Check for updates

ADVANCING EARTH AND SPACE SCIENCES

JGR Planets

COMMENTARY

10.1029/2024JE008378

Estimating π Using the Topography of the Terrestrial Planets

Jérémie Vasseur[1] and **Fabian B. Wadsworth**[2]

[1]Earth and Environmental Sciences, Ludwig-Maximilians-Universität, München, Germany, [2]Earth Sciences, Durham University, Durham, UK

Key Points:

- Every year March 14th is international π-day when we celebrate by finding fun ways to compute the number ~3.14
- The topography of the terrestrial planets is variable and can be used to compute π in slices of constant latitude
- Fun ways to compute π using planetary science methods are educational and can support quantitative learning in a range of fields

Correspondence to:

J. Vasseur,
j.vasseur@lmu.de

Citation:

Vasseur, J., & Wadsworth, F. B. (2024). Estimating π using the topography of the terrestrial planets. *Journal of Geophysical Research: Planets*, *129*, e2024JE008378. https://doi.org/10.1029/2024JE008378

Received 3 MAR 2024
Accepted 8 MAR 2024

Author Contributions:

Conceptualization: Jérémie Vasseur, Fabian B. Wadsworth
Formal analysis: Jérémie Vasseur
Investigation: Jérémie Vasseur
Methodology: Jérémie Vasseur, Fabian B. Wadsworth
Visualization: Jérémie Vasseur, Fabian B. Wadsworth
Writing – original draft: Fabian B. Wadsworth
Writing – review & editing: Jérémie Vasseur

© 2024. The Authors.
This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

Abstract It is π-day 2024 (March 14th) and to celebrate, we ask the question: can π be estimated accurately using the shape of the terrestrial planets and the Moon? We proceed by using models for the planet topographies in slices of constant latitude and computing π as the ratio of the circumference to the diameter in each slice. We define the circumference as the topographically undulating perimeter of the slice (which deviates from the circular section of the reference ellipsoid). We define the diameter as the distance connecting two opposite points on the topography model through the center of the slice. This gives us a distribution of π estimates for each latitude slice and overall for each planetary body. The distribution and accuracy of the π estimates we compute are a representation of how smooth and circular latitude slices of the terrestrial planets are. We find that Mercury and Venus are particularly good at predicting π because they are relatively smooth and the topographic deviations from the reference ellipsoid are small compared with Mars, Earth, and the Moon. In the spirit of π-day, this mathematical exercise motivates an interest in planetary geophysics and shape.

Plain Language Summary 14th March is π-day: an annual celebration of the number π and a fun opportunity to raise the profile of mathematics in all fields of science and of numeracy in general. One way to define the number π is by taking the ratio of a circle's circumference to its diameter. And so, to celebrate π-day this year, we take the terrestrial worlds of our solar system and we cut them into near-circular slices. We take the ratio of a slice circumference to its diameter to see how close it is to the number π. This results in some good estimates of π, but with inaccuracies because the planets are not perfectly smooth spheres. Nevertheless, it turns out that Mars, Earth, the Moon, Venus, and Mercury all slice up in such a way as to provide accurate estimates of π, and Venus is the best among them all.

1. Introduction: π-Day

π-day is an annual celebration of the number π held on March 14th. Each year it is both fun and educational to find interesting methods by which the value of π can be computed. For example, in 2023 Matt Parker (See Matt Parker's π-day exploits here: https://www.youtube.com/user/standupmaths) used the arcing skidding behavior of an out-of-control car to estimate π and arrived at a value of 3.12. The joy of these fun approaches is often that they are built on a number of assumptions and uncertainties related to measurement, such that if the resulting π estimate is at all close to the true value, we are set to be amazed. A popular method that is easy to execute at home is the "Buffon's needle problem" which utilizes the probability that a randomly dropped stick (e.g., a toothpick) will land across equally spaced lines on a sheet of paper (Klyve, 2019).

In the Geosciences, Wadsworth et al. (2022) used early estimates of the Earth's radius R_E and circumference C_E from antiquity to estimate π using the circle identity $\pi = C_E/(2R_E)$. This method yielded $\pi \approx 3$, which is impressive given the rudimentary methods by which C_E and R_E were found in the third century BCE. and c.1000 CE., respectively. Since those early days of metrology of our planet, things have moved on. For π-day 2024, herein we aim to bring the geosciences contribution to π-day into the 21st century by using the shape of the terrestrial planets (and the Moon) from our solar system to estimate π.

2. Method

The approximate overall shapes of the rocky terrestrial bodies in our solar system are defined by reference ellipsoids. In the case of Venus, Mercury and Earth's Moon, these are usually approximated with a single mean planetary radius, whereas for Earth and Mars, the reference ellipsoids are characterized by different equatorial and polar radii (Wieczorek, 2015). In all cases, any slice of constant latitude through the planet will give a reference circle with a well-defined center point. Unlike the reference ellipsoids, the surfaces of planets are characterized by topography that deviates at multiple scales from idealized shapes. This topography can be approximated to high spatial resolution (on planetary scales) by so-called high-order spherical harmonics models. The spherical harmonics equation used here is

$$H(\lambda,\phi) = \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} P_{\ell m}[\sin(\phi)]\,[C_{\ell m}\cos(m\lambda) + S_{\ell m}\sin(m\lambda)] \tag{1}$$

where $H(\lambda,\phi)$ is the topography relative to the reference ellipsoid as a function of the longitude λ and the co-latitude ϕ, $P_{\ell m}$ are the normalized associated Legendre functions of degree ℓ (to a maximum degree $\ell_{\max}$), and order m, and $C_{\ell m}$ and $S_{\ell m}$ are suites of coefficients. These models are usually fit to high-resolution topographic data (Hirt & Rexer, 2015; Hirt et al., 2012; Wieczorek, 2015). While Earth's surface is arguably the best-studied of the planets, vegetation, ice cover, and uncertainties in ocean floor bathymetry present complications (Hirt et al., 2012). We use the following models: (a) Earth2014 for the Earth (Hirt & Rexer, 2015); (b) MoonTopo2600p, MarsTopo2600, and VenusTopo719, for the Moon, Mars, and Venus, respectively (Wieczorek, 2015); and (c) GTMES150 for Mercury (Goossens et al., 2022; Neumann et al., 2016). In each of the references given, there are the relevant lists of coefficients $C_{\ell m}$ and $S_{\ell m}$. To solve Equation 1 we use pyshtools which is an archive of Python software that can be used for spherical harmonic transforms, spectral analyses, expansions of gridded data into Slepian basis functions, and standard operations on global gravitational and magnetic field data (Wieczorek & Meschede, 2018). pyshtools makes use of grid formats that accommodate exact quadrature, including regularly spaced grids that satisfy sampling theorems (Driscoll & Healy, 1994).

The above methodology results in a topography output for each of the bodies considered here. We then sample slices of constant latitude (the equator is shown as an example in Figures 1b, 1f, 1j, 1n, and 1r) and perform two levels of analysis:

- First, we take the length of the topographic model curve and take that to be a pseudo total circumference C_i for that latitude slice i. We then take the diameter of the reference ellipsoid (a circle for latitudes) in that slice D_i. The resultant $\pi = C_i/D_i$ estimate is shown for each latitude in Figures 1c, 1g, 1k, 1o, and 1s.
- Second, with the same C_i for each latitude slice, we take a local diameter by measuring the length through the reference ellipsoid slice center that connects to opposite points on the topography model D_{ij} (where the additional index j refers to the point being taken on the topography model). By assuming $\pi = C_i/D_{ij}$, we collect many estimates for π for each latitude. These estimates are shown in histograms in Figures 1d, 1h, 1l, 1p, and 1t.

3. Result and Discussion

Overall, we find that existing reference ellipsoids and topography for the terrestrial worlds can be used to estimate π to high accuracy, especially when compared with the first simple estimates of the Earth's circumference and radius (Wadsworth et al., 2022). In Table 1 we give the mean and standard deviation of our estimates for each planetary body studied here. It is clear that Venus performs the best as a planetary π-computer, being accurate to within 0.0025%. By contrast, the Moon performs relatively poorly with a 0.61% deviation from the true value of π (Table 1). These results are consistent with the fact that Venus is a particularly smooth planet in terms of its topography, while the Moon is heavily cratered and has relatively extreme highlands and lowlands for its size. In this way, the π estimates given here can be interpreted as a proxy for planetary smoothness, where the smoother a planet is, the more precise the π estimate will be.

The average person will generally think of planets as spherical objects with a single radius and circumference. Geo- and planetary scientists like to remind us that this is not the case and that the Earth, Moon, and Mars, have specific non-spherical shapes called the geoid, selenoid, and arenoid, respectively. These deviations from purely spherical shapes are clearly not found in slices of constant latitude, where the reference ellipsoid is always circular and the true 2D slice shape is quite close to being circular (hence why the π estimates given here are so accurate; Table 1). However, if we had taken slices of constant longitude, then the non-circularity of the sections would be more pronounced and the associated π estimates would be less good and would be consistent with the fact that we know some terrestrial worlds are flattened at the poles relative to the equators.

21699100, 2024, 3, Downloaded from https://agupubs.onlinelibrary.wiley.com/doi/10.1029/2024JE008378, Wiley Online Library on [04/06/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

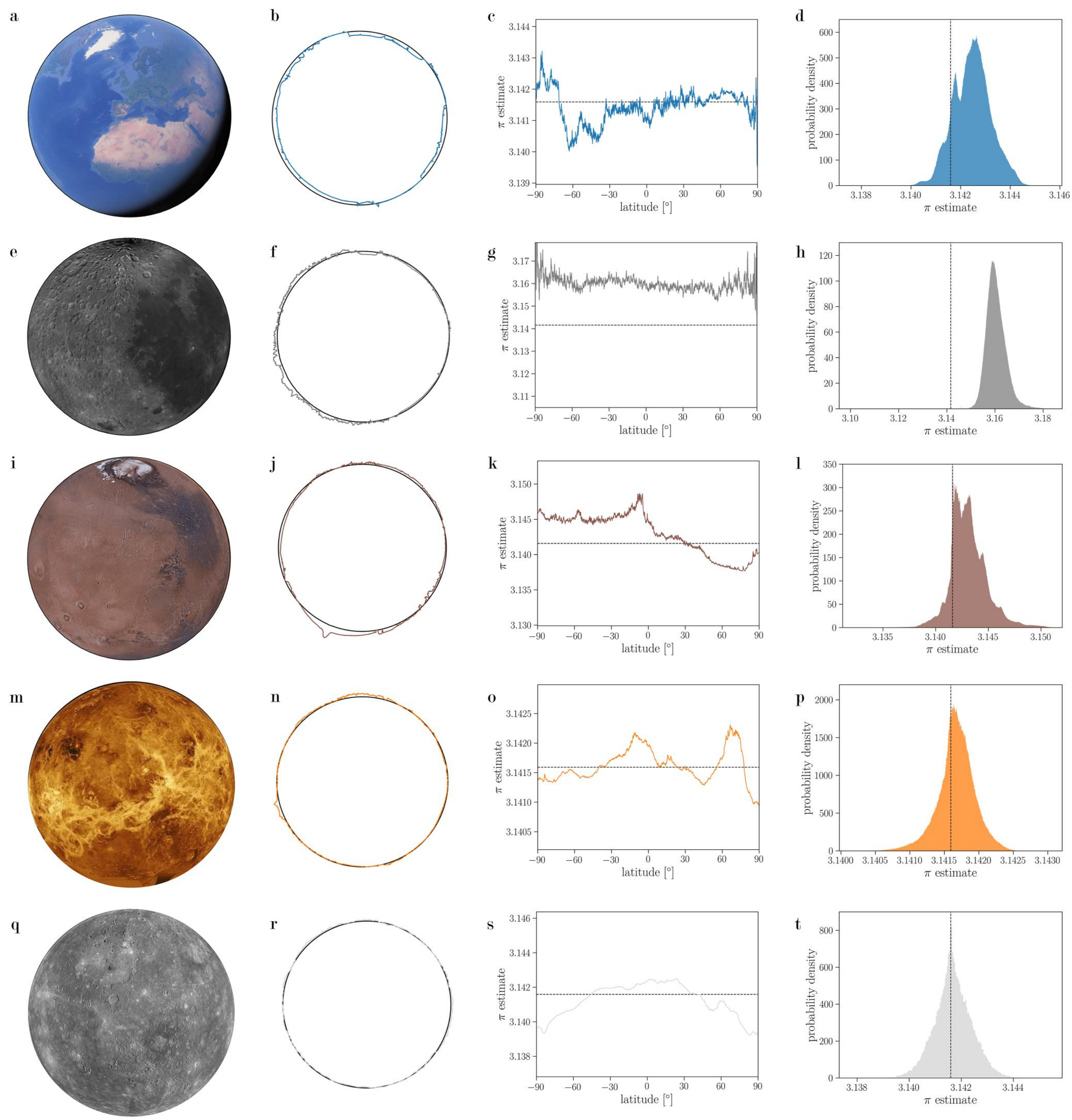

Figure 1. Estimating the value of π using (a–d) Earth, (e–h) the Moon, (i–l) Mars, (m–p) Venus, and (q–t) Mercury. Panels (a), (e), (i), (m), and (q) are images of the planetary bodies used where (a), (e), and (i) are from Google™ project GoogleEarth™, and (m) and (q) are from NASA. Panels (b), (f), (j), (n), and (r) are the equatorial slices showing the reference ellipsoid (circular in latitude slices) and the undulating topography with vertical exaggeration factors of 65, 18, 35, 65, and 25, respectively. Panels (c), (g), (k), (o), and (s) show the π estimate found by taking the ratio of the computed topographic perimeter to the diameter of the reference ellipsoid for each latitude slice (i.e., this would be perfectly accurate if there were no topographic deviation from the reference ellipsoid). Panels (d), (h), (l), (p), and (t) represent the distribution of π estimates for all slices, where we additionally allow the diameter to take a range of values for each slice (i.e., a single diameter is the distance through the slice center point that connects to antipodal points on the topography). In these final panels, the variance of the distribution is an approximate measure of how smooth the surface of the planet is and the number of sampled estimates is $n = 18{,}675{,}362$, $n = 27{,}055{,}602$, $n = 27{,}055{,}602$, $n = 2{,}072{,}160$, and $n = 90{,}902$, respectively.

21699100, 2024, 3, Downloaded from https://agupubs.onlinelibrary.wiley.com/doi/10.1029/2024JE008378, Wiley Online Library on [04/06/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

21699100, 2024, 3, Downloaded from https://agupubs.onlinelibrary.wiley.com/doi/10.1029/2024JE008378, Wiley Online Library on [04/06/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Table 1
Estimates of π Found Using the Topography of Planets

Body	π estimate	Standard deviation on the π estimate	% deviation from true π	The number of latitude slices used[a]	ℓ_{max}
Earth	3.142464	0.000748	0.027748	4,321	2,160
The Moon	3.160762	0.007716	0.610188	5,201	2,600
Mars	3.143096	0.001737	0.047852	5,201	2,600
Venus	3.141672	0.000269	0.002517	1,439	719
Mercury	3.141683	0.000714	0.002881	301	150

[a]Given by the gridding algorithm in pyshtools (Wieczorek & Meschede, 2018).

Our fun exercise to find π using the inner Solar System planetary bodies shows that, in general, deviations from clean circular latitude slices are clearly relatively minor because all π estimates are satisfyingly good, and far better than most π-day exercises found online each year. We conclude that the topography is a minor contribution to non-circularity in these sections and that $\pi = C_i/D_{ij}$ holds even when accounting for topography. As the geo- and planetary sciences grow, increasing numeracy in these fields is key (Manduca et al., 2008) and we posit that quantitative π-day exercises such as shown here represent fun but engaging and challenging entry-points to the mathematics required to understand our solar system.

Acknowledgments
We thank Matt Parker for his engaging Youtube™ video content, some of which inspired this contribution, and we thank Iphigenia Anagnostopoulos and Lucia Pérez-Diaz both for stimulating conversations associated with previous π-day celebrations (Wadsworth et al., 2022). Open Access funding enabled and organized by Projekt DEAL.

References

Driscoll, J. R., & Healy, D. M. (1994). Computing Fourier transforms and convolutions on the 2-sphere. *Advances in Applied Mathematics*, *15*(2), 202–250. https://doi.org/10.1006/aama.1994.1008

Goossens, S., Genova, A., James, P. B., & Mazarico, E. (2022). Estimation of crust and lithospheric properties for Mercury from high-resolution gravity and topography. *Planetary Science Journal*, *3*(6), 145. https://doi.org/10.3847/PSJ/ac703f

Hirt, C., Kuhn, M., Featherstone, W. E., & Göttl, F. (2012). Topographic/isostatic evaluation of new-generation GOCE gravity field models. *Journal of Geophysical Research*, *117*(B5), B05407. https://doi.org/10.1029/2011JB008878

Hirt, C., & Rexer, M. (2015). Earth2014: 1 arc-min shape, topography, bedrock and ice-sheet models—Available as gridded data and degree-10,800 spherical harmonics. *International Journal of Applied Earth Observation and Geoinformation*, *39*, 103–112. https://doi.org/10.1016/j.jag.2015.03.001

Klyve, D. (2019). How to calculate pi: Buffon's needle (Calculus version).

Manduca, C. A., Baer, E., Hancock, G., Macdonald, R. H., Patterson, S., Savina, M., & Wenner, J. (2008). Making undergraduate geoscience quantitative. *Eos, Transactions American Geophysical Union*, *89*(16), 149–150. https://doi.org/10.1029/2008EO160001

Neumann, G. A., Perry, M. E., Mazarico, E., Ernst, C. M., Zuber, M. T., Smith, D. E., et al. (2016). Mercury shape model from laser altimetry and planetary comparisons. In *47th annual lunar and planetary science conference* (p. 2087).

Wadsworth, F. B., Vasseur, J., Anagnostopoulos, I., & Pérez-Díaz, L. (2022). Estimating pi using geoscience. *Nature Geoscience*, *15*(4), 246–247. https://doi.org/10.1038/s41561-022-00928-z

Wieczorek, M. A. (2015). Gravity and topography of the terrestrial planets. *Treatise on Geophysics*, *10*, 165–206.

Wieczorek, M. A., & Meschede, M. (2018). SHTools: Tools for working with spherical harmonics. *Geochemistry, Geophysics, Geosystems*, *19*(8), 2574–2592. https://doi.org/10.1029/2018GC007529