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Asymptotic Completeness in a Class of Massive Wedge-Local Quantum Field Theories in any Dimension

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Abstract: A recently developed n-particle scattering theory for wedge-local quantum field theories is applied to a class of models described and constructed by Grosse, Lechner, Buchholz, and Summers. In the BLS-deformation setting we establish explicit expressions for n-particle wave operators and the S-matrix of ordered asymptotic states, and we show that ordered asymptotic completeness is stable under the general BLS-deformation construction. In particular, the (ordered) Grosse–Lechner S-matrices are non-trivial also beyond two-particle scattering and factorize into 2-particle scattering processes, which is an unusual feature in space-time dimension $d > 1+1$. Most notably, the Grosse–Lechner models provide the first examples of relativistic (wedge-local) QFT in space-time dimension $d > 1 + 1$ which are interacting and asymptotically complete.

Contents

1. Introduction

Typical scattering theory has two steps. The first is a construction of two wave operators, incoming and outgoing, which describe the propagation at large times $t \to \pm\infty$ in terms of a simpler asymptotic dynamics. The second step, more difficult, is asymptotic completeness (AC), which usually means that essentially all states in the Hilbert space are included in the range of the wave operators. This conceptual question was already raised early on during the development of the quantum theory of fields [Gre61,Ru62]. But even after many decades of research, mathematical results on AC of interacting QFT models have remained rather scarce (e.g. [CD82,DG99,IM06,Le08]) and there are various obstacles of physical and technical nature (cf. [DyG13]). In fact, the first complete proof of asymptotic completeness of an interacting QFT has been established rather recently: It was obtained by Lechner [Le06,Le08] for certain two-dimensional models with factorizing S-matrix, such as the Sinh-Gordon model.

In the present work we provide, to the best of our knowledge, the first full proof of AC in a class of interacting wedge-local QFT models on Minkowski space-times of dimension $d > 1 + 1$. The models in question, found by Grosse, Lechner, Buchholz and Summers in [GL07,BS08,BLS11], may not be local, but have the weaker property of wedge locality.[1] The constructive procedure developed in [BLS11], which we will call a BLS-deformation, preserves the general structure of a relativistic wedge-local theory and in addition introduces interaction. For example, the BLS-deformation of a free field theory, which we will call a GL-model, has a non-trivial two-particle scattering matrix [GL07,BS08]. However, collision processes involving $n \geq 3$ particles were not investigated in these works, since n-particle scattering theory was not available in the wedge-local setting back then. Such a scattering theory has been developed meanwhile in [Du18], so that the question of asymptotic completeness can now be posed and positively answered for these models. More than that, by adapting and generalizing methods from [BLS11,DT11] we will establish explicit expressions for the general effect of BLS-deformations on asymptotic states and scattering data. In this manner we also prove stability of asymptotic completeness under BLS-deformations for general wedge-local models with massive particles.

To construct n-particle scattering states according to [Du18], we define Haag–Ruelle operators $B_{k\tau}(f_k)$, $\tau \in \mathbb{R}$, $1 \leq k \leq n$, which create the desired one-particle states from the vacuum $\Omega \in \mathscr{H}$. Then outgoing and incoming velocity-ordered scattering states are

$$\Psi^{\pm} := \lim_{\tau \to \pm\infty} B_{1\tau}(f_1) \dots B_{n\tau}(f_n)\Omega. \tag{1}$$

The momentum-space configuration of such scattering states is specified by a family of regular Klein-Gordon solutions f_k. For *wedge-ordered velocity configurations* (as defined below in (2)), the wedge-local Haag–Ruelle theorem establishes convergence in (1) and Fock structure of the resulting asymptotic states [Du18]. This result holds in general wedge-local theories with massive particle spectrum, given that all $B_{k\tau}(f_k)$ are constructed from a common localization wedge $\mathcal{W}$. For the concrete case of the standard wedge $\mathcal{W} = \mathcal{W}_{\mathrm{R}} = \{(t, \mathbf{x}) \in \mathbb{R}^d : |t| < x^1\}$, the velocity ordering of outgoing states ($\tau \to +\infty$) reads

$$\mathcal{V}_{f_n} \prec_{\mathcal{W}_{\mathrm{R}}} \mathcal{V}_{f_{n-1}} \prec_{\mathcal{W}_{\mathrm{R}}} \dots \prec_{\mathcal{W}_{\mathrm{R}}} \mathcal{V}_{f_1}. \tag{2}$$

[1] Asymptotically complete fully non-local models have been constructed, e.g. [BW84].

Here $\mathcal{V}_{f_k} := \{(1, \mathbf{v}_m(\mathbf{p})) \in \mathbb{R}^d : \mathbf{p} \in \operatorname{supp} \tilde{f}_k\}$ denote the supports of the Klein-Gordon wave packets

$$f_k(t, \mathbf{x}) := \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \, \mathrm{e}^{-\mathrm{i}\omega_m(\mathbf{k})t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tilde{f}_k(\mathbf{k}), \quad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}, \tag{3}$$

with respect to velocity $\mathbf{v}_m(\mathbf{p}) := \mathbf{p}/\omega_m(\mathbf{p})$. Further, the *precursor* ordering relation in (2) is defined by $\mathcal{V}_{f_k} \prec_{\mathcal{W}_\mathrm{R}} \mathcal{V}_{f_{k-1}} :\Longleftrightarrow \mathcal{V}_{f_{k-1}} - \mathcal{V}_{f_k} \subseteq \mathcal{W}_\mathrm{R}$ and corresponds to the geometrical requirement that all relative velocities $\mathbf{v}_{k-1} - \mathbf{v}_k \in \mathcal{V}_{f_{k-1}} - \mathcal{V}_{f_k}$ yield a positive directed separation of wave packets f_k for $t \to +\infty$ with respect to the space-like opening direction of $\mathcal{W}_\mathrm{R}$, i.e. $(\mathbf{v}_{k-1} - \mathbf{v}_k)^1 > 0$ for all $2 \leq k \leq n$. In the present work we prove, in particular, for GL-models that the velocity-ordered scattering states constructed according to (1) and (2) for each wedge $\mathcal{W}$ span dense sets $\mathscr{H}^{\pm}_{\mathcal{W}}$ in the full Hilbert space $\mathscr{H}$ of the interacting model. In this case, we will say that a wQFT model has the property of *(ordered) asymptotic completeness*.

We now briefly explain the basic ideas of the present analysis of deformed scattering data and asymptotic completeness in non-technical terms. In the deformation construction from [BS08,BLS11] the new deformed model is generated by observables constructed as formal spectral integrals

$$A_Q := \int \mathrm{d}E_{(H,\boldsymbol{P})}(p) \, \alpha_{Qp}(A) = \int \alpha_{Qp}(A) \, \mathrm{d}E_{(H,\boldsymbol{P})}(p) =: {}_Q A. \tag{4}$$

In the terminology of [BS08,BLS11], the (smooth) wedge-local operators A of the initial model are *'warped'* with respect to the spectral measure $\mathrm{d}E = \mathrm{d}E_{(H,\boldsymbol{P})}$ of the energy-momentum operators $P = (H, \boldsymbol{P})$. Presently, $Q \in \mathbb{R}^{d\times d}$ denotes a fixed parameter matrix satisfying certain geometrical properties [GL07]. Later it will be chosen as a mapping depending on the localization wedge of the operator A, as prescribed by Grosse and Lechner, to yield wedge-locality and Poincaré covariance (if applicable) of the deformed model.

Let us now apply the wedge-local n-particle scattering theory from [Du18] and compare the scattering states

$$\Psi_0^+ := \lim_{\tau\to\infty} B_{1\tau}(f_1) \ldots B_{n\tau}(f_n)\Omega, \tag{5}$$

$$\Psi_Q^+ := \lim_{\tau\to\infty} B_{1Q\tau}(f_1) \ldots B_{nQ\tau}(f_n)\Omega, \tag{6}$$

where $B_{kQ\tau}(f_k)$, $1 \leq k \leq n$, obtained from (4) are a corresponding family of creation operators associated to the deformed model. If we proceed on the heuristic level of (4) and insert these formal definitions of the warped convolution into (6), we see directly that

$$\Psi_Q^+ := \lim_{\tau\to\pm\infty} \int \mathrm{d}E(p_1)\alpha_{Qp_1}(B_{1\tau}(f_1)) \ldots \int \mathrm{d}E(p_n)\alpha_{Qp_n}(B_{n\tau}(f_n))\Omega. \tag{7}$$

Provided we can justify exchanging these formal warped-convolution integrations with operator products and the scattering theoretic limit, we find that Ψ_Q^+ can be written as a superposition of scattering states from the undeformed model of the form

$$\Psi^+_{0;p_1,\ldots,p_n} := \lim_{\tau\to\infty} \alpha_{Qp_1}(B_{1\tau}(f_1))\alpha_{Qp_2}(B_{2\tau}(f_2)) \ldots \alpha_{Qp_n}(B_{n\tau}(f_n))\Omega, \tag{8}$$

where operators are additionally modified by space-time translations depending on parameters $p_1, \ldots, p_n \in \mathbb{R}^d$. As the action of space-time translations on the one-particle states can be made explicit, namely $\alpha_y(B_\tau(f))\Omega = B_\tau(f^y)\Omega$, where $f^y(x) = f(x-y)$, we expect for a general deformed wedge-local model that the corresponding subspaces of scattering states of the deformed and undeformed model will coincide, that is, for outgoing scattering states,

$$\mathscr{H}^+_{Q,\prec\mathcal{W}} = \mathscr{H}^+_{0,\prec\mathcal{W}} \subseteq \mathscr{H}. \tag{9}$$

Here $\mathscr{H}$ denotes the full Hilbert space of the interacting model, and one has similarly for incoming states $\mathscr{H}^-_{Q,\prec\mathcal{W}} = \mathscr{H}^-_{0,\prec\mathcal{W}}$.

These heuristic arguments already suggest our main result, which establishes stability of asymptotic completeness of ordered scattering states under BLS-deformations. Asymptotic completeness of the Grosse–Lechner models then follows from AC of the free massive scalar field theory with respect to ordered scattering states. Indeed, using that for local QFTs the space spanned by wedge-ordered scattering states $\mathscr{H}^+_{0,\prec\mathcal{W}}$ coincides with the full space of scattering states $\mathscr{H}^+_0$, and exploiting AC of the free field, $\mathscr{H}^+_0 = \mathscr{H}_0$, we obtain for the GL-model

$$\mathscr{H}^\pm_{Q,\prec\mathcal{W}} = \mathscr{H}^\pm_{0,\prec\mathcal{W}} = \mathscr{H}. \tag{10}$$

Hence these wedge-local QFT models are asymptotically complete.

The main results of the present paper make the above heuristic arguments leading from (7) to (10) mathematically precise. Our analysis is based on oscillatory integral techniques from [BLS11] and the recent wedge-local n-particle scattering theory [Du18]. We also profit from the previous analysis of a scattering in BLS-deformed massless two-dimensional models by Dybalski and Tanimoto [DT11]. Let us note that the analysis of the models treated here is technically more involved compared to [DT11] due to the presence of dispersion. Without dispersion the problem of AC reduces to two-body scattering of left- and right-movers, and hence wedge-local scattering theory with $n \leq 2$ particles suffices. Two-particle scattering theory has been studied much earlier [BBS01, Le03, GL07, BS08] via direct application of standard Haag-Ruelle theory. The swapping method as introduced in [Du18] to establish wedge-local scattering theory for arbitrary particle numbers is not needed in these simpler cases.[2]

To conclude this introduction we should remark that physical intuition familiar from local QFT makes (10) very plausible on one hand. On the other, such intuition may fail in the much larger class of general wedge-local models. Interesting examples can already be found among recently constructed free product models [LTU19, Sec. 5]. There it was shown that ordered two-particle scattering states are not sufficient for two-particle asymptotic completeness. In contrast to a previous work on clustering properties [Sol14], where scattering theory is also mentioned briefly, our results do not assume locality of the underlying wQFT model and in this regard the results of the present work are much more widely applicable within the general wedge-local framework.

2. Preliminaries on Wedge-Local QFT and Scattering Theory

2.1. Operator-algebraic framework for wedge-local QFT. We will work in an operator-algebraic setting of wedge-local quantum field theory on Minkowski space-time $\mathbb{R}^d$, and

[2] We note that 1+1-dimensional BLS-deformed massive models are included in our analysis. For those and similar models, where localization in bounded space-time regions is not established or is likely not available, swapping is in fact required also in 1+1 dimensions.

our results are valid in arbitrary spatial dimension $s := d - 1$. The family of wedge regions is defined as the orbit $\mathcal{P}\mathcal{W}_{\mathrm{R}} := \{\lambda\mathcal{W}_{\mathrm{R}} = \Lambda\mathcal{W}_{\mathrm{R}} + x, \lambda = (x, \Lambda) \in \mathcal{P}\}$ of the conventional *Rindler wedge* $\mathcal{W}_{\mathrm{R}} := \{(t, \mathbf{x}) \in \mathbb{R}^d : |t| < x^1\}$ (also, *standard wedge* or *right wedge*) under the action of the Poincaré group $\mathcal{P} = \mathbb{R}^d \rtimes \mathcal{L}$.

A wedge-local quantum field theory model is specified by mathematical objects $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$, where $\mathscr{H}$ is the Hilbert space of pure states containing the *vacuum* as a distinguished unit vector $\Omega \in \mathscr{H}$. The wedge-localization of observables is described by a family of von Neumann algebras $\mathfrak{A}(\mathcal{W}) \subseteq \mathcal{B}(\mathscr{H})$ associated to wedge regions $\mathcal{W}$. Poincaré symmetry acts on the wedge-local algebras $\mathfrak{A}(\mathcal{W})$ by a given group of isomorphisms α_λ and we denote by $\lambda = (x, \Lambda) \in \mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow$ the elements of the proper orthochronous Poincaré group. In this paper we are working mostly with space-time translations of some given operator $A \in \mathfrak{A}(\mathcal{W})$ by $x \in \mathbb{R}^d$, also denoted by $\alpha_x(A)$.

Guided by physical intuition one asks that these objects satisfy wedge-local variants of the Haag–Kastler postulates, which are concerned with the algebraic and representation-theoretic properties of $\mathfrak{A}$. Firstly, for any choice of wedge regions $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2$ one has

$$\textbf{Isotony} \quad \mathfrak{A}(\mathcal{W}_1) \subseteq \mathfrak{A}(\mathcal{W}_2) \text{ for } \mathcal{W}_1 \subseteq \mathcal{W}_2, \tag{HK1}$$

$$\textbf{Locality} \quad \mathfrak{A}(\mathcal{W}_1) \subseteq \mathfrak{A}(\mathcal{W}_2)' \text{ for } \mathcal{W}_1 \subseteq \mathcal{W}_2', \tag{HK2}$$

$$\textbf{Wedge-Duality} \quad \mathfrak{A}(\mathcal{W}') = \mathfrak{A}(\mathcal{W})', \tag{HK2$^\sharp$}$$

$$\textbf{Translation-Covariance} \quad \alpha_x(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}(\mathcal{W} + x), \quad x \in \mathbb{R}^d, \tag{HK3}$$

$$\textbf{Poincaré-Covariance} \quad \alpha_\lambda(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}(\lambda\mathcal{W}), \quad \lambda \in \mathcal{P}_+^\uparrow. \tag{HK3$^\sharp$}$$

Here the Minkowski causal complement $\mathcal{W}' = (\Lambda\mathcal{W}_{\mathrm{R}} + x)' = -\Lambda\mathcal{W}_{\mathrm{R}} + x$ of $\mathcal{W}$ is also a wedge region and $\mathfrak{A}(\mathcal{W})'$ denotes the commutant of $\mathfrak{A}(\mathcal{W})$ relative to $\mathcal{B}(\mathscr{H})$.

On the representation-theoretic side it is further assumed that translations are unitarily implemented on the vacuum Hilbert space $\mathscr{H}$ by a strongly continuous $s+1$-parameter group, $\alpha_x(A) = U(x)AU(x)^*$. The representing unitaries are generated by the *energy-momentum operators* via $U(x) = U(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$, whose joint spectral resolution in terms of projection-operator-valued measures will be denoted $\Delta \longmapsto E_{(H,\boldsymbol{P})}(\Delta)$ and abbreviated as $E(\Delta)$ for Borel sets $\Delta \subseteq \mathbb{R}^d$. Focusing on the analysis of scattering, we impose the following standard assumptions concerned with the vacuum representation and its one-particle spectrum,

$$\textbf{Uniqueness of }\boldsymbol{\Omega} \quad E(\{0\})\mathscr{H} = \mathbb{C}\Omega, \tag{HK4}$$

$$\textbf{Cyclicity of }\boldsymbol{\Omega} \quad \overline{\mathfrak{A}(\mathcal{W})\Omega} = \mathscr{H}, \tag{HK5}$$

$$\textbf{Spectral Condition} \quad \operatorname{supp} E \subseteq \bar{V}^+, \tag{HK6}$$

$$\textbf{Haag–Ruelle Mass Gap Condition} \quad H_m \subseteq \operatorname{supp} E \subseteq \{0\} \cup H_m \cup \operatorname{conv}(H_M), \tag{HK6$^\sharp$}$$

for some $M > m > 0$, where $\bar{V}^+ := \{(\omega, \mathbf{p}) : |\mathbf{p}| \leq \omega\}$ denotes the positive energy cone, $H_m := \{(\omega_m(\mathbf{p}), \mathbf{p}) : \mathbf{p} \in \mathbb{R}^s\}$, $\omega_m(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}$, is the (positive) hyperboloid of mass $m > 0$ and we write $\operatorname{conv}(H_M) := \{(\omega, \mathbf{p}) : \mathbf{p} \in \mathbb{R}^s, \omega \geq \omega_M(\mathbf{p})\}$ for the convex hull of H_M.

Definition 1. A wedge-local quantum field theory is a tuple $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ as above, which satisfies the basic assumptions (HK1)–(HK6).

This summarizes the axiomatic operator-algebraic formalism for wedge-local QFTs providing the basis of our present investigations. Our choice of framework serves the purpose of accommodating scattering-theoretic reasoning and capturing the requirements of the wedge-local Haag–Ruelle theory [Du18]. In the literature also the closely related concept of a *causal Borchers triple* $(\mathcal{R}, U, \Omega)$, corresponding to $\mathcal{R} := \mathfrak{A}(\mathcal{W}_{\mathrm{R}})$ in the present framework, is often studied. Such a framework is equivalent to the above setting including Poincaré covariance (HK3$^\sharp$). For further historical background and other aspects of wedge-local Quantum Field Theory, we refer to [BS08,Le15,BLS11,Du18].

Regarding the scattering-theoretic analysis, Poincaré covariance (HK3$^\sharp$) is not essential. The important properties for us are the existence of isolated mass shells (HK6$^\sharp$) and the well-established wedge duality condition (HK2$^\sharp$), which strengthen (HK6) and (HK2), respectively. As such, these will be standing assumptions of this work. Whereas (HK6) merely demands positivity of the Hamiltonian in any Lorentz frame, the Haag–Ruelle mass gap condition (HK6$^\sharp$) physically amounts to non-triviality of the one-particle subspace $\mathscr{H}_1 := E(H_m)\mathscr{H}$. The upper mass gap $M > m$ has the same technical purpose as in traditional Haag–Ruelle scattering theory: it enables the efficient separation of one-particle states from the remaining energy-momentum spectrum.

2.2. Warped convolutions and the Grosse–Lechner model. Our scattering-theoretic analysis is concerned with a general class of massive wedge-local QFT models, which are constructed via the deformation method introduced by Buchholz, Lechner, and Summers [BLS11]. The Grosse–Lechner models were introduced and studied from a non-commutative geometry perspective in [GL07]. Here we adopt the operator-algebraic approach from [BS08,BLS11] and study, in particular, the GL-models as constructed by applying BLS-deformations to the free scalar field.

The starting point of the BLS-construction is a general wedge-local model $(\mathfrak{A}^0, \alpha, \mathscr{H}, \Omega)$. With $\kappa \geq 0$ and, in dimension $d = 3 + 1$ additionally $\eta \in \mathbb{R}$, as parameters of the deformation, a family $\mathcal{W} \longmapsto Q_{\mathcal{W}}$ of *warping matrices* is now defined according to [BS08,GL07] by either

$$Q_{\mathcal{W}_{\mathrm{R}}} := \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}, \ (s = 3), \quad Q_{\mathcal{W}_{\mathrm{R}}} := \begin{pmatrix} 0 & \kappa & 0 & \cdots & 0 \\ \kappa & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ (\text{for general } s \geq 1), \tag{11}$$

for the warping matrix of the reference wedge $\mathcal{W}_{\mathrm{R}} = \{x \in \mathbb{R}^d : |x^0| < x^1\}$ and its translates. For general wedges given by $\mathcal{W} = \Lambda\mathcal{W}_{\mathrm{R}}+x$, $(x, \Lambda) \in \mathcal{P}^{\uparrow}$, the warping matrix is defined by Poincaré-covariance[3] as $Q_{\mathcal{W}} := \Lambda Q_{\mathcal{W}_{\mathrm{R}}} \Lambda^{-1}$. These warping matrices are antisymmetric with respect to the scalar product defined by the Lorentzian metric g with signature $(+, -, \ldots, -)$.

[3] We write Q as linear map, or $(1, 1)$-tensor $Q = (Q^{\mu}{}_{\nu})_{\mu\nu}$, as in [BS08]. In [GL07] and other works, especially in contexts of non-commutative spacetime, it is also very common to write $Q_{\mathcal{W}_{\mathrm{R}}}$ in $(2, 0)$ or $(0, 2)$-tensor notation.

Let now $Q \in \mathbb{R}^{d\times d}$ be any warping matrix. The *warped convolution* of an operator A as introduced in [BS08] is denoted by

$$A_Q := \int \mathrm{d}E_{(H,\boldsymbol{P})}(p)\ \alpha_{Qp}(A). \tag{12}$$

Here we will use the mathematically rigorous definition of such a priori formal spectral integrals given in subsequent work of Buchholz et al. [BLS11] using oscillatory integral methods. To this end, one starts with operators A on which space-time translations act smoothly and defines A_Q in terms of its action on vectors from the dense domain

$$\mathcal{D} := \bigcup_{\substack{\Delta\subseteq\mathbb{R}^d\\ \text{compact}}} E_{(H,\boldsymbol{P})}(\Delta)\mathscr{H} \tag{13}$$

of finite energy states. We denote by $\mathcal{C}^\infty$ the algebra of all *regular* operators $A \in \mathcal{B}(\mathscr{H})$, defined by the requirement that $\mathbb{R}^d \ni x \longmapsto \alpha_x(A) = U(x)AU(x)^*$ is arbitrarily often differentiable with respect to the operator norm topology. By standard mollification arguments, the regular subalgebra $\mathfrak{A}^{0r}(\mathcal{W}) := \mathcal{C}^\infty \cap \mathfrak{A}^0(\mathcal{W})$, for any wedge $\mathcal{W}$, is weakly dense in the full wedge algebra $\mathfrak{A}^0(\mathcal{W})$ of the initial model.

Definition 2 (*Warped convolution of a regular operator* [BLS11]). The warped convolution A_Q of a bounded regular operator $A \in \mathcal{C}^\infty$ with respect to the spectral measure dE by a warping matrix Q is defined by its action on finite-energy vectors $\Psi \in \mathcal{D}$ as limit of strong integrals of oscillatory type,

$$A_Q\Psi := \lim_{\epsilon\to 0} \frac{1}{(2\pi)^d} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \eta(\epsilon x, \epsilon y)\, \mathrm{e}^{-\mathrm{i}x\cdot y} U(x)\alpha_{Qy}(A)\Psi \tag{14}$$

with $\eta \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$, such that $\eta(0, 0) = 1$, serving as a regularizing function.

Here the technical assumptions of regularity of A and the consideration of only finite energy states are needed for the existence of the limit $\epsilon \to 0$. In addition to existence of this limit, it is shown in [BLS11] that A_Q extends to a bounded operator, and is independent of the choice of η. Let us summarize some helpful properties of warped convolutions from the literature. For standard results about Bochner integrals we refer to [Zaa67, Ch. 6 §31].

Lemma 3 [BLS11]. *The warped convolution A_Q of any regular operator $A \in \mathcal{C}^\infty$ with respect to any warping matrix Q, as given by* (14)*, is well-defined. In particular it does not depend on the specific choice of the regularizing function η. The deformed operator extends to a bounded operator $A_Q \in \mathcal{B}(\mathscr{H})$, which satisfies*

(i) $A_Q\Omega = A\Omega$,
(ii) $(A_Q)^* = (A^*)_Q$,
(iii) $A_0 = A$,
(iv) $[A_Q, A'_{-Q}] = 0$, *whenever* $QV^+ \subseteq \mathcal{W}_\mathrm{R}$, $(-Q)V^+ \subseteq \mathcal{W}_\mathrm{L}$, $A \in \mathfrak{A}^{0r}(\mathcal{W}_\mathrm{R})$, *and* $A' \in \mathfrak{A}^{0r}(\mathcal{W}_\mathrm{L})$.
(v) *Warping commutes with space-time translations,*

$$\alpha_x(A_Q) = (\alpha_x(A))_Q.$$

(vi) *If the underlying model satisfies Poincaré-covariance* (HK3$^\sharp$) *one has further*

$$\alpha_\lambda(A_Q) = (\alpha_\lambda(A))_{\Lambda Q \Lambda^{-1}}, \ \textit{for}\ \lambda = (x, \Lambda) \in \mathcal{P}_+^\uparrow.$$

It is important to emphasize that the definition (14) of A_Q is only mathematically rigorous when working with regular operators $A \in \mathfrak{A}^{0r}$. To obtain the full warped wedge-local QFT model in the sense of Sect. 2.1, we additionally pass to the weak closure of the set of warped regular operators.

Definition 4 (*Warped wedge-local model* [BS08,BLS11]). Let $Q_\mathcal{W}$ be given by (11) for $\mathcal{W} = \mathcal{W}_\mathrm{R}$ and let $Q_\mathcal{W} := \Lambda Q_{\mathcal{W}_\mathrm{R}} \Lambda^{-1}$ for general wedges $\mathcal{W} = \Lambda \mathcal{W}_\mathrm{R} + x$, $(x, \Lambda) \in \mathcal{P}^\uparrow$. The warped wedge-local algebras are defined as

$$\mathfrak{A}^Q(\mathcal{W}) := \{A_{Q_\mathcal{W}} : \ A \in \mathfrak{A}^{0r}(\mathcal{W})\}'' \subset \mathcal{B}(\mathscr{H}), \tag{15}$$

where the bicommutant is taken with respect to $\mathcal{B}(\mathscr{H})$.

Theorem 5 [BLS11, Thm. 4.2]. $(\mathfrak{A}^Q, \alpha, \Omega, \mathscr{H})$ *defines a wedge-local QFT model satisfying the wedge-local Haag–Kastler postulates* (HK1)–(HK6) *from the initial model* $(\mathfrak{A}, \alpha, \Omega, \mathscr{H})$. *Further, the deformed model satisfies the Haag–Ruelle spectral condition* (HK6$^\sharp$)*, wedge-duality* (HK2$^\sharp$)*, or Poincaré covariance* (HK3$^\sharp$)*, respectively, if and only if the initial model satisfies the respective conditions.*

We further note that [BLS11] study wedge-local quantum field theories in the framework of a *(causal) Borchers triple* $(\mathfrak{A}(\mathcal{W}_\mathrm{R}), \alpha, \Omega)$. There one specifies only an observable algebra for the single wedge region $\mathcal{W}_\mathrm{R}$. The full family of wedge algebras is then obtained using Poincaré symmetry (HK3$^\sharp$). This symmetry of the wQFT model is of course an additional requirement, which is natural from the perspective of physics and a useful constraint for constructive efforts. In particular, it is a non-trivial feature that BLS-deformations can be used to construct Poincaré covariant models. However, the BLS-deformation method also applies to the present framework, where Poincaré covariance is optional. The proofs of Lemma 3 and Theorem 5 can be extracted without significant modifications from the results of [BLS11].

2.3. Scattering states in wedge-local QFT. We briefly review the n-particle Haag–Ruelle scattering theory for wedge-local QFTs. To construct one-particle states via the wedge-local Haag–Ruelle method [Du18], one has to first fix a wedge $\mathcal{W}$.[4] Then one chooses operators $A \in \mathfrak{A}(\mathcal{W})$ such that $E(H_m)A\Omega \neq 0$. The existence of such operators follows from (HK5). Let $\chi \in \mathscr{S}(\mathbb{R}^d)$ be a Haag–Ruelle auxiliary function supported within a sufficiently small neighborhood of the mass shell H_m, disjointly from the remaining energy-momentum spectrum. Then the operator

$$B := A(\chi) := \int \mathrm{d}^d x \ \chi(x)\, \alpha_x(A) \tag{16}$$

solves the one-particle problem in the sense that $B\Omega$ is in the one-particle space $\mathscr{H}_1 := E(H_m)\mathscr{H}$ while B still has certain unsharp wedge-localization properties (see below,

[4] It may be helpful for guiding intuition to take $\mathcal{W} = \mathcal{W}_\mathrm{R}$ on first reading. As we are also interested in a comparison of the scattering data constructed from different wedges, we use here the coordinate-free formulation with a general wedge $\mathcal{W}$. Lastly, this is also important for studying non-Poincaré covariant models and their deformations, as the status of AC then can depend on the choice of $\mathcal{W}$.

e.g. (28), (29) or [Du18, Lemma 7]). In terms of the Fourier transform defined in the relativistic unitary convention by

$$\hat{\chi}(p) = \hat{\chi}(p^0, \mathbf{p}) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}} \, \mathrm{e}^{\mathrm{i}p^0 x^0 - \mathrm{i}\mathbf{p}\cdot\mathbf{x}} \, \chi(x^0, \mathbf{x}), \tag{17}$$

this follows from the identity $B\Omega = (2\pi)^{d/2} \hat{\chi}(H, \boldsymbol{P}) A\Omega$, obtained by spectral calculus and translation-invariance of the vacuum. Hence $B\Omega$ is in the one-particle space $\mathscr{H}_1 := E(H_m)\mathscr{H}$, as a consequence of the support of the Haag–Ruelle auxiliary function $\hat{\chi}$ intersecting the energy-momentum spectrum supp $E_{(H,\boldsymbol{P})}$ of the theory only on subsets of the mass shell H_m (by construction, and here in particular the mass gap assumption (HK6$^\sharp$) is used, see e.g. [A, Ch. 5] or [Dyb17, Sec. 2.1]).

Proceeding towards the n-particle problem we define similarly for a family of $A_j \in \mathfrak{A}(\mathcal{W})$, $1 \leq j \leq n$, the operators $B_j := A_j(\chi)$. Further we consider (positive-energy) Klein–Gordon solutions

$$f_j(t, \mathbf{x}) = \int \frac{\mathrm{d}^s k}{(2\pi)^s} \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t} \, \tilde{f}_j(\mathbf{k}), \quad (1 \leq j \leq n), \tag{18}$$

as scattering-theoretic comparison dynamics, with relativistic dispersion $\omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}$ for mass $m > 0$, and recall that we abbreviate $s := d - 1$. For usual technical reasons, the *wave packets* are assumed to be *regular*, that is, $\tilde{f}_j \in C_c^\infty(\mathbb{R}^s)$. By a stationary phase analysis, these regular Klein–Gordon solutions vanish rapidly in all space- and time-like directions away from the classical propagation cones

$$\Upsilon_{f_j} := \{(t, \mathbf{x}) \in \mathbb{R}^d : \exists \mathbf{k}_j \in \operatorname{supp} \tilde{f}_j : t\,\mathbf{k}_j = \omega_m(\mathbf{k}_j)\,\mathbf{x}\}. \tag{19}$$

We note that these cones describe the scattering geometry of the single-particle wave packets. Geometrically, scattering situations are concerned with phenomena at very large distances, and it is convenient to introduce the centering of $\mathcal{W} = \Lambda\mathcal{W}_\mathrm{R} + x$, denoted by $\mathcal{W}_\mathrm{c} := \Lambda\mathcal{W}_\mathrm{R}$.

The construction of n-particle states is now accomplished by means of wedge-frame adapted Haag–Ruelle creation-operator approximants

$$B^\Lambda_{j,\tau}(f_j) := \int \mathrm{d}^s x \; f_j(\Lambda(\tau, \mathbf{x})) \, \alpha_{(\Lambda(\tau,\mathbf{x}))}(B_j), \quad (\tau \in \mathbb{R}). \tag{20}$$

Here, the boost $\Lambda \in \mathcal{L}_+^\uparrow$ specifies an auxiliary Lorentz frame used for the construction. The simplest cases are that the operators $A_j \in \mathfrak{A}(\mathcal{W})$ are localized in a wedge $\mathcal{W}$ which is a rotation or spatial reflection of $\mathcal{W}_\mathrm{R}$. Then one can simply take $\Lambda = \mathbb{1}$, corresponding to the integration in (20) ranging over equal-time hyperplanes. For general wedges $\mathcal{W}$ it is technically preferable to choose this boost from

$$\mathcal{L}^*(\mathcal{W}) := \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda\mathcal{W}_\mathrm{R} = \mathcal{W}_c\}. \tag{21}$$

Regarding the causal geometry of wedges, we recall here that in dimension $d \geq 2 + 1$ a general wedge region $\mathcal{W}$ can be written as $\mathcal{W} = \lambda\mathcal{W}_\mathrm{R} := \Lambda\mathcal{W}_\mathrm{R} + x$ in terms of the standard Rindler wedge $\mathcal{W}_\mathrm{R}$ and some Poincaré transformation $\lambda = (x, \Lambda)$ with Lorentz transformation part $\Lambda \in \mathcal{L}_+^\uparrow$ and translation $x \in \mathbb{R}^d$. Our analysis also applies to the special case of dimension $d = 1 + 1$. There, the set of all wedges splits into the disjoint orbits under the proper orthochronous Poincaré group, reducing to space-time translates of the right wedge $\mathcal{W}_\mathrm{R}$ and left wedge $\mathcal{W}_\mathrm{L} := \mathcal{W}_\mathrm{R}{}' = -\mathcal{W}_\mathrm{R}$ and we can choose $\Lambda = \mathbb{1}$.

The choice of Λ enters in the space-time localization of the operators $B^{\Lambda}_{j,\tau}(f_j)$, and keeping track of the latter is important for the Haag–Ruelle method. Namely, the localization depends on τ and the wedge $\mathcal{W}$ of localization of A_j, translated to

$$\mathcal{W} + \tau \mathcal{V}^{\Lambda}_{f_j} \subseteq \mathbb{R}^d, \quad \mathcal{V}^{\Lambda}_{f_j} := \Upsilon_{f_j} \cap \Lambda T_1, \ \Lambda \in \mathcal{L}^{\uparrow}_{+}. \tag{22}$$

Here $T_1 := \{(1, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^s\}$ is the standard space-like hyperplane at $\tau = 1$ and $\mathcal{V}^{\Lambda}_{f_j}$ is the *velocity support* of f_j with respect to the Lorentz frame specified by Λ. As centered wedges $\mathcal{W}_{\mathrm{c}}$ are convex cones with $\mathcal{W}_{\mathrm{c}} + \mathcal{W}_{\mathrm{c}} \subseteq \mathcal{W}_{\mathrm{c}}$, the *precursor* relation

$$\mathcal{O}_1 \prec_{\mathcal{W}} \mathcal{O}_2 :\Longleftrightarrow \mathcal{O}_2 - \mathcal{O}_1 \subseteq \mathcal{W}_{\mathrm{c}}, \tag{23}$$

defined for non-empty regions $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathbb{R}^d$, is transitive, and Poincaré covariant as a partial ordering in the sense that $\mathcal{O}_1 \prec_{\mathcal{W}} \mathcal{O}_2 \Longleftrightarrow \lambda\mathcal{O}_1 \prec_{\Lambda\mathcal{W}} \lambda\mathcal{O}_2, \ \lambda = (x, \Lambda) \in \mathcal{P}^{\uparrow}_{+}$.

To construct n-particle scattering states we consider ordered configurations of wave packet velocities

$$\mathcal{V}^{\Lambda}_{f_n} \prec_{\mathcal{W}} \mathcal{V}^{\Lambda}_{f_{n-1}} \prec_{\mathcal{W}} \ldots \prec_{\mathcal{W}} \mathcal{V}^{\Lambda}_{f_1}, \tag{24}$$

for the outgoing limit $\tau \to +\infty$, and the reversed ordering

$$\mathcal{V}^{\Lambda}_{f_n} \succ_{\mathcal{W}} \mathcal{V}^{\Lambda}_{f_{n-1}} \succ_{\mathcal{W}} \ldots \succ_{\mathcal{W}} \mathcal{V}^{\Lambda}_{f_1}, \tag{25}$$

for the incoming scattering limit $\tau \to -\infty$, respectively. These conventions provide the correct, consistent, and coordinate free[5] generalization of the familiar velocity-ordering relations from the 1+1-dimensional form-factor programme to higher space-time dimensions, as argued in [Du18]. In this work the convergence of

$$\Psi^{\Lambda}_n(\tau) := B^{\Lambda}_{1\tau}(f_1) \ldots B^{\Lambda}_{n\tau}(f_n)\Omega, \tag{26}$$

for $\tau \to \pm\infty$ is proved when all underlying wedge-local operators $A_j \in \mathfrak{A}(\mathcal{W})$ are localizable in a common wedge $\mathcal{W}$ with the respective ordering from (24) or (25), and Fock structure of the limits is established. This approach is distinct from conventional Haag–Ruelle theory, where convergence and Fock structure proofs use that the commutators $[B^{\Lambda}_{j\tau}(f_j), B^{\Lambda}_{k\tau}(f_k)]$ vanish rapidly in norm for $j \neq k$ when $\tau \to \pm\infty$. Such stronger estimates are obtained from locality of the QFT and disjoint velocity supports of the Klein-Gordon solutions. Applying this reasoning in a wedge-local context leads to a more restrictive setup. Firstly we should take $A \in \mathfrak{A}(\mathcal{W})$ and $A^{\perp} \in \mathfrak{A}(\mathcal{W}^{\perp})$ for an *opposite wedge* $\mathcal{W}^{\perp} := \mathcal{W}' + x, x \in \mathbb{R}^d$. Additionally, the Klein–Gordon solutions $f, f^{\perp}$ must be chosen such that the velocity supports are not merely disjoint, but satisfy the stronger geometrical ordering property

$$\mathcal{V}^{\Lambda}_{f^{\perp}} \prec_{\mathcal{W}} \mathcal{V}^{\Lambda}_{f}. \tag{27}$$

[5] In the literature on integrable QFT models in 1+1 dimensions such velocity orderings are usually expressed by rapidities $\theta_j = \sinh^{-1}(k^1_j/m)$, ordered by $\theta_1 < \theta_2 < \ldots < \theta_n$ for outgoing states, and $\theta_1 > \theta_2 > \ldots > \theta_n$ for incoming states, respectively, and operator localizations are fixed by convention to be in the left wedge, see e.g. [Le06]. This is feasible for 1+1 dimensions, where there are only two distinct centered wedges given by the left and right wedge. Although the 1+1-dimensional case is included in our considerations, our main focus is on the cases of higher dimensions where there are infinitely many distinct centered wedges and similarly distinct precursor ordering relations $\prec_{\mathcal{W}}$. This makes the coordinate-free formulation of (23)–(25) preferable for the analysis of higher-dimensional wQFT models, especially due to our interest in the behavior of the scattering data under changing the wedge $\mathcal{W}$.

Then one obtains by wedge-locality and the decay properties described in (19) and (22) a rapid decay for large $|\tau|$,

$$\left\|[B_\tau^\Lambda(f), B_\tau^{\perp\Lambda}(f^\perp)]\right\| \leq \frac{C_N}{\tau^N}, \quad (\tau > 0), \tag{28}$$

$$\left\|[B_\tau^\Lambda(f^\perp), B_\tau^{\perp\Lambda}(f)]\right\| \leq \frac{C_N}{(-\tau)^N}, \quad (\tau < 0). \tag{29}$$

Here it should again be noted that the geometrical configuration depends on whether the outgoing ($\tau > 0$) or incoming ($\tau < 0$) regime is considered. Pairs of opposite wedge configurations were used in the previous constructions of two-particle scattering states in the wedge-local context [GL07,BS08], and also in studies of wedge-local aspects of local QFTs, such as the foundational work of Borchers et al. [BBS01] on the existence and properties of *polarization-free generators*.

In the multi-particle generalization (26) opposite wedges appear only indirectly. Namely, we work with one-particle states that can be generated from the vacuum within two opposite wedges $\mathcal{W}$, $\mathcal{W}^\perp = \mathcal{W}' + x$

$$\Psi = A\Omega = A^\perp\Omega, \quad A \in \mathfrak{A}(\mathcal{W}),\ A^\perp \in \mathfrak{A}(\mathcal{W}^\perp), \tag{30}$$

for some $x \in \mathbb{R}^d$, depending on Ψ. In this case we call Ψ *swappable* (with respect to the wedge $\mathcal{W}$). Swappable vectors with $\mathcal{W}^\perp = \mathcal{W}'$ are dense in the full Hilbert space $\mathscr{H}$ as a consequence of wedge duality (HK2$^\sharp$), see [Du18, App. B]. For us the abbreviation $\mathcal{W}^\perp$ always refers to translates of $\mathcal{W}'$, which may or may not overlap $\mathcal{W}$.[6] Swappable one-particle states are obtained by projecting swappable vectors onto the one-particle space $\mathscr{H}_1$. In this way we obtain from (16) and (20) an oppositely localized pair of Haag–Ruelle creation-operator approximants for each one-particle state ($1 \leq k \leq n$)

$$\Psi_1^k := B_{k\tau}^\Lambda(f_k)\Omega = B_{k\tau}^{\perp\Lambda}(f_k)\Omega, \tag{31}$$

where both expressions involving Haag–Ruelle operators are τ-independent by construction, so that $\tau \to \pm\infty$ limits can be dropped. We note that the scattering states are defined without explicit use of swapping. However, the swapped operators $B_{k\tau}^{\perp\Lambda}(f_k)$ are the main tool for proving the wedge-local Haag–Ruelle theorem:

Theorem 6. [Du18] *Let* $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ *be a wedge-local quantum field theory satisfying wedge duality* (HK2$^\sharp$) *and the mass gap condition* (HK6$^\sharp$). *Let* $\Lambda \in \mathcal{L}_+^\uparrow$ *and* $\Psi_1^j = E(H_m)A_j\Omega$ *with* $A_j \in \mathfrak{A}(\mathcal{W})$ *be swappable one particle states, and define* $B_k := A_k(\chi)$ *with an auxiliary function* χ *as in* (16).

(i) *For regular positive-energy Klein–Gordon solutions* f_j *satisfying*

$$\mathcal{V}_{f_n}^\Lambda \prec_\mathcal{W} \mathcal{V}_{f_{n-1}}^\Lambda \prec_\mathcal{W} \dots \prec_\mathcal{W} \mathcal{V}_{f_1}^\Lambda, \tag{32}$$

the scattering state approximants $\Psi_n^\Lambda(\tau) := B_{1\tau}^\Lambda(f_1)B_{2\tau}^\Lambda(f_2)\dots B_{n\tau}^\Lambda(f_n)\Omega$ *converge rapidly in norm for* $\tau \to \infty$. *More precisely, for any* $N \in \mathbb{N}$ *there exists a* $C_N > 0$ *such that*

$$\left\|\Psi_n^\Lambda(\tau) - \Psi_n^{+,\Lambda}\right\| \leq \frac{C_N}{\tau^N}, \quad (\tau > 0). \tag{33}$$

[6] Allowing swapping with wedge overlaps makes swapping trivially realizable in local models and their deformations (see Lemma 3 (i)). It is also natural from the perspective of scattering theory, where it serves the additional purpose of making the compatibility to local Haag–Ruelle theory manifest, where applicable, as mentioned in [Du18]. In the present work admitting such overlapping swapping partners will also become technically useful (see e.g. Corollary 9).

(ii) *For* $\Lambda \in \mathcal{L}^*(\mathcal{W})$ *scalar products of* $\Psi_n^{+,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^{\Lambda}(f_1)\dots B_{n\tau}^{\Lambda}(f_n)\Omega$, *and* $\Psi_{n'}^{\prime +,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^{\prime\Lambda}(f_1^{\prime})\dots B_{n'\tau}^{\prime\Lambda}(f_{n'}^{\prime})\Omega$, *constructed both with respect to the same wedge* $\mathcal{W}$, *satisfy*

$$\left\langle \Psi_n^{+,\Lambda}, \Psi_{n'}^{\prime +,\Lambda} \right\rangle = \delta_{nn'} \prod_{j=1}^{n} \left\langle B_{j\tau}^{\Lambda}(f_j)\Omega, B_{j\tau}^{\prime\Lambda}(f_j^{\prime})\Omega \right\rangle. \tag{34}$$

Here the right hand side is again τ*-independent by construction.*

Analogous statements hold for incoming scattering states, assuming opposite ordering.

For the later discussion of the scattering matrix, let us note that all asymptotic data in wedge-local models, including wave operators, must be defined depending on a localization wedge $\mathcal{W}$ from which the scattering states have been prepared. This is an unusual feature of wedge-local models. But we note that, of course, the $\mathcal{W}$-dependence can be trivial. This happens in 1+1 dimensions, where our results apply as well. In this case there are only two centered wedges $\mathcal{W}_{\mathrm{R}}$ and $\mathcal{W}_{\mathrm{L}}$, whose associated observables and scattering states can be related by swapping symmetry (see Proposition 26). In general this certainly does not imply that the models will be local. Presently the status of existence or non-existence of local observables even in 1+1-dimensional GL-models appears to be still open. In the present formalism for higher dimensions, the existence of a local QFT model underlying the wedge-local model under consideration implies a certain trivial $\mathcal{W}$-dependence of scattering states, which is discussed in Sect. 5.

The possibility of more general wedge-dependences of scattering states is an interesting feature of wedge local quantum field theories in higher dimensions. In particular, we will use the formalism introduced in [Du18] to describe this wedge dependence in a more transparent manner for two-particle and n-particle scattering reactions. One of the main aims of the present paper is to illustrate this wedge dependence in BLS-deformed wQFT models and in particular for the special case of the Grosse–Lechner models (see Sects. 3.3 and 5).

3. *N*-Particle Scattering in BLS-Deformed Wedge-Local QFTs

Using the results and notation described in the previous sections, we can now state our results in precise form. Let $(\mathfrak{A}^0, \alpha, \mathscr{H}, \Omega)$ be a given wedge-local quantum field theory satisfying wedge duality (HK2$^\sharp$) and the mass gap condition (HK6$^\sharp$), in addition to (HK1)–(HK6). Let $(\mathfrak{A}^Q, \alpha, \mathscr{H}, \Omega)$ be the model constructed by BLS-deformation with some fixed warping parameter Q.

Our main aim will be to prove asymptotic completeness of the deformed model. On the technical side, this will be achieved by establishing a direct relation between a scattering state

$$\lim_{\tau\to\pm\infty} B_{1,\tau}^{Q,\Lambda}(f_1) B_{2,\tau}^{Q,\Lambda}(f_2)\dots B_{n,\tau}^{Q,\Lambda}(f_n)\Omega \tag{35}$$

of the deformed model, and the scattering states of the initial model,

$$\lim_{\tau\to\pm\infty} B_{1,\tau}^{\Lambda}(f_1) B_{2,\tau}^{\Lambda}(f_2)\dots B_{n,\tau}^{\Lambda}(f_n)\Omega. \tag{36}$$

Here we write $B_k^Q = A_k^Q(\chi)$ with $\chi \in \mathscr{S}(\mathbb{R}^d)$, as before, and $A_k^Q \in \mathfrak{A}^Q(\mathcal{W})$ for $1 \le k \le n$ with some fixed centered wedge $\mathcal{W} = \Lambda\mathcal{W}_{\mathrm{R}}$. In the following we will for simplicity suppress dependencies on the fixed reference frame boost Λ and on the wedge in our notation, writing for example $\prec$, $\mathcal{V}_{f_k}$ and $B^{Q}_{k,\tau}(f_k)$, instead of $\prec_{\mathcal{W}}$, $\mathcal{V}^{\Lambda}_{f_k}$ and $B^{Q,\Lambda}_{k,\tau}(f_k)$, etc., when these dependencies can be clearly seen from the context.

3.1. Wave operators in BLS-deformed wedge-local QFTs. Our introductory heuristic considerations (see Eqs. (7) and (8)) suggested that the deformation preserves outgoing and incoming particle numbers. This makes our approach feasible. As we can in principle deform general models, such as $P(\phi)_2$, the scope of our result is not restricted to particularly simple or integrable deformed models. However, we can infer similarly that the deformed scattering state (35) cannot be directly related to a scattering state in the undeformed model with the same simple product form (36). This is addressed here by using the formulation of the scattering data in terms of wave operators $\mathbb{W}^{\pm}_{\mathcal{W}}$, as defined in [Du18, Sec. 5]. To recall the formal construction of the wave operators we start by introducing the full (unsymmetrized) Fock space by

$$\Gamma^{u}(\mathscr{H}_1) := \bigoplus_{n=0}^{\infty} \mathscr{H}_1^{\otimes n}, \tag{37}$$

over the one particle space $\mathscr{H}_1 := E_{(H,\boldsymbol{P})}(H_m)\mathscr{H}$. Here it is convenient to define the velocity ordering $\succ_{\mathcal{W}}$ also as partial order on $\mathscr{H}_1$, by writing

$$\Psi_1 \prec_{\mathcal{W}} \Psi_1' :\Longleftrightarrow \mathcal{V}^{\Lambda}_{\Psi_1} \prec_{\mathcal{W}} \mathcal{V}^{\Lambda}_{\Psi_1'}, \tag{38}$$

where the velocity support $\mathcal{V}^{\Lambda}_{\Psi_1}$ of a one-particle vector Ψ_1 is defined analogously to (19) and (22), replacing the support of $\tilde{f}_j$ by the spectral support of the momentum operator in the state Ψ_1. Natural domains for the wave operators in the wedge-local setting are the velocity-ordered Fock spaces $\Gamma^{\succ}(\mathscr{H}_1)$ and $\Gamma^{\prec}(\mathscr{H}_1)$, which are defined as the closures of the spans

$$\Gamma_0^{\succ}(\mathscr{H}_1) := \operatorname{span}\{\Psi^1 \otimes \ldots \otimes \Psi^n : n \in \mathbb{N}_0,\ \Psi^1, \ldots, \Psi^n \in \mathscr{H}_1,\ \Psi^1 \succ \Psi^2 \succ \ldots \succ \Psi^n\}, \tag{39}$$

$$\Gamma_0^{\prec}(\mathscr{H}_1) := \operatorname{span}\{\Psi^1 \otimes \ldots \otimes \Psi^n : n \in \mathbb{N}_0,\ \Psi^1, \ldots, \Psi^n \in \mathscr{H}_1,\ \Psi^1 \prec \Psi^2 \prec \ldots \prec \Psi^n\}, \tag{40}$$

of finite linear combinations, for outgoing- and incoming scattering states, respectively. We define analogous algebraic spans $\Gamma_0^{\succ}(\mathscr{H}_1')$ and $\Gamma_0^{\prec}(\mathscr{H}_1')$ for any dense subsets of one-particle states $\mathscr{H}_1' \subset \mathscr{H}_1$, to accommodate the technical requirements from Theorem 6 for the construction of scattering states. To make this more precise let us now recall the construction of the wave operators $\mathbb{W}^{\pm}_{Q,\mathcal{W}}$ as given in [Du18] by means of this theorem.

Our main technical results concern the wave operators $\mathbb{W}^{\pm}_{Q,\mathcal{W}}$ of the deformed model, and we will now discuss their construction as given in [Du18]. Here we will also include some additional details which are of particular importance for us. First we recall that the wave operators are constructed by means of Theorem 6. In particular this means that, on the technical side, we are in fact working with a smaller subset of one-particle states, namely all which can be written in the form

$$\Psi_1 = B^{\Lambda}_{\tau}(f)\Omega = B^{\perp\Lambda}_{\tau}(f)\Omega. \tag{41}$$

We call such Ψ_1 swappable (with respect to $\mathcal{W}$) one-particle states of bounded energy, and we let $\mathscr{H}^{\mathcal{W}}_{1c}$ be the (non-closed) linear space spanned by them. Alternatively, using spectral calculus, one sees that the above generating set[7] of $\Psi_1 \in \mathscr{H}^{\mathcal{W}}_{1c}$ can also be

[7] A general vector $\Psi_1 \in \mathscr{H}^{\mathcal{W}}_{1c}$ is a sum of vectors of the form (41) or (42).

characterized by the existence of $A \in \mathfrak{A}(\mathcal{W})$, $A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, and $\tilde{f} \in C_c^\infty(\mathbb{R}^s)$, such that

$$\Psi_1 = \tilde{f}(\boldsymbol{P})E_{(H,\boldsymbol{P})}(H_m)A\Omega = \tilde{f}(\boldsymbol{P})E_{(H,\boldsymbol{P})}(H_m)A^\perp\Omega. \tag{42}$$

Proposition 7. *In a wQFT satisfying wedge duality* (HK2$^\sharp$)*, the density* $\overline{\mathscr{H}_{1c}^{\mathcal{W}}} = \mathscr{H}_1$ *holds for any wedge* $\mathcal{W}$*.*

Proof. Let $\Psi_1 \in \mathscr{H}_1$ and $\epsilon > 0$. By [Du18] App. B, there exist $A \in \mathfrak{A}(\mathcal{W})$ and $A^\perp \in \mathfrak{A}(\mathcal{W}')$, such that $\Psi' := A\Omega = A^\perp\Omega$ and $\left\|\Psi' - \Psi_1\right\| < \epsilon/2$. Let now $f \in C_c^\infty(\mathbb{R}^s)$ with $\tilde{f}(0) = 1$. Then for any $\delta > 0$ we have $\Psi_\delta := \tilde{f}(\delta\boldsymbol{P})E(H_m)\Psi' \in \mathscr{H}_{1c}^{\mathcal{W}}$ by (42). By spectral calculus $\Psi_\delta \longrightarrow E(H_m)\Psi'$ when we take $\delta \to 0$. In particular there exists $\delta' > 0$ s.t. $\|\Psi_{\delta'} - E(H_m)\Psi'\| < \epsilon/2$. Together we get $\|\Psi_{\delta'} - \Psi_1\| \leq \left\|\Psi_{\delta'} - E(H_m)\Psi'\right\| + \left\|E(H_m)(\Psi' - \Psi_1)\right\| \leq \epsilon/2 + \|E(H_m)\| \left\|\Psi' - \Psi_1\right\| \leq \epsilon$. □

Concerning this technical detail, let us add a brief side remark on the considerations which motivated our specific choice of $\mathscr{H}_1'$ by describing in a general manner the technical properties these dense sets of one-particle states should have.

Remark 8. The main technical subtlety which arises in comparison to the standard construction of bosonic and fermionic Fock spaces is that the resulting Fock space should be independent of possibly different technically motivated choices of $\mathscr{H}_1'$. Yet, for ordered Fock spaces in wedge-local models it is not completely trivial to find a suitable $\mathscr{H}_1'$, due to the possible interplay of swapping and smoothness with the ordering conditions. A suitable choice is, what we may call a *momentum resolving* subspace $\mathscr{H}_1' \subseteq \mathscr{H}_1$, namely, that for any $\Psi \in \mathscr{H}_1$ there exists a sequence $(\Psi_n)_{n\in\mathbb{N}} \subset \mathscr{H}_1'$ such that $\Psi_n \to \Psi$ and $\operatorname{supp} E_{(H,\boldsymbol{P})}\Psi_n \subseteq \operatorname{supp} E_{(H,\boldsymbol{P})}\Psi$. We note that from Proposition 7 and (42) it follows by choosing $\tilde{f}$ suitably that the linear spaces $\mathscr{H}_{1c}^{\mathcal{W}}$ are momentum resolving.

By construction, the energy-momentum operators of the initial and deformed model coincide. In particular, the one-particle spaces of the two models are identical. Hence we can directly compare the scattering data of the two models on the present abstract level in terms of the respective wave operators, as they are defined on the same ordered Fock spaces for both the deformed and initial model. For a direct comparison of the two wave operators, we will make use of a strengthened technical result on the density of one-particle states. Note that here also our admission of overlaps for opposite wedges in the swapping relation becomes useful from a technical perspective.

Corollary 9. *Any swappable one-particle state* $\Psi_1 \in \mathscr{H}_{1c}^{\mathcal{W}}$ *can be generated by swapping pairs* $\tilde{A} \in \mathfrak{A}^r(\mathcal{W})$ *and* $\tilde{A}^\perp \in \mathfrak{A}^r(\tilde{\mathcal{W}}^\perp)$ *of regular operators, where the opposite wedge* $\tilde{\mathcal{W}}^\perp$ *depends on* Ψ_1*.*

Proof. For swappable one-particle states $\Psi_1 \in \mathscr{H}_{1c}^{\mathcal{W}}$ there exist by definition an operator $A \in \mathfrak{A}(\mathcal{W})$, an opposite wedge $\mathcal{W}^\perp = \mathcal{W}' + x$, a swapping partner $A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ and a regular wave packet $\tilde{f}$ such that

$$\Psi_1 = \tilde{f}(\boldsymbol{P})E_{(H,\boldsymbol{P})}(H_m)A\Omega = \tilde{f}(\boldsymbol{P})E_{(H,\boldsymbol{P})}(H_m)A^\perp\Omega. \tag{43}$$

We write

$$\Psi_1 = \hat{\chi}(\omega_m(\boldsymbol{P}), \boldsymbol{P})^{-1}\tilde{f}(\boldsymbol{P})E_{(H,\boldsymbol{P})}(H_m)\hat{\chi}(H, \boldsymbol{P})A\Omega, \tag{44}$$

where $\chi \in C_c^\infty(\mathbb{R}^d)$ is a compactly supported function with $\hat{\chi}(\omega_m(\mathbf{p}), \mathbf{p})$ non-vanishing for all $\mathbf{p} \in \operatorname{supp} \tilde{f}$. Then $\tilde{f}'(\mathbf{p}) := \hat{\chi}(\omega_m(\mathbf{p}), \mathbf{p})^{-1} \tilde{f}(\mathbf{p})$ defines a new regular Klein-Gordon wave packet and $\hat{\chi}(H, \boldsymbol{P})A\Omega = (2\pi)^{-d/2}A(\chi)\Omega =: A_1\Omega$ (see eq. (17)) shows that the vector part is obtained from the vacuum as image of a regular operator $A_1 \in \mathfrak{A}^r(\mathcal{W}_1)$ for some wedge $\mathcal{W}_1 \supset \mathcal{W} + \operatorname{supp} \chi$. An analogous calculation yields $A_1^\perp \in \mathfrak{A}^r(\mathcal{W}_1^\perp)$ with $\mathcal{W}_1^\perp \supset \mathcal{W}^\perp + \operatorname{supp} \chi$. Finally we obtain the desired pair of operators with $\tilde{A} \in \mathfrak{A}^r(\mathcal{W})$ and $\tilde{A}^\perp \in \mathfrak{A}^r(\tilde{\mathcal{W}}^\perp)$ by translating the just constructed operator A_1 back to $\mathcal{W}$, and also replacing $A_1^\perp$ and $\mathcal{W}_1^\perp$ with the respective translates by the same vector. These translations can be compensated by absorbing the resulting phase in (44) into the wave packet. □

Definition 10 (*Wave operators of initial and deformed model*). Let $(\mathfrak{A}^0, \alpha, \mathscr{H}, \Omega)$ be a wedge-local quantum field theory. The incoming and outgoing wave operators $\mathbb{W}^\pm_{0,\mathcal{W}}$ associated to centered wedge regions $\mathcal{W}$, $\Lambda \in \mathcal{L}^*(\mathcal{W})$, in the initial model are the maps defined by

$$\begin{aligned} \mathbb{W}^+_{0,\mathcal{W}} &: \begin{cases} \Gamma^{\succ\mathcal{W}}(\mathscr{H}_1) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau\to\infty} B^\Lambda_{1\tau}(f_1)\ldots B^\Lambda_{n\tau}(f_n)\Omega, \end{cases} \\ \mathbb{W}^-_{0,\mathcal{W}} &: \begin{cases} \Gamma^{\prec\mathcal{W}}(\mathscr{H}_1) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau\to-\infty} B^\Lambda_{1\tau}(f_1)\ldots B^\Lambda_{n\tau}(f_n)\Omega, \end{cases} \end{aligned} \tag{45}$$

via linear and continuous extension from product states in $\Gamma_0^{\succ\mathcal{W}/\prec\mathcal{W}}(\mathscr{H}_{1c}^{\mathcal{W}})$, where $B^\Lambda_{k\tau}(f_k)\Omega = \Psi_1^k$. Similarly the wave operators of the deformed model $(\mathfrak{A}^Q, \alpha, \mathscr{H}, \Omega)$ are

$$\begin{aligned} \mathbb{W}^+_{Q,\mathcal{W}} &: \begin{cases} \Gamma^{\succ\mathcal{W}}(\mathscr{H}_1) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau\to\infty} B^\Lambda_{1Q\tau}(f_1)\ldots B^\Lambda_{nQ\tau}(f_n)\Omega, \end{cases} \\ \mathbb{W}^-_{Q,\mathcal{W}} &: \begin{cases} \Gamma^{\prec\mathcal{W}}(\mathscr{H}_1) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau\to-\infty} B^\Lambda_{1Q\tau}(f_1)\ldots B^\Lambda_{nQ\tau}(f_n)\Omega. \end{cases} \end{aligned} \tag{46}$$

Here we already made use of Corollary 9, which allows us to fully analyze the scattering in the deformed model while restricting to creation-operator approximants $B^\Lambda_{kQ\tau}(f_k)$ constructed from warpings $(A_k)_Q$ of regular swapping pairs A_k, $A_k^\perp$, as provided by Corollary 9, instead of having to admit general $A_k^Q \in \mathfrak{A}^Q(\mathcal{W})$ as in eq. (35).

For simplicity we will refer to $\mathbb{W}^\pm_{Q,\mathcal{W}}$ as the *deformed wave operators*. We note that, strictly speaking, this terminology is justified only in retrospective after the results of the present paper are established. Namely, our main result shows the wave operators $\mathbb{W}^\pm_{Q,\mathcal{W}}$, as defined using the general construction in the deformed model, can be regarded as a "deformation" of the wave operators $\mathbb{W}^\pm_{0,\mathcal{W}}$ from the underlying "undeformed" model.

Theorem 11. *The wave operators of the deformed model* $(\mathfrak{A}^Q, \alpha, \mathscr{H}, \Omega)$ *can be expressed in terms of the wave operators of the initial model* $(\mathfrak{A}^0, \alpha, \mathscr{H}, \Omega)$ *via*

$$\mathbb{W}^+_{Q,\mathcal{W}} = \mathbb{W}^+_{0,\mathcal{W}} S^{\succ\mathcal{W}}_{Q\mathcal{W}}, \qquad \mathbb{W}^-_{Q,\mathcal{W}} = \mathbb{W}^-_{0,\mathcal{W}} S^{\prec\mathcal{W}}_{Q\mathcal{W}}, \tag{47}$$

where $S_{Q_{\mathcal{W}}}^{\succ \mathcal{W}/\prec \mathcal{W}}$ *are restrictions to* $\Gamma^{\succ \mathcal{W}/\prec \mathcal{W}}$ *of* $S_{Q_{\mathcal{W}}} : \Gamma^u(\mathscr{H}_1) \to \Gamma^u(\mathscr{H}_1)$ *defined by*

$$S_{Q_{\mathcal{W}}} \Psi_1^1 \otimes \ldots \otimes \Psi_1^n := \prod_{1 \leq i < j \leq n} \mathrm{e}^{\mathrm{i} P_i \cdot Q_{\mathcal{W}} P_j} \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \tag{48}$$

for $\Psi_1^1, \ldots, \Psi_1^n \in \mathscr{H}_1$*. Here, for* $i \in \mathbb{N}$*,* P_i *denotes the self-adjoint operator defined on* $\Gamma^u(\mathscr{H}_1)$ *in terms of the energy-momentum operator* $P = (H, \boldsymbol{P})$ *of the model by*

$$P_i(\Psi_1^1 \otimes \ldots \otimes \Psi_1^n) := \Psi_1^1 \otimes \ldots \otimes P\Psi_1^i \otimes \ldots \otimes \Psi_1^n \tag{49}$$

for $n \geq i$*,* $\Psi_1^1, \ldots, \Psi_1^n \in \mathscr{H}_1$ *with* $\Psi_1^i \in D(P)$ *and* $P_i(\Psi_1^1 \otimes \ldots \otimes \Psi_1^n) := 0$ *if* $n < i$*.*

We note that P_i can be regarded as i-th unordered second quantization of the energy-momentum operator P. Essential self-adjointness of P_i on the domain of vectors of finite particle number follows from standard arguments (see e.g. [RS1, Sec. VIII.10]).

3.2. Proof of the wave operator identity. We start with some preparations. First we will recall how to re-express the warped convolutions as convergent $\mathscr{H}$-valued integrals. This is done by using standard oscillatory integral methods [BLS11,DT13,LW16].

Lemma 12. *Let* $\mathbb{R}^{2d} \ni (x, y) \longmapsto \Psi(x, y) \in \mathscr{H}$ *be a map with uniformly bounded derivatives for all multi-indices* $\beta \in \mathbb{N}_0^{2d}$ *up to order* $|\beta| \leq 4d$.[8] *Then we have*

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \eta(\epsilon x, \epsilon y) \mathrm{e}^{-\mathrm{i} x \cdot y} \Psi(x, y) \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \mathrm{e}^{-\mathrm{i} x \cdot y} D_{\mathrm{reg}}(\partial_x, \partial_y) \Psi(x, y). \end{aligned} \tag{50}$$

Here $D_{\mathrm{reg}}(\partial_x, \partial_y) := \prod_{j=0}^{s} D_j(\partial_{x_j}, \partial_{y_j})^2$ *is a product of auxiliary, mutually commuting partial differential operators defined by*

$$D_j(\partial_{x_j}, \partial_{y_j}) \Phi(x, y) := (1 - \partial_{x_j}^2 - \partial_{y_j}^2) \frac{1}{1 + y_j^2 + x_j^2} \; \Phi(x, y). \tag{51}$$

In particular, the limit (50) *exists and it is independent of the choice of the regularizing function* $\eta \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ *with* $\eta(0, 0) = 1$*.*

Proof. The operators D_j are constructed such that their formal adjoints satisfy

$$D_j^* \mathrm{e}^{\pm \mathrm{i} x_j y_j} = \frac{1}{1 + y_j^2 + x_j^2} (1 - \partial_{x_j}^2 - \partial_{y_j}^2) \mathrm{e}^{\pm \mathrm{i} x_j y_j} = \mathrm{e}^{\pm \mathrm{i} x_j y_j}. \tag{52}$$

Inserting this identity into the oscillatory integral with finite $\epsilon > 0$ twice for every $1 \leq j \leq d$, using integration by parts, and writing $\eta_\epsilon(x, y) := \eta(\epsilon x, \epsilon y)$, we get

$$\frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \mathrm{e}^{-\mathrm{i} x \cdot y} \eta_\epsilon(x, y) \Psi(x, y)$$

[8] Our strategy mostly follows [BLS11]. With our choice of D_{reg}, integrability estimates are split into pairs (x_j, y_j), $1 \leq j \leq d$, and thereby they become slightly more explicit. On the other hand, our smoothness requirements are not optimal. For a regularizing differential operator requiring only $|\beta| \leq d + 1$ derivatives of $\Psi(x, y)$, we refer to [BLS11].

$$= \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \mathrm{e}^{-\mathrm{i}x\cdot y} D_{\mathrm{reg}}(\partial_x, \partial_y) \eta_\epsilon(x, y) \Psi(x, y). \tag{53}$$

In this form the $\epsilon \to 0$ limit can now be carried out: it follows by explicit calculation and induction that

$$\left\| D_{\mathrm{reg}}(\partial_x, \partial_y) \Phi(x, y) \right\| \leq \prod_{j=0}^{d-1} \frac{C}{(1 + x_j^2 + y_j^2)^2} \|\Phi\|_{\mathscr{C}^{4d}(\mathbb{R}^{2d}, \mathscr{H})}, \tag{54}$$

where the norm is defined by

$$\|\Phi\|_{\mathscr{C}^k(\mathbb{R}^{2d}, \mathscr{H})} := \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^d \\ |\alpha| + |\beta| \leq k}} \sup_{x, y \in \mathbb{R}^d} \left\| \partial_x^\alpha \partial_y^\beta \Phi(x, y) \right\|_{\mathscr{H}}. \tag{55}$$

From (54) we now obtain an integrable majorant for the rewritten integral (53), by estimating $\|\eta_\epsilon \Psi\|_{\mathscr{C}^{4d}(\mathbb{R}^{2d}, \mathscr{H})} \leq C \|\eta_\epsilon\|_{\mathscr{C}^{4d}(\mathbb{R}^{2d})} \|\Psi\|_{\mathscr{C}^{4d}(\mathbb{R}^{2d}, \mathscr{H})}$ and, for $0 < \epsilon < 1$, $\|\eta_\epsilon\|_{\mathscr{C}^{4d}(\mathbb{R}^{2d})} \leq \|\eta\|_{\mathscr{C}^{4d}(\mathbb{R}^{2d})}$ using the definition of η_ϵ and the chain rule. Hence, by dominated convergence, it is sufficient to verify pointwise convergence of the integrand. To this end we write

$$D_{\mathrm{reg}}(\partial_x, \partial_y) \eta_\epsilon(x, y) \Psi(x, y) = \eta_\epsilon(x, y) D_{\mathrm{reg}}(\partial_x, \partial_y) \Psi(x, y) + R_\epsilon(x, y). \tag{56}$$

In each term of the product rule expansion of the remainder R_ϵ there is at least one derivative with respect to x or y acting on η_ϵ. Thus this remainder is proportional to ϵ and vanishes for $\epsilon \to 0$. The first term converges pointwise to $D_{\mathrm{reg}}(\partial_x, \partial_y)\Psi(x, y)$ and from this the claim (50) follows. □

For our scattering theoretic purposes it is useful that this method can be readily extended to obtain similar integral representations by iterating Lemma 12. In this manner we can further strengthen the decay of the integrand for large $x, y \in \mathbb{R}^d$.

Lemma 13. *Let $M \in \mathbb{N}$ and let $\mathbb{R}^{2d} \ni (x, y) \longmapsto \Psi(x, y) \in \mathscr{H}$ be a map with uniformly bounded derivatives for all multi-indices $\beta \in \mathbb{N}_0^{2d}$ up to order $|\beta| \leq 4Md$. Then we have*

$$\lim_{\epsilon \to 0} \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \eta(\epsilon x, \epsilon y) \mathrm{e}^{-\mathrm{i}x\cdot y} \Psi(x, y)$$
$$= \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \mathrm{e}^{-\mathrm{i}x\cdot y} D_{\mathrm{reg}}(\partial_x, \partial_y)^M \Psi(x, y). \tag{57}$$

The integrand satisfies the integrable norm bound

$$\left\| D_{\mathrm{reg}}(\partial_x, \partial_y)^M \Psi(x, y) \right\| \leq \prod_{j=0}^{d-1} \frac{C_M}{(1 + x_j^2 + y_j^2)^{2M}} \|\Psi\|_{\mathscr{C}^{4Md}(\mathbb{R}^{2d}, \mathscr{H})}. \tag{58}$$

Next, we use the convergent integral representation to check that warped convolutions can be exchanged with the smearing operations used to define the creation-operator approximants in the deformed model.

Lemma 14. *Let $\Psi \in \mathcal{D}$, $A \in \mathfrak{A}^{0r}(\mathcal{W})$, $\chi \in \mathscr{S}(\mathbb{R}^d)$, and let f be a regular Klein–Gordon solution. Then*

$$B_Q\Psi := A_{Q\mathcal{W}}(\chi)\Psi = (A(\chi))_{Q\mathcal{W}}\Psi, \tag{59}$$

and

$$B_{Q\tau}(f)\Psi = (B_\tau(f))_{Q\mathcal{W}}\Psi. \tag{60}$$

Proof. We can apply Lemma 12 to warped convolutions, as the corresponding integrand $\Psi(x, y) = \alpha_{Qx}(A)U(y)\Psi$ is arbitrarily often differentiable for $A \in \mathcal{C}^\infty$ and $\Psi \in \mathcal{D}$, with $\left\|\partial_x^\alpha \partial_y^\beta \Psi(x, y)\right\| \leq C_{\alpha\beta}$ for suitable constant depending on the multi-indices $\alpha, \beta \in \mathbb{N}_0^d$. By translation covariance of warped convolutions from Lemma 3 (v) we have

$$\begin{aligned} A_{Q\mathcal{W}}(\chi)\Psi &= \int \mathrm{d}^d z\, \chi(z)\alpha_z(A_{Q\mathcal{W}})\Psi = \int \mathrm{d}^d z\, \chi(z)(\alpha_z A)_{Q\mathcal{W}}\Psi \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d z\, \chi(z) \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{e}^{-\mathrm{i}x\cdot y} D_{\mathrm{reg}}(\partial_x, \partial_y)U(x)\alpha_{Qy+z}(A)\Psi. \end{aligned} \tag{61}$$

Using the decay of $\chi \in \mathscr{S}(\mathbb{R}^d)$ and estimate (54), the above integrand has integrable norm with respect to the product Lebesgue measure $\mathrm{d}^{3d}(x, y, z)$. Hence the order of integrations can be exchanged by Fubini's theorem. Taking the x- and y-dependent translation operators outside the inner strong integral, we obtain (59). The proof of (60) is analogous. □

Definition 15. For $\Psi, \Psi' \in \Gamma^u(\mathscr{H}_1)$ and $Q \in \mathbb{C}^{d\times d}$ let

$$\Psi \otimes_Q \Psi' := \mathrm{e}^{\mathrm{i}P_1\cdot QP_2}\Psi \otimes \Psi'. \tag{62}$$

Here, $P_1 = P \otimes \mathbb{1}$ is the energy-momentum operator acting on the first argument only, and similarly $P_2 = \mathbb{1} \otimes P$ acts on the second argument.

Before we continue, let us remark that this deformed tensor product preserves the ordered subspaces in the sense that if $\Psi, \Psi' \in \Gamma^{\succ}(\mathscr{H}_1)$, with $\Psi \succ \Psi'$, then also $\Psi \otimes_Q \Psi' \in \Gamma^{\succ}(\mathscr{H}_1)$. Further, the deformed tensor product is clearly linear in its arguments and associative. Its noncommutative structure strongly resembles the Zamolodchikov-Faddeev relations from integrable QFT models (see e.g. [Le03]), in contrast to the canonical commutation relations and the corresponding commutative symmetrized tensor product structure underlying the usual free scalar QFT. Let us note cautiously that $\otimes_Q$ is in general not mixed-associative in combination with ordinary tensor products, that is, $\Psi_1 \otimes_Q (\Psi_2 \otimes \Psi_3) \neq (\Psi_1 \otimes_Q \Psi_2) \otimes \Psi_3$. We also note that the definition is consistent with the fact that on Fock spaces $\Psi \otimes \Omega = \Psi = \Omega \otimes \Psi$ are identified for any Ψ.

We can now establish the main technical lemma for proving the wave operator identity from Theorem 11.

Lemma 16. *Let $\Psi_1^k = B_{k\tau}(f_k)\Omega = B_{k\tau}^{\perp}(f_k)\Omega$, $1 \leq k \leq n$, be swappable one-particle states, s.t. $A_k \in \mathfrak{A}^{0r}(\mathcal{W})$, $A_k^{\perp} \in \mathfrak{A}^{0r}(\mathcal{W}^{\perp})$ are regular ($1 \leq k \leq n$), and $B_k = A_k(\chi)$ where $\chi \in \mathscr{S}(\mathbb{R}^d)$ is an admissible Haag–Ruelle auxiliary function. For ordered velocity supports $\mathcal{V}_1 \succ \ldots \succ \mathcal{V}_n$ we have*

$$\lim_{\tau\to\infty} B_{1Q\tau}(f_1)B_{2Q\tau}(f_2)\ldots B_{nQ\tau}(f_n)\Omega = \mathbb{W}_{0,\mathcal{W}}^{+}\Psi_1^1 \otimes_{Q\mathcal{W}} \ldots \otimes_{Q\mathcal{W}} \Psi_1^n. \tag{63}$$

With incoming ordering $\mathcal{V}_1 \prec \ldots \prec \mathcal{V}_n$ we have analogously for the incoming limit $\tau \to -\infty$

$$\lim_{\tau\to-\infty} B_{1Q\tau}(f_1)B_{2Q\tau}(f_2)\ldots B_{nQ\tau}(f_n)\Omega = \mathbb{W}^-_{0,\mathcal{W}}\Psi^1_1 \otimes_{Q_\mathcal{W}} \ldots \otimes_{Q_\mathcal{W}} \Psi^n_1. \quad (64)$$

Proof. We consider only the case $\tau \to \infty$. We first note that $\Psi_{k\tau} := B_{kQ\tau}(f_k)\ldots B_{nQ\tau}(f_n)\Omega \in \mathcal{D}$ for $1 \le k \le n$, due to the compact energy-momentum transfer of the $B_{jQ\tau}(f_j)$, $1 \le j \le n$ (see [Du18] Lemma 7). Thus Lemmas 14 and 12 apply and we obtain

$$\begin{aligned}\Psi_\tau := \Psi_{1\tau} &= B_{1\tau}(f_1)_{Q_\mathcal{W}} B_{2\tau}(f_2)_{Q_\mathcal{W}} \ldots B_{n\tau}(f_n)_{Q_\mathcal{W}}\Omega \\ &= \int \frac{\mathrm{d}^d x_1\, \mathrm{d}^d y_1 \ldots \mathrm{d}^d x_n\, \mathrm{d}^d y_n}{(2\pi)^{nd}} \prod_{j=1}^{n} (\mathrm{e}^{-\mathrm{i}x_j\cdot y_j} D_{\mathrm{reg}}(\partial_{x_j}, \partial_{y_j})^M) \\ &\qquad U(x_1)\alpha_{Qy_1}(B_{1\tau}(f_1))\ldots U(x_n)\alpha_{Qy_n}(B_{n\tau}(f_n))\Omega, \end{aligned} \quad (65)$$

where the constant $M \in \mathbb{N}$ will be chosen below. We rewrite the vector part of the integrand using that $U(x_1)\alpha_{Qy_1}(B_{1\tau}(f_1)) = \alpha_{Qy_1+x_1}(B_{1\tau}(f_1))U(x_1)$ as

$$\begin{aligned}\Psi_{\underline{x},\underline{y},\tau} &:= U(x_1)\alpha_{Qy_1}(B_{1\tau}(f_1))\ldots U(x_n)\alpha_{Qy_n}(B_{n\tau}(f_n))\Omega \\ &= \alpha_{Qy_1+x_1}(B_{1\tau}(f_1))\alpha_{Qy_2+x_1+x_2}(B_{2\tau}(f_2))\ldots\alpha_{Qy_n+x_1+\ldots+x_n}(B_{n\tau}(f_n))\Omega, \end{aligned} \quad (66)$$

with $\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^{nd}$, and we abbreviate $Q := Q_\mathcal{W}$.

Now we split the integration in (65) into an integral over the region

$$R^\uparrow_{\rho\tau} := \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2nd} : |x^0_k| + |\mathbf{x}_k| \le \rho\tau, |y^0_k| + |\mathbf{y}_k| \le \rho\tau\} \quad (67)$$

and its complement, where the constant $\rho > 0$ is for now fixed but arbitrary, and will be specified later. The vector resulting from the integral over $R^\uparrow_{\rho\tau}$ will be denoted $\Psi^\uparrow_{1\tau}$. By means of applying (58) iteratively for all pairs (x_k, y_k), $1 \le k \le n$, we verify that with an appropriately strong power $M \in \mathbb{N}$ of the regularizing operator D_{reg}, the remainder integral over the complement $R^\downarrow_{\rho\tau} := \mathbb{R}^{2nd} \backslash R^\uparrow_{\rho\tau}$ becomes small for large τ. That is,

$$\begin{aligned}\left\|\Psi^\downarrow_\tau\right\| &:= \left\|\Psi_\tau - \Psi^\uparrow_\tau\right\| \\ &\le \int_{R^\downarrow_{\rho\tau}} \frac{\mathrm{d}^{nd}\underline{x}\, \mathrm{d}^{nd}\underline{y}}{(2\pi)^{nd}} \left\| \left(\prod_{k=1}^{n} D_{\mathrm{reg}}(\partial_{x_k}, \partial_{y_k})^M\right) \Psi_{\underline{x},\underline{y},\tau} \right\| \qquad (68) \\ &\le \int_{R^\downarrow_{\rho\tau}} \frac{\mathrm{d}^{nd}\underline{x}\, \mathrm{d}^{nd}\underline{y}}{(2\pi)^{nd}} \left(\prod_{k=1}^{n}\prod_{j=0}^{d-1} \frac{1}{(1 + x^2_{k,j} + y^2_{k,j})^{2M}} \right) C(1 + |\tau|^{ns/2}). \qquad (69)\end{aligned}$$

In the last step we estimated

$$\begin{aligned}&\left\| D_{\mathrm{reg}}(\partial_{x_1}, \partial_{y_1})^M \cdots D_{\mathrm{reg}}(\partial_{x_n}, \partial_{y_n})^M \Psi_{\underline{x},\underline{y},\tau} \right\| \\ &\qquad \le C \prod_{k=1}^{n}\prod_{j=0}^{d-1} \frac{1}{(1 + x^2_{k,j} + y^2_{k,j})^{2M}} \left\| \Psi_{\underline{x},\underline{y},\tau} \right\|_{\mathscr{C}^{4Mnd}(\mathbb{R}^{2nd},\mathscr{H})}. \end{aligned} \quad (70)$$

The derivative norm of the vector part was then bounded by expanding it into individual differentiated terms $\partial_{\underline{x}}^{\underline{\alpha}}\partial_{\underline{y}}^{\underline{\beta}}\Psi_{\underline{x},\underline{y},\tau}$, which in turn can be expanded by the product rule into terms with differential operators acting on the translated creation-operator approximants $B_{k\tau}(f_k)$. Due to the assumed norm differentiability of A_k, the differentiated $B_{k\tau}(f_k)$ can be rewritten as creation operator approximants constructed using the differentiated A_k. These will be denoted here by $\tilde{B}_{k\tau}(f_k)$ and $\tilde{A}_k$, respectively. Using this rewriting, the terms from the expanded differentiated vector parts from (68) can each be bounded using the standard estimate $\|\tilde{B}_{k\tau}(f_k)\| \leq C_{\chi,f_k}\|\tilde{A}_k\|(1+|\tau|^{s/2})$ to obtain $\|\alpha_{z_1}(\tilde{B}_{1\tau}(f_1))\dots\alpha_{z_n}(\tilde{B}_{n\tau}(f_n))\Omega\| \leq \prod_{k=1}^n \|\tilde{B}_{k\tau}(f_k)\| \leq C(1+|\tau|^{ns/2})$. Taking all these terms together we obtain the last step from (69), where the new constant C depends on M, on all wave packages and on the norms of derivatives of the A_k operators up to order $4Mnd$. To obtain the decay estimate for sufficiently large τ we proceed to estimate

$$\left\|\Psi_\tau^{\downarrow}\right\| \leq C\,|\tau|^{ns/2}\left(\sup_{(\underline{x},\underline{y})\in R_{\rho\tau}^{\downarrow}}\prod_{k=1}^{n}\prod_{j=0}^{d-1}\frac{1}{(1+x_{k,j}^2+y_{k,j}^2)^{2M-2}}\right)\prod_{k=1}^{n}\prod_{j=0}^{d-1}\int_{\mathbb{R}^2}\frac{\mathrm{d}x_{k,j}\mathrm{d}y_{k,j}}{(1+x_{k,j}^2+y_{k,j}^2)^2}. \tag{71}$$

Here constant factors such as the convergent integrals can be absorbed into the constant C. To bound the supremum we note that by definition of $R_{\rho\tau}^{\uparrow}$ we have for any point $(\underline{x},\underline{y})$ in the complement $\mathbb{R}^{2nd}\backslash R_{\rho\tau}^{\uparrow}$ at least one vector with $|x_{k^*}^0|+|\mathbf{x}_{k^*}| > \rho\tau$ or $|y_{k^*}^0|+|\mathbf{y}_{k^*}| > \rho\tau$. This in turn implies that at least one coordinate $|x_{k^*,j^*}|$ or $|y_{k^*,j^*}|$, respectively, is larger than $\rho\tau/\sqrt{4\,s}$. As all other factors in the supremum are bounded from above by one, we obtain

$$\left\|\Psi_\tau^{\downarrow}\right\| \leq C|\tau|^{ns/2+4-4M}. \tag{72}$$

Choosing $M \in \mathbb{N}$ large enough, this contribution becomes arbitrarily small for large $\tau > 0$.

In particular, for establishing (63) it is sufficient to consider the limit of $\Psi_\tau^{\uparrow}$. To address this convergence, let $\tilde{\rho}>0$ denote the minimum over all such constants from Lemma 17 for the families of $(\tilde{B}_{j\tau}(f_j))_{1\leq j\leq n}$, where $\tilde{B}_{j\tau}(f_j) = \partial_{x_j}^{\beta_j}\alpha_{x_j}(B_{j\tau}(f_j))\big|_{x_j=0}$ stand for all possible combinations of derivatives of these operators up to order $|\beta_j| \leq 4Mnd$, $\beta_j \in \mathbb{N}_0^d$. It is used here that $\partial_{x_j}^{\beta_j}\alpha_{x_j}(B_{j\tau}(f_j))\Omega = \partial_{x_j}^{\beta_j}\alpha_{x_j}(B_{j\tau}^{\perp}(f_j))\Omega$ for all multi-indices $\beta_j \in \mathbb{N}_0^d$, which shows by setting $x_j = 0$ that all such $\tilde{B}_{j\tau}(f_j)\Omega$ are swappable, as needed to apply Lemma 17 to these differentiated families. Further the derivatives with respect to y_k can also be written as multiples of such differentiated operators by using the chain rule. Thus we set $\rho := \tilde{\rho}\cdot(1+\|Q\|)^{-1}/(n+1) < \infty$. With this choice we have for any $1 \leq k \leq n$ and $(\underline{x},\underline{y}) \in R_{\rho\tau}^{\uparrow}$ that the vectors $z_k := Qy_k + x_1 + \ldots + x_k$ are contained in the double cone of radius $k\rho\,|\tau| + \|Q\|\rho\tau \leq \tilde{\rho}\,|\tau|$ for $1 \leq k \leq n$. Considering the asymptotically dominant part

$$\Psi_\tau^{\uparrow} := \int_{R_{\rho\tau}^{\uparrow}}\frac{\mathrm{d}^{nd}\underline{x}\;\mathrm{d}^{nd}\underline{y}}{(2\pi)^{nd}}\left(\prod_{j=1}^{n}\mathrm{e}^{-\mathrm{i}x_j\cdot y_j}D_{\mathrm{reg}}(\partial_{x_j},\partial_{y_j})^M\right)\Psi_{\underline{x},\underline{y},\tau}, \tag{73}$$

we can obtain a τ-uniform integrable bound by applying (70) and noting that instead of the coarse bound (72) we can now estimate

$$\mathbb{1}_{R^{\uparrow}_{\rho\tau}}(\underline{x},\underline{y})\left\|\Psi_{\underline{x},\underline{y},\tau}\right\|_{\mathscr{C}^{4Mnd}(\mathbb{R}^{2nd},\mathscr{H})}\leq C. \tag{74}$$

Here we make use of Lemma 17, noting that this uniform bound applies to each of the terms resulting from the expansion

$$\left\|\Psi_{\underline{x},\underline{y},\tau}\right\|_{\mathscr{C}^{4Mnd}(\mathbb{R}^{2nd},\mathscr{H})}=\sum_{\substack{\underline{\alpha},\underline{\beta}\in\mathbb{N}_0^{nd},\\ |\underline{\alpha}|+|\underline{\beta}|\leq 4Mnd}}\|\partial^{\underline{\alpha}}_{\underline{x}}\partial^{\underline{\beta}}_{\underline{y}}\Psi_{\underline{x},\underline{y},\tau}\| \tag{75}$$

by our choice of ρ. Due to the restriction to $R^{\uparrow}_{\rho\tau}$ we can estimate the differentiable vector norm by means of Lemma 17, after expanding it into individual terms with fixed derivatives acting on the creation-operators $\tilde{B}_{k\tau}(f_k)$. As before these can be written as Haag–Ruelle-type operators in terms of the differentiated A_k, so that Lemma 17 applies. To summarize we note that the proof strategy here is in fact analogous to the above method used for the outside region $R^{\downarrow}_{\rho\tau}$. However in $R^{\uparrow}_{\rho\tau}$ the use of the clustering bound of Lemma 17 is geometrically permitted and yields the much stronger τ-uniform estimate on the vector part.

By dominated convergence we obtain

$$\lim_{\tau\to\infty}\Psi_\tau\overset{(72)}{=}\lim_{\tau\to\infty}\Psi^{\uparrow}_\tau=\int\frac{\mathrm{d}^{nd}\underline{x}\,\mathrm{d}^{nd}\underline{y}}{(2\pi)^{nd}}\lim_{\tau\to\infty}\prod_{j=1}^{n}(\mathrm{e}^{-\mathrm{i}x_j\cdot y_j}D_{\mathrm{reg}}(\partial_{x_j},\partial_{y_j})^M)\Psi_{\underline{x},\underline{y},\tau}, \tag{76}$$

where we already used that the characteristic function $\mathbb{1}_{R^{\uparrow}_{\rho\tau}}(\underline{x},\underline{y})\to 1$ pointwise for $\tau\to\infty$. Here the Haag–Ruelle limit can be exchanged with the regularizing differential operators by explicit computation: we expand everything into differentiated translated Haag–Ruelle operators, for which the right hand side of (76) converges to the scattering state generated by the differentiated and translated operators, which are again Haag–Ruelle type creation-operator approximants. Collecting the Haag–Ruelle limits again after performing them, we have by linearity of the wave operator

$$\lim_{\tau\to\infty}\left(\prod_{j=1}^{n}D_{\mathrm{reg}}(\partial_{x_j},\partial_{y_j})^M\right)\Psi_{\underline{x},\underline{y},\tau} \tag{77}$$

$$=\mathbb{W}^{+}_{0,\mathcal{W}}\left(\prod_{j=1}^{n}D_{\mathrm{reg}}(\partial_{x_j},\partial_{y_j})^M\right)(U(Qy_1+x_1)\Psi_1^1)\otimes(U(Qy_2+x_1+x_2)\Psi_1^2)\otimes\ldots$$
$$\otimes(U(Qy_n+x_1+\ldots+x_n)\Psi_1^n)$$
$$=\mathbb{W}^{+}_{0,\mathcal{W}}\left(\prod_{j=1}^{n}D_{\mathrm{reg}}(\partial_{x_j},\partial_{y_j})^M\right)U(x_1)\Big\{(U(Qy_1)\Psi_1^1)\otimes U(x_2)\left\{(U(Qy_2)\Psi_1^2)\otimes\ldots\right.$$
$$\left.\otimes U(x_n)\left\{(U(Qy_n)\Psi_1^n)\right\}\ldots\right\}\Big\}. \tag{78}$$

In the last equality we have written the one-particle state translations again in groups using the second quantized translations on the unordered Fock space. In this form we can now apply the warped-convolution-type integral representation of the deformed tensor product (Proposition 20). For this purpose we note that the wave operator is bounded on the range of the integrand and can therefore be taken outside the strong integral. Thus the introduction of the regularizing differential operators can be undone by iterative application of Lemma 13 and we obtain

$$\lim_{\tau\to\infty} \Psi_\tau = \mathbb{W}^+_{0,\mathcal{W}} \left(\lim_{\epsilon_1,\dots,\epsilon_n\to 0} \int \frac{\mathrm{d}\underline{x}\,\mathrm{d}\underline{y}}{(2\pi)^{nd}} \left(\prod_{j=1}^{n} \eta(\epsilon_j x_j, \epsilon_j y_j) \mathrm{e}^{-\mathrm{i}x_j\cdot y_j} \right) \right.$$
$$\left. U(x_1)\left\{ (U(Qy_1)\Psi_1^1) \otimes U(x_2)\left\{ (U(Qy_2)\Psi_1^2) \otimes \ldots \otimes U(x_n)\left\{ (U(Qy_n)\Psi_1^n) \right\} \cdots \right\}\right\} \right). \tag{79}$$

Regrouping the convergent integrals, using Fubini and continuity of the tensor product, and applying Proposition 20 iteratively, we obtain deformed tensor products, as claimed in (63). We note that after replacing all tensor products by Q-deformed tensor products the right-associative grouping from (79) becomes inessential and can be dropped. The proof of the statement for the incoming limit $\tau \to -\infty$ is analogous. □

We used the following auxiliary result concerning the norm of scattering-state approximants involving certain restricted translations of each operator.

Lemma 17. *For any family of swappable operators $B_{k\tau}(f_k)$, $1 \le k \le n$, with regular Klein-Gordon wave packets satisfying the outgoing ordering $\mathcal{V}_1 \succ \ldots \succ \mathcal{V}_n$ there exists a constant $\rho > 0$ such that for all $\tau \ge 0$ and $x_1, \dots, x_n \in \mathscr{C}_{\rho|\tau|}$*

$$\left\| \alpha_{x_1}(B_{1\tau}(f_1)) \ldots \alpha_{x_n}(B_{n\tau}(f_n))\Omega \right\| \le C, \tag{80}$$

where $\mathscr{C}_r := \{x = (x^0, \mathbf{x}) \in \mathbb{R}^d : |x|_c := |x^0| + |\mathbf{x}| < r\} = r\mathscr{C}_1$ denotes the double cone of radius $r > 0$. With opposite ordering $\mathcal{V}_1 \prec \ldots \prec \mathcal{V}_n$ an analogous bound holds for incoming times $\tau < 0$.

Before proving this lemma let us recall the following useful technical result from [Du18], which concerns the approximation of Haag–Ruelle creation-operator approximants by wedge-local operators.

Lemma 18 [Du18, Lemma 9]. *Let $A \in \mathfrak{A}(\mathcal{W})$. For any $\tau \in \mathbb{R}$ and $\delta > 0$ the corresponding $B_\tau := B_\tau(f)$ can be approximated by $B_\tau^{(\delta)} \in \mathfrak{A}(\tau\mathcal{V}_f + \mathscr{C}_{\delta|\tau|} + \mathcal{W})$, $(\delta > 0)$, such that for any $N \in \mathbb{N}$*

$$\left\| B_\tau^{(\delta)} - B_\tau \right\| \le \frac{C_N^\delta}{1 + |\tau|^N}, \tag{81}$$

where the constants C_N^δ depend on f, A and χ.

For the proof of Lemma 17 we use a corresponding version of the commutator estimate, which will be formulated as a separate lemma to be proven first. For this estimate certain translations of oppositely localized pairs are admitted, similarly to the corresponding translations appearing in Lemma 17.

Lemma 19. *Let $B = A(\chi)$, $B^{\perp} = A^{\perp}(\chi)$ with $A \in \mathfrak{A}(\mathcal{W})$, $A^{\perp} \in \mathfrak{A}(\mathcal{W}^{\perp})$, for a pair of opposite wedges $\mathcal{W}, \mathcal{W}^{\perp}$, $\chi \in \mathscr{S}(\mathbb{R}^d)$, and let f, $f^{\perp}$ be regular Klein-Gordon solutions ordered by $\mathcal{V}_{f^{\perp}} \prec_{\mathcal{W}} \mathcal{V}_f$. Then there exists a constant $\rho > 0$ and for any $N \in \mathbb{N}$ a constant $C_N > 0$ such that for any $\tau > 0$ and $x, y \in \mathscr{C}_{\rho\tau}$,*

$$\left\| \left[\alpha_y(B_\tau^{\perp}(f^{\perp})), \alpha_x(B_\tau(f)) \right] \right\| \leq \frac{C_N}{1 + |\tau|^N}. \tag{82}$$

This rapid decay extends to the case that either or both operators are replaced by their adjoint. Further these estimates also hold for $\tau < 0$ given the opposite ordering $\mathcal{V}_{f^{\perp}} \succ_{\mathcal{W}} \mathcal{V}_f$.

In the proof presented here we will focus on the arguments needed to generalize the corresponding commutator estimate of Corollary 10 from [Du18] to the present statement. More details can be found in Appendix A of [Du18].

Proof. We only consider the outgoing case $\tau > 0$ and note that it is sufficient to prove (82) for all $\tau \geq \tau_0$ with some fixed $\tau_0 > 0$. Let us begin by assuming for simplicity that $y = 0$. For $\delta > 0$ we obtain families of wedge-local approximants $B_\tau^{(\delta)}$, $B_\tau^{\perp(\delta)}$ to the respective Haag–Ruelle operators via Lemma 18. These satisfy $B_\tau^{(\delta)} \in \mathfrak{A}(\tau\mathcal{V}_f + \mathscr{C}_{\delta|\tau|} + \mathcal{W})$, with $\|B_\tau^{(\delta)} - B_\tau(f)\| \leq C_M^{\delta}/(1+|\tau|^M)$, and analogously $B_\tau^{\perp(\delta)} \in \mathfrak{A}(\tau\mathcal{V}_{f^{\perp}} + \mathscr{C}_{\delta|\tau|} + \mathcal{W}^{\perp})$, with $\|B_\tau^{\perp(\delta)} - B_\tau^{\perp}(f^{\perp})\| \leq C_M'^{\delta}/(1+|\tau|^M)$ with constants $C_M^{\delta}, C_M'^{\delta} > 0$, provided for all $M \in \mathbb{N}$ by Lemma 18.

Proceeding towards (82) we obtain from translation covariance that $\|\alpha_x(B_\tau^{(\delta)}) - \alpha_x(B_\tau(f))\| \leq C_M'^{\delta}/(1+|\tau|^M)$ and $\alpha_x(B_\tau^{(\delta)}) \in \mathfrak{A}(\tau\mathcal{V}_f + \mathscr{C}_{\delta|\tau|} + \mathcal{W} + x)$. To obtain (82) by means of the wedge-locality of these two approximating operators we have to choose $\delta > 0$ and subsequently $\rho > 0$ sufficiently small so that the localization regions are space-like separated. These causality considerations can be simplified for large enough τ by absorbing any finite translations into the growing double cones and rewriting as a region involving a single growing double cone: $\mathscr{C}_{\delta\tau} + x \subseteq \mathscr{C}_{\delta\tau} + \mathscr{C}_{\rho\tau} \subseteq \mathscr{C}_{(\delta+\rho)\tau}$. Further we can write $\mathcal{W} = \mathcal{W}_c + x_{\mathcal{W}}$, $\mathcal{W}^{\perp} = \mathcal{W}_c' + x_{\mathcal{W}^{\perp}}$ for some $x_{\mathcal{W}}, x_{\mathcal{W}^{\perp}} \in \mathbb{R}^d$. Choosing for simplicity $\rho = \delta$ and assuming $\tau > \tau_0 := (|x_{\mathcal{W}}|_c + \left|x_{\mathcal{W}^{\perp}}\right|_c)/\delta$ we obtain

$$\begin{aligned} M_1^{\tau} &:= \tau\mathcal{V}_f + \mathscr{C}_{\delta\tau} + \mathcal{W} + x \subseteq \tau\mathcal{V}_f + \mathscr{C}_{3\delta\tau} + \mathcal{W}_c, \text{ and} \\ M_2^{\tau} &:= \tau\mathcal{V}_{f^{\perp}} + \mathscr{C}_{\delta\tau} + \mathcal{W}^{\perp} \subseteq \tau\mathcal{V}_{f^{\perp}} + \mathscr{C}_{3\delta\tau} + \mathcal{W}_c'. \end{aligned} \tag{83}$$

By the ordering assumption we have that $\mathcal{V}_f - \mathcal{V}_{f^{\perp}}$ is a compact subset of the open set $\mathcal{W}_c$. In particular there exists an $\epsilon > 0$ with $\mathcal{V}_f - \mathcal{V}_{f^{\perp}} + \mathscr{C}_{\epsilon} \subseteq \mathcal{W}_c$. Then for $\tau > 0$ also $\tau\mathcal{V}_f - \tau\mathcal{V}_{f^{\perp}} + \mathscr{C}_{\epsilon\tau} \subseteq \mathcal{W}_c$. Thus we can choose $\rho = \delta := \epsilon/6$ and it then follows that the two sets M_1^{τ} and M_2^{τ} are space-like separated for $\tau > \tau_0$, because for any $x_1 \in M_1^{\tau}$, $x_2 \in M_2^{\tau}$ we have

$$x_1 - x_2 \in \tau(\mathcal{V}_f - \mathcal{V}_{f^{\perp}}) + \mathscr{C}_{6\delta\tau} + \mathcal{W}_c - \mathcal{W}_c' \subseteq \mathcal{W}_c,$$

where we used that $-\mathcal{W}_c' = \mathcal{W}_c$ and $\mathcal{W}_c + \mathcal{W}_c = \mathcal{W}_c$ contains only space-like vectors.

We now obtain from locality that for all $\tau \geq \tau_0$ and $x \in \mathscr{C}_{\rho\tau}$ we have $[B_\tau^{\perp(\delta)}, \alpha_x(B_\tau^{(\delta)})] = 0$, which implies the uniform commutator estimate by expanding

$$\left\| [B_\tau^{\perp}(f^{\perp}), \alpha_x(B_\tau(f))] \right\| = \left\| [B_\tau^{\perp}(f^{\perp}) - B_\tau^{\perp(\delta)} + B_\tau^{\perp(\delta)}, \alpha_x(B_\tau(f) - B_\tau^{(\delta)} + B_\tau^{(\delta)})] \right\|$$

$$\le \left\| [B_\tau^\perp(f^\perp) - B_\tau^{\perp(\delta)}, \alpha_x(B_\tau(f) - B_\tau^{(\delta)} + B_\tau^{(\delta)})] \right\| + \left\| [B_\tau^{\perp(\delta)}, \alpha_x(B_\tau(f) - B_\tau^{(\delta)})] \right\| + \left\| [B_\tau^{\perp(\delta)}, \alpha_x(B_\tau^{(\delta)})] \right\|, \quad (84)$$

where $\|[B_\tau^\perp(f^\perp) - B_\tau^{\perp(\delta)}, \alpha_x(B_\tau(f))]\| \le 2C_{N'}^\delta C/(1+|\tau|^{N'}) \cdot (1+|\tau|)^{s/2} \le C'_N \tau^{-N}$ by estimating the commutator via the two operator norms, using Lemma 18 for its first and the standard polynomially growing norm estimate for its second argument, and analogously for the other non-vanishing commutator.

Finally, if $y \neq 0$ we can write by translation-covariance of the operator norm that

$$\left\| \left[\alpha_y(B_\tau^\perp(f^\perp)), \alpha_x(B_\tau(f)) \right] \right\| = \left\| \left[B_\tau^\perp(f^\perp), \alpha_{x-y}(B_\tau(f)) \right] \right\| \le C_N/(1+|\tau|^N), \quad (85)$$

for all $x - y \in \mathscr{C}_{\rho\tau}$. We can thus set $\tilde{\rho} := \rho/2$ to obtain (85) for all $x, y \in \mathscr{C}_{\tilde{\rho}\tau}$, using that then $x - y \in \mathscr{C}_{\rho\tau/2} - \mathscr{C}_{\rho\tau/2} \subseteq \mathscr{C}_{\rho\tau}$. This establishes (82).

Concerning the extensions to adjoints and $\tau < 0$ we note that the commutator estimates involving adjoints follow using the same approximation argument, noting that Lemma 18 also directly yields wedge-local approximants of adjoint operators by the C^*-property of the operator norm. For $\tau < 0$ the geometric situation in (83) is inverted, but the same arguments work if the opposite ordering $\mathcal{V}_{f^\perp} \succ_{\mathcal{W}} \mathcal{V}_f$ holds. □

Proof of Lemma 17. For the outgoing case, let $\rho = \rho_n > 0$ be the minimum over the constants $\rho_{k,j}$ from Lemma 19 for the pairs $B_{k\tau}(f_k)$ and $B_{j\tau}^\perp(f_j)$, $1 \le k < j \le n$.

For $n = 1$, estimate (80) follows directly from the fact that $\left\| \Psi_1^{x_1} \right\| := \left\| \alpha_{x_1}(B_\tau(f))\Omega \right\| = \|U(x_1)B_\tau(f)\Omega\|$ does not depend on the translation vector x_1 due to translation invariance of the norm and the vacuum, neither on τ by construction of the Haag–Ruelle operators.

For the case $n \ge 2$, let $\Psi_n^{\underline{x}_n}(\tau) := \alpha_{x_1}(B_{1\tau}(f_1)) \ldots \alpha_{x_n}(B_{n\tau}(f_n))\Omega$ with $\underline{x}_n := (x_1, \ldots, x_n) \in \mathbb{R}^{nd}$. To simplify the notation we will drop the obvious wave packet dependences and write $B_{k\tau}^{x_k} := \alpha_{x_k}(B_{k\tau}(f_k))$, so that $\Psi_n^{\underline{x}_n}(\tau) = B_{1\tau}^{x_1} \ldots B_{n\tau}^{x_n}\Omega$ and for later use $B_{k\tau}^{\perp x_k} := \alpha_{x_k}(B_{k\tau}^\perp(f_k))$. To give an inductive argument we can write using the swapping property

$$\begin{aligned}
\left\| \Psi_n^{\underline{x}_n}(\tau) \right\|^2 &= \langle \Omega, B_{n\tau}^{x_n *} \ldots B_{1\tau}^{x_1 *} B_{1\tau}^{x_1} \ldots B_{n\tau}^{x_n} \Omega \rangle \\
&= \left\langle \Omega, B_{n\tau}^{x_n *} \ldots B_{1\tau}^{x_1 *} B_{1\tau}^{x_1} \ldots B_{n-1,\tau}^{x_{n-1}} B_{n\tau}^{\perp x_n} \Omega \right\rangle \\
&= \left\langle \Omega, B_{n\tau}^{x_n *} B_{n\tau}^{\perp x_n} B_{n-1,\tau}^{x_{n-1} *} \ldots B_{1\tau}^{x_1 *} B_{1\tau}^{x_1} \ldots B_{n-1,\tau}^{x_{n-1}} \Omega \right\rangle \\
&\quad + \left\langle \Omega, B_{n\tau}^{x_n *} \left[B_{n-1,\tau}^{x_{n-1} *} \ldots B_{1\tau}^{x_1 *} B_{1\tau}^{x_1} \ldots B_{n-1,\tau}^{x_{n-1}}, B_{n\tau}^{\perp x_n} \right] \Omega \right\rangle. \quad (86)
\end{aligned}$$

The first term can be bounded using the clustering property of the Haag–Ruelle operators [Du18, Prop. 8 (vi)], which yields

$$\begin{aligned}
&\left\langle \Omega, B_{n\tau}^{x_n *} B_{n\tau}^{\perp x_n} B_{n-1,\tau}^{x_{n-1} *} \ldots B_{1\tau}^{x_1 *} B_{1\tau}^{x_1} \ldots B_{n-1,\tau}^{x_{n-1}} \Omega \right\rangle \\
&\quad = \left\langle \Omega, B_{n\tau}^{x_n *} B_{n\tau}^{\perp x_n} \Omega \right\rangle \left\langle \Omega, B_{n-1,\tau}^{x_{n-1} *} \ldots B_{1\tau}^{x_1 *} B_{1\tau}^{x_1} \ldots B_{n-1,\tau}^{x_{n-1}} \Omega \right\rangle \\
&\quad = \left\| B_{n\tau}^{x_n} \Omega \right\|^2 \|\Psi_{n-1}^{\underline{x}_{n-1}}(\tau)\|^2 \le C. \quad (87)
\end{aligned}$$

Here the estimate is obtained uniformly for all $x_1, \dots, x_n \in \mathscr{C}_{\rho_n \tau}$ by using that $x_1, \dots, x_{n-1} \in \mathscr{C}_{\rho_n \tau} \subseteq \mathscr{C}_{\rho_{n-1}\tau}$ due to the induction hypothesis, and bounding the first factor as for $n = 1$. Note that here we use $\rho_n \leq \rho_{n-1}$, which follows directly by the definition

$$\rho_n = \min_{1 \leq k < j \leq n} \rho_{k,j} \leq \min_{1 \leq k < j \leq n-1} \rho_{k,j} = \rho_{n-1}, \tag{88}$$

where $\rho_{k,j}$ denote the constants from Lemma 19 for the pairs $B_{k\tau}(f_k)$ and $B^{\perp}_{j\tau}(f_j)$, $1 \leq k < j \leq n$.

Finally, the commutator term from (86) is also seen to be bounded in τ (in fact, rapidly decreasing), uniformly for $x_1, \dots, x_n \in \mathscr{C}_{\rho_n \tau}$, by means of Lemma 19: we expand the big commutator in (86) into a sum of terms of vacuum expectation values of operators of the form

$$\begin{aligned} &B^{x_n *}_{n\tau} \dots B^{x_1 *}_{1\tau} B^{x_1}_{1\tau} \dots \left[B^{x_k}_{k\tau}, B^{\perp x_n}_{n\tau} \right] \dots B^{x_{n-1}}_{n-1,\tau}, \quad \text{or} \\ &\qquad B^{x_n *}_{n\tau} \dots \left[B^{x_k *}_{k\tau}, B^{\perp x_n}_{n\tau} \right] \dots B^{x_1 *}_{1\tau} B^{x_1}_{1\tau} \dots B^{x_{n-1}}_{n-1,\tau}. \end{aligned} \tag{89}$$

Each of those terms is now estimated using that $\left\| \left[B^{x_k *}_{k\tau}, B^{\perp x_n}_{n\tau} \right] \right\|$ and $\left\| \left[B^{x_k}_{k\tau}, B^{\perp x_n}_{n\tau} \right] \right\|$ can for $1 \leq k < n$ and $x_k, x_n \in \mathscr{C}_{\rho_n \tau}$ by definition of ρ_n be bounded by $C_M (1 + \tau)^{-M}$. Choosing $M \in \mathbb{N}$ sufficiently large we absorb the growth of the simple norm estimates $\| B^{x_j *}_{j\tau} \| = \| B^{x_j}_{j\tau} \| \leq C_j (1 + |\tau|^{s/2})$ used for the remaining operators with $1 \leq j \leq n$ in the respective terms of the expansion. Together the desired uniform bound on (86) is obtained. □

Proposition 20. *Let $\Psi_1, \Psi_2 \in \Gamma^u(\mathscr{H}_1)$ be vectors of bounded energy-momentum. Then for any warping matrix $Q \in \mathbb{R}^{d \times d}$ their Q-deformed tensor product has the oscillatory integral representation*

$$\Psi_1 \otimes_Q \Psi_2 = \lim_{\epsilon \to 0} \int \frac{\mathrm{d}^d x \, \mathrm{d}^d y}{(2\pi)^d} \eta(\epsilon x, \epsilon y) \mathrm{e}^{-\mathrm{i} x \cdot y} U(x) \{ (U(Qy)\Psi_1) \otimes \Psi_2 \}, \tag{90}$$

where $\eta \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\eta(0,0) = 1$.

Proof. First we can infer from Lemma 12 that the limit in (90) exists and is independent of η within the specified restrictions. We rewrite the expression under the limit on the right-hand side via the spectral calculus of the energy-momentum operators as

$$\begin{aligned} &\int \frac{\mathrm{d}^d x \, \mathrm{d}^d y}{(2\pi)^d} \eta(\epsilon x, \epsilon y) \mathrm{e}^{-\mathrm{i} x \cdot y} \int \mathrm{d}E(p) \, \mathrm{e}^{\mathrm{i} p \cdot x} \left(\left(\int \mathrm{d}E(q) \, \mathrm{e}^{\mathrm{i} q \cdot Qy} \Psi_1 \right) \otimes \Psi_2 \right) \\ &\quad = \int \mathrm{d}E_P(p) \mathrm{d}E_{P_1}(q) \left(\Psi_1 \otimes \Psi_2 \cdot \int \frac{\mathrm{d}^d x \, \mathrm{d}^d y}{(2\pi)^d} \eta(\epsilon x, \epsilon y) \mathrm{e}^{-\mathrm{i} x \cdot y + \mathrm{i} p \cdot x + \mathrm{i} q \cdot Qy} \right). \end{aligned} \tag{91}$$

Here the Fubini theorem for exchanging the order of integrations applies, where integrability with respect to the product measure follows from the bounded energy-momentum of Ψ_1, Ψ_2 and the rapid decay of η. Here P denote the energy-momentum operators on the full Fock space and $P_1(\Psi_1 \otimes \Psi_2) := (P\Psi_1) \otimes \Psi_2$ acts only on the first component. Let us denote the inner scalar integral from (91) by $I_\epsilon(p, q)$. By the uniqueness result of Lemma 12 we can proceed to concretely choose $\eta(x, y) := \mathrm{e}^{-|x|_e^2 - |y|_e^2}$, where $|\cdot|_e$

denote the Euclidean norm. Then an elementary calculation (see Proposition 21) shows that $I_\epsilon(p,q) \longrightarrow \mathrm{e}^{-\mathrm{i}p\cdot Qq}$ pointwise for $\epsilon \to 0$, and in addition $I_\epsilon(p,q)$ is bounded uniformly in p, q for small $\epsilon > 0$. By dominated convergence we obtain that (91) yields for $\epsilon \to 0$

$$\int \mathrm{d}E_P(p)\,\mathrm{d}E_{P_1}(q)\ \Psi_1 \otimes \Psi_2 \cdot \mathrm{e}^{-\mathrm{i}p\cdot Qq} = \mathrm{e}^{-\mathrm{i}P\cdot QP_1}\Psi_1 \otimes \Psi_2 = \mathrm{e}^{-\mathrm{i}P_2\cdot QP_1}\Psi_1 \otimes \Psi_2$$
$$= \mathrm{e}^{\mathrm{i}P_1\cdot QP_2}\Psi_1 \otimes \Psi_2 = \Psi_1 \otimes_Q \Psi_2, \tag{92}$$

where we used that $P = P_1 + P_2$ and $P_1 \cdot QP_1 = 0$. □

Proposition 21. *For any $p, q \in \mathbb{R}^d$ and any warping matrix $Q \in \mathbb{R}^{d\times d}$ we have*

$$\lim_{\epsilon\to 0}\int \frac{\mathrm{d}^d x\ \mathrm{d}^d y}{(2\pi)^d}\mathrm{e}^{-|\epsilon x|_e^2-|\epsilon y|_e^2}\mathrm{e}^{-\mathrm{i}x\cdot y+\mathrm{i}p\cdot x+\mathrm{i}q\cdot Qy} = \mathrm{e}^{-\mathrm{i}p\cdot Qq}. \tag{93}$$

Proof. We let $\varepsilon := \epsilon^2$ and use anti-symmetry of Q with respect to the Minkowski scalar product to write

$$\int \frac{\mathrm{d}^d x\ \mathrm{d}^d y}{(2\pi)^d}\mathrm{e}^{-\varepsilon|x|_e^2-\varepsilon|y|_e^2}\ \mathrm{e}^{-\mathrm{i}x\cdot y+\mathrm{i}p\cdot x+\mathrm{i}q\cdot Qy} = \int \frac{\mathrm{d}^d x\ \mathrm{d}^d y}{(2\pi)^d}\mathrm{e}^{-\varepsilon|x|_e^2-\varepsilon|y|_e^2}\ \mathrm{e}^{-\mathrm{i}x\cdot y+\mathrm{i}p\cdot x-\mathrm{i}(Qq)\cdot y}$$
$$=: J_\varepsilon^d(p, Qq). \tag{94}$$

Let us express the Minkowski products in terms of ordinary scalar products involving the Minkowski metric g. Then we can calculate component-wise by Fubini,

$$J_\varepsilon^d(p,p') = \int \frac{\mathrm{d}^d x\ \mathrm{d}^d y}{(2\pi)^d}\mathrm{e}^{-\varepsilon|x|_e^2-\varepsilon|y|_e^2}\ \mathrm{e}^{-\mathrm{i}x^T gy+\mathrm{i}p^T gx-\mathrm{i}p'^T gy} = \prod_{\mu=0}^{d-1} J_\varepsilon^1(p^\mu, p'_\mu), \tag{95}$$

with

$$J_\varepsilon^1(p,p') := \int \frac{\mathrm{d}x\ \mathrm{d}y}{2\pi}\mathrm{e}^{-\varepsilon x^2-\varepsilon y^2}\mathrm{e}^{-\mathrm{i}(xy+px-p'y)}. \tag{96}$$

Here we have substituted $x = x'_\mu := g_{\mu\nu}x^\nu$. By elementary calculation one obtains

$$J_\varepsilon^1(p,p') = \frac{1}{\sqrt{1+4\varepsilon^2}}\mathrm{e}^{\frac{-\mathrm{i}pp'-\varepsilon(p^2+p'^2)}{1+4\varepsilon^2}} \overset{\varepsilon\searrow 0}{\to} \mathrm{e}^{-\mathrm{i}pp'} \tag{97}$$

from which the claim follows. □

Proof of Theorem 11. We consider the outgoing case for a fixed wedge $\mathcal{W}$. By definition the linear combinations of ordered product states

$$\Psi_n = \Psi_1^1 \otimes \ldots \otimes \Psi_1^n, \quad \Psi_1^1 \succ_{\mathcal{W}} \ldots \succ_{\mathcal{W}} \Psi_1^n, \quad (n \in \mathbb{N}) \tag{98}$$

are dense in $\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1)$. Further by Proposition 9 such states can be approximated with arbitrarily small error by vectors

$$\tilde{\Psi}_n = \tilde{\Psi}_1^1 \otimes \ldots \otimes \tilde{\Psi}_1^n = B_{1\tau}(f_1)\Omega \otimes \ldots \otimes B_{n\tau}(f_n)\Omega$$
$$= B_{1Q\tau}(f_1)\Omega \otimes \ldots \otimes B_{nQ\tau}(f_n)\Omega, \quad \mathcal{V}_{f_1} \succ_{\mathcal{W}} \ldots \succ_{\mathcal{W}} \mathcal{V}_{f_n}, \tag{99}$$

generated by swappable Haag–Ruelle approximants $B_{k\tau}(f_k)\Omega = B^{\perp}_{k\tau}(f_k)\Omega = \tilde{\Psi}^k_1 \in \mathscr{H}^{\mathcal{W}}_{1c}$, which are constructed from regular $A_k \in \mathfrak{A}^{0r}(\mathcal{W})$, $A^{\perp}_k \in \mathfrak{A}^{0r}(\mathcal{W}^{\perp})$. Here the further restriction from swappable states to swappable states generated by regular operators is used to assure that the warped convolutions are well defined and we have just seen that the linear combinations of the vectors (99) are also dense in $\Gamma^{\succ\mathcal{W}}(\mathscr{H}_1)$ even with this additional smoothness requirement. By Lemma 3 (i) it is clear that they yield the same one-particle vectors $B_{kQ\tau}(f_k)\Omega = B^{\perp}_{kQ\tau}(f_k)\Omega = \tilde{\Psi}^k_1$ for all $1 \leq k \leq n$. For simplicity of notation we will be dropping the tilde for the remainder of the proof.

By definition of the wave operator we now obtain

$$\mathbb{W}^{+}_{Q,\mathcal{W}}\Psi_n = \lim_{\tau\to\infty} B_{1Q\tau}(f_1)\ldots B_{nQ\tau}(f_n)\Omega = \mathbb{W}^{+}_{0,\mathcal{W}}\Psi^1_1 \otimes_Q \ldots \otimes_Q \Psi^n_1, \quad (100)$$

where the last equality follows from Lemma 16. By induction we obtain that

$$\begin{aligned}\Psi^1_1 \otimes_Q \ldots \otimes_Q \Psi^n_1 &= \Psi^1_1 \otimes_Q (\Psi^2_1 \otimes_Q \ldots \otimes_Q \Psi^n_1) = \Psi^1_1 \otimes_Q S_{Q\mathcal{W}}(\Psi^2_1 \otimes \ldots \otimes \Psi^n_1)\\ &= e^{iP_1\cdot Q_{\mathcal{W}}(P_2+\ldots+P_n)}\Psi^1_1 \otimes S_{Q\mathcal{W}}(\Psi^2_1 \otimes \ldots \otimes \Psi^n_1)\\ &= S_{Q\mathcal{W}}(\Psi^1_1 \otimes \Psi^2_1 \otimes \ldots \otimes \Psi^n_1). \end{aligned} \quad (101)$$

Hence the claimed identity holds on a dense subspace and thus by continuity of the wave operators and S_Q on the full domain $\Gamma^{\succ\mathcal{W}}(\mathscr{H}_1)$ of $\mathbb{W}^{+}_{Q,\mathcal{W}}$. The argument for $\mathbb{W}^{-}_{Q,\mathcal{W}}$ is analogous. □

3.3. Scattering data and wedge-transition matrix elements. The two-particle S-matrix of higher-dimensional deformed or GL-type models was worked out in [GL07,BS08] for any fixed wedge $\mathcal{W}$. Even if the initial and thus also the deformed model were Poincaré covariant, these authors observed that the two-particle S-matrix is in fact not fully Lorentz covariant in higher dimensions $d > 1 + 1$. In [Du18] it was proposed to make this observation more precise by specifying the dependence of wave operators and all related asymptotic data on the localization wedge $\mathcal{W}$ of the Haag–Ruelle operators explicitly. Due to translation covariance of the S-matrix this reduces to a dependence on a wedge modulo translations, or equivalently, a dependence on a centered wedge.

Definition 22. Let $\mathcal{W}_{\rm f}$, $\mathcal{W}_{\rm i}$ be centered wedges. Following [Du18] we define S-matrices of the initial and deformed wQFT model, respectively, as maps between the incoming ordered Fock space $\Gamma^{\prec\mathcal{W}_{\rm i}}(\mathscr{H}_1)$ and the outgoing space $\Gamma^{\succ\mathcal{W}_{\rm f}}(\mathscr{H}_1)$ by

$$S^{\mathcal{W}_{\rm f}\mathcal{W}_{\rm i}}_{0,{\rm f\,i}} := (\mathbb{W}^{+}_{0,\mathcal{W}_{\rm f}})^*\mathbb{W}^{-}_{0,\mathcal{W}_{\rm i}}, \qquad S^{\mathcal{W}_{\rm f}\mathcal{W}_{\rm i}}_{Q,{\rm f\,i}} := (\mathbb{W}^{+}_{Q,\mathcal{W}_{\rm f}})^*\mathbb{W}^{-}_{Q,\mathcal{W}_{\rm i}}. \quad (102)$$

Similarly we define wedge-transition maps between two final or two initial states, respectively, as

$$\begin{aligned} S^{\mathcal{W}_2\mathcal{W}_1}_{0,{\rm f\,f}} &:= (\mathbb{W}^{+}_{0,\mathcal{W}_2})^*\mathbb{W}^{+}_{0,\mathcal{W}_1}, \qquad & S^{\mathcal{W}_2\mathcal{W}_1}_{Q,{\rm f\,f}} &:= (\mathbb{W}^{+}_{Q,\mathcal{W}_2})^*\mathbb{W}^{+}_{Q,\mathcal{W}_1},\\ S^{\mathcal{W}_2\mathcal{W}_1}_{0,{\rm i\,i}} &:= (\mathbb{W}^{-}_{0,\mathcal{W}_2})^*\mathbb{W}^{-}_{0,\mathcal{W}_1}, \qquad & S^{\mathcal{W}_2\mathcal{W}_1}_{Q,{\rm i\,i}} &:= (\mathbb{W}^{-}_{Q,\mathcal{W}_2})^*\mathbb{W}^{-}_{Q,\mathcal{W}_1}, \end{aligned} \quad (103)$$

for any two centered wedges $\mathcal{W}_1$, $\mathcal{W}_2$.

As a direct consequence of Theorem 11 we obtain similar expressions for the S-matrices and wedge-transition matrix elements in deformed wedge-local models.

Corollary 23. *Let $\mathcal{W}_{\mathrm{f}}, \mathcal{W}_{\mathrm{i}}, \mathcal{W}_1, \mathcal{W}_2$ be arbitrary centered wedges. The S-matrices and wave operators of a BLS-deformed wQFT model can be expressed in terms of the corresponding objects of the undeformed model:*

(i) $S^{\mathcal{W}_{\mathrm{f}}\mathcal{W}_{\mathrm{i}}}_{Q,\mathrm{f\,i}} = (S^{\succ\mathcal{W}_{\mathrm{f}}}_{Q_{\mathcal{W}_{\mathrm{f}}}})^* S^{\mathcal{W}_{\mathrm{f}}\mathcal{W}_{\mathrm{i}}}_{0,\mathrm{f\,i}} S^{\prec\mathcal{W}_{\mathrm{i}}}_{Q_{\mathcal{W}_{\mathrm{i}}}}$,

(ii) $S^{\mathcal{W}_2\mathcal{W}_1}_{Q,\mathrm{f\,f}} = (S^{\succ\mathcal{W}_2}_{Q_{\mathcal{W}_2}})^* S^{\mathcal{W}_2\mathcal{W}_1}_{0,\mathrm{f\,f}} S^{\succ\mathcal{W}_1}_{Q_{\mathcal{W}_1}}$, *and* $S^{\mathcal{W}_2\mathcal{W}_1}_{Q,\mathrm{i\,i}} = (S^{\prec\mathcal{W}_2}_{Q_{\mathcal{W}_2}})^* S^{\mathcal{W}_2\mathcal{W}_1}_{0,\mathrm{i\,i}} S^{\prec\mathcal{W}_1}_{Q_{\mathcal{W}_1}}$.

Proof. We obtain the expression for the S-matrix (i) from the short computation

$$\begin{aligned} S^{\mathcal{W}_f\mathcal{W}_i}_{Q,\mathrm{f\,i}} &= (\mathbb{W}^+_{Q,\mathcal{W}_{\mathrm{f}}})^*\mathbb{W}^-_{Q,\mathcal{W}_{\mathrm{i}}} = (\mathbb{W}^+_{0,\mathcal{W}_{\mathrm{f}}} S^{\prec\mathcal{W}_{\mathrm{f}}}_Q)^*(\mathbb{W}^-_{0,\mathcal{W}_{\mathrm{i}}} S^{\succ\mathcal{W}_{\mathrm{i}}}_Q) \\ &= (S^{\prec\mathcal{W}_{\mathrm{f}}}_Q)^*(\mathbb{W}^+_{0,\mathcal{W}_{\mathrm{f}}})^*\mathbb{W}^-_{0,\mathcal{W}_{\mathrm{i}}} S^{\succ\mathcal{W}_{\mathrm{i}}}_Q \\ &= (S^{\prec\mathcal{W}_{\mathrm{f}}}_Q)^* S^{\mathcal{W}_{\mathrm{f}}\mathcal{W}_{\mathrm{i}}}_{0,\mathrm{f\,i}} S^{\succ\mathcal{W}_{\mathrm{i}}}_Q. \end{aligned} \tag{104}$$

The calculations for the wedge-transition formulas (ii) are analogous. □

Remark 24. Expressions (i) and (ii) here simplify further by noting that it follows immediately from the definition of $S_{Q_\mathcal{W}}$ and its restrictions $S^{\prec\mathcal{W}/\succ\mathcal{W}}_{Q_\mathcal{W}}$ that

$$(S^{\succ\mathcal{W}}_{Q_\mathcal{W}})^* = S^{\succ\mathcal{W}}_{-Q_\mathcal{W}} = S^{\succ\mathcal{W}}_{Q_{\mathcal{W}'}}, \tag{105}$$

where in the last equality the definition of $Q_\mathcal{W}$ was used. This further implies that these deformation maps are unitary, by writing

$$(S^{\succ\mathcal{W}}_{Q_\mathcal{W}})^* S^{\succ\mathcal{W}}_{Q_\mathcal{W}} = S^{\succ\mathcal{W}}_{-Q_\mathcal{W}} S^{\succ\mathcal{W}}_{Q_\mathcal{W}} = S^{\succ\mathcal{W}}_{-Q_\mathcal{W}+Q_\mathcal{W}} = \mathbb{1}, \tag{106}$$

and analogously $S^{\succ\mathcal{W}}_{Q_\mathcal{W}}(S^{\succ\mathcal{W}}_{Q_\mathcal{W}})^* = \mathbb{1}$.

Let us note that we can further identify the scattering data for opposite wedges by making use of the swapping property. For this consideration it is immaterial whether we are in the deformed or undeformed model, so we will drop the corresponding indices from the wave operators. We start with an outgoing scattering state given by

$$\begin{aligned} \Psi^+_n &= \lim_{\tau\to\infty} B_{1\tau}(f_1)\ldots B_{n\tau}(f_n)\Omega \\ &= \mathbb{W}^+_\mathcal{W}(B_{1\tau}(f_1)\Omega)\otimes\ldots\otimes(B_{n\tau}(f_n)\Omega) \in \mathbb{W}^+_\mathcal{W}\Gamma^{\succ\mathcal{W}}(\mathscr{H}_1). \end{aligned} \tag{107}$$

By an analogous argument as in (86) and (89) we write

$$\begin{aligned} B_{1\tau}(f_1)\ldots B_{n\tau}(f_n)\Omega &= B_{1\tau}(f_1)\ldots B_{n-1\tau}(f_{n-1})B^\perp_{n\tau}(f_n)\Omega \\ &= B^\perp_{n\tau}(f_n)B_{1\tau}(f_1)\ldots B_{n-1\tau}(f_{n-1})\Omega + (\text{commutators}), \end{aligned} \tag{108}$$

where the commutator terms are rapidly decreasing faster than any polynomial in $\tau > 0$. Iterating this swapping argument we obtain

$$\begin{aligned} \Psi^+_n &= \lim_{\tau\to\infty} B^\perp_{n\tau}(f_n)\ldots B^\perp_{1\tau}(f_1)\Omega \\ &= \mathbb{W}^+_{\mathcal{W}'}(B_{n\tau}(f_n)\Omega)\otimes\ldots\otimes(B_{1\tau}(f_1)\Omega) \in \mathbb{W}^+_{\mathcal{W}'}\Gamma^{\succ\mathcal{W}'}(\mathscr{H}_1). \end{aligned} \tag{109}$$

In the last equality we already used that by swapping $B^\perp_{k\tau}(f_k)\Omega = B_{k\tau}(f_k)\Omega$ and that

$$\mathcal{V}_{f_1} \succ_\mathcal{W} \mathcal{V}_{f_2} \succ_\mathcal{W} \ldots \succ_\mathcal{W} \mathcal{V}_{f_n} \Longleftrightarrow \mathcal{V}_{f_n} \succ_{\mathcal{W}'} \mathcal{V}_{f_{n-1}} \succ_{\mathcal{W}'} \ldots \succ_{\mathcal{W}'} \mathcal{V}_{f_1} \tag{110}$$

by the definition of the precursor relation and from the fact that $\mathcal{W}' = -\mathcal{W}$ for any centered wedge $\mathcal{W}$. Let us therefore define $Z : \Gamma^u(\mathscr{H}_1) \to \Gamma^u(\mathscr{H}_1)$ by its action on n-particle states,

$$Z\Psi_1^1 \otimes \ldots \otimes \Psi_1^n := \Psi_1^n \otimes \ldots \otimes \Psi_1^1, \tag{111}$$

for any $n \in \mathbb{N}$ and $\Psi_1^1, \ldots, \Psi_1^n \in \mathscr{H}_1$ and we note that Z is a unitary and self-adjoint involution. Summarizing these considerations we obtain:

Proposition 25. *For any centered wedge $\mathcal{W}$ we have as subspaces of $\Gamma^u(\mathscr{H}_1)$,*

(i) $\Gamma^{\succ_{\mathcal{W}'}}(\mathscr{H}_1) = \Gamma^{\prec_{\mathcal{W}}}(\mathscr{H}_1)$,
(ii) $Z\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) = \Gamma^{\succ_{\mathcal{W}'}}(\mathscr{H}_1)$, *and analogously* $Z\Gamma^{\prec_{\mathcal{W}}}(\mathscr{H}_1) = \Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1)$.

Proposition 26. *For any centered wedge $\mathcal{W}$ the wave operators associated to complementary wedges can be identified by*

$$\mathbb{W}^+_{\mathcal{W}} = \mathbb{W}^+_{\mathcal{W}'}Z, \qquad \mathbb{W}^-_{\mathcal{W}} = \mathbb{W}^-_{\mathcal{W}'}Z. \tag{112}$$

This identification appears to take us somewhat away from the localization properties appearing in the Haag–Ruelle construction. Yet it shows that the scattering matrices for the two simplest choices $\mathcal{W}_\mathrm{f} = \mathcal{W}_\mathrm{i}$ and $\mathcal{W}_\mathrm{f} = \mathcal{W}_\mathrm{i}'$ contain the same scattering theoretic data. These choices correspond to ones made in [GL07,BS08] for the analysis of two-particle scattering. In these works asymptotic two-particle states are constructed following more closely the methods of Haag–Ruelle theory from local QFT. Hence a mixed localization is used, where

$$\Psi_2^+ = \lim_{\tau\to\infty} B_{1\tau}^{\perp}(f_1)B_{2\tau}(f_2)\Omega = \lim_{\tau\to\infty} B_{2\tau}(f_2)B_{1\tau}^{\perp}(f_1)\Omega, \quad \mathcal{V}_{f_2} \succ_{\mathcal{W}} \mathcal{V}_{f_1}. \tag{113}$$

The independence of the operator order is reminiscent of Haag–Ruelle scattering theory for bosonic local QFT. In the general wedge-local scattering theory it is a special feature appearing for the case of two particles. It was already remarked in [Du18] that this definition of two-particle scattering states captures the same information as our analysis restricted to the level of two-particle states. This can be seen by swapping the corresponding sides,

$$\Psi_2^+ = \lim_{\tau\to\infty} B_{1\tau}^{\perp}(f_1)B_{2\tau}^{\perp}(f_2)\Omega = \lim_{\tau\to\infty} B_{2\tau}(f_2)B_{1\tau}(f_1)\Omega. \tag{114}$$

Thereby we have not only illustrated the symmetry of Proposition 26 at the two-particle level. We also see that the compatibility of our analysis with earlier calculations from [GL07,BS08] is in fact a corollary of an intermediate step of the proof of Proposition 26.

4. Asymptotic Completeness of BLS-Deformed Wedge-Local QFT

Given a wedge-local model $(\mathfrak{A}^0, \alpha, \mathscr{H}, \Omega)$, we can now proceed to our second main objective and study the completeness of asymptotic states

$$\begin{aligned}\mathscr{H}^{\pm}_{0,\mathcal{W}} &:= \mathbb{W}^{\pm}_{0,\mathcal{W}}\Gamma^{\succ_{\mathcal{W}}/\prec_{\mathcal{W}}},\\ \mathscr{H}^{\pm}_{Q,\mathcal{W}} &:= \mathbb{W}^{\pm}_{Q,\mathcal{W}}\Gamma^{\succ_{\mathcal{W}}/\prec_{\mathcal{W}}}\end{aligned} \tag{115}$$

in the common Hilbert space $\mathscr{H}$ of the initial and BLS-deformed wQFT model. We will use a notion of asymptotic completeness for wQFT which directly generalizes the corresponding standard definition from local QFT.

Definition 27. We say that a wave operator $\mathbb{W}^{\pm}_{\mathcal{W}}$ of a wQFT model for a centered localization wedge $\mathcal{W}$ is *asymptotically complete* iff the subspace of velocity-ordered scattering states

$$\mathscr{H}^{+}_{\mathcal{W}} := \mathbb{W}^{+}_{\mathcal{W}} \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1), \text{ or}, \qquad \mathscr{H}^{-}_{\mathcal{W}} := \mathbb{W}^{-}_{\mathcal{W}} \Gamma^{\prec \mathcal{W}}(\mathscr{H}_1), \tag{116}$$

respectively, is equal to the full Hilbert space $\mathscr{H}$ of the wQFT. We say that a wQFT model satisfies the property of *ordered asymptotic completeness* (more precisely, *asymptotic completeness with respect to velocity-ordered scattering states*) iff both $\mathbb{W}^{+}_{\mathcal{W}}$ and $\mathbb{W}^{-}_{\mathcal{W}}$ are asymptotically complete for any wedge $\mathcal{W}$.

The results of Sect. 3 give an explicit representation of the deformed wave operators $\mathbb{W}^{\pm}_{Q,\mathcal{W}}$ in terms of the undeformed wave operators $\mathbb{W}^{\pm}_{0,\mathcal{W}}$. This directly yields a general result regarding the stability of asymptotic completeness of wedge-local theories under BLS-deformations.

Theorem 28. *A wave operator of a deformed wQFT model $\mathbb{W}^{\pm}_{Q,\mathcal{W}}$ is asymptotically complete if and only if the wave operator of the underlying "undeformed" model $\mathbb{W}^{\pm}_{0,\mathcal{W}}$ is asymptotically complete.*

Proof. By Theorem 11, we have $\mathbb{W}^{+}_{Q,\mathcal{W}} = \mathbb{W}^{+}_{0,\mathcal{W}} S^{\succ \mathcal{W}}_{Q\mathcal{W}}$, and $S^{\succ \mathcal{W}}_{Q\mathcal{W}} \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1) = \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1)$ by unitarity of $S^{\succ \mathcal{W}}_{Q\mathcal{W}}$ (see Remark 24). Hence

$$\mathbb{W}^{+}_{Q,\mathcal{W}} \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1) = \mathbb{W}^{+}_{0,\mathcal{W}} S^{\succ \mathcal{W}}_{Q\mathcal{W}} \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1) = \mathbb{W}^{+}_{0,\mathcal{W}} \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1) \tag{117}$$

yields the equivalence of ordered asymptotic completeness of deformed and initial model for outgoing states. The argument for the incoming case is analogous. □

In the following final section we will discuss the application of our results to GL-type models, which are constructed by applying BLS-deformations to a free field. For the scattering theoretic analysis of these models we have to work with only wedge-ordered states, as stated explicitly in (116) and dictated by the scope of the wedge-local scattering theory. In GL-type models we will see this restriction to wedge-ordered states is inessential for the particle interpretation and still yields a dense set of scattering states. This may of course be expected on grounds of the bosonic statistics of the underlying free theory. Let us also note that there are wedge-local models, which do not satisfy ordered asymptotic completeness at the two-particle level. Examples of such models have been obtained by applying a von Neumann operator-algebraic free product construction to the free field [LTU19].

5. Application to Grosse–Lechner Models

In the work of Grosse and Lechner [GL07] an interesting class of wedge-local models in any space-time dimension $d \geq 1+1$ is constructed in a wedge-local variant of the Wightman framework. A closely related class of models is obtained in the operator-algebraic framework by applying the BLS-deformation construction to the standard scalar free field [BS08,BLS11]. The scalar field has a canonical wedge-local description given by the von Neumann algebras

$$\mathfrak{A}^{0}(\mathcal{W}) := \{W(f) : f \in \mathscr{S}(\mathbb{R}^d, \mathbb{R}),\ \mathrm{supp}\, f \subseteq \mathcal{W}\}'' \tag{118}$$

generated by the Weyl operators $W(f) = \mathrm{e}^{\mathrm{i}\phi(f)}$, where $\phi(f)$ is the standard free scalar Wightman field (see e.g. [Dyb17,Dim,IZ]). Local algebras $\mathfrak{A}^0(\mathcal{O})$ are defined analogously, requiring that supp $f \subseteq \mathcal{O}$ for bounded open regions $\mathcal{O}$ in Minkowski spacetime. The algebras $\mathfrak{A}(\mathcal{W})$ and $\mathfrak{A}(\mathcal{O})$ act on the bosonic Fock space $\mathscr{H} = \Gamma^b(\mathscr{H}_1)$ over the scalar one-particle space $\mathscr{H}_1 = L^2(\mathbb{R}^s)$, and we write $\Omega_F \in \mathscr{H}$ for the Fock vacuum. The net is covariant with respect to the standard second quantized scalar representation of the proper orthochronous Poincaré group. For any non-vanishing warping matrix $Q \in \mathbb{R}^{d\times d}$ of the form (11) we will call the BLS-deformed theory $(\mathfrak{A}^Q, \alpha, \mathscr{H}, \Omega_F)$ a *Grosse–Lechner model*. Let us now apply our general results from Sects. 3 and 4 to evaluate the scattering theoretic content of these models and establish their ordered asymptotic completeness.

It is a well-known fact that in the case of the free field also the conventional local Haag–Ruelle scattering theory applies (see e.g. [A,Dyb17]). It yields wave operators $\mathbb{W}^\pm : \Gamma^b(\mathscr{H}_1) \to \mathscr{H}$, defined on the bosonic Fock space. For the free model we have $\mathscr{H} = \Gamma^b(\mathscr{H}_1)$ and the wave operators are trivial in the sense that $\mathbb{W}^+ = \mathbb{W}^- = \mathbb{1}$.[9] This also directly implies asymptotic completeness of the free theory in the conventional sense with respect to the standard Haag–Ruelle scattering theory. That is, $\mathbb{W}^\pm\Gamma^b(\mathscr{H}_1) = \mathscr{H}$ and in these cases we will say similarly that $\mathbb{W}^\pm$, respectively, are asymptotically complete.

For the free field, the present velocity-ordered wedge-local formalism for scattering theory applies as well. The wedge-local wave operators $\mathbb{W}^\pm_{0,\mathcal{W}}$ can be efficiently determined from their local counterparts in cases for which the latter exist. For this purpose we define the embeddings

$$I^{\succ\mathcal{W}/\prec\mathcal{W}} : \begin{cases} \Gamma^{\succ\mathcal{W}/\prec\mathcal{W}}(\mathscr{H}_1) \longrightarrow \Gamma^b(\mathscr{H}_1), \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto a^*(\Psi_1^1)\ldots a^*(\Psi_1^n)\Omega_F = \sqrt{n!}\cdot \Psi_1^1 \otimes_s \ldots \otimes_s \Psi_1^n, \end{cases} \tag{119}$$

which map wedge-ordered n-particle vectors to the corresponding bosonic symmetrized tensor product. In terms of the norm on $\Gamma^b(\mathscr{H}_1)$ we get

$$\left\| \sqrt{n!}\cdot \Psi_1^1 \otimes_s \ldots \otimes_s \Psi_1^n \right\|^2 = \sum_{\pi\in\mathfrak{S}_n} \prod_{k=1}^n \langle \Psi_1^k, \Psi_1^{\pi(k)} \rangle = \prod_{k=1}^n \left\| \Psi_1^k \right\|^2. \tag{120}$$

In the second equality it is used that the one-particle states $\Psi_1^k \in \mathscr{H}_1$ $(1 \le k \le n)$ with ordered velocity supports are pairwise orthogonal, so that only the identity permutation yields a nonzero contribution. Thus $I^{\succ\mathcal{W}/\prec\mathcal{W}}$ are well defined by linear continuous extension from (119) and yield isometries on $\Gamma^{\succ\mathcal{W}/\prec\mathcal{W}}(\mathscr{H}_1)$.

Theorem 29. *The wedge-local wave operators of a local quantum field theory with isolated mass shell are well-defined and can be expressed using local Haag–Ruelle wave operators* $\mathbb{W}^\pm : \Gamma^b(\mathscr{H}_1) \to \mathscr{H}$ *as*

$$\mathbb{W}^+_{\mathcal{W}} = \mathbb{W}^+ I^{\succ\mathcal{W}}, \quad \mathbb{W}^-_{\mathcal{W}} = \mathbb{W}^- I^{\prec\mathcal{W}}. \tag{121}$$

Further, the S-matrix and wedge-transition maps in the case of a local quantum field theory are given by

$$S^{\mathcal{W}_1\mathcal{W}_2}_{\mathrm{f\,i}} = (I^{\succ\mathcal{W}_1})^* S_{\mathrm{f\,i}} I^{\prec\mathcal{W}_2}, \text{ and,} \tag{122}$$

[9] The argument for the two-dimensional case can be extracted from [Le06] Lemma 6.1.1, applied for the (trivial) special case $S = 1$.

$$S_{\mathrm{f\,f}}^{\mathcal{W}_1\mathcal{W}_2} = (I^{\succ_{\mathcal{W}_1}})^* I^{\succ_{\mathcal{W}_2}}, \qquad S_{\mathrm{i\,i}}^{\mathcal{W}_1\mathcal{W}_2} = (I^{\prec_{\mathcal{W}_1}})^* I^{\prec_{\mathcal{W}_2}}, \tag{123}$$

where $S_{\mathrm{f\,i}} = (\mathbb{W}^+)^*\mathbb{W}^-$ *denotes the usual scattering matrix from local Haag–Ruelle theory.*

Proof. It is sufficient to establish (121), from which the other statements follow. The wedge-transition matrix identities (123) then follow from the Fock structure identities $(\mathbb{W}^+)^*\mathbb{W}^+ = \mathbb{1} = (\mathbb{W}^-)^*\mathbb{W}^-$ of scattering states of the local Haag–Ruelle theory.

We consider the case of the outgoing wave operator. We may restrict to one-particle vectors of the form $\Psi_k = B_{k\tau}(f_k)\Omega$ defined in terms of local operators $A_k \in \mathfrak{A}(\mathcal{O})$ and regular Klein-Gordon solutions f_k for $1 \leq k \leq n$. In a local QFT model such vectors yield a dense subset of the one-particle space by standard arguments. We further assume that the one-particle states (and similarly the wave packets) are velocity ordered $\Psi_1 \succ_{\mathcal{W}} \Psi_2 \succ_{\mathcal{W}} \ldots \succ_{\mathcal{W}} \Psi_n$. Lastly, these states trivially satisfy the swapping property with $A_k^{\perp} := A_k$ for some overlapping wedge-regions $\mathcal{W} \supset \mathcal{O}$, $\mathcal{W}^{\perp} \supset \mathcal{O}$. Thus, we have on one hand by definition of the wedge-local wave operators that

$$\lim_{\tau\to\infty} B_{1\tau}(f_1)\ldots B_{n\tau}(f_n)\Omega = \mathbb{W}^+_{\mathcal{W}}\Psi_1 \otimes \Psi_2 \otimes \ldots \otimes \Psi_n. \tag{124}$$

On the other hand, we obtain from standard local Haag–Ruelle theory that

$$\lim_{\tau\to\infty} B_{1\tau}(f_1)\ldots B_{n\tau}(f_n)\Omega = \mathbb{W}^+ a^*(\Psi_1)\ldots a^*(\Psi_n)\Omega_F = \mathbb{W}^+ I^{\succ_{\mathcal{W}}}\Psi_1 \otimes \Psi_2 \otimes \ldots \otimes \Psi_n. \tag{125}$$

Equating we obtain (121) on a total subset and thus by continuity on $\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1)$. The argument for the incoming case is analogous. □

Using that the spectrum of the momentum operators is Lebesgue absolutely continuous on the one-particle space, $I^{\succ_{\mathcal{W}}/\prec_{\mathcal{W}}}$ are in fact surjective. Such continuity clearly holds in the free example (118). In a general context it is known to follow from locality and the spectrum condition [BF82, Prop. 2.2], or, from Poincaré covariance [Mai68].

Proposition 30. *For a local QFT model the following statements are equivalent:*

(i) $\mathbb{W}^+$ *is asymptotically complete,*
(ii) $\mathbb{W}^+_{\mathcal{W}}$ *is asymptotically complete for one wedge* $\mathcal{W}$*,*
(iii) $\mathbb{W}^+_{\mathcal{W}}$ *are asymptotically complete for all wedges* $\mathcal{W}$*,*

and analogously for the completeness of incoming wave operators.

Proof. The equivalence of (i) and (ii) for any wedge $\mathcal{W}$ follows from $I^{\succ_{\mathcal{W}}/\prec_{\mathcal{W}}}$ being a surjective isometry and (121). As $\mathcal{W}$ was arbitrary, this implies (iii). □

Corollary 31. *The wedge-local wave operators of the free scalar field are given by*

$$\mathbb{W}^+_{0,\mathcal{W}} = I^{\succ_{\mathcal{W}}}, \quad \mathbb{W}^-_{0,\mathcal{W}} = I^{\prec_{\mathcal{W}}}, \tag{126}$$

In particular, the free scalar field satisfies the property of ordered asymptotic completeness.

Theorem 32. *The Grosse–Lechner models are asymptotically complete with respect to velocity-ordered scattering states for any warping matrix* Q *(as defined in* (11)*).*

Proof. We consider a GL-model for any fixed warping matrix Q. Theorem 29 shows that the wedge-local wave operators $\mathbb{W}^{\pm}_{0,\mathcal{W}}$ of the scalar free field are asymptotically complete for any wedge $\mathcal{W}$. By Theorem 28 we obtain ordered asymptotic completeness of the wave operator $\mathbb{W}^{\pm}_{Q,\mathcal{W}}$ for any $\mathcal{W}$ and, thereby, ordered asymptotic completeness of the considered GL-model. □

To conclude these investigations, let us note the explicit n-particle scattering matrix for the case of the Grosse–Lechner models. We will focus on the two cases of equal or opposite initial and final wedges, which correspond to the earlier investigations from [GL07,BS08]. The case $\mathcal{W}_{\mathrm{f}} = \mathcal{W}'_{\mathrm{i}}$ can perhaps be regarded as slightly more natural due to the coincidence of $\Gamma^{\prec\mathcal{W}}(\mathscr{H}_1) = \Gamma^{\succ\mathcal{W}'}(\mathscr{H}_1)$ as subspaces of $\Gamma^{u}(\mathscr{H}_1)$, which is a consequence of the equivalence of the ordering relations $\succ_{\mathcal{W}'}$ and $\prec_{\mathcal{W}}$.

Proposition 33. *Let $\mathcal{W}$ be a fixed initial wedge. Then for the case of $\mathcal{W}_{\mathrm{f}} := \mathcal{W}'$ we obtain the Grosse–Lechner S-matrix as unitary operator on $\Gamma^{\prec\mathcal{W}}(\mathscr{H}_1)$ given by*

$$S^{\mathcal{W}'\mathcal{W}}_{Q,\mathrm{f\,i}} = (S^{\succ\mathcal{W}'}_{Q_{\mathcal{W}'}})^* S^{\prec\mathcal{W}}_{Q_{\mathcal{W}}} = (S^{\prec\mathcal{W}}_{Q_{\mathcal{W}}})^2 = S^{\prec\mathcal{W}}_{2Q_{\mathcal{W}}}. \tag{127}$$

Further we have for any ordered n-particle state $\Psi_n = \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \in \Gamma^{\prec\mathcal{W}}(\mathscr{H}_1)$ that

$$S^{\prec\mathcal{W}}_{2Q_{\mathcal{W}}} \Psi_n = \prod_{1\le i<j\le n} \mathrm{e}^{2\mathrm{i}P_i\cdot Q_{\mathcal{W}} P_j} \Psi_n. \tag{128}$$

In particular, the Grosse–Lechner S-matrix is non-trivial and factorizing.

This follows directly from Corollary 23 and Theorem 29, together with the triviality of the scattering matrix of the free field. Further we also used that $\succ_{\mathcal{W}'}$ and $\prec_{\mathcal{W}}$ are equivalent, giving $(S^{\succ\mathcal{W}'}_{Q_{\mathcal{W}'}})^* = S^{\prec\mathcal{W}}_{-Q_{\mathcal{W}'}} = S^{\prec\mathcal{W}}_{Q_{\mathcal{W}}}$ by Remark 24 and the definition of $Q_{\mathcal{W}}$. For the case of equal initial and final wedges we obtain similarly

$$S^{\mathcal{W}\mathcal{W}}_{Q,\mathrm{f\,i}} = (S^{\succ\mathcal{W}}_{Q_{\mathcal{W}}})^*(I^{\succ\mathcal{W}})^* I^{\prec\mathcal{W}} S^{\prec\mathcal{W}}_{Q_{\mathcal{W}}} = S^{\succ\mathcal{W}}_{-Q_{\mathcal{W}}} Z S^{\prec\mathcal{W}}_{Q_{\mathcal{W}}} = Z(S^{\prec\mathcal{W}}_{Q_{\mathcal{W}}})^2 = Z S^{\prec\mathcal{W}}_{2Q_{\mathcal{W}}}. \tag{129}$$

We note that the same result is obtained when using Proposition 26 and (127).

6. Conclusions

In this paper we showed stability of ordered asymptotic completeness under BLS-deformations in wedge-local QFT. We concluded that the Grosse–Lechner model is interacting and asymptotically complete in any spacetime dimension. We also showed that this model has a factorizing S-matrix, which is an unusual feature in higher dimensions.

Although we focused on models obtained by BLS deformations, our approach should also apply to wedge-local factorizing models in two dimensions. Even in the cases in which strict locality is still open, such as the non-linear sigma models [AL17] and for models with fermions [BC21], our strategy may give asymptotic completeness and factorization of the S-matrix. We leave detailed analysis of these problems to future investigations.

Another natural direction is a generalization of our results to theories of massless particles. The BLS-deformations remain valid for such theories, but so far neither interaction nor asymptotic completeness have been studied for $d > 1 + 1$. Such an investigation would require a massless variant of wedge-local scattering theory from [Du18]. It may be difficult to develop such a theory at the same level of generality as its local counterpart [Bu77], since the decay of correlations in massless wedge-local models may be very slow. Also the energy bounds [Bu90], which simplify more recent constructions of massless scattering states [DH15, AD17], are not available for wedge-local theories. But under some natural assumptions on the decay of correlations massless wedge-local scattering theory appears to be within reach. It should apply, in particular, to the massless Grosse–Lechner model.

We mention as an aside, that such a scattering theory could also help to understand recent computations of infraparticle scattering amplitudes in four-dimensional string-local models [GRT21]. An apparent breakdown of unitarity of scattering amplitudes was found in this reference after adapting a formula from a two-dimensional context [DM21]. This problem may have its roots in our limited understanding of collisions of massless Wigner particles in string- and wedge-local theories.

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Declarations

Conflict of interest The authors have no conflicts of interest to declare.

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