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The Value of the One Value: *Exactly True Logic* revisited

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Abstract
In this paper we re-assess the philosophical foundation of *Exactly True Logic* ($\mathcal{ETL}$), a competing variant of *First Degree Entailment* ($\mathcal{FDE}$). In order to do this, we first rebut an argument against it. As the argument appears in an interview with Nuel Belnap himself, one of the fathers of $\mathcal{FDE}$, we believe its provenance to be such that it needs to be taken seriously. We submit, however, that the argument ultimately fails, and that $\mathcal{ETL}$ cannot easily be dismissed. We then proceed to give an overview of the research that was inspired by this logic over the last decade, thus providing further motivation for the study of $\mathcal{ETL}$ and, more generally, of $\mathcal{FDE}$-related logics that result from semantical analyses alternative to Belnap's canonical one. We focus, in particular, on philosophical questions that these developments raise.

Keywords Exactly true logic · FDE · Truth value gluts · Super-belnap logics · Designated values

1 Introduction

In this paper we want to consider and rebut an argument that we distill from an interview with Nuel Belnap, one of the fathers of the well-known four-valued logic known as *First Degree Entailment* ($\mathcal{FDE}$).[1] This argument criticizes *Exactly True Logic* ($\mathcal{ETL}$), which is a four-valued sibling of and rival to $\mathcal{FDE}$. We introduced $\mathcal{ETL}$ in a paper called *Nothing but the Truth*, published in this journal [22]. The title of the

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present piece is a homage to João Marcos's *The Value of the Two Values* [17], which appeared slightly earlier than [22] and discussed a conservative expansion of $\mathcal{ETL}$, albeit motivated by rather different informal considerations (Marcos's title is in turn a nod to Arieli and Avron's *The Value of the Four Values* [3]).[2]

In a nutshell, the philosophical charge that inspired the proposal of $\mathcal{ETL}$ is this: *Inconsistent data is not good enough to be preserved by a logic that serves as the epistemic engine of a data processing unit.* In the logical setting of $\mathcal{FDE}$ and $\mathcal{ETL}$, this corresponds to the question whether the inconsistent value **b** (Both True and False) should be designated, and this is precisely what sets these two logics apart from each other (for a more detailed exposition and discussion of the two logics, see Subsections 2.1 to 2.3).

Crammed into a similarly small nutshell, the claim of the present paper is that this basic and fundamental point is not outweighed by other concerns. We make this claim in response to the above-mentioned argument, which we will survey in the next section (Section 2).

After giving our reasons for rejecting the argument (Subsections 2.4 and 2.5), we aim to further strengthen the case for taking $\mathcal{ETL}$ seriously by showing what fruitful new areas of research its introduction has opened up. To that end, we review the research in algebraic logic and proof theory that $\mathcal{ETL}$ has prompted (Section 3), and the directions in which these developments may further be extended. We end by drawing special attention to open philosophical questions this logic invites us to ask (Section 4).

2 The Interview and the Argument

As we mentioned above, the impetus for this paper comes from an interesting interview with Nuel Belnap that recently appeared in a volume [21] edited by Hitoshi Omori and Heinrich Wansing. The questions were asked by Wansing.

The topic, both of the book and of the interview, is *First Degree Entailment* ($\mathcal{FDE}$), a logic introduced and first investigated by Belnap and collaborators more than four decades ago [6, 7].

$\mathcal{FDE}$ goes beyond classical logic in adding two new truth values to the classical **t** (True) and **f** (False). These values are called **b** (Both True and False) and **n** (Neither True nor False). The central question this paper is concerned with is which of these values should be designated. $\mathcal{FDE}$ is obtained by designating **t** and **b**, but this proposal is contested by the proponents of $\mathcal{ETL}$. Here is the part of the interview that touches on the question:

> H.W. [Heinrich Wansing]: [...] In the truth-table semantics for $\mathcal{FDE}$ both the values **t** and **b** are designated values. Recently, in a paper by Andreas Pietz (Kapsner) and Umberto Rivieccio [22], it is suggested to consider a variant of $\mathcal{FDE}$, which they call "exactly true logic", $\mathcal{ETL}$, where one only has **t** as a designated value. So we take the same truth tables, but designate only **t** ("just

2 We had not been aware of Marcos's work, which in general seems to have had less exposure, probably due to the less accessible venue.

> told True"). But the logic is not paraconsistent, it's not a relevance logic. Would you say that this feature overrides the intuitive appeal of designating just **t** instead of both **t** and **b**?
> N.B. [Nuel Belnap]: I just don't have a feeling about that. I never thought about that in that way at all, and I can't come up with a comment.
> H.W.: Ok, but I would guess that the fact that the logic is not paraconsistent and that it's also not a relevance logic is a drawback in your view, or isn't it?
> N.B.: It surely is. ([21], p. 109-110)

The interview then moves on to other topics, leaving the reader with a sense that spending further energy on thinking about $\mathcal{ETL}$ might not be worthwhile.

We think this impression would be mistaken, and we would like to take this opportunity to assess $\mathcal{ETL}$'s standing and further development a decade after its introduction.

Although the interview moves swiftly at the relevant part quoted above, a discernible argument shines through its lines:

The argument

1. There are intuitively appealing reasons to treat **b** as an *un*designated value.
2. These, however, are outweighed by the loss of paraconsistency and relevance.
3. Therefore, **b** should be designated.

This argument seems defective to us, and we would like to spell out why that is so. In order to do so, we will fill in the blanks in the argument, starting with the reasons for and against designating **b**. The reason for designating **b**, namely the relevance requirement, has been introduced and often endorsed by Belnap himself. By contrast, the reasons *against* designating **b** are not clear to him, as is apparent from the first answer he gave. Our point is to spell out clearly what we take to be a natural understanding of the underlying argument as we sketched it above, and it is not to argue that Belnap fully endorses it, or would endorse it if he were to take more time to think matters through. The danger that we might be battling straw men is therefore quite real; we have, however, heard versions of this argument in conversation several times, though we have never seen it in print before. What is more, it is the *only* philosophical argument that we are aware of against [22]'s critique of $\mathcal{FDE}$.

In order to analyze the argument, we think it useful to consider a conceptual distinction that plays a role in the interview; though it comes up at a different point of the conversation, it is pertinent to our topic. It is the distinction between *pure* and *applied* semantics that J. Copeland used to criticize certain semantics of relevant logics (see [12]).

It is helpful to locate the argument explicitly at either the pure or the applied level. We believe that, at the pure level, we are dealing with a more or less plausible argument. However, we also believe that the argument should better be evaluated at the applied level, and there it has little plausibility at all.

What, then, is this distinction? A pure semantics is a mathematical structure that characterizes the valid inferences of a logic that is syntactically specified. An applied semantics goes beyond that in that it supplies an explanation of what the abstract elements of a pure semantics *mean*. It gives an interpretation of, say, what abstract points in a Kripke model might be, for example a possible world. This interpretation, potentially, has a lot to add; the step from a single point to a whole world is, when you think about it, quite a big one. A bit closer to our case, an applied semantics might supply an interpretation of what non-classical values in a many-valued logic might stand for.

In principle, the development of an applied semantics can proceed either by working out the pure semantics and then looking for an interpretation, the other way around, or it could be a back and forth between the two. When it comes to $\mathcal{FDE}$, the historical development of which is nicely gone over in the interview, the case is best described by the first option.

Here it is in a nutshell: The first steps were taken by finding fault with classical logic for certain inferences that were believed to be descriptively inaccurate for our reasoning patterns, because the premises were irrelevant to the conclusion (see the next section). Then, proof systems were developed that captured, arguably, those inferences that are left when one removes the irrelevant ones. A family of pure semantics was developed in many-valued, algebraic and other forms. Lastly, an interpretation was found that turned these semantic ideas into an applied semantics, namely Belnap's informational semantics to be discussed below.

We will retrace some aspects of this development and then come back to the question how **b** should be dealt with.

2.1 $\mathcal{FDE}$

Relevantists think that certain classically valid inferences are objectionable, because to endorse them is to misunderstand the nature of logical consequence. The inferences they dislike are those in which the premises are irrelevant to the conclusion. A prime example of such an irrelevant inference is the following principle, which is usually called *Explosion*:

$$p \wedge \neg p \vdash q$$

To see why it might be strange to hold this as a valid inference, let p be the statement "Cows give milk" and q "Mars is the morning star". We seem to be reasoning from a strange claim about cows to a not so strange, but false claim about planets. The problem is that no one would draw such an inference, because the two statements have nothing at all to do with one another.

The logic $\mathcal{FDE}$ is a prominent way to get rid of many such irrelevant inferences. Algebraically, the four values of $\mathcal{FDE}$ can be arranged in the following (bi)lattice[3] (Fig. 1):

To get a logic from the Belnap lattice, we interpret conjunction as the meet,

[3] It is a bi-lattice because there is a second lattice order on the elements (from **n** to **b**) which, however, we will not be concerned with here.

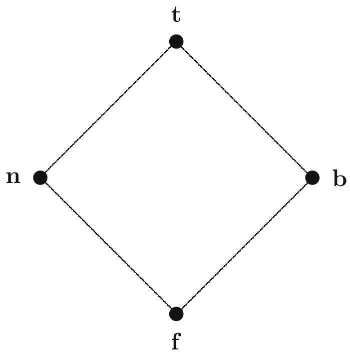

Fig. 1 The (bi)lattice $FOUR$

disjunction as the join, and negation as an operator that flips **t** and **f** but leaves **b** and **n** fixed. This results in the following truth tables:

¬	
t	**f**
b	**b**
n	**n**
f	**t**

∧	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

∨	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

Finally, and this is the main point here, we need to decide which values are to be designated. $\mathcal{FDE}$'s choice is to designate **t** and **b**, and the resulting consequence relation coincides with the logic that preserves the lattice order of $FOUR$ (see Remark 3.1 in Section 3).

So far, so good, at least at the level of pure semantics. As we suggested above, though, the main force of the attack of $\mathcal{ETL}$ comes at the applied level. As Belnap said in the interview, "there certainly can be applications for many-valued logic, but they have to tell me what the applications are" [21, p. 107]. On to the next subsection, then.

2.2 Belnap's Computer Interpretation

Here is Wansing's summary in the interview of the applied interpretation that Belnap found for $\mathcal{FDE}$[4]:

> H.W.: [...] Let us now also talk about the four values and the particular interpretation you gave to them in 'How a computer should think' and 'A useful four-valued logic.' There you explain that you are thinking of a reasoner who is supposed to use the four-valued logic as an "artificial information processor; that is, a (programmed) computer". Moreover, you suppose the information

[4] This sets up a question about the historical development of that idea. This is interesting, especially given that it was developed in the 1970s, when thinking about the cognitive capacities of computers was not a widespread activity.

processor to be a question-answering system being able to answer questions not only based on information given to or stored by the system, but also on the basis of deductions. Another assumption is that the information processor may or may not receive information concerning the truth or falsity of atomic propositions from different, in general trustworthy sources, none of which, however, can be assumed to be a universal truth-teller. Then, of course, the problem of processing inconsistent information and absence of information arises" [21, p. 101].

Here is how this interpretation works in a little more detail: Belnap thinks of the valuations as recording the information that a computer has received about different statements. The computer is given input in the form of atomic statements that are labeled as true or false. As many people are supposed to be building up the database of the computer, it is not impossible that one person might enter a statement as true, while another enters it as false. We then end up with four possibilities for each statement:

n : The computer received no information pertaining to the statement;

f : The computer received the information that the statement is false;

t : The computer received the information that the statement is true;

b : The computer received the information that the statement is true and the information that it is false.

Now the job of the computer is to compute the values of complex statements and draw suitable inferences.[5] Note that, even though they are often referred to as "truth values", it is not clear that these four values need to have much to do with any substantial notion of *truth* at all. After all, the computer might have been fed false data. Belnap calls his semantic values "epistemic" values, and marks the distinction between them and what he calls "ontological" truth. The distinction comes out very clearly in the value we are focusing on, **b**, which is sometimes called a truth value glut. Though others have ventured into ontological interpretations of truth value gluts, in the interview Belnap makes clear that he never had much sympathy with the idea [21, p. 102].

Now, given this interpretation, what does it mean to choose **t** and **b** as designated values, as Belnap did? It means to choose a logic that preserves information that has been labeled as true, even if it might have been labeled false, as well. The idea is that logic is concerned with preserving truth, and both **t** and **b** contain (some) truth. More precisely, Belnap's logic preserves information that is labeled at least as true (and possibly as false, as well).

Belnap also takes the fact (see Remark 3.1) that designating **t** and **n** leads to the same consequence relation as further justification for it: Just as important as preserving

[5] In writing our paper, we just took for granted that the definitions of negation, conjunction and disjunction are *prima facie* plausible on this interpretation, but a referee asks us to expand on this. A conjunction receives value **t** iff it has been told that the first conjunct is true, that the second conjunct is true and it has not been told that either is false. That makes sense, and also the fact that the computer might now know more than its informants: Anne might have told it that it is raining in Amsterdam, Ben that it is raining in Berlin. Both might be agnostic about the claim that it is raining in Amsterdam and in Berlin, but the computer seems to be entitled to draw that conclusion, as long as no conflicting information is present. Dual thoughts underwrite disjunction, and negation seems make sense, too: If some source has told the computer that a statement is false, it should be allowed to conclude that its negation is true, and vice versa. Of course, *prima facie* plausible definitions like that might lead to difficulties under closer inspection, such as the problem discussed below in Section 2.6.

truth is preserving non-falsity. In terms of undesirable properties, the absence of truth and the presence of falsity are on a par. That said, it is usually the value **b** that we find among the designated values of $\mathcal{FDE}$, not **n**.

2.3 Against Designating b: $\mathcal{ETL}$

Why only t should be designated

Let us now hear the reasons against designating informational gluts. Here is a quote from [22]:

> [I]f you want to put your faith in some statements but not others, based on what the computer tells you about its database, then you should be reluctant to do so in the cases of propositions that are "told false", no matter whether they are "told true" or not. Furthermore, if you let the computer draw inferences for you, you should have it choose those pieces of information that you would be willing to put your faith in and reason from those to others that you would be equally willing to rely on. In sum, the prudent computer should choose those pieces of information that are univocally supported and infer conclusions that are similarly univocally supported.

That thought still seems right to us, but it might be useful to illustrate it further than we did in [22]. Imagine using a rather crude GPS system that guides you by displaying text messages, and that furthermore runs on a Belnapian four-valued database. As you get to a crossing, it displays the following three messages:

- Straight ahead is wrong.
- Turning left is wrong.
- Turning left is correct.

What should you do? Going straight seems like a very bad idea. But going either right or left does not sound much better. Sure, you haven't read something bad about the way to the right, but is that enough to be confident that it will lead you to your destination? And though you read something in favor of going left, you also read that it is the wrong way, which should give you pause. Maybe it is time to check whether the good old paper map is still somewhere in the trunk or to look for a pedestrian who might give you better guidance.

That shows that information that is labeled **t** is pragmatically important in ways in which the alternatives are not. A many-valued logic, of course, might designate all kinds of values if the interest is only of a formal nature. Why not, for example, designate only **f** and see what happens? But the point of giving an applied interpretation is that one is interested in matters that go beyond such formal curiosity. The logic should preserve certain qualities from premises to conclusions. And in these examples, these qualities (being apt to base your actions on, to assert to others without reservation, etc.) are possessed by **t**, and none of the other values.[6]

[6] This thought is reminiscent of Dummett's argument that nothing much is gained by categorizing sentences into true and false ones if we do not proceed to explain what the pragmatic difference between the two is (see [13]). Logic is then used to preserve the positive effects.

Accordingly, we argued that the only value that should be designated is **t**. In other words, we asked for a consequence relation that preserves *truth-and-non-falsity* (see [22], p 128). Both features of truth and non-falsity were, as we've seen at the end of the last section, recognized as important by Belnap. However, we stress that having one feature and not the other is not enough. We call for a consequence relation that leads from premises that are true and not false to conclusions that are true and not false. It turns out that under this conception of logical consequence, there are correct inferences that $\mathcal{FDE}$ does not capture.

In our paper, we went on to discuss the logic that results from this choice of a single designated value and called it *Exactly True Logic* ($\mathcal{ETL}$). This logic, as Wansing noted in his question, is not paraconsistent, and therefore not a relevant logic. Let us look a bit closer at it.

$\mathcal{ETL}$ and its properties

From a proof-theoretic point of view, $\mathcal{ETL}$ may be obtained from a Hilbert-style presentation of $\mathcal{FDE}$ (see e.g [28]) by adding the *Disjunctive Syllogism* (DS) rule[7]:

$$p, \neg p \vee q \vdash q$$

which can of course be viewed as a version of *Modus Ponens* with respect to the implication defined by $p \to q := \neg p \vee q$. The presence of the Disjunctive Syllogism entails that $\mathcal{ETL}$ also validates the principle of *ex contradictione quodlibet*:

$$p \wedge \neg p \vdash q.$$

However, the following inference

$$(p \wedge \neg p) \vee (q \wedge \neg q) \vdash r$$

fails in $\mathcal{ETL}$, as does the following stronger contradiction principle, which we may dub *Congruential Explosion*:

$$(p \wedge \neg p) \vee q \vdash q.$$

Indeed, as observed in [28], if we add Congruential Explosion to $\mathcal{ETL}$ (or even to $\mathcal{FDE}$) we obtain straight away the strong Kleene three-valued logic (see Subsection 3.1 and Fig. 4).

$\mathcal{ETL}$ thus lives in a place that is somewhat in the vicinity of paraconsistency, but not quite there; this pattern is somewhat strange, and in [22] we made sure to acknowledge that. We also drew attention to the fact that $\mathcal{ETL}$ "allows for theories that contain disjunctions, but cannot consistently be expanded by *either* disjunct!" Having dubbed this feature *anti-primeness*, we made no pretense that it might be a desirable one when thinking about applications. We also noted that "[...] the inference from $p \vdash r$ and

[7] We here adopt the terminology of [22], which in turn derived from [27]. Another tradition calls the following slightly different rule *disjunctive syllogism*: $\neg p, p \vee q \vdash q$.

$q \vdash r$ to $p \vee q \vdash r$ fails. Presumably, this will not make it easy to find a nice sequent calculus for this logic." We will get back to the merits of that prediction below.

Before we move further on, let us make a comment about the features highlighted above that goes beyond what we wrote in [22]. The behaviour of the disjunction in $\mathcal{ETL}$, unusual as it may be, is shared by disjunction connectives of other non-classical logics. Some substructural logics (e.g. Linear logic, Łukasiewicz logic) employ two distinct disjunction connectives: one (the "additive disjunction") corresponds to the usual lattice join on the algebras ($\vee$), while the other is the "multiplicative disjunction" usually denoted by $\oplus$ and defined by $p \oplus q := \neg p \rightarrow q$. Now, in Łukasiewicz logic, for instance, while one contradiction ($p \wedge \neg p$) or a $\vee$-disjunction of contradictions can never be satisfied, the $\oplus$-disjunction of two contradictions (e.g. $(p \wedge \neg p) \oplus (q \wedge \neg q)$, or even $(p \wedge \neg p) \oplus (p \wedge \neg p)$) is indeed satisfiable.

Such a phenomenon is not uncommon in resource-sensitive logics[8]: what is particularly unusual in $\mathcal{ETL}$ is that it is the only existing (lattice) disjunction the one exhibiting a resource-sensitive behaviour.

We will return to this connection to resource-sensitive logics later on. For now let us get back to the main question of this first part of our present paper: Given that relevance is lost, should we dismiss the argument that motivated the development of $\mathcal{ETL}$?

2.4 An Illustrative Example

We want to argue that, at the level of applied semantics, the fact that relevance is lost by designating only **t** does not speak against this move if you are convinced by the arguments for it. Here is a parallel example to illustrate the point, one that employs a very different interpretation of the truth values we find in FOUR.[9]

Consider statements that contain names and definite descriptions and two classes of cases in which things go wrong semantically: First, when an existential presupposition for a definite description is not met, as in:

The present King of France is bald,

and second, for cases in which a name fails to pick out a suitable object, such as:

Vulcan is large.

Suppose we have concluded that it would be good to have separate truth values for instances of these two classes of semantic failures; let us call them **d** and **e**, respectively. Suppose further that we want to treat complex statements exactly parallel to the way they are treated in $\mathcal{FDE}$.

[8] The emphasis on *resources* rather than truth or proof is, in fact, the main motivation behind the introduction of Linear logic: see e.g. the Stanford Encyclopedia entry (https://plato.stanford.edu/entries/logic-linear/)

[9] Our example is inspired by [18], but it is adapted for our purposes and should not be confused with the actual proposal Martínez and Martí make.

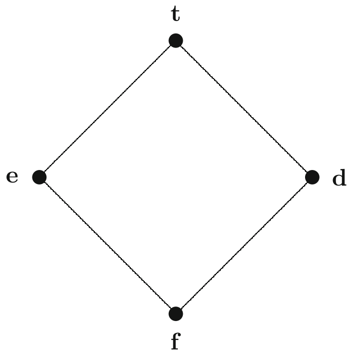

Fig. 2 Bilattice $FOUR$, Denotation Failure Interpreation

Here is how the lattice would look, with conjunction, disjunction and negation treated as above (Fig. 2):

To cash in on the illustrative value of the example, let us get to our main question: Which values should be designated under this interpretation? Clearly **t** should be designated, and equally clearly **f** should not be. What about **e**, the value that attaches to statements with empty names? A statement that purports to be about an existing object that is not there is nothing one should assert, base one's opinions or actions on, etc. So, we should not designate this value. But the same seems true about sentences with denotationless definite descriptions. Therefore, **d** should likewise be undesignated. Once again, we are left with **t** as the only designated value.

But what should we say if someone should object that this choice does not lead to relevance, and that we therefore must treat exactly one of **e** or **d** as designated? We think the appropriate response to that argument is clearly *not* to comply, but to point out that the interpretation is just not the right one to satisfy relevantist purposes. It might be an interesting and potentially fruitful interpretation, but relevant logicians in search for an applied semantics will have to look elsewhere.

2.5 The Lesson to be Learned

Now, let us apply the lesson from the hypothetical example of the last section to our actual problem, namely how to deal with Belnap's informational interpretation. If you are swayed by the arguments against designating **b**, your response should be just the same as the one at the end of the last section: The informational interpretation is interesting and useful, but it is not useful for a relevantist looking for an intuitive explanation of their base logic. This will have to be found elsewhere, in a different interpretation that makes the assignments of values to complex formulas along $\mathcal{FDE}$'s lines plausible and also calls for the designation of exactly one of the non-classical values (either **b** or **n**).

On the other hand, what if someone is *not* swayed by the arguments against designating **b**? Then they should give their reasons - *on the merits of the interpretation* *alone* - for designating **b**. That this move leads to relevance can't be among those reasons, just like it can't be among the reasons for designating **e** (or **d**) in the illustrative example above. This is an argument to be had at the applied level, not the pure one.

The only situation in which the question of relevance should be the final clincher is one in which, given a particular interpretation, it really did not otherwise matter whether one of the "middle" values is designated or not. Given that questions about designation are just about as fundamentally important to logic as any, this situation is hard to conceive of. It certainly doesn't strike us as the situation we are in regarding Belnap's informational interpretation.

What, finally, should the relevantist do who sees the merit of the argument against designating **b**, but likes $\mathcal{FDE}$ and wishes to use it as her base logic? She will have to look for other interpretations. Maybe ontological gluts, applied in the realm of paradox resolution hold some hope. Or, maybe more likely, there is an interpretation that doesn't make the middle values any type of gaps and gluts at all, and readily suggests that one should be designated while the other should not. We don't know what that interpretation might be (surely not the one in Section 2.4), but it might be worthwhile to look for it.

A reviewer points us, in particular, to J.C. Beall's essay *FDE as the One True Logic* ([4]. The idea of that essay is that FDE might be suitable as a very general base logic, in fact the most general base logic of all (hence the one true logic). By implementing certian requirements, one can force FDE to behave like classical logic, hence all theories that are governed by classical logic can be dealt with. However, there is also the possibility to allow theories that have truth value gaps, theories that have truth value gluts, or both. Beall does not discuss the question of designation, but the examples for theories that call for gluts give a hint. He is thinking about the semantic paradoxes and religious theories, such as the Christian idea "that the god-human figure is both divine and human (with all of the apparent contradictions entailed therein)". We want to voice no opinion on the viability of these examples. That Belnap was never in favor of the former kind of application is documented,[10] what he thinks about Christ, we do not know.[11]

To sum up: At the level of pure semantics, the argument has some sway. The relevantist has no reason to be interested in the valid inferences of $\mathcal{ETL}$ at the level of pure semantics, and neither, we would think, would have many other people. So, if there is any fault at all to find with the argument as we reconstructed it on page 2 when viewed solely from the pure-semantics angle, it would be that to speak of "intuitively appealing" reasons for designating only **t** is quite misleading. There are no such intuitively appealing reasons from this point of view.

[10] See, again, the relevant part of the interview at [21, p. 102].

[11] In an appendix to the paper, Beall very tentatively suggests another interpretation: To read the values as judgments of the following kind: "There's truth in that", "There's truth and falsity in that", "There's falsity in that" and "There's neither truth nor falsity in that". Though he himself acknowledges that the proposal is "terribly sketchy" (p. 125), this might indeed be an interpretation that makes designating **b** natural, and moreover one that would be congenial to Belnap's philosophical outlook.

However, at the level of *applied* semantics, the argument ceases to be a good one if we have the informational interpretation in mind, just as the parallel argument concerning the interpretation in Section 2.4 was a bad one.[12]

At the applied level, the only course an argument for $\mathcal{FDE}$ and against $\mathcal{ETL}$ could take is to show why the informational interpretation makes it intuitively natural to designate contradictory information. As long as that version of the argument is not presented, the challenge of $\mathcal{ETL}$ stands. And, at least to our knowledge, an attempt to present this version of the argument has not yet been made.

2.6 More Values?

While we know of no arguments that show that designating **t** and **b** is intuitively more plausible than designating only **t**, we must admit that $\mathcal{ETL}$ is not without intuitive problems (problems, we hasten to add, that $\mathcal{FDE}$ also has).

A perceptive reviewer asks us, seeing our hypotheitical example in Section 2.4, to consider the statement

(1) The present King of France is bald or Vulcan is large.

That statement, on the proposed semantics, receives value **t**. The reviewer notes, correctly, that this is as unacceptable as either of its disjuncts.

It is an old wound that the reviewer rubs salt into, and we can't pretend that it doesn't hurt. The point, of course, carries over from our hypothetical example to the actual informational interpretation we are interested in. A statement that consists of a disjunction of a statement that the computer has been given no information about and one that it has received contradictory information about should ostensibly not be designated. That it is on the semantics of both $\mathcal{ETL}$ and $\mathcal{FDE}$ is the famous oddity that Anderson and Belnap had already mentioned back in [2, p. 518].

The reviewer asked us to discuss an idea proposed by Kapsner in [16], presumably not knowing that this was pointing us to our own prior work. Be that as it may, the idea is to overcome the oddity by adding two more values, labeled **t¿** and **f¿**, leading to the following six-valued lattice (Fig. 3):

The value **t¿**, intuitively read as "contestedly true", signifies that there is a conflict that corresponds to our intuitive rejection of a disjunction of kind (1) above. Whether Kapsner succeeded in giving an intuitively satisfying interpretation beyond that might be debatable, but he shows that designating only **t** in this setting leads to Strong Kleene logic, a result that might be satisfying as its consequence reltation is less exotic as that of $\mathcal{ETL}$.

[12] One could argue that the reasons for disliking $\mathcal{ETL}$ go beyond the loss of relevance, and thus we give those who want to reject $\mathcal{ETL}$ short shrift in focusing on relevance and paraconsistency. As discussed above, as a consequence relation it has more unattractive features than the mere validity of Explosion, and indeed, its problematic features had been already pointed out in [22]. However, dialectically, that does not seem to us to change much. Think again about the illustrative example in Section 2.4. The logic that results from not designating **e** and **d** has, of course, all of these bad features, as it coincides with $\mathcal{ETL}$ in all but the names and interpretations of the values. And still, this is not reason enough to accept that **e** or **d** should be designated.

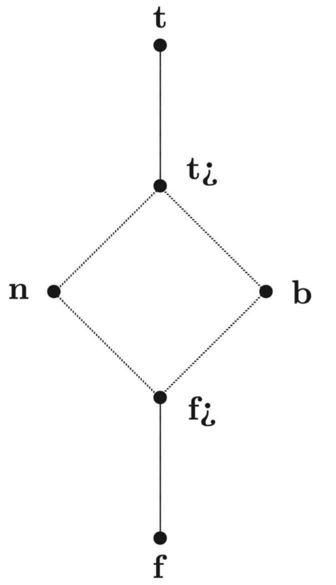

Fig. 3 Lattice SIX

The six-valued semantics has been sporadically discussed before (see references in [16]), and it has recently featured prominently in work by Melvin Fitting (such as [14]), who has other applications of it in mind that do not rely on explicit interpretations of the values.[13]

However, beyond that it has not received much attention, and it seems too early to say whether it has a bright future ahead of it. At least at this stage, $\mathcal{ETL}$ has received much more attention, not to speak of $\mathcal{FDE}$. We will not delve further into these matters, but only make the following refinement of our claim:

Assuming that we want to restrict ourselves to the *four* values employed in $\mathcal{FDE}$ and $\mathcal{ETL}$ and compare the two logics, the best fit to our intuitions about the informational interpretation is achieved by $\mathcal{ETL}$. This is because the choice of designated values makes more sense to us, and because any problems that one of the two logics has with complex statements, the other has as well.

[13] Prof. Fitting has, however, suggested a very interesting interpretation in private conversation, and kindly allowed us to mention it here. It is, broadly speaking, a constructive interpretation in which the values are understood, from top to bottom:

1. The proposition has a proof, and hence no counter-model.
2. The proposition has no counter-model, but I haven't yet established that it has a proof.
3. The proposition has neither a proof nor a counter-model.
4. I haven't yet established that it has a proof, and I haven't yet established that it has a counter-model.
5. The proposition has no proof, but I haven't yet established that it has a counter-model.
6. The proposition has a counter-model, and hence no proof.

3 Research Inspired by *Nothing but the Truth*

This concludes our assessment of the argument distilled from the interview, and our conclusion is that the loss of relevance is no reason to ignore $\mathcal{ETL}$.

There is, however, another way in which one can pay attention to the semantics of $\mathcal{ETL}$ and judge the fruitfulness of this logic, much further removed from questions of intuitive interpretations: that is to view the semantics as an algebraic structure and see how many doors it opens in the exploration of the algebraic realm. Here $\mathcal{ETL}$ proved highly valuable as well, as it led to a substantial amount of interesting research over the last decade. Indeed, beyond the algebraic realm, $\mathcal{ETL}$ can also be seen as the origin of interesting new developments in proof theory. In this section we provide further motivation for (the choice of designated values leading to) $\mathcal{ETL}$ in the form of a brief account on some research trends which, over the last decade, have been stimulated or influenced by *Nothing but the truth*.

3.1 The Lattice of Super-Belnap Logics

A line of research that mostly attracted the interest of algebraic logicians originated with the following question. The fact that $\mathcal{ETL}$ had been shown in [22] to be a logic intermediate between $\mathcal{FDE}$ and classical logic (indeed, even intermediate between $\mathcal{FDE}$ and strong Kleene logic) suggests that there may be other distinct logics lying in an interval where one might have naively assumed there were none (cf. Footnote 19). The latest reply to this question, as given in [26] – namely, that there are infinitely many (at least a continuum of) logics in the interval between $\mathcal{FDE}$ and classical logic – might come as a surprise, especially to those who overlook the peculiar features of logics construed as set-formula consequence relations, as is nowadays standard in the algebraic logic community, rather than sets of theorems. The systems that are intermediate between $\mathcal{FDE}$ and classical logic have been dubbed *super-Belnap logics* in [28], a paper which also contains the first proof that there are infinitely (countably) many of them.

The term "super-Belnap" was inspired by the standard term for logics intermediate between the intuitionistic and the classical, known as "super-intuitionistic logics". As far as cardinality goes, super-Belnap logics are indeed not unlike super-intuitionistic logics, for of the latter there has long been known to be continuum many as well. From an algebraic logic point of view, however, the analogy between extensions of $\mathcal{FDE}$ and super-intuitionistic logics does not go much further, which makes the result on super-Belnap logics more unexpected.

In fact, the algebraic counterpart of intuitionistic logic is the (equationally definable) class of Heyting algebras, which is known to have continuum many equational subclasses. If one regards a logic as the syntactical *alter ego* of an equational class of algebras – classical logic corresponding to Boolean algebras, Łukasiewicz logic to MV-algebras, and so on – one should not be surprised at the observation that super-intuitionistic logics are in one-to-one correspondence with equational subclasses of Heyting algebras. By contrast, the algebraic counterpart of $\mathcal{FDE}$ is the class of De Morgan algebras, which, far from having infinitely many equational subclasses, is well known to have (besides the trivial one) only two of them: Kleene algebras[14] and Boolean algebras. This mismatch provides perhaps one of the few non-artificial examples in the logical literature which illustrate two important aspects of the behaviour of general logical systems.

First of all, one should stress that, from an algebraic perspective, a logic is best viewed as the syntactical counterpart of a *quasi-equational* class of algebras (one definable by means quasi-equations, i.e., first-order formulas consisting of an implication with a finite conjunction of equations as a premise and a single equation as a conclusion) rather than an equational one; or indeed, in case the logic is not compact, of a more general class of algebras (definable by first-order implications having possibly infinite conjunctions of premises).

A second observation, to a certain extent overlapping with the first, is that one may naively assume that there is but one way of associating a logical system to a given class of algebras. This is certainly not the case: for instance, a classical result which can be traced back to V. Glivenko shows that Heyting algebras may be used (if in a somewhat unnatural way) as an algebraic semantics for classical (rather than intuitionistic) logic. More specifically (and even more naively), one might think that the standard way of defining a logic from a class of algebras – which gives the expected result for classical and intuitionistic logic and for $\mathcal{FDE}$, but not for Łukasiewicz, to give but one example – is the one that employs the definable partial order on the algebras (see Remark 3.1 below).

The above-mentioned assumption, in turn, may lead one to take it for granted that arbitrary logics are congruential (for all logics of order indeed are) in the sense that, whenever two formulas p and q are inter-derivable in the logic, then so must be $p \vee r$ and $q \vee r$, as well as $p \wedge r$ and $q \wedge r$ (for any formula r), and so on[15]. This is dramatically false in the setting of super-Belnap logics (and indeed it had to be so, given the above considerations): only two non-trivial super-Belnap logics are congruential, namely classical logic and Kleene's logic of order (we note that the latter is not the standard "strong Kleene logic" usually considered in the literature: see below). In $\mathcal{ETL}$, in particular, we have for instance $p \wedge \neg p \dashv\vdash q \wedge \neg q$, but not $(p \wedge \neg p) \vee r \dashv\vdash (q \wedge \neg q) \vee r$. (To see this, consider a valuation v such that $v(p) = v(r) = \mathbf{n}$ and $v(q) = \mathbf{b}$.)

Remark 3.1 *It appears that, indeed, $\mathcal{FDE}$ was originally introduced and motivated in [6, 7] as the consequence relation preserving the lattice order. Formally, this consequence coincides with the logic obtained by designating* $\mathbf{t}$ *and* $\mathbf{n}$ *(i.e. preserving non-falsity) as well as with the logic that preserves truth: any of these three alternative definitions may thus be employed to further substantiate the claim that $\mathcal{FDE}$ is – at least from a formal point of view – the "right" logic to be associated to the Belnap lattice. We should however like to point out that this coincidence relies essentially on*

14 A formal definition of Kleene algebras can be found in [28]; this usage of the term can be traced back at least to the paper [9]. To avoid confusion, it may be useful to point out that another time-honoured tradition (see e.g. [11]) calls "Kleene algebras" algebraic structures that are unrelated to $\mathcal{FDE}$ (and indeed to non-classical logics in general).

15 In the algebraic logic literature, a synonymous, more standard (albeit more opaque) term for 'congruential' is *self-extensional*.

the (algebraically rather weak) language of $\mathcal{FDE}$ (which is not expressive enough to formally distinguish **b** *from* **n***), and therefore doesn't apply to the most well-known expansions of $\mathcal{FDE}$ (e.g. Nelson's and bi-lattice logics). Once these definitions of consequence come apart, it is not at all clear that the definition in terms of the lattice order has, philosophically, primacy over the definitions in terms of designated values. In fact, considering the lattice that results from removing either* **n** *or* **b** *from FOUR, we may observe that the logics* ***K3*** *and* ***LP****, definable by designating one or two of the upper values, have received much more attention than the logic that corresponds to the order (called* Kleene's logic of order *in [28] and later algebraic studies, also known as the first degree fragment of* ***RM****).*

We further note that order-preserving logics do not often display very nice mathematical properties (such as algebraizability): the standard fuzzy logics, for instance (Łukasiewicz, product logic, Hájek's Basic Logic), are not the logics of order of the corresponding classes of algebras; the latter are less well-known and, at least from an algebraic point of view, less well-behaved.

The extensions (i.e. strengthenings) of a given logic may naturally be ordered by setting $\mathcal{L}_1 \leq \mathcal{L}_2$ whenever $\mathcal{L}_2$ extends $\mathcal{L}_1$. The resulting partially ordered "set" (actually a proper class) is in fact a lattice, that is, it possesses finite meets and joins. One can thus ask questions about this lattice, not only on its cardinality but also regarding its order structure, such as: is it a distributive (or modular) lattice? Does it have (and, if so, which logics are its) atoms? Etc. These are often hard questions, making the techniques used to answer them and the corresponding results mathematically interesting. In this respect, the lattice of super-Belnap logics is no exception to the above-mentioned trend, for it has a complex, interesting and not yet fully understood structure.

A powerful toolkit in the study of lattices of logics is universal algebra, which can often be used to transfer information obtained on the lattice of sub-quasi-varieties of a given class of algebras (the algebraic counterpart of a given logic $\mathcal{L}$) to the lattice of extensions of $\mathcal{L}$. This strategy is guaranteed to work for *algebraizable* logics, i.e. for those logics whose consequence relation is equivalent, in a strong sense, to the equational consequence of some class of algebras (see [8] for the relevant formal definition). In this respect the case of $\mathcal{FDE}$ is quite interesting.

On the one hand, $\mathcal{FDE}$ is not algebraizable, which means that we cannot simply obtain a description of the lattice of super-Belnap logics by a routine application of general results known for algebraizable logics. On the other hand, as shown in [28], it is indeed possible (if by *ad hoc* methods) to import useful results from the study of the lattice of sub-quasi-varieties of De Morgan algebras; this lattice, however, is itself quite complex and its structure is, to this day, not fully understood[16]. Recently, further help was obtained thanks to the discovery of an interesting connection between the algebraic theory of De Morgan algebras and graph theory [1, 26]: this unexpected

[16] We add another curious technical observation for the interested reader. The lattice of sub-quasi-varieties of De Morgan *lattices* (i.e. algebras in the language $\{\wedge, \vee, \neg\}$ not including the lattice bounds) is fairly simple and finite, whereas the lattice of sub-quasi-varieties of De Morgan *algebras* (which expand the language of De Morgan lattices with constants 0 and 1 representing the two lattice bounds) is the complex infinite one we are referring to. However, as shown in [28], even if we do not include truth constants in the language of super-Belnap logics, we can still show that there are at least countably many of them by importing certain results from the theory of sub-quasi-varieties of De Morgan algebras.

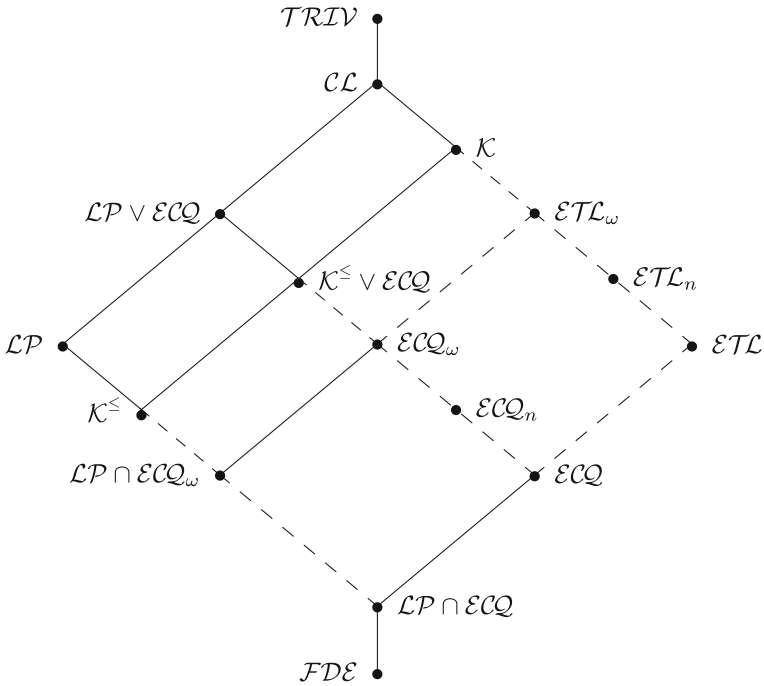

Fig. 4 The lattice of super-Belnap logics

insight made it possible to establish a number of significant results, of which we shall cite a few below.

Let us have a look at the diagram of the lattice Ext($\mathcal{FDE}$) of super-Belnap logics (Figure 4)[17]. As a lattice, Ext($\mathcal{FDE}$) is non-modular [1, Thm. 4.1] – hence, *a fortiori*, also non-distributive – and has a unique atom (as well as a unique co-atom, if we include the trivial logic $\mathcal{TRIV}$ into the picture). Some parts of the diagram are finite and show all the logics that are there, while others (corresponding to the dotted lines) indicate the presence of infinitely many logics within the given interval. Thus, for instance, there are only five proper non-trivial extensions of the logic $\mathcal{K}^\leq$ (Kleene's logic of order), all of them shown in the diagram, while there are at least countably many logics included between $\mathcal{ETL}$ and $\mathcal{ETL}_\omega$, and likewise between $\mathcal{ECQ}$ and $\mathcal{ECQ}_\omega$.

The logic $\mathcal{ECQ}$ – named after the explosion principle known as *Ex Contradictione Quodlibet* – was first singled out by A. Přenosil, who proved that $\mathcal{ECQ}$ is the weakest explosive (i.e. non-paraconsistent) extension of $\mathcal{FDE}$ [1, 26]. Indeed, ECQ is precisely the logic obtained by adding to FDE the *Ex Contradictione Quodlibet* rule:

$$p \wedge \neg p \vdash q.$$

[17] The figure displays the lattice of super-Belnap logics considered in the language that includes the truth constants **t** and **f**. If the constants were not included, then we would have an extra logic included between $\mathcal{CL}$ and $\mathcal{K}$. To avoid confusion, we also note that $\mathcal{FDE}$ is denoted $\mathcal{B}$ – for Belnap – in the papers [1, 26, 28].

A denumerable sequence of stronger logics

$$\mathcal{FDE} < \mathcal{ECQ}_1 < \mathcal{ECQ}_2 < \ldots < \mathcal{ECQ}_n < \ldots \mathcal{ECQ}_\omega$$

can be further obtained by adding to $\mathcal{FDE}$ rules of the following type:

$$(p_1 \wedge \neg p_1) \vee \ldots \vee (p_n \wedge \neg p_n) \vdash q.$$

On the other hand, if we add the *Congruential Explosion* rule $(p \wedge \neg p) \vee q \vdash q$ to $\mathcal{ETL}$ (or, indeed, even to $\mathcal{FDE}$), we obtain strong Kleene's logic $\mathcal{K}$ straight away.

In $\mathcal{ECQ}$, one contradiction explodes but a disjunction of two contradictions may not; likewise in $\mathcal{ECQ}_n$ a disjunction of n contradictions explodes but a disjunction of $n+1$ may not. A similar reasoning applies to the sequence going from $\mathcal{ETL}$ ($= \mathcal{ETL}_1$) to $\mathcal{ETL}_\omega$, except that all the logics in the chain $\mathcal{ETL} < \mathcal{ETL}_2 < \ldots < \mathcal{ETL}_n < \ldots \mathcal{ETL}_\omega$ further satisfy the *Disjunctive Syllogism* rule ($p, \neg p \vee q \vdash q$).

To conclude this brief overview on super-Belnap logics, let us mention a few interesting results that were obtained thanks to the above-mentioned connection with algebra and graph theory:

1. Some (finite-valued) super-Belnap logics are not finitely axiomatizable by means of Hilbert-style calculi [28, Thm. 15].
2. Some super-Belnap logics are not finitary, i.e., not compact [1, Prop. 4.9].
3. For each $2 \leq n < \omega$, the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ are not finite-valued (in the sense of e.g. [10]), that is, they are not complete with respect to any finite set of finite matrices [1, Prop. 4.10]. Thus, for these logics – similarly, for instance, to the intutionistic – we cannot fix a standard semantics given by a finite truth table. This suggests that one cannot reason semantically, as we have done with $\mathcal{FDE}$ and $\mathcal{ETL}$, in terms of the interpretations to be assigned to a fixed (finite) set of truth values.
4. Each of the intervals of logics $[\mathcal{LP} \cap \mathcal{ECQ}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $[\mathcal{ETL}, \mathcal{CL}]$ has at least the cardinality of the continuum [1, Thm. 4.13].

3.2 Proof-Theoretic Studies on $\mathcal{ETL}$ and Its Cousins

Another line of research worth mentioning is the proof-theoretical study of $\mathcal{ETL}$ and related logics. This trend was apparently prompted by our statement (from *Nothing but the Truth*) that "the inference from $p \vdash r$ and $q \vdash r$ to $p \vee q \vdash r$ fails [in $\mathcal{ETL}$]. Presumably, this will not make it easy to find a nice sequent calculus for this logic" ([22], p. 129). The above remark must have sounded like a challenge to S. Wintein and R. Muskens, who proceeded in [36] to show that a sequent calculus for $\mathcal{ETL}$ could be easily obtained by adapting previous (but unknown to us at the time) work by the same authors on the proof theory of $\mathcal{FDE}$. The sequent calculus presented in [36] is analytic and cut-free, but it is non-standard in that its objects are not ordinary sequents (i.e. ordered pairs (Γ, Δ) where Γ and Δ are sets of formulas, perhaps endowed with further structure – e.g. multisets etc.) but labelled sequents, the (four) possible labels ranging over two-element subsets of FOUR.

Two further remarks from [36] deserve, in our opinion, to be highlighted. Firstly, as the authors observe [36, p. 464-5], their non-standard axiomatization of $\mathcal{ETL}$ allows one to explain away – at least from a proof-theoretic point of view – the puzzling feature called anti-primeness in [22], namely the phenomenon that one contradiction explodes in $\mathcal{ETL}$ whereas the disjunction of two contradictions need not[18]. We note that a similar behavior is exhibited by each logic $\mathcal{ETL}_n$ and each $\mathcal{ECQ}_n$ (for $n < \omega$)

Secondly, the authors of [36] also observe that, defining $p \to q := \neg p \vee q$, one obtains an implication $\to$ that satisfies *modus ponens* with respect to the consequence of $\mathcal{ETL}$: this does not hold in $\mathcal{FDE}$, and may indeed be regarded as a further desirable feature one gains when abandoning $\mathcal{FDE}$ for $\mathcal{ETL}$. However, it is also true that the above-defined implication does not (and could not possibly) enjoy the Deduction Theorem, for $\mathcal{ETL}$ (like $\mathcal{FDE}$) is a logic without theorems. If one is bothered by this, one can add a new implication $\supset$ to obtain a conservative expansion of $\mathcal{ETL}$ that will (by design) satisfy the Deduction Theorem [36, p. 459]. Besides this new implication, Wintein and Muskens also axiomatize other expansions of $\mathcal{ETL}$ obtained by adding, for instance, the bilattice connectives $\otimes$ and $\oplus$ (corresponding to the *information* order on FOUR, i.e. the one having **n** as least element and **b** as the greatest; see the tables below).

$\supset$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**t**	**t**	**t**
n	**t**	**t**	**t**	**t**
f	**t**	**t**	**t**	**t**

$\otimes$	**t**	**b**	**n**	**f**
t	**t**	**t**	**n**	**n**
b	**t**	**b**	**n**	**f**
n	**n**	**n**	**n**	**n**
f	**n**	**f**	**n**	**f**

$\oplus$	**t**	**b**	**n**	**f**
t	**t**	**b**	**t**	**b**
b	**b**	**b**	**b**	**b**
n	**t**	**b**	**n**	**f**
f	**b**	**b**	**f**	**f**

In a subsequent paper [37], the same authors prove the important interpolation property for ETL, also pointing out that this result could not be obtained via the standard (Maehara-style) method. Interpolation theorems are established for various other extensions of $\mathcal{FDE}$ in A. Přenosil's paper [24], which contains a general proof-theoretic study of super-Belnap logics.

The paper [35] by Wintein alone also deals with $\mathcal{ETL}$ both from a semantical and a proof-theoretic point of view. The main focus in this case is the study of several possible consequence relations that may be defined on FOUR through different choices of designated elements, corresponding to the idea that derivations should preserve 'truth', 'exact truth', 'non-falsity' etc.

The same approach is pursued in a series of subsequent papers by Y. Shramko, A. Přenosil and others [25, 29, 33]; the main focus of [29], for instance, is the "non-falsity" logic ($\mathcal{NFL}$), i.e. the logic determined on FOUR by designating the set $\{\mathbf{n}, \mathbf{b}, \mathbf{t}\}$. The intuitive idea is that there may be situations in which a logic might be suitable that avoids pure falsity, but is accepting all the other possibilities. From an algebraic point of view, this "dual-$\mathcal{ETL}$" logic is unusual in that the designated

[18] This feature depends on the following technical remark. Many logics (classical, intuitionistic, $\mathcal{FDE}$) may be defined through lattice-ordered algebraic structures by taking *prime filters* as sets of designated elements. While the set $\{\mathbf{t}, \mathbf{b}\}$ is a prime filter on the Belnap lattice, the set $\{\mathbf{t}\}$ is not, and it cannot be equivalently replaced by any family of prime filters.

elements do not form a lattice filter but only an upward set with respect to the usual lattice order on FOUR (so logical theories need not be closed under conjunctions). This observation opens the perspective of a more general study of logics determined by upward sets on De Morgan algebras, or even on Boolean algebras: such a route is indeed explored in the forthcoming [23].

Another very recent paper by Shramko [30], which appears to be an extension of the earlier [34], contains a systematic investigation of super-Belnap logics parallel to the one considered in the preceding subsection, the main difference being that the setting of the papers [30, 34] is that of formula-formula (Hilbert-style) axiomatizations: this amounts to saying that only rules of type $\phi \vdash \psi$ are permitted (the premise being required to be a single formula), whereas [1, 26, 28] employ the more standard rules of type $\Gamma \vdash \psi$, where the premise is assumed to be a finite set of formulas.

Lastly, let us mention that the study of exactly-true and non-falsity preservation logics of [34] has been recently extended in [5] to yet other systems related to $\mathcal{FDE}$, namely logics defined by four-valued truth tables that differ from those of $\mathcal{FDE}$ in that some non-classical values exhibit an "infectious" behavior, as in the well-known Bochvar-Kleene logic.

To add a concluding consideration on the research trends considered in this subsection, it seems to us that a main impact the introduction of $\mathcal{ETL}$ had on the community of logicians interested in $\mathcal{FDE}$-related logics may be summarized as follows: *Nothing but the Truth* conveyed the idea that, besides Belnap's canonical and time-honored proposal leading to $\mathcal{FDE}$, other FOUR-valued consequence relations, based on alternative choices of the values to be preserved, are also worth playing with and investigating – at the very least from a formal point of view. [19]

4 Philosophical Reflections on $\mathcal{ETL}$-extensions

We end by some philosophical reflections inspired by $\mathcal{ETL}$ and the subsequent developments summarized in the last section. These are just meant as very selective and illustrative examples, there are surely many more. In particular, the new proof techniques developed to deal with $\mathcal{ETL}$ and its cousins might lead to interesting questions

[19] Given the playfulness of many well-known researchers in the area, it is perhaps a little surprising that such experiments had not been performed even earlier. Also, in view of our argument that **t** is the natural choice for being the only designated value, one has to ask why it has taken well over thirty years for Marcos and us to investigate the idea almost simultaneously.

As a partial explanation, we can relate that prior to writing our paper, we asked many of the leading experts in the field what would happen if **b** was dropped as a designated value. We found that many thought that the result would be Strong Kleene logic, a natural but, of course, false assumption. Making this assumption might have left many with the impression that there was nothing interestingly new to be explored here.

Also, we should mention that we later found that Section 3.5 of the *Stanford Encyclopedia of Philosophy* entry on "Many-Valued Logic" (due to S. Gottwald) contains a very brief reference to the idea of designating **t** alone within FOUR: this possibility is mentioned just in passing but, interestingly, it is said that "for computer science applications it is natural to have **t** as the only designated degree". In private conversation, Prof. Gottwald told us that he was talking to a colleague in computer science who had not heard of FDE and immediately objected to the choice of designating **b**.

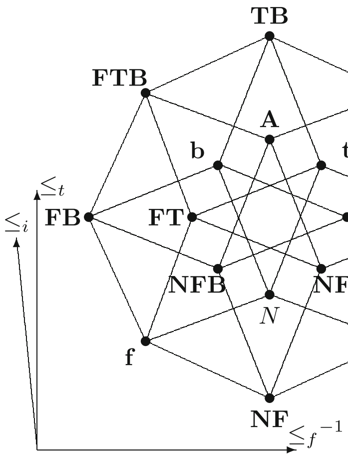

Fig. 5 The trilattice $SIXTEEN_3$

in proof-theoretical semantics and other philosophical applications in which proof theory plays a role, but we will not comment on those.

4.1 Exporting the Philosophical Argument

The philosophical challenge that gave rise to the development of $\mathcal{ETL}$ is not tied to any specific feature of the (bi)lattice FOUR, but can be put more generally as a requirement on any logic that deals with truth gaps and gluts. To give a concrete example, consider the logic based on the trilattice SIXTEEN introduced by Shramko, Dunn and Takenaka [31] and developed much further by Shramko and Wansing [32][20] (Fig. 5).

It was first introduced as a kind of constructive logic[21] and later as a logic that deals with information flows in computer networks. We will concentrate on the latter interpretation, as it is much closer to the informational interpretation that we have discussed above.

Shramko and Wansing took Belnap's question, "How should a computer think?", a step further and asked "How should a computer *network* think?" In such a scenario, reasoning takes place inside a server that gets information from many computers that are set up in Belnapian fashion. That is, the server receives information about statements, and the connected computers will report either **t**, **n**, **b** or **f**, depending on what they have been told. As the server gets information from many sources, every combination of these values is possible.[22]

[20] See also [19] for a proof-theoretical study of the logic of SIXTEEN. A note on the diagram: we have relabeled the original four values in accordance with our labels in this paper and left the twelve additional values as they appear in [32].

[21] [31]. See also [15] for extended discussion.

[22] N.B.: This includes the case where it receives no feedback from the connected computers at all, denoted by the value N in the diagram above. This should not be confused with value **n**, which means that some of the computers have reported back, saying that they have no information about the statement in question.

Now, what would it mean to bring the conceptual motivation of $\mathcal{ETL}$ into this setting? Shramko and Wansing do not define their logic in terms of designated values, but we can ask what the right choice of designated values in SIXTEEN would be under the computer network interpretation. And, expanding on the reasoning above, a very plausible answer seems to be: Just t. This is the only value that reflects that we can, given what was told to the connected computers, unreservedly put our trust in the statement's truth. All other values seem to have some kind of problem that puts the statement in doubt in one way or another. In particular, the value BT has a problem: At least one computer was informed that the statement in question is false (as it happens, it was also informed that it is true).

If the lattice order is the one depicted above, then designating only **t** would lead to a weird logic that would seem to be virtually useless (as it would not validate many familiar rules, such as disjunction introduction, conjunction elimination, etc.). This is because **t** is not the top element, as it is strictly smaller than BT.

In a sense, the $\mathcal{ETL}$-inspired challenge is setting in at an even earlier point than the question of designation, namely at the question of why and whether BT should be above **t** for this application. Due to the symmetrical nature of SIXTEEN, it is possible to choose an alternative lattice structure that, indeed, has **t** as top element. Shramko and Wansing [32] mention this as one of the possible orders: it is the one having **t** as greatest and NFB as least element, respectively. Let us denote it by $\leq_{\mathcal{ETL}}$, to reflect our intuition that the argument that inspired $\mathcal{ETL}$ would lead one to consider this order, with **t** as sole designated element, as a very natural system to pair with the network interpretation. Whether or not this will lead to the same consequence relation as $\mathcal{ETL}$, however, will depend on the choice of our connectives.

Having endowed SIXTEEN with the lattice operations (meet and join, interpreting conjunction and disjunction) determined by the order $\leq_{\mathcal{ETL}}$, we can further define a negation operator which allows us to view SIXTEEN as an algebra in the same language as the Belnap (bi)lattice FOUR. The choice of such a negation is by no means unique; even if we further postulate that the resulting algebra must satisfy the De Morgan and the double negation laws (as FOUR does), we still have more than one possible candidate (for these calculations, the matrix representation of the elements of SIXTEEN introduced by Odintsov [20] is particularly helpful). The latter limitation, however, allows us to make the following observation: the logic obtained by endowing the lattice $(SIXTEEN, \leq_{\mathcal{ETL}})$ with a De Morgan negation (having only **t** as designated) will be one of the logics in the interval $[\mathcal{ETL}, \mathcal{CL}]$ displayed in Figure 4 of Subsection 3.1. This easily follows from the characterization of the algebraic models of $\mathcal{ETL}$ established in [28]; it is also, in particular, easy to define negation operators on SIXTEEN whose resulting logics are the weakest and the strongest possible ones (that is, $\mathcal{ETL}$ and $\mathcal{CL}$, respectively).

We leave a full exploration of these thoughts to later research in order to turn to another topic of philosophical interest.

4.2 The Infinite Hierarchy

We'd like to bring our considerations to an end by making some philosophical observations on the infinite hierarchy of logics covered in Subsection 3.1. When we look at that infinite hierarchy and ask what philosophical sense we can possibly make of the situation, we are faced with the following picture: We have an infinite chain of logics that get more and more explosive, or, viewed the other way around, less and less paraconsistent[23]. In the following two subsections, we will ask whether there is some applied sense to be made of this, and also what more abstract philosophical problems it might pose.

4.2.1 A Computational Interpretation

From a more applied point of view, it might seem hard to see any sense that could be made of the hierarchy, and in particular the fact that there is an ever increasing number of disjoined contradictions that becomes satisfiable in one logic after the other. But, given the computational nature of Belnap's early interpretation, we tentatively suggest that, as a kind of logical tool box, there might be something useful in this kind of behavior, after all. Our suggestion is broadly similar to the interpretation of linear logic in that it pays attention and makes the logic sensitive to the available resources a reasoning system has at a given time. As we noted above, linear logic is among those logics that also share a certain kinship to $\mathcal{ETL}$ in the behavior of (one of) its disjunction(s). However, linear logic does not, at least not in an obvious fashion, give rise to an infinite hierarchy of logics as the super-Belnapian ones.

In any case, here is the sketch of an interpretation: Imagine that you have a system that has bounded resources. Depending on system load, it can, up to a point, sort out corrupted data on the fly before it moves on. However, once this threshold is surpassed, it must move on without resolving the conflicts.

Let's say that at the present moment, the system is able to deal with "error messages" comprised of disjunctions of up to three disjunctions ("either p or q or r is true and false"). It will employ a logic in which this is not satisfiable and sort through the matter before moving on. However, if an error message comprised of four disjoined contradictions comes in, it will not react and move on. Hence, it will employ the logic in which four disjoined contradictions become satisfiable.

Obviously, we have here only sketched an idea of where some kind of applied interpretation might lie. It is not necessarily in harmony with the rest of the story so far (in Belnap's story, the system was only fed atomic data, hence there would not be a case in which a disjunction of contradictions was assumed by the system without the ability to immediately check the database for the culprits). Our main purpose here is to suggest that the seemingly arbitrary feature that distinguishes the logics in the hierarchy, namely the number of disjoined contradictions each one is able to tolerate, might have some applicational sense further down the line.

[23] In general, it may not make sense to ask of two arbitrary logics whether one is more paraconsistent than the other. In this case, however, the logics form a chain, which is moreover obtained by adding only explosion rules to a base logic. In this sense, every logic in the chain is clearly more explosive (hence, less paraconsistent) than all the weaker logics below it.

4.2.2 Expressing the Principle of Non-Contradiction

Leaving that applied perspective, the hierarchy also invites interesting reflections about how intuitive ideas can be formalized, and the limitations of that endeavor. In particular, it shines an interesting light on the debate between defenders and critics of Explosion.

One way to argue for Explosion is to say that it captures a normative principle in the object language of propositional logic. This principle is, roughly: “Don’t ever believe in a contradiction, or else you are committed to triviality”.

But $\mathcal{ETL}$ shows a problem with this line of argument. By requiring Explosion, someone who dislikes contradictions might *think* he has formalized his normative stance. But of course, it would be very strange if he was happy with rational agents believing a disjunction of two contradictions. But $\mathcal{ETL}$ shows that requiring Explosion alone is not enough to formalize his normative outlook on contradictions.

And things get worse: the infinite hierarchy of Subsection 3.1 shows that he can not easily fix this situation by adding a new axiom to his set of requirements, or even any finite number of new axioms, as long as these take the form of disjunctions of contradictions in the premise and an arbitrary consequence.[24]

This is the kind of dynamic apt to be the topic of ancient myths. We present an excerpt of an exchange between the maybe greatest pair of ancient debaters, Achilles and the Tortoise. We enter the scene after Achilles has, with great difficulty, convinced the Tortoise that contradictions should not be asserted, and that the normative penalty for doing so is to be committed to everything whatsoever. However, just when he thought the debate was won, the conversation took a disconcerting turn...

> Achilles: “… but just a minute ago you agreed that a contradiction can never be true, and that everything would follow if it were!"
>
> Tortoise: “Yes. And?”
>
> Achilles: “And then you went right on to state a disjunction of two contradictions! You can’t do that, everything follows from that!”
>
> Tortoise: “No it doesn’t. Here, I’ll draw a little logical diagram in the sand with my beak.” (Proceeds to explain $\mathcal{ETL}$)
>
> Achilles: “Well, the values seem to make some sense, and I surely won’t argue with your choice of designated values. Something must be wrong here, but I can’t put my finger on it.
>
> Look, between you and me, I’m not here by chance. Helen of Troy said she would give me a kiss if only I could talk some sense into you. How about I give you the honey-dipped plums I have packed for lunch, and you just accept that a disjunction of two contradictions explodes without further argument?”

[24] For a critic of Explosion (say, on grounds of lack of relevance), the hierarchy presents no great obstacle. From $\mathcal{ETL}$ onwards, things get worse and worse from her point of view, but $\mathcal{ETL}$ is bad enough. She says no to Explosion, and so she says no to the whole hierarchy.

> Tortoise: “Sure, why not? These plums look quite tasty, and I’m not a very principled animal.”
>
> Achilles: “Here you go then."
>
> Tortoise: “Man, these are good! Any other logical principles you want me to accept? For a plum each, I’ll accept any number of them. I’ll even accept infinitely many, provided you have an infinite number of plums for me!"
>
> Achilles: “What a strange remark from an utterly strange animal. I’m just glad we’re done here. For the record, you agree now that a disjunction of two contradictions can never be true, and that everything would follow from one?”
>
> Tortoise: “Of course, that’s what we agreed to."
>
> Achilles: “Thank gods."
>
> Tortoise: “But you know what *could* be true?”
>
> Achilles: “Oh no, this can’t be good..."

The situation might seem hopeless for Achilles, as he seems in for a loss of infinitely many plums (one for each n disjoined contradictions that the Tortoise will accept explodes). However, there is, in the present logical environment, a potential way out: Achilles could try to convince the Tortoise to accept what we called above Congruential Explosion, $(p \wedge \neg p) \vee q \vdash q$ (see Subsections 2.3 and 3.1). As we observed there, this principle will immediately ensure that any number of disjoined cotradictions explodes.

So, what is the lesson here? Is it that disputes about formalized normative principles about contradictions should concentrate on the rule of Congruential Explosion instead of Explosion? In this case, the study of $\mathcal{ETL}$ has had merit in pointing us towards correcting a sizeable chunk of the literature in philosophical logic.

On the other hand, Congruential Explosion is surely lacking the same intuitively plausible connection to the normative principle of non-contradiction that Explosion, arguably, has. And even if we get over that sense of unnaturalness, we might have become wary by this experience. Without the reflections on $\mathcal{ETL}$, it would have been hard to see why Explosion does not necessarily capture all that the enemy of contradictions wanted to say. How can we be sure that a similar problem might not arise later on that shows that $(p \wedge \neg p) \vee q \vdash q$ does not necessarily express all that needs to be expressed?

If the latter is the lesson we draw, then this presents an interesting reflection about the limits of formalizing philosophical problems. Even if these are *problems about logic itself*, we can maybe never be sure to capture them unequivocally with our formalizations. Of course, we know what someone is apt to actually mean when he calls for Explosion. But the dream of formalizing such things, in one of its forms, is to be able to express things so clearly that speculation about what the speaker “actually means” becomes obsolete.

Maybe nowadays most people have abandoned such dreams for other reasons anyways, but the reflections on $\mathcal{ETL}$ and the infinite hierarchy of super-Belnap logics above might help determine more precisely how closely we can approach this elusive goal.

5 Summary

In this essay, we hope to have shown that $\mathcal{ETL}$ is a logic that merits attention, be it because of the philosophical challenges that it embodies or because it is starting point for many interesting technical explorations that, in turn, lead to challenging philosophical questions.

In particular, we have argued that the loss of relevance is not a good reason to ignore $\mathcal{ETL}$, at least if one is thinking in terms of applied semantics. We have further explored the work in algebraic logic that has been inspired by $\mathcal{ETL}$, such as the discovery of super-Belnap logics. We have also taken a look at the techniques in proof theory that have been employed in order to deal with $\mathcal{ETL}$ and its cousins. In the last part, we have tried to show how the guiding idea of $\mathcal{ETL}$ can be exported to other settings, giving the logic determined by SIXTEEN as an example. We have also taken a deeper look at the infinite hierarchy of super-Belnap logics from a philosophical perspective.

In many of these observations, we have merely scratched the surface, but that just emphasizes the point we are trying to make: $\mathcal{ETL}$ and the questions it inspires us to ask are worth exploring further and are sure to lead to many more interesting insights in the future.

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