*-derivability.

4 Adding Modalities

We now move to extend the results of the previous sections to a multimodal system where modalities are of the same type as $\mathcal{O}$, namely let $\Box$ be an overriding non-normative modality, and let it in addition be normal, captured by the rule, due to [7] (also adapted to create rule LR-K in [17]):

$$\frac{\Pi \Rightarrow A}{\Box\Pi, \Gamma \Rightarrow \Delta, \Box A}\ LR\text{-}\Box$$

(Necessitation is obtained when Π, Γ and Δ are all empty, K when the premise is a statement of *modus ponens*.)

It is straightforward to establish that

Theorem 4.1 *Any of the multimodal systems G3xD+$\Box$, combining one of G3xD and LR-$\Box$, retains all the structural properties, including admissibility of contraction and cut.*

Proof Routine, so we just illustrate on the example of the new case of cut where the cut formula is principal in both premises of the cut and of the form $\Box A$. The cut is then of the form:

$$\dfrac{\dfrac{\Pi_1 \Rightarrow A}{\Box\Pi_1, \Gamma \Rightarrow \Delta, \Box A}\ LR\text{-}\Box \qquad \dfrac{A, \Pi_2 \Rightarrow B}{\Box A, \Box\Pi_2, \Gamma' \Rightarrow \Delta', \Box B}\ LR\text{-}\Box}{\Box\Pi_1, \Box\Pi_2, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \Box B}\ Cut$$

This is then transformed into:

$$\dfrac{\dfrac{\Pi_1 \Rightarrow A \qquad A, \Pi_2 \Rightarrow B}{\Pi_1, \Pi_2, \Rightarrow B}\,Cut}{\Box\Pi_1, \Box\Pi_2, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \Box B}\,LR\text{-}\Box$$ □

With this in place, we can now show that these multimodal systems do not allow either □-Ought or Ought-□ arguments. To show this it is enough to show that

Lemma 4.2 (Generalized separation lemma) *Lemma 3.8 extends to* $G3xD + \Box$. *Namely, if* $\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3$ *is derivable in G3xD+□, where* Γ_1 *and* Δ_1 *are all purely descriptive,* Γ_2 *and* Δ_2 *are all purely normative, and* Γ_3 *and* Δ_3 *are all purely □, then either* $\Gamma_1 \Rightarrow \Delta_1$, $\Gamma_2 \Rightarrow \Delta_2$ *or* $\Gamma_3 \Rightarrow \Delta_3$ *are derivable in G3xD.*

Proof Straightforwardly extends the proof of Lemma 3.8. The new case of LR-□ is the same as the step for LR-K. □

One example of such a modality can be found in doxastic logic, where a "type 1" reasoner [26, 27] is one who believes all tautologies (Necessitation) and whose beliefs are closed under modus ponens (K). Therefore, applying the separation lemma, and assuming belief constitutes a separate (doxastic) polarity, we learn (in addition to what was already established in Theorem 3.9, which still holds here) that there are no (non-trivial) conclusion between beliefs and norms, and likewise for facts and beliefs.

We could, on the other hand, treat belief as a descriptive operator, which is reasonable since it offers a (descriptive) account of a certain part of reasoner's mind, regardless of its contents. Even if a believer held that something ought to be the case, the fact that that is what she believes is itself (overridingly) descriptive. In this case Lemma 4.2 simply collapses into Lemma 3.8, and we once again learn there are no Is-Ought nor Ought-Is arguments, even when the description of the world is enriched with contents of minds of logically (or in any case propositionally) perfect reasoners.

4.1 Bridge Principles

So far we have been interested only in KD and weaker systems. We now explore arguably the simplest stronger system, one where the T-schema is added. Since it does not intuitively hold for $\mathcal{O}$ (we live in a sadly imperfect world, where not all that ought to be the case actually is), we add it for □.

$$\dfrac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\,T$$

One example of such an operator can be found in epistemic logic, of a non-luminous [29] (i.e. where neither positive introspection, $\Box A \rightarrow \Box\Box A$, nor negative introspection, $\neg\Box A \rightarrow \Box\neg\Box A$, hold), but nonetheless factive (i.e. where $\Box A \rightarrow A$ holds) knowledge.

For the addition of an operator captured by the rules LR-$\Box$ and T (regardless of the interpretation we may adopt), it is easy to see that

Lemma 4.3 *All the structural properties still hold in G3xD+$\Box$+T. Namely, weakening and contraction are height-preserving admissible, all the propositional rules, as well as T, are height-preserving invertible, and cut is admissible.*

Proof Straightforward for most of the properties. We only illustrate for a new case of cut, where cut formula is principal in both premises and specifically principal in T in the right premise. The application of cut then has the following form:

$$\dfrac{\dfrac{\Pi \Rightarrow A}{\Box\Pi, \Gamma \Rightarrow \Delta, \Box A}\,LR\text{-}\Box \qquad \dfrac{A, \Box A, \Gamma' \Rightarrow \Delta'}{\Box A, \Gamma' \Rightarrow \Delta'}\,T}{\Box\Pi, \Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,Cut$$

This is transformed into:

$$\dfrac{\dfrac{\Pi \Rightarrow A \qquad \dfrac{\Box\Pi, \Gamma \Rightarrow \Delta, \Box A \qquad A, \Box A, \Gamma' \Rightarrow \Delta'}{\Box\Pi, \Gamma, A, \Gamma' \Rightarrow \Delta', \Delta}\,Cut_1}{\Pi, \Box\Pi, \Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,Cut_2}{\Box\Pi, \Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,T$$

Where Cut$_1$ is of lower height, and Cut$_2$ is of lower weight. □

With this simple addition (let us assume here $\Box$ is not normative),

Proposition 4.4 *There exists a $\Box$-Ought argument in G3xD+$\Box$+T, namely, (T_{br}): $\Box\mathcal{O}A \Rightarrow \mathcal{O}A$:*

$$\dfrac{\mathcal{O}A, \Box\mathcal{O}A \Rightarrow \mathcal{O}A}{\Box\mathcal{O}A \Rightarrow \mathcal{O}A}\,T$$

Therefore, the addition of this rule creates a bridge principle between different polarities.

We now explore a version of the T schema which can be added while retaining the results of Theorem 3.9. Consider a weaker version of the rule T:

$$\dfrac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\,T'$$

where A does not contain $\mathcal{O}$.

It is easy to see that structural rules still hold (the proof is the same as with the unrestricted case). It follows that the bridge principle T_{Br} is no longer derivable, since the last rule can either be one of the rules for $\mathcal{O}$, depending on the variant we're using, but all of those add an $\mathcal{O}$ formula on the left, or T_{Br}, but it does not allow to introduce $\Box$ over an $\mathcal{O}$ formula.

Assuming $\Box$ is overridingly descriptive and not constituting a separate polarity, the bridge principle T_{Br} constitutes an Is-Ought argument, which T' doesn't allow for. Therefore that particular derivation is blocked.

But a more general result is likewise available, namely we can then show that

Theorem 4.5 *No Is-Ought argument is derivable in G3xD+$\Box$+T'.*

Proof By extending the proof of Lemma 3.8. A new case we need to check is when the last step is T':

$$\frac{A, \Box A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\Box A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\, T'$$

According to the limitation on the rule T', A does not contain $\mathcal{O}$, and is therefore purely descriptive. So, we can apply the inductive hypothesis to the premise and obtain either

1. $A, \Box A, \Gamma_1 \Rightarrow \Delta_1$, or
2. $\Gamma_2 \Rightarrow \Delta_2$.

In the second case we are done and otherwise we can derive

$$\frac{A, \Box A, \Gamma_1 \Rightarrow \Delta_1}{\Box A, \Gamma_1 \Rightarrow \Delta_1}\, T'$$

where all formulas are purely descriptive. □

It is easy to show that G3xD+□+T' is a subsystem of G3xD+□+T, since the rule T' is just a special case of T, and thus admissible in the latter system. We can also easily see that

Lemma 4.6 (Conservativity of T over T') *G3xD+□+T is a conservative extension of G3xD+□+T'. Namely, if $\Gamma \Rightarrow \Delta$ is derivable in the former, and contains no $\mathcal{O}$ within the scope of □, then it is also derivable in the latter.*

Proof We simply rewrite the derivation of $\Gamma \Rightarrow \Delta$, replacing every instance of T by one of T'. Since by stipulation no $\mathcal{O}$ occurs within the scope of □, the side condition for T' will be met. The resulting derivation is of $\Gamma \Rightarrow \Delta$, but in G3xD+□+$T'$. □

Let us return to the previous epistemic interpretation of □ as merely factive knowledge, where then the limitation to T' can be understood, in accordance with Lemma 4.6, as a restriction of knowledge to purely descriptive knowledge (note that if A contains no $\mathcal{O}$, then it is purely descriptive), including possibly introspective knowledge (though of course it is not guaranteed to be introspective, either positively or negatively). Then Theorem 4.5 shows us that no amount of descriptive facts, including descriptive knowledge (and that potentially encompassing some, though not necessarily all, introspective knowledge) suffices to derive any norms unless the norms are already included in the premises, i.e. the antecedent of the sequent. Even if we introduce bridge principles, like with the addition of T instead of T' to G3xD+□, the argument will have to already contain some normativity in its premises. Not even adding this bridge principle will yield normativity if none was already present. We finally note that, naturally, this generalizes beyond the illustrative epistemic example to any overriding (-ly descriptive) modality properly captured by these rules, since it was only the form of the rules that gave us this result.

5 Conclusions and Future Work

In this paper we have examined a series of deontic logics of the SDL family and the motivations that led to them being proposed and adopted, thereby providing a solid philosophical connection between the more technical work and its practical utility. On the big-picture view, this work affords an illustration of the usefulness of proof theory in examining the issues in philosophical logic, and especially when it comes to the questions of non-derivability (or of *non sequitur*).

When it comes to the technical details, we took over the useful formulation of interesting deontic logics from their G3-family presentation in [17]. The reason for this is that the G3 approach allows for a demonstration of the required structural properties which is very straightforward and, as importantly, modular, thus enabling us to establish them in one swoop for a range of systems, and thereby allowing us to examine a whole family of deontic logics together. Of course, of these structural properties the most important one is the admissibility of cut, as it provides the subformula property (that all the formulas appearing in the derivation are a subformula of some formula in its endsequent), and that is the cornerstone of the results of this paper.

The subformula property is also a pillar of analyticity (whereby nothing appears in a proof that is not contained in its conclusion), and this is, as has been argued e.g. in [19], precisely what is constitutive of a "good" proof system. We have in this paper seen another useful role for it, and consequently for the theorem of cut admissibility it follows from, in establishing the crucial separation lemma. This lemma has shown us that the purely descriptive and purely normative sentences (categories which are in our paper captured on solely syntactic grounds) can be kept apart in any derivable sequent. Therefore, in purported Is-Ought, as well as Ought-Is, arguments, the 'Ought' and the 'Is' parts can be separated, thereby trivializing at least one of them. And so, we have come to learn, by a property of given logics (a "good"-making one no less) all such arguments are purely trivial. Instead of a methodological principle, this is a consequence of syntax of our logic, and in particular the way these logics conceptualize (i.e. capture by rules) the modality 'Ought'.

However, a big caveat that should be added is that the Special Hume Thesis of Definition 1.2 does not exhaust Hume's observation on the connection between an 'ought' and an 'is'. A stronger principle, the *General Hume Thesis* (GHT), has also received significant attention in the literature (see e.g. [22, 23]). GHT states that if a mixed formula is derivable from purely descriptive premises, then so is a formula where those occurrences of predicates that lie within the scope of 'ought' are replaced by arbitrary predicates (of the same arity). The conclusion is therefore "completely Ought-irrelevant" [23]. GHT is inherently tied to first-order logic, whereas we began this avenue of research by investigating the inferential usefulness of structural proof theory within the propositional deontic base. Therefore, we leave delving into this more general principle for (near) future research. We expect that the proof there will rely in part on an FOL extension of the separation lemma and therefore fit into the syntactic framework introduced in this paper.

But of course, the framework of G3-systems in which the present result has been established is itself, despite its clear utility and straightforward metatheory, not without proof-theoretic limitations, and it is obvious that the separation lemma is formulation-dependent. One consequence of this is that it puts some restrictions on the deontic logics we can employ, as future exploration of rules for axioms stronger than T (e.g. in [28]), unlike those presented in [17], will not always allow for overriding to do as much work as it did here. This in turn leads to a question for further research: whether the separation lemma can be extended to more powerful proof-theoretically interesting systems, such as e.g. labelled calculi, where the stronger systems are exceedingly easy to capture [14, 16].

Another conseqence of the formulation-dependence is a limit on the discussion of bridge principles, of which there could potentially be an unlimited number – to mention just one famous one, $\Box(A \rightarrow B) \rightarrow (\mathcal{O}A \rightarrow \mathcal{O}B)$, with the operator $\Box$ naturally interpreted as the universal modality. Therefore, another question that exceeds the confines of this paper, but appears to be a fruitful avenue of future research, is the examination of the effects of adding further bridge principles, possibly to a different proof-theoretic base (connecting to the previous question above) than the one used here.

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References

1. Brown, C. (2014). Minding the Is-Ought gap. *Journal of Philosophical Logic*, *43*, 53–69.
2. Chellas, B. F. (1980). *Modal logic: An introduction*. Cambridge: Cambridge University Press.
3. Fine, K. (2017). Truth-maker content I. *Journal of Philosophical Logic*, *46*, 625–674.
4. Fine, K. (2017). Truth-maker content II. *Journal of Philosophical Logic*, *46*, 675–702.
5. Fine, K. (2021). Truthmaking and the Is–Ought gap. *Synthese*, *198*(2), 887–914.
6. Fuhrmann, A. (2017). Deontic modals: Why abandon the default approach. *Erkenntnis*, *82*(6), 1351–1365.

7. Hakli, R., & Negri, S. (2012). Does the deduction theorem fail for modal logic? *Synthese*, *187*(3), 849–867.
8. Horty, J. F. (2007). Reasons as defaults. *Philosophers' Imprint*, *7*(3), 1–28.
9. Horty, J. F. (2012). *Reasons as defaults*. New York: Oxford University Press.
10. Humberstone, L. (2019). Recent thought on is and ought: Connections, confluences and rediscoveries. *FLAP*, *6*(6), 1373–1446.
11. Hume, D. A. (1978). *Treatise on Human Nature*, 2nd edn. New York: Oxford University Press.
12. McNamara, P., & Van De Putte, F. (2021). Deontic logic. In E. N. Zalta (Ed.) *The Stanford encyclopedia of philosophy*. Spring 2021. Stanford: Metaphysics Research Lab, Stanford University.
13. Morscher, E. (2012). *Normenlogik - Grundlagen, Systeme, Anwendungen*. Paderborn: Mentis Verlag.
14. Negri, S., & von Plato, J. (1998). Cut elimination in the presence of axioms. *Bulletin of Symbolic Logic*, *4*(4), 418–435.
15. Negri, S., & von Plato, J. (2001). *Structural proof theory*. Cambridge: Cambridge University Press.
16. Negri, S., & von Plato, J. (2011). *Proof analysis: A contribution to Hilbert's last problem*. Cambridge: Cambridge University Press.
17. Orlandelli, E. (2014). Proof analysis in deontic logics. In F. Cariani, D. Grossi, J. Meheus, & X. Parent (Eds.) *Deontic logic and normative systems, DEON 2014, LNCS*, (Vol. 8554 pp. 198–216). Cham: Springer.
18. Pigden, C. (Ed). (2010). *Hume on Is and Ought*. London: Palgrave Macmillan.
19. Poggiolesi, F. (2012). On the importance of being analytic: The paradigmatic case of the logic of proofs. *Logique et Analyse*, *55*(219), 443–461.
20. Prior, A. (1960). The autonomy of ethics. *Australian Journal of Philosophy*, *83*(3), 199–206.
21. Restall, G., & Russell, G. (2010). Barriers to implication. In C. Pigden (Ed.) *Hume on Is and Ought* (pp. 243–260). London: Palgrave Macmillan.
22. Russell, G. (2022). How to prove Hume's law. *Journal of Philosophical Logic*, *51*(3), 603–632.
23. Schurz, G. (1997). *The is-ought problem: An investigation in philosophical logic, vol. 1 of Trends in Logic*. Dordrecht: Springer.
24. Schurz, G. (2010). Non-trivial versions of Hume's Is-Ought thesis. In *Hume on Is and Ought* (pp. 198–216). London: Palgrave Macmillan.
25. Singer, D. J. (2015). Mind the Is-Ought gap. *Journal of Philosophy*, *4*, 193–210.
26. Smullyan, R. M. (1986). Logicians who reason about themselves. In J. Halpern (Ed.) *Theoretical aspects of reasoning about knowledge,* (pp. 341–352). Los Altos: Morgan Kaufmann.
27. Smullyan, R. M. (1987). *Forever Undecided A Puzzle Guide to Gödel*. New York: Alfred A. Knopf Inc.
28. Troelstra, A., & Schwichtenberg, H. (2000). Basic Proof Theory. In *Cambridge tracts in theoretical computer science*. 2nd edn. Cambridge: Cambridge University Press.
29. Williamson, T. (2002). Knowledge and its Limits. Oxford University Press on Demand.