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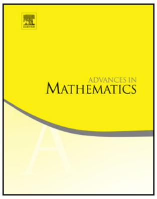

Endpoint Schatten class properties of commutators

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ABSTRACT

We study the trace ideal properties of the commutators $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ of a power of the Laplacian with the multiplication operator by a function f on $\mathbb{R}^d$. For a certain range of $\epsilon \in \mathbb{R}$, we show that this commutator belongs to the weak Schatten class $\mathcal{L}_{\frac{d}{1-\epsilon},\infty}$ if and only if the distributional gradient of f belongs to $L_{\frac{d}{1-\epsilon}}$. Moreover, in this case we determine the asymptotics of the singular values. Our proofs use, among other things, the tool of Double Operator Integrals.

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1. Introduction and main results

1.1. Background and motivations

In this paper we are interested in spectral properties of commutators of the form

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \qquad \text{in } L_2(\mathbb{R}^d)\,.$$

Here ϵ is a parameter that we typically fix in one of the intervals $(0,1]$ and $(-\frac{d}{2},0)$, and M_f is the operator of multiplication by a function $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. In Section 3 we will recall that the operator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ is always well defined from $C_c^\infty(\mathbb{R}^d)$ to the space of distributions $(C_c^\infty(\mathbb{R}^d))'$.

The spectral properties that we are interested in are boundedness, compactness and membership to a trace ideal, as well as the computation of the asymptotics of its singular values. The trace ideals in question are Schatten classes $\mathcal{L}_p$ and weak Schatten classes $\mathcal{L}_{p,\infty}$, whose definition we will recall in Section 2.

Our goal is to investigate these spectral properties under minimal assumptions on the function f. In particular, we will derive the asymptotics of the singular values under the sole assumption that the asymptotic coefficient is finite. We will also be interested in converse theorems, where spectral properties of the commutators imply certain properties of the function f.

To set the stage, let us recall a prototypical result in this area, due to Coifman, Rochberg and Weiss [17] with later contributions by Janson [26] and Uchiyama [52] and many others. In a certain sense this result corresponds to the case $\epsilon = 0$ of our problem. Let $d \geq 2$ and let $R_j = -i\partial_j(-\Delta)^{-\frac{1}{2}}$, $j = 1, \ldots, d$, be a Riesz transform. Then

$$[R_j, M_f] \qquad \text{in } L_2(\mathbb{R}^d)$$

is bounded if and only if $f \in BMO(\mathbb{R}^d)$, the class of functions of bounded mean oscillations. In addition, there is a two-sided bound between the operator norm of $[R_j, M_f]$ and the seminorm of f in $BMO(\mathbb{R}^d)$. Furthermore, the operator is compact if and only if $f \in VMO(\mathbb{R}^d)$, the class of functions of vanishing mean oscillations. Concerning trace ideals, Janson and Wolff [29] characterized membership of $[R_j, M_f]$ to Schatten classes $\mathcal{L}_p$ with $d < p < \infty$ in terms of membership of f to a homogeneous fractional Sobolev space, and they showed that the operator does not belong to $\mathcal{L}_d$ unless f is constant; for alternative proofs see [44,27]. Connes, Sullivan and Teleman, together with Semmes, [19] added a characterization in the endpoint case of the weak Schatten class $\mathcal{L}_{d,\infty}$, namely membership to a certain homogeneous first order Sobolev space. In the recent papers [35,24] we have revisited and extended the latter result.

Thus, there is a scale of nested function spaces, parametrized by p, namely the homogeneous fractional Sobolev spaces, such that if the regularity of f improves (in the sense of it belonging to a smaller one of these spaces), then the trace ideal properties of $[R_j, M_f]$ improve (in the sense of it belonging to a smaller trace ideal space). At the same time, there is a saturation effect in the sense that if the regularity properties of f improve beyond membership to the corresponding first order Sobolev space, then the trace ideal properties no longer improve. The existence of a critical exponent p_* with this property, here $p_* = d$, is sometimes referred to as the 'Janson–Wolff phenomenon' or 'cut-off phenomenon'. It has been observed in many other problems of this kind, involving commutator or Hankel operators, see for instance, [2,22,3,41,28,21] and references therein.

The present paper is a continuation of [24], but can be read independently. Our goal is to prove the analogue of the before-mentioned results for the operators $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ for $\epsilon \neq 0$. Several results about these operators were obtained by Murray [40] (concerning boundedness) and by Janson and Peetre [27] (concerning boundedness and trace ideal properties) and we review them in detail later on in this introduction. Here we just mention that for every $\epsilon \in (-\frac{d}{2}, 0) \cup (0, 1)$ there is a Janson–Wolff phenomenon with critical exponent $p_* = \frac{d}{1-\epsilon}$. The existing results characterize membership to $\mathcal{L}_p$ for $p > p_*$ in terms of f belonging to some homogeneous fractional Sobolev space. It is also known that $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ does not belong to $\mathcal{L}_{p_*}$ unless f is constant. Our new results characterize membership to the endpoint space, namely the weak Schatten ideal $\mathcal{L}_{p_*,\infty}$, in terms of f belonging to some homogeneous first order Sobolev space. Moreover, we will compute the asymptotics of the singular values under the sole assumption of membership to this Sobolev space.

We will also obtain some results in the case $\epsilon = 1$, which is somewhat different. Note that the exponent $p_* = \frac{d}{1-\epsilon}$ tends to $+\infty$ as $\epsilon \to 1^-$. Among other things, we show that $[(-\Delta)^{\frac{1}{2}}, M_f]$ is never compact, unless f is constant. Moreover, we provide a short proof that a well-known criterion of Calderón [9] for boundedness is not only sufficient, but also necessary.

1.2. Main results

We now come to a precise formulation of our main results in the following three theorems. The necessary notation related to function spaces and trace ideals, as well as the precise meaning of the operator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ can be found in the following two sections.

Our first result concerns the case $0 < \epsilon < 1$.

Theorem 1.1. *Let $d \geq 1$ and let $0 < \epsilon < 1$.*

(i) *If $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$, then $[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_{\frac{d}{1-\epsilon},\infty}$ and*

$$\left\| [(-\Delta)^{\frac{\epsilon}{2}}, M_f] \right\|_{\frac{d}{1-\epsilon},\infty} \leq c_{d,\epsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}.$$

(ii) *If* $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$, *then*

$$\lim_{t\to\infty} t^{\frac{1-\epsilon}{d}} \mu\Big(t, [(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big) = \kappa_{d,\epsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}$$

with

$$\kappa_{d,\epsilon} := |\epsilon| \left((2\pi)^{-d} d^{-1} \int\limits_{\mathbb{S}^{d-1}} |\omega_d|^{\frac{d}{1-\epsilon}}\, d\omega \right)^{\frac{1-\epsilon}{d}}. \qquad (1)$$

(iii) *If* $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ *and if* $[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_{\frac{d}{1-\epsilon},\infty}$, *then* $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$ *and*

$$\|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)} \le c'_{d,\epsilon} \Big\| [(-\Delta)^{\frac{\epsilon}{2}}, M_f] \Big\|_{\frac{d}{1-\epsilon},\infty}.$$

Our second result concerns the case $-\frac{d}{2} < \epsilon < 0$.

Theorem 1.2. *Let* $d \ge 2$ *and let* $-\frac{d}{2} < \epsilon < 0$.

(i) *If* $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$, *then* $[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_{\frac{d}{1-\epsilon},\infty}$ *and*

$$\Big\| [(-\Delta)^{\frac{\epsilon}{2}}, M_f] \Big\|_{\frac{d}{1-\epsilon},\infty} \le c_{d,\epsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}.$$

(ii) *If* $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$, *then*

$$\lim_{t\to\infty} t^{\frac{1-\epsilon}{d}} \mu\Big(t, [(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big) = \kappa_{d,\epsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}$$

with $\kappa_{d,\epsilon}$ *given by* (1).

(iii) *If* $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ *and if* $[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_{\frac{d}{1-\epsilon},\infty}$, *then* $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$ *and*

$$\|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)} \le c'_{d,\epsilon} \Big\| [(-\Delta)^{\frac{\epsilon}{2}}, M_f] \Big\|_{\frac{d}{1-\epsilon},\infty}.$$

Note that in this theorem we assume $\epsilon > -\frac{d}{2}$ and $d \ge 2$. For the case $\epsilon \le -\frac{d}{2}$, see Lemma 3.3 below. Concerning the assumption $d \ge 2$, see the discussion in the following subsection.

Our third result concerns the case $\epsilon = 1$.

Theorem 1.3. *Let $d \geq 1$.*

(i) *If $f \in \dot{W}^1_\infty(\mathbb{R}^d)$, then $[(-\Delta)^{\frac{1}{2}}, M_f]$ is bounded and*

$$\left\| [(-\Delta)^{\frac{1}{2}}, M_f] \right\|_\infty \leq c_d \, \|f\|_{\dot{W}^1_\infty(\mathbb{R}^d)}.$$

(ii) *If $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and if $[(-\Delta)^{\frac{1}{2}}, M_f]$ is bounded, then $f \in \dot{W}^1_\infty(\mathbb{R}^d)$ and*

$$\|f\|_{\dot{W}^1_\infty(\mathbb{R}^d)} \leq \left\| [(-\Delta)^{\frac{1}{2}}, M_f] \right\|_\infty .$$

(iii) *If $f \in \dot{W}^1_\infty(\mathbb{R}^d)$ and if $[(-\Delta)^{\frac{1}{2}}, M_f]$ is compact, then $f \equiv \mathrm{const}$.*

Part (i) of this theorem is a celebrated result of Calderón [9], which we restate here only for the sake of completeness. Parts (ii) and (iii) for $d = 1$ are due to Janson and Peetre [27, Section 6, Example 7]. Part (ii) appeared recently in [13, Theorem 1.5 with $\Omega \equiv 1$], but we present an alternative, shorter proof for the case at hand. Part (iii) for $d \geq 2$ can be considered as new.

1.3. Comparison with known results

Let us formulate precisely the previous results on the operators $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$. We will not define the relevant function spaces, since they will not play any role in the remainder of this paper. They are the fractional Sobolev spaces $\dot{W}^s_p(\mathbb{R}^d)$ with $0 < s < 1$, which are equal to a special case of Besov spaces, namely $\dot{W}^s_p(\mathbb{R}^d) = \dot{B}^s_{p,p}(\mathbb{R}^d)$. For these spaces see, for instance, [33, Chapter 17]. Less standard spaces are the BMO-Sobolev spaces $(-\Delta)^{\frac{s}{2}} BMO(\mathbb{R}^d)$, $0 < s < 1$, which are studied in [47]. As discussed, for instance, in [54], these spaces coincide with a special case of Triebel–Lizorkin spaces, namely $(-\Delta)^{\frac{s}{2}} BMO(\mathbb{R}^d) = \dot{F}^s_{\infty,2}(\mathbb{R}^d)$. For these spaces see [51, Subsection 2.3.4].

Case $\epsilon \in (0,1)$

It is shown by Murray [40] in dimension $d = 1$ and by Janson and Peetre [27] in general dimension d that

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \text{ is bounded iff } f \in (-\Delta)^{\frac{\epsilon}{2}} BMO(\mathbb{R}^d).$$

We are not aware of a published result characterizing compactness of $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$, but it is natural to guess that the relevant space is the closure of $C^\infty_c(\mathbb{R}^d)$ in the space $(-\Delta)^{\frac{\epsilon}{2}} BMO(\mathbb{R}^d)$. Concerning trace ideals it is shown in [27] that for any $\frac{d}{1-\epsilon} < p < \infty$ one has

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_p \qquad \Longleftrightarrow \qquad f \in \dot{W}^{\epsilon+\frac{d}{p}}_p(\mathbb{R}^d),$$

as well as

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_{\frac{d}{1-\epsilon}} \qquad \Longleftrightarrow \qquad f \equiv \text{const}\,.$$

Our Theorem 1.1 completes these results, namely by characterizing membership of the commutator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ to the weak Schatten class $\mathcal{L}_{\frac{d}{1-\epsilon},\infty}$ in terms of the condition $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$. Moreover, we compute the asymptotics of the singular values under the sole assumption of membership to this Sobolev space.

Case $\epsilon \in (-\frac{d}{2}, 0)$

Janson and Peetre [27] have shown that for any $\frac{d}{1-\epsilon} < p < \frac{d}{-\epsilon}$ one has

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \mathcal{L}_p \qquad \Longleftrightarrow \qquad f \in \dot{W}_p^{\epsilon+\frac{d}{p}}(\mathbb{R}^d)\,.$$

We are not aware of any results characterizing membership of $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ to $\mathcal{L}_p$ with $p \geq \frac{d}{-\epsilon}$ or its boundedness or compactness. Thus, the results for $\epsilon < 0$ are far less complete than those for $\epsilon > 0$. Our Theorem 1.2 completes the existing results at the endpoint $p = \frac{d}{1-\epsilon}$ in a similar way as in the case $\epsilon > 0$. However, now we need the additional assumption $d \geq 2$.

The origin of this extra-assumption can be seen from the fact that the integrability exponent $\frac{d}{1-\epsilon}$ of the Sobolev space $\dot{W}^1_{\frac{d}{1-\epsilon}}$ is less than 1 for $d = 1$ and $\epsilon < 0$. In this way the restriction to $d \geq 2$ arises from a technical point of view in our proofs. We think, however, that this restriction is not only technical but that the results are significantly different for $d = 1$. More specifically, while it is probably still true that for $f \in C_c^\infty(\mathbb{R})$, the operator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ belongs to $\mathcal{L}_{\frac{1}{1-\epsilon}}$ and its singular values satisfy the asymptotics in (ii), it is conceivable that there are f in $L_{1,\mathrm{loc}}(\mathbb{R})$ with $f' \in L_{\frac{1}{1-\epsilon}}(\mathbb{R})$ for which $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ does not belong to $\mathcal{L}_{\frac{1}{1-\epsilon}}$. This is a subject for further investigation.

Case $\epsilon = 1$

As we have already mentioned, the boundedness of $[(-\Delta)^{\frac{1}{2}}, M_f]$ under the assumption $f \in \dot{W}^1_\infty(\mathbb{R}^d)$ was proved in an influential paper of Calderón [9]. For alternative proofs and extensions we refer, for instance, to [16,10,15].

Janson and Peetre [27] prove that the condition $f \in \dot{W}^1_\infty(\mathbb{R}^d)$ is also necessary for boundedness in dimension $d = 1$ and mention without proof that a referee of their paper has told them that this condition is also necessary for $d \geq 2$. We provide a proof of this claim. A different proof has appeared recently in [13, Theorem 1.5 with $\Omega \equiv 1$]. The latter paper deals with a much larger class of operators than we do, but for the problem at hand our proof has the advantage of being much more direct.

Finally, Janson and Peetre [27] prove that in dimension $d = 1$ the operator $[(-\Delta)^{\frac{1}{2}}, M_f]$ is not compact unless f is constant. Our Theorem 1.3 shows that this remains valid in any dimension.

1.4. Outline of the paper

We end this introduction by giving a quick overview over this paper. For simplicity we restrict ourselves to the proofs of Theorems 1.1 and 1.2; that of Theorem 1.3 appears in Section 9 and uses related arguments.

When proving Theorems 1.1 and 1.2, the methods that we are using are rather different for part (i) on the one hand and parts (ii) and (iii) on the other hand.

Let us begin with discussing our proof of part (i) in Theorems 1.1 and 1.2, which appears in Section 4. It is of operator-theoretic nature and based on the technique of Double Operator Integrals (DOIs). As two of us have demonstrated in [30,42,35], DOIs are an excellent tool for studying commutators. Earlier uses of DOIs in connection with spectral properties of pseudodifferential operators can be found, for instance, in the work of Birman and Solomyak [7].

DOIs allow us to write the commutators $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ as a certain transformation applied to the simple commutator $[\nabla, M_f] = M_{\nabla f}$. Note that trace ideal properties of the operators $(-\Delta)^{\frac{\epsilon-1}{4}}[\nabla, M_f](-\Delta)^{\frac{\epsilon-1}{4}}$ and generalizations thereof can be obtained from Cwikel's theorem [20] and its generalizations.

Therefore, the proof of our results is reduced to studying the mapping properties of the transformation that allows us to write $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ in terms of $[\nabla, M_f]$. This transformation depends on the parameter ϵ. The mapping properties in the case $\epsilon \in [-1, 1]$ are relatively straightforward, given the previous results in [42] by one of us. To prove the mapping properties for $\epsilon < -1$ we use a certain 'renormalization procedure', where we extract a finite number (depending on ϵ) of extra terms, before being able to apply again the results in [42].

We now turn to a discussion of the methods used to prove parts (ii) and (iii) in Theorems 1.1 and 1.2. For part (ii) the analysis is divided into two steps. In a first step we derive the spectral asymptotics in the smooth case, that is, for $f \in C_c^\infty(\mathbb{R}^d)$ and with $-\Delta$ replaced by $1 - \Delta$. In a second step we use the uniform a priori bounds from part (i) to extend the asymptotics to the maximal class of functions f.

The first step here uses an approximation result for commutators that is in the spirit of parametrix constructions in the theory of pseudodifferential operators, but we present it in an elementary fashion; see Section 5. Once we have this approximation, we can apply the results about spectral properties that we developed in our previous paper [24]; see Section 7. To be more precise, the methods from [24] are applicable, but the results are not, at least not in an obvious way. Instead of referring the reader to redo the arguments in [24], in Section 6 we present a method to deduce from [24] the results we need *without* redoing the argument. We believe that this argument is interesting in its own right and illustrates, once again, the power of DOI techniques.

Part (iii) of Theorems 1.1 and 1.2 is proved in Section 8. The argument is related to that in the proof of part (ii). Namely, we show that if $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ has certain trace ideal properties, then the same is true for the corresponding operator with a regularized f. For the latter operator we can use (a localized version of) the spectral asymptotics in part (ii) to get a bound on the Sobolev norm of the regularized version of f, uniform in the regularization parameter. This allows us to conclude the f itself has to be sufficiently regular.

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2. Preliminaries

All considered functions are complex-valued, unless otherwise specified.

Let (Ω, Σ, ν) be a measure space with a σ-finite measure ν, defined on a σ-algebra Σ, and let $L(\Omega)$ be the algebra of all classes of equivalent measurable complex-valued functions on (Ω, Σ, ν). Denote by $L_0(\Omega)$ the subalgebra of $L(\Omega)$ consisting of all functions f such that $\nu(\{|f| > s\}) < \infty$ for some $s > 0$. For every $f \in L_0(\Omega)$, its non-increasing rearrangement is defined by

$$\mu(t, f) := \inf\{s > 0 : \nu(\{|f| > s\}) \leq t\},\ t > 0.$$

For $0 < p < \infty$ the space $L_p(\Omega, \Sigma, \nu)$ (resp. $L_{p,\infty}(\Omega, \Sigma, \nu)$) consists of all elements $f \in L_0(\Omega)$ for which

$$\|f\|_p := \left(\int_0^\infty \mu(t, f)^p dt\right)^{1/p} < \infty, \quad \left(\text{resp. } \|f\|_{p,\infty} := \sup_{t>0} t^{1/p}\mu(t, f) < \infty\right).$$

As always, the measure spaces $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$, $\mathbb{N} = \{1, 2, 3, \cdots\}$ are equipped with counting measures and the Euclidean space $\mathbb{R}^d$ with Lebesgue measure.

For detailed information concerning classical function and sequence spaces such as $L_p(\mathbb{R}^d)$, $L_{p,\infty}(\mathbb{R}^d)$, $\ell_p = \ell_p(\mathbb{Z}_+)$ and $\ell_{p,\infty} = \ell_{p,\infty}(\mathbb{Z}_+)$, we refer the reader for instance to [32,36,37].

In this paper, ∇ is the *self-adjoint* gradient operator on $L_2(\mathbb{R}^d)$, that is

$$\nabla = (\frac{1}{i}\frac{\partial}{\partial t_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial t_d}).$$

Definition 2.1. The homogeneous Sobolev space $\dot{W}^1_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, consists of functions $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ whose distributional gradient belongs to $L_p(\mathbb{R}^d)$. This is a space of functions modulo constants. We set

$$\|f\|_{\dot{W}^1_p(\mathbb{R}^d)} := \|\nabla f\|_p\,.$$

The following result is available, see e.g. [33, Theorem 12.9].

Theorem 2.2 *(Sobolev embedding theorem). Let* $1 \le p < d$ *and let* $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$*. We have* $\dot{W}^1_p(\mathbb{R}^d) \subset L_q(\mathbb{R}^d)$*. This should be understood in the sense that in the class of functions modulo constants there is one that belongs to* $L_q(\mathbb{R}^d)$*.*

Let H be a complex separable Hilbert space, and let $\mathcal{B}(H)$ denote the set of bounded operators on H. The standard uniform operator norm on $\mathcal{B}(H)$ is denoted by $\|\cdot\|_\infty$.

Let $\mathcal{K}(H)$ denote the ideal of compact operators on H. Given $T \in \mathcal{K}(H)$, the singular value function $t \mapsto \mu(t, T)$ is defined by the formula

$$\mu(t,T) := \inf\{\|T - R\|_\infty : \quad \operatorname{rank}(R) \le t\}, \quad t \ge 0.$$

We denote by $\mu(T)$ the sequence $\{\mu(n,T)\}_{n=0}^\infty$. Equivalently, $\mu(T)$ is the sequence of eigenvalues of $|T|$ arranged in nonincreasing order with multiplicities.

The following basic properties of singular values will be frequently used in the sequel:

$$\mu(t+s, A+B) \le \mu(t,A) + \mu(s,B), \quad t,s \ge 0\,, \tag{2}$$

and, for any $0 < p < \infty$,

$$\mu(t,A) = \mu(t,A^*) = \mu(t,|A|^p)^{\frac{1}{p}}, \quad t \ge 0\,; \tag{3}$$

see e.g. [36, Section 2.3].

Let $p \in (0,\infty)$. The Schatten class $\mathcal{L}_p$ is the set of operators T in $\mathcal{K}(H)$ such that $\mu(T)$ is p-summable, that is, belongs to the sequence space ℓ_p. If $1 \le p < \infty$, then the $\mathcal{L}_p$ norm is defined by

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \Big(\sum_{n=0}^\infty \mu(n,T)^p\Big)^{1/p}.$$

With this norm $\mathcal{L}_p$ for $1 \le p < \infty$ is a Banach space, and a Banach ideal of $\mathcal{B}(H)$.

Analogously, for $0 < p < \infty$, the weak Schatten class $\mathcal{L}_{p,\infty}$ is the set of operators $T \in \mathcal{K}(H)$ such that $\mu(T)$ is in the weak L_p-space $\ell_{p,\infty}$, with quasi-norm

$$\|T\|_{p,\infty} := \sup_{n\ge 0}\,(n+1)^{1/p}\mu(n,T) < \infty.$$

As with the $\mathcal{L}_p$ spaces, $\mathcal{L}_{p,\infty}$ is an ideal of $\mathcal{B}(H)$.

For more details on ideals of compact operators and singular value sequences, we refer the reader to [36,37,45].

We will frequently use the following two trace ideal bounds for operators of the form $M_f g(\nabla)$ on $L^2(\mathbb{R}^d)$. To formulate them, we use the Birman–Solomyak spaces $\ell_p(L_q)(\mathbb{R}^d)$ and $\ell_{p,\infty}(L_q)(\mathbb{R}^d)$ (see [7], [45, Chapter 4] or [4, Subsection 5.6]), as well as their analogue $\ell_{2,\log}(L_\infty)(\mathbb{R}^d)$ in [34, Definition 5.5]. The only fact about these spaces that we will be using is that they all contain $C_c(\mathbb{R}^d)$.

Theorem 2.3. *Let* $d \geq 1$.

(i) If $2 \leq p \leq \infty$ *and* $f, g \in L_p(\mathbb{R}^d)$, *then* $M_f g(\nabla) \in \mathcal{L}_p(L_2(\mathbb{R}^d))$ *and*

$$\|M_f g(\nabla)\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))} \leq c_{p,d} \|f\|_{L_p(\mathbb{R}^d)} \|g\|_{L_p(\mathbb{R}^d)}.$$

(ii) If $0 < p < 2$ *and* $f, g \in \ell_p(L_2)(\mathbb{R}^d)$, *then* $M_f g(\nabla) \in \mathcal{L}_p(L_2(\mathbb{R}^d))$ *and*

$$\|M_f g(\nabla)\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))} \leq c_{p,d} \|f\|_{\ell_p(L_2)(\mathbb{R}^d)} \|g\|_{\ell_p(L_2)(\mathbb{R}^d)}.$$

Part (i) is known as the Kato–Seiler–Simon inequality; for a proof see, e.g., [45, Theorem 4.1]. Part (2) is due to Birman and Solomyak [6, Theorem 11.1]; for a proof for $p \geq 1$ see also [45, Theorem 4.5]. For a strengthening see [34, Theorem 1.4].

Theorem 2.4. *Let* $d \geq 1$.

(i) If $2 < p < \infty$ *and* $f \in L_p(\mathbb{R}^d)$, $g \in L_{p,\infty}(\mathbb{R}^d)$, *then* $M_f g(\nabla) \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ *and*

$$\|M_f g(\nabla)\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))} \leq c_{p,d} \|f\|_{L_p(\mathbb{R}^d)} \|g\|_{L_{p,\infty}(\mathbb{R}^d)}.$$

(ii) If $p = 2$ *and* $f \in \ell_{2,\log}(L_\infty)(\mathbb{R}^d)$, $g \in \ell_{2,\infty}(L_4)(\mathbb{R}^d)$, *then* $M_f g(\nabla) \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ *and*

$$\|M_f g(\nabla)\|_{\mathcal{L}_{2,\infty}(L_2(\mathbb{R}^d))} \leq c_{2,d} \|f\|_{\ell_{2,\log}(L_\infty)(\mathbb{R}^d)} \|g\|_{\ell_{2,\infty}(L_4)(\mathbb{R}^d)}.$$

(iii) If $0 < p < 2$ *and* $f \in \ell_p(L_2)(\mathbb{R}^d)$, $g \in \ell_{p,\infty}(L_2)(\mathbb{R}^d)$, *then* $M_f g(\nabla) \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ *and*

$$\|M_f g(\nabla)\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))} \leq c_{p,d} \|f\|_{\ell_p(L_2)(\mathbb{R}^d)} \|g\|_{\ell_{p,\infty}(L_2)(\mathbb{R}^d)}.$$

Part (1) is known as the Cwikel inequality and due to [20]; for proofs see also, for instance, [45, Theorem 4.2] and [23,34]. Part (2) is due to two of us [34, Theorem 1.3]. Part (3) is due to Simon [45, Theorem 4.6] for $p > 1$ and appears in [4, Subsection 5.7] in general. For a strengthening see [34, Theorem 1.4].

3. Domain and codomain of the commutator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$

We begin by clarifying in which sense the commutator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ is understood for $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and $\epsilon \in (-d, 1]$.

Lemma 3.1. *Let $\epsilon \in (-d, 1]$ and let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$ there are unique distributions, denoted by*

$$(-\Delta)^{\frac{\epsilon}{2}} M_f \phi \quad \text{and} \quad M_f (-\Delta)^{\frac{\epsilon}{2}} \phi ,$$

such that for all $\psi \in C_c^\infty(\mathbb{R}^d)$ one has

$$\langle (-\Delta)^{\frac{\epsilon}{2}} M_f \phi, \psi \rangle = \langle f\phi, (-\Delta)^{\frac{\epsilon}{2}} \psi \rangle \qquad \text{and} \qquad \langle M_f (-\Delta)^{\frac{\epsilon}{2}} \phi, \psi \rangle = \langle (-\Delta)^{\frac{\epsilon}{2}} \phi, \overline{f} \psi \rangle .$$

Here $\langle \cdot, \cdot \rangle$ on the left sides denotes the sesquilinear duality pairing between $(C_c^\infty)'$ and C_c^∞, and on the right sides that between L_1 and L_∞ (L_∞ and L_1, respectively). The distribution $M_f(-\Delta)^{\frac{\epsilon}{2}}\phi$ is regular (that is, given by an $L_{1,\mathrm{loc}}$-function), and so is, for $\epsilon < 0$, the distribution $(-\Delta)^{\frac{\epsilon}{2}} M_f \phi$.
Moreover, if $\{f_n\}_{n \geq 0} \subset L_{1,\mathrm{loc}}(\mathbb{R}^d)$ is such that $f_n \to f$ in $L_{1,\mathrm{loc}}(\mathbb{R}^d)$, then for every $\phi \in C_c^\infty(\mathbb{R}^d)$, we have

$$(-\Delta)^{\frac{\epsilon}{2}} M_{f_n} \phi \to (-\Delta)^{\frac{\epsilon}{2}} M_f \phi, \quad M_{f_n} (-\Delta)^{\frac{\epsilon}{2}} \phi \to M_f (-\Delta)^{\frac{\epsilon}{2}} \phi, \quad n \to \infty,$$

in $(C_c^\infty(\mathbb{R}^d))'$.

As a consequence of this lemma, the mappings

$$M_f (-\Delta)^{\frac{\epsilon}{2}} : C_c^\infty(\mathbb{R}^d) \to L_{1,\mathrm{loc}}(\mathbb{R}^d)$$

and

$$(-\Delta)^{\frac{\epsilon}{2}} M_f : C_c^\infty(\mathbb{R}^d) \to \begin{cases} L_{1,\mathrm{loc}}(\mathbb{R}^d) & \text{if } \epsilon \in (-d, 0) , \\ (C_c^\infty(\mathbb{R}^d))' & \text{if } \epsilon \in (0, 1] . \end{cases}$$

are well defined.

Proof. We begin with the case $\epsilon \in (-d, 0)$. It is well known (see, e.g., [46, Chapter V, Theorem 1]) that $(-\Delta)^{\frac{\epsilon}{2}} : L_1(\mathbb{R}^d) \to L_{\frac{d}{d+\epsilon},\infty}(\mathbb{R}^d)$. If $\phi \in C_c^\infty(\mathbb{R}^d)$, then $f\phi \in L_1(\mathbb{R}^d)$ and, therefore,

$$\big((-\Delta)^{\frac{\epsilon}{2}} M_f\big)(\phi) = (-\Delta)^{\frac{\epsilon}{2}} (f\phi) \in L_{\frac{d}{d+\epsilon},\infty}(\mathbb{R}^d) \subset L_{1,\mathrm{loc}}(\mathbb{R}^d).$$

Moreover, by duality, $(-\Delta)^{\frac{\epsilon}{2}} : L_{-\frac{d}{\epsilon},1}(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d)$. Therefore,

$$\big(M_f(-\Delta)^{\frac{\epsilon}{2}}\big)(\phi) = f \cdot (-\Delta)^{\frac{\epsilon}{2}}\phi \in L_{1,\mathrm{loc}}(\mathbb{R}^d) \cdot L_\infty(\mathbb{R}^d) \subset L_{1,\mathrm{loc}}(\mathbb{R}^d).$$

Thus, both $(-\Delta)^{\frac{\epsilon}{2}} M_f\phi$ and $M_f(-\Delta)^{\frac{\epsilon}{2}}\phi$ are regular distributions.

Now let $\epsilon \in (0,1]$. Then $(-\Delta)^{\frac{\epsilon}{2}} : C_c^\infty(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d)$. Thus,

$$(M_f(-\Delta)^{\frac{\epsilon}{2}})(\phi) = f \cdot (-\Delta)^{\frac{\epsilon}{2}}\phi \in L_{1,\mathrm{loc}}(\mathbb{R}^d) \cdot L_\infty(\mathbb{R}^d) \subset L_{1,\mathrm{loc}}(\mathbb{R}^d).$$

Moreover, by duality, $(-\Delta)^{\frac{\epsilon}{2}} : (L_\infty(\mathbb{R}^d))' \to (C_c^\infty(\mathbb{R}^d))'$, where on $(L_\infty(\mathbb{R}^d))'$ we consider the weak star topology. Since $f\phi \in L_1(\mathbb{R}^d) \subset (L_\infty(\mathbb{R}^d))'$, we have

$$((-\Delta)^{\frac{\epsilon}{2}} M_f)(\phi) = (-\Delta)^{\frac{\epsilon}{2}}(f\phi) \in (C_c^\infty(\mathbb{R}^d))'.$$

This proves the existence of the two distributions. The uniqueness is clear. It remains to prove the convergence statement. For the distribution $(-\Delta)^{\frac{\epsilon}{2}} M_f\phi$ we use the fact that $f_n\phi \to f\phi$ in L_1 and therefore

$$\langle f_n\phi, (-\Delta)^{\frac{\epsilon}{2}}\psi\rangle \to \langle f\phi, (-\Delta)^{\frac{\epsilon}{2}}\psi\rangle\,.$$

The proof for $M_f(-\Delta)^{\frac{\epsilon}{2}}\phi$ is similar, using $\overline{f_n}\psi \to \overline{f}\psi$ in L_1. $\square$

Lemma 3.2. *Let* $\epsilon \in (-\frac{d}{2}, 1]$ *and let* $f \in L_\infty(\mathbb{R}^d)$ *and* $\phi \in C_c^\infty(\mathbb{R}^d)$. *For* $m \geq 1$ *let* $P_m = \chi_{(\frac{1}{m}, m)}(-\Delta)$. *Then*

$$(P_m(-\Delta)^{\frac{\epsilon}{2}})M_f P_m\phi \to (-\Delta)^{\frac{\epsilon}{2}} M_f\phi, \quad P_m M_f(P_m(-\Delta)^{\frac{\epsilon}{2}})\phi \to M_f(-\Delta)^{\frac{\epsilon}{2}}\phi, \quad m \to \infty,$$

in $(C_c^\infty(\mathbb{R}^d))'$. *Here* $(-\Delta)^{\frac{\epsilon}{2}} M_f\phi$ *and* $M_f(-\Delta)^{\frac{\epsilon}{2}}\phi$ *are defined as in Lemma 3.1.*

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$. Then, with $\langle\cdot,\cdot\rangle$ denoting the L_2-inner product,

$$\begin{aligned}\langle((P_m(-\Delta)^{\frac{\epsilon}{2}}) \cdot M_f \cdot P_m)\phi, \psi\rangle &= \langle f \cdot P_m\phi, (P_m(-\Delta)^{\frac{\epsilon}{2}})\psi\rangle,\\ \langle(P_m \cdot M_f \cdot (P_m(-\Delta)^{\frac{\epsilon}{2}})\phi, \psi\rangle &= \langle(P_m(-\Delta)^{\frac{\epsilon}{2}})\phi, \overline{f} \cdot P_m\psi\rangle.\end{aligned}$$

Let χ denote either ϕ or ψ. We claim that

$$\chi, (-\Delta)^{\frac{\epsilon}{2}}\chi \in L_2(\mathbb{R}^d). \tag{4}$$

Indeed, for χ this is clear. When $\epsilon \in (0,1]$ it is also clear for $(-\Delta)^{\frac{\epsilon}{2}}\chi$. Thus let $\epsilon \in (-\frac{d}{2}, 0)$. As observed in the proof of Lemma 3.1, we have

$$(-\Delta)^{\frac{\epsilon}{2}} : L_1(\mathbb{R}^d) \to L_{\frac{d}{d+\epsilon},\infty}(\mathbb{R}^d), \quad (-\Delta)^{\frac{\epsilon}{2}} : L_{-\frac{d}{\epsilon},1}(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d).$$

Therefore,

$$(-\Delta)^{\frac{\epsilon}{2}} : L_1(\mathbb{R}^d) \cap L_{-\frac{d}{\epsilon},1}(\mathbb{R}^d) \to L_{\frac{d}{d+\epsilon},\infty}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \subset L_2(\mathbb{R}^d).$$

In particular, $(-\Delta)^{\frac{\epsilon}{2}}\chi \in L_2(\mathbb{R}^d)$. This proves (4).

It follows from (4) that $P_m\chi \to \chi$ and $P_m(-\Delta)^{\frac{\epsilon}{2}}\chi \to (-\Delta)^{\frac{\epsilon}{2}}\chi$ in $L_2(\mathbb{R}^d)$. This, together with $f \in L_\infty(\mathbb{R}^d)$, implies

$$\begin{aligned}\langle f \cdot P_m\phi, (P_m(-\Delta)^{\frac{\epsilon}{2}})\psi\rangle &\to \langle f \cdot \phi, (-\Delta)^{\frac{\epsilon}{2}}\psi\rangle\,,\\ \langle (P_m(-\Delta)^{\frac{\epsilon}{2}})\phi, \overline{f} \cdot P_m\psi\rangle &\to \langle (-\Delta)^{\frac{\epsilon}{2}}\phi, \overline{f}\cdot\psi\rangle\,.\end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L_2-inner product. Alternatively, it can be interpreted as the duality pairing between L_1 and L_∞ (L_∞ and L_1, respectively), and then, by Lemma 3.1, the right sides coincide with $\langle(-\Delta)^{\frac{\epsilon}{2}}M_f\phi,\psi\rangle$ and $\langle M_f(-\Delta)^{\frac{\epsilon}{2}}\phi,\psi\rangle$, respectively, where now $\langle\cdot,\cdot\rangle$ denotes the sesquilinear duality pairing between $(C_c^\infty(\mathbb{R}^d))'$ and $C_c^\infty(\mathbb{R}^d)$. This proves the claimed convergence in $C_c^\infty(\mathbb{R}^d)$. □

Having defined the operator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f] : C_c^\infty(\mathbb{R}^d) \to (C_c^\infty)'(\mathbb{R}^d)$, we are now interested in when it extends to a bounded operator on $L_2(\mathbb{R}^d)$ and belongs to certain operator ideal. The following theorem provides a negative result in a certain range of negative ϵ's.

Lemma 3.3. *If $\epsilon \in (-d, -\frac{d}{2}]$ and if $0 \neq f \in L_1(\mathbb{R}^d)$ is compactly supported, then the operator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ does not map $L_2(\mathbb{R}^d)$ to itself.*

Proof. The operator $(-\Delta)^{\frac{\epsilon}{2}}$ is an integral operator with an integral kernel $|t-s|^{-d-\epsilon}$ (up to a multiplicative constant). Hence, the operators $M_f(-\Delta)^{\frac{\epsilon}{2}}, (-\Delta)^{\frac{\epsilon}{2}}M_f : C_c^\infty(\mathbb{R}^d) \to L_{1,\mathrm{loc}}(\mathbb{R}^d)$ (those operators are well defined, see the proof of Lemma 3.1) are also integral ones. Thus, $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ is an integral operator with the integral kernel given, up to a multiplicative constant, by

$$(t,s) \mapsto \frac{f(t)-f(s)}{|t-s|^{d+\epsilon}}, \quad t,s \in \mathbb{R}^d.$$

Suppose f is supported in the ball $B(0,R)$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be such that

$$\int\limits_{\mathbb{R}^d} (f\phi)(s)ds \neq 0.$$

We have

$$\Big(\big([(-\Delta)^{\frac{\epsilon}{2}}, M_f]\big)\phi\Big)(t) = -\int\limits_{|s|\le R} \frac{(f\phi)(s)}{|t-s|^{d+\epsilon}}ds, \quad |t| > R.$$

It follows that

$$\Big(\big([(-\Delta)^{\frac{\epsilon}{2}}, M_f]\big)\phi\Big)(t) = -|t|^{-d-\epsilon} \cdot \int\limits_{|s|\le R} (f\phi)(s)ds + O(|t|^{-d-\epsilon-1}), \quad |t| > 2R.$$

The function on the right hand side does not belong to $L_2(\mathbb{R}^d \backslash B(0,2R))$. Hence, we have $\big([(-\Delta)^{\frac{\epsilon}{2}}, M_f]\big)\phi \notin L_2(\mathbb{R}^d)$. □

Let now $-\frac{d}{2} < \epsilon < 0$. By the weak Young inequality (see, e.g., [46, Chapter V, Theorem 1]), Hölder's inequality and Sobolev's inequality (see Theorem 2.2) we find

$$\|M_f(-\Delta)^{\frac{\epsilon}{2}}\|_\infty,\ \|(-\Delta)^{\frac{\epsilon}{2}} M_f\|_\infty \leq c^{(1)}_{d,\epsilon}\|f\|_{-\frac{d}{\epsilon}} \leq c^{(2)}_{d,\epsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}.$$

Hence, for $-\frac{d}{2} < \epsilon < 0$ and $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}$ the commutator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ is a difference of two bounded operators and is, therefore, bounded. Moreover,

$$\Big\|[(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big\|_\infty \leq c_{d,\epsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}, \quad f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d), \quad -\frac{d}{2} < \epsilon < 0. \tag{5}$$

4. Proof of Theorem 1.1 (I) and Theorem 1.2 (I)

If $\mathbf{A} = (A_1, \ldots, A_d)$ is a d-tuple of mutually commuting self-adjoint operator on a Hilbert space and ϕ is a sufficiently regular function on $\mathbb{R}^d \times \mathbb{R}^d$, then the symbol $T^{\mathbf{A}}_\phi$ denotes the corresponding Double Operator Integral (DOI). We refer the reader to [30,12] for the notion of a DOI with respect to tuples of mutually commuting operators. The following lemma shows that for certain ϕ this DOI can be reduced to one for a single operator. It is the exact analogue of [42, Lemma 8] and its proof is exactly the same as in [42], so we omit it.

Lemma 4.1. *Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function and let $h : \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function. Set*

$$\phi(\lambda, \mu) = \Psi(h(\lambda), h(\mu)), \quad \lambda, \mu \in \mathbb{R}^d.$$

For every tuple $\mathbf{A} = (A_1, \cdots, A_d)$ of commuting self-adjoint operators we have

$$T^{\mathbf{A}}_\phi = T^{h(\mathbf{A})}_\Psi.$$

For a fixed $\epsilon \in \mathbb{R}$, define the function ϕ_ϵ on $\mathbb{R}^d \times \mathbb{R}^d$ by setting

$$\phi_\epsilon(\lambda, \mu) := \begin{cases} \frac{|\lambda|^\epsilon - |\mu|^\epsilon}{|\lambda| - |\mu|} \cdot |\lambda|^{\frac{1-\epsilon}{2}} |\mu|^{\frac{1-\epsilon}{2}} & \text{if } |\lambda| \neq |\mu|, \\ \epsilon & \text{if } |\lambda| = |\mu|. \end{cases} \tag{6}$$

Lemma 4.2. *Let $\epsilon \in [-1, 1]$ and let ϕ_ϵ be as in (6). Then for any $1 < p < \infty$ the operator $T^{\nabla}_{\phi_\epsilon}$ is bounded on $\mathcal{L}_p$ and on $\mathcal{L}_{p,\infty}$*

Proof. For $\epsilon = \pm 1$, ϕ_ϵ is a constant and there is nothing to prove. Suppose $\epsilon \in (-1, 1)$. By Lemma 4.1 applied with $h(\lambda) = |\lambda|$, we have

$$T^{\nabla}_{\phi_\epsilon} = T^{(-\Delta)^{\frac{1}{2}}}_{\Psi_\epsilon}$$

with

$$\Psi_\epsilon(s,t) := \frac{s^\epsilon - t^\epsilon}{s-t} \cdot s^{\frac{1-\epsilon}{2}} t^{\frac{1-\epsilon}{2}}, \quad s,t>0.$$

Thus,

$$\Psi_\epsilon(s,t) = h_\epsilon(\frac{s}{t}), \quad s,t>0,$$

with the function h_ϵ given by the formula

$$(h_\epsilon \circ \exp)(t) = \frac{e^{\epsilon t}-1}{e^t-1} \cdot e^{\frac{(1-\epsilon)t}{2}} = \frac{\sinh(\frac{\epsilon t}{2})}{\sinh(\frac{t}{2})}, \quad t \in \mathbb{R}.$$

In particular, $h_\epsilon \circ \exp$ is a Schwartz function. By [42, Lemma 9], we have

$$T^{(-\Delta)^{\frac{1}{2}}}_{\Psi_\epsilon} : \mathcal{L}_1 \to \mathcal{L}_1, \quad T^{(-\Delta)^{\frac{1}{2}}}_{\Psi_\epsilon} : \mathcal{L}_\infty \to \mathcal{L}_\infty.$$

Therefore, by interpolation,

$$T^{(-\Delta)^{\frac{1}{2}}}_{\Psi_\epsilon} : \mathcal{L}_p \to \mathcal{L}_p, \quad T^{(-\Delta)^{\frac{1}{2}}}_{\Psi_\epsilon} : \mathcal{L}_{p,\infty} \to \mathcal{L}_{p,\infty}, \quad 1<p<\infty,$$

proving the assertion. □

For $\epsilon \in \mathbb{R}$ and $n \in \mathbb{N}$, define the function $\theta_{n,\epsilon}$ on $\mathbb{R}^d \times \mathbb{R}^d$ by setting

$$\theta_{n,\epsilon}(\lambda,\mu) := \sum_{l=1}^{n} (\frac{|\lambda|}{|\mu|})^{\frac{\epsilon+2l-1}{2}} + \sum_{l=1}^{n} (\frac{|\mu|}{|\lambda|})^{\frac{\epsilon+2l-1}{2}}, \ \lambda,\mu \in \mathbb{R}^d, \ \lambda,\mu \neq 0.$$

Lemma 4.3. *Let $n \in \mathbb{N}$ and let $\epsilon \in [-2n-1,-2n+1)$. We have*

$$\phi_\epsilon(\lambda,\mu) = \phi_{\epsilon+2n}(\lambda,\mu) - \theta_{n,\epsilon}(\lambda,\mu), \quad \lambda,\mu \in \mathbb{R}^d, \quad \lambda,\mu \neq 0.$$

Proof. For every $\epsilon \in \mathbb{R}$, we have

$$\begin{gathered}\phi_\epsilon(\lambda,\mu) - \phi_{\epsilon+2}(\lambda,\mu) = \\ = \frac{|\lambda|^\epsilon - |\mu|^\epsilon}{|\lambda|-|\mu|} \cdot |\lambda|^{\frac{1-\epsilon}{2}} |\mu|^{\frac{1-\epsilon}{2}} - \frac{|\lambda|^{\epsilon+2} - |\mu|^{\epsilon+2}}{|\lambda|-|\mu|} \cdot |\lambda|^{-\frac{1+\epsilon}{2}} |\mu|^{-\frac{1+\epsilon}{2}} = \\ = \frac{|\lambda|^{-\frac{1+\epsilon}{2}} |\mu|^{-\frac{1+\epsilon}{2}}}{|\lambda|-|\mu|} \cdot \Big(|\lambda||\mu|(|\lambda|^\epsilon - |\mu|^\epsilon) - (|\lambda|^{\epsilon+2} - |\mu|^{\epsilon+2}) \Big).\end{gathered}$$

Note that

$$|\lambda||\mu|(|\lambda|^{\epsilon}-|\mu|^{\epsilon})-(|\lambda|^{\epsilon+2}-|\mu|^{\epsilon+2})=$$
$$=-|\lambda|\cdot|\mu|^{\epsilon+1}+|\mu|\cdot|\mu|^{\epsilon+1}-|\lambda|\cdot|\lambda|^{\epsilon+1}+|\mu|\cdot|\lambda|^{\epsilon+1}=$$
$$=-(|\lambda|-|\mu|)(|\lambda|^{\epsilon+1}+|\mu|^{\epsilon+1}).$$

Thus,

$$\phi_\epsilon(\lambda,\mu)-\phi_{\epsilon+2}(\lambda,\mu)=$$
$$=-|\lambda|^{-\frac{1+\epsilon}{2}}|\mu|^{-\frac{1+\epsilon}{2}}\cdot(|\lambda|^{\epsilon+1}+|\mu|^{\epsilon+1})=-(\frac{|\lambda|}{|\mu|})^{\frac{\epsilon+1}{2}}-(\frac{|\mu|}{|\lambda|})^{\frac{\epsilon+1}{2}}.$$

The assertion follows now by induction on n. □

Define the function ψ_k, $1\le k\le d$, on $\mathbb{R}^d\times\mathbb{R}^d$ by setting

$$\psi_k(\lambda,\mu):=\begin{cases}\frac{|\lambda|-|\mu|}{|\lambda-\mu|^2}\cdot(\lambda_k-\mu_k) & \text{if } \lambda\neq\mu\,,\\ 0 & \text{if } \lambda=\mu\,.\end{cases}\tag{7}$$

Lemma 4.4. *Let $1\le k\le d$ and let ψ_k be as in (7). For all $1<p<\infty$ the operator $T^{\nabla}_{\psi_k}$ is bounded on $\mathcal{L}_p$ and on $\mathcal{L}_{p,\infty}$.*

Proof. Boundedness of $T^{\nabla}_{\psi_k}:\mathcal{L}_p\to\mathcal{L}_p$, $1<p<\infty$, is established in [12, Corollary 5.2]; see also [11, Theorem 5.1]. By interpolation, $T^{\nabla}_{\psi_k}:\mathcal{L}_{p,\infty}\to\mathcal{L}_{p,\infty}$, $1<p<\infty$, is bounded as well. □

For a tuple $\mathbf{A}=(A_1,\dots,A_d)$ we write

$$|\mathbf{A}|=(A_1^2+\ldots+A_d^2)^{\frac{1}{2}}\,.$$

In the next two lemmas we derive representations for commutators $[|\mathbf{A}|^{\epsilon},B]$ as DOIs.

Lemma 4.5. *Let $\epsilon\in\mathbb{R}$ and let ϕ_ϵ and ψ_k be as in (6) and (7), respectively. If $B\in\mathcal{L}_2$ and if $\mathbf{A}$ is a tuple of bounded self-adjoint, mutually commuting operators such that $|\mathbf{A}|$ is bounded from below by a strictly positive constant, then*

$$[|\mathbf{A}|^{\epsilon},B]=\sum_{k=1}^{d}T^{\mathbf{A}}_{\phi_\epsilon\cdot\psi_k}(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}).$$

Proof. Let g_k be the function on $\mathbb{R}^d\times\mathbb{R}^d$ given by the formula

$$g_k(\lambda,\mu)=|\lambda|^{\frac{\epsilon-1}{2}}(\lambda_k-\mu_k)|\mu|^{\frac{\epsilon-1}{2}},\quad \lambda,\mu\in\mathbb{R}^d.$$

Note that g_k is bounded when μ,λ range over compact sets in $\mathbb{R}^d\setminus\{0\}$. Then we have

$$|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}=T^{\mathbf{A}}_{g_k}(B).$$

It follows that

$$\sum_{k=1}^{d}T^{\mathbf{A}}_{\phi_\epsilon\cdot\psi_k}(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}})=$$

$$=\sum_{k=1}^{d}T^{\mathbf{A}}_{\phi_\epsilon\cdot\psi_k}\Big(T^{\mathbf{A}}_{g_k}(B)\Big)=\sum_{k=1}^{d}T^{\mathbf{A}}_{\phi_\epsilon\cdot\psi_k\cdot g_k}(B)=T^{\mathbf{A}}_{\sum_{k=1}^{d}\phi_\epsilon\cdot\psi_k\cdot g_k}(B).$$

Since

$$(\sum_{k=1}^{d}\phi_\epsilon\cdot\psi_k\cdot g_k)(\lambda,\mu)=|\lambda|^\epsilon-|\mu|^\epsilon,\quad \lambda,\mu\in\mathbb{R}^d,$$

we have

$$T^{\mathbf{A}}_{\sum_{k=1}^{d}\phi_\epsilon\cdot\psi_k\cdot g_k}(B)=|\mathbf{A}|^\epsilon B-B|\mathbf{A}|^\epsilon,$$

which proves the assertion. □

Lemma 4.6. *Let $n\in\mathbb{N}$, let $\epsilon\in[-2n-1,-2n+1)$ and let ϕ_ϵ and ψ_k be as in (6) and (7), respectively. If $B\in\mathcal{L}_2$ and if $\mathbf{A}$ is a tuple of bounded self-adjoint, mutually commuting operators such that $|\mathbf{A}|$ is bounded from below by a strictly positive constant, then*

$$\begin{aligned}[|\mathbf{A}|^\epsilon,B]=&\sum_{k=1}^{d}T^{\mathbf{A}}_{\phi_{\epsilon+2n}\cdot\psi_k}\Big(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\Big)\\&-\sum_{k=1}^{d}T^{\mathbf{A}}_{\psi_k}\Big(\sum_{l=1}^{n}|\mathbf{A}|^{\epsilon+l-1}[A_k,B]|\mathbf{A}|^{-l}+\sum_{l=1}^{n}|\mathbf{A}|^{-l}[A_k,B]|\mathbf{A}|^{\epsilon+l-1}\Big)\end{aligned}$$

Proof. By Lemma 4.5 and Lemma 4.3, we have

$$[|\mathbf{A}|^\epsilon,B]=\sum_{k=1}^{d}T^{\mathbf{A}}_{(\phi_{\epsilon+2n}-\theta_{n,\epsilon})\cdot\psi_k}(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}})=$$

$$=\sum_{k=1}^{d}T^{\mathbf{A}}_{\phi_{\epsilon+2n}\cdot\psi_k}\Big(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\Big)-\sum_{k=1}^{d}T^{\mathbf{A}}_{\psi_k}\Big(T^{\mathbf{A}}_{\theta_{n,\epsilon}}\Big(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\Big)\Big).$$

It is immediate from the definition of $\theta_{n,\epsilon}$, given before Lemma 4.3, that

$$T^{\mathbf{A}}_{\theta_{n,\epsilon}}\Big(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\Big)=$$

$$=\sum_{l=1}^{n}|\mathbf{A}|^{\frac{\epsilon+2l-1}{2}}\cdot|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\cdot|\mathbf{A}|^{-\frac{\epsilon+2l-1}{2}}+$$

$$+\sum_{l=1}^{n}|\mathbf{A}|^{-\frac{\epsilon+2l-1}{2}}\cdot|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k,B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\cdot|\mathbf{A}|^{\frac{\epsilon+2l-1}{2}}=$$

$$=\sum_{l=1}^{n}|\mathbf{A}|^{\epsilon+l-1}[A_k,B]|\mathbf{A}|^{-l}+\sum_{l=1}^{n}|\mathbf{A}|^{-l}[A_k,B]|\mathbf{A}|^{\epsilon+l-1}.$$

This proves the claimed representation. □

Our goal is to apply the representation formulas in Lemmas 4.5 and 4.6 to (regularizations of) $A=\nabla$ and $B=M_f$. To that end, we now derive trace ideal properties of the operators that appear. It is exactly in the following lemma that we are using the assumption $\epsilon>-\frac{d}{2}$.

Lemma 4.7. *Let $\epsilon\in(-\frac{d}{2},1)$ with $\epsilon>0$ when $d=1$. Let $f\in L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$.*

(i) *We have*

$$\left\|(-\Delta)^{\frac{\epsilon-1}{4}}M_f(-\Delta)^{\frac{\epsilon-1}{4}}\right\|_{\frac{d}{1-\epsilon},\infty}\leq c_{d,\epsilon}\|f\|_{\frac{d}{1-\epsilon}}.$$

(ii) *If $n\in\mathbb{N}$ is such that $\epsilon\in[-2n-1,-2n+1)$, then we have*

$$\left\|(-\Delta)^{\frac{\epsilon+l-1}{2}}M_f(-\Delta)^{-\frac{l}{2}}\right\|_{\frac{d}{1-\epsilon},\infty}\leq c'_{d,\epsilon}\|f\|_{\frac{d}{1-\epsilon}},\quad 1\leq l\leq n.$$

Proof. In the proof of (i), we are only using the assumption $\epsilon\in(1-d,1)$. We write

$$(-\Delta)^{\frac{\epsilon-1}{4}}M_f(-\Delta)^{\frac{\epsilon-1}{4}}=(-\Delta)^{\frac{\epsilon-1}{4}}M_{|f|^{\frac{1}{2}}}\cdot M_{\mathrm{sgn}(f)}\cdot M_{|f|^{\frac{1}{2}}}(-\Delta)^{\frac{\epsilon-1}{4}}.$$

By Hölder's inequality for weak $\mathcal{L}_p$ ideals (see [8, Theorem 11.6.9] and, concerning sharp constants, [50]), we write

$$\left\|(-\Delta)^{\frac{\epsilon-1}{4}}M_f(-\Delta)^{\frac{\epsilon-1}{4}}\right\|_{\frac{d}{1-\epsilon},\infty}\leq$$

$$\leq c^{(1)}_{d,\epsilon}\left\|(-\Delta)^{\frac{\epsilon-1}{4}}M_{|f|^{\frac{1}{2}}}\right\|_{\frac{2d}{1-\epsilon},\infty}\cdot\left\|M_{|f|^{\frac{1}{2}}}(-\Delta)^{\frac{\epsilon-1}{4}}\right\|_{\frac{2d}{1-\epsilon},\infty}.$$

Since $\epsilon\in(1-d,1)$, it follows that $\frac{2d}{1-\epsilon}>2$. Note that the function $t\mapsto|t|^{\frac{\epsilon-1}{2}}$, $t\in\mathbb{R}^d$, falls into $L_{\frac{2d}{1-\epsilon},\infty}(\mathbb{R}^d)$. Hence, Theorem 2.4 is applicable to the functions $|f|^{1/2}$ and $g(t)=|t|^{\frac{\epsilon-1}{2}}$ and we can write

$$\left\|(-\Delta)^{\frac{\epsilon-1}{4}}M_{|f|^{\frac{1}{2}}}\right\|_{\frac{2d}{1-\epsilon},\infty},\left\|M_{|f|^{\frac{1}{2}}}(-\Delta)^{\frac{\epsilon-1}{4}}\right\|_{\frac{2d}{1-\epsilon},\infty}\leq c^{(2)}_{d,\epsilon}\||f|^{\frac{1}{2}}\|_{\frac{2d}{1-\epsilon}}=c^{(2)}_{d,\epsilon}\|f\|^{\frac{1}{2}}_{\frac{d}{1-\epsilon}}.$$

Combining these estimates, we obtain (i).

In the proof of (ii), we note that ϵ is negative and that $1-\epsilon > n \geq l$, $1 \leq l \leq n$. We write

$$(-\Delta)^{\frac{\epsilon+l-1}{2}} M_f (-\Delta)^{-\frac{l}{2}} = \\ = (-\Delta)^{\frac{\epsilon+l-1}{2}} M_{|f|^{1-\frac{l}{1-\epsilon}}} \cdot M_{\mathrm{sgn}(f)} \cdot M_{|f|^{\frac{l}{1-\epsilon}}} (-\Delta)^{-\frac{l}{2}}.$$

Using Hölder inequality for weak $\mathcal{L}_p$ ideals, we write

$$\left\| (-\Delta)^{\frac{\epsilon+l-1}{2}} M_f (-\Delta)^{-\frac{l}{2}} \right\|_{\frac{d}{1-\epsilon},\infty} \leq \\ \leq c_{l,d,\epsilon} \left\| (-\Delta)^{\frac{\epsilon+l-1}{2}} M_{|f|^{1-\frac{l}{1-\epsilon}}} \right\|_{\frac{d}{1-\epsilon-l},\infty} \left\| M_{|f|^{\frac{l}{1-\epsilon}}} (-\Delta)^{-\frac{l}{2}} \right\|_{\frac{d}{l},\infty}.$$

Note that

$$\frac{d}{l} \geq \frac{d}{n} > 2, \quad \frac{d}{1-\epsilon-l} \geq \frac{d}{-\epsilon} > 2.$$

Note that the function $t \mapsto |t|^{\epsilon+l-1}$, $t \in \mathbb{R}^d$ (respectively, the function $t \mapsto |t|^{-l}$, $t \in \mathbb{R}^d$) falls into $L_{\frac{d}{1-\epsilon-l},\infty}(\mathbb{R}^d)$ (respectively, into $L_{\frac{d}{l},\infty}(\mathbb{R}^d)$). Hence, Theorem 2.4 is applicable and we can write

$$\left\| (-\Delta)^{\frac{\epsilon+l-1}{2}} M_{|f|^{1-\frac{l}{1-\epsilon}}} \right\|_{\frac{d}{1-\epsilon-l},\infty} \leq c^{(3)}_{d,\epsilon} \| |f|^{1-\frac{l}{1-\epsilon}} \|_{\frac{d}{1-\epsilon-l}} = c^{(3)}_{d,\epsilon} \|f\|^{1-\frac{l}{1-\epsilon}}_{\frac{d}{1-\epsilon}},$$

$$\left\| M_{|f|^{\frac{l}{1-\epsilon}}} (-\Delta)^{-\frac{l}{2}} \right\|_{\frac{d}{l},\infty} \leq c^{(4)}_{d,\epsilon} \| |f|^{\frac{l}{1-\epsilon}} \|_{\frac{d}{l}} = c^{(4)}_{d,l} \|f\|^{\frac{l}{1-\epsilon}}_{\frac{d}{1-\epsilon}}.$$

Combining these estimates, we obtain (ii). □

The following corollary follows immediately by applying Lemma 4.7 with f replaced by $D_k f$.

Lemma 4.8. *Let $\epsilon \in (-\frac{d}{2}, 1)$ with $\epsilon > 0$ when $d = 1$, and let $n \in \mathbb{Z}_+$ with $\epsilon \in [-2n-1, -2n+1)$. For $f \in C_c^\infty(\mathbb{R}^d)$ set*

$$X_k = (-\Delta)^{\frac{\epsilon-1}{4}} M_{D_k f} (-\Delta)^{\frac{\epsilon-1}{4}}, \quad 1 \leq k \leq d,$$

$$Y_k = \sum_{l=1}^{n} \left((-\Delta)^{\frac{\epsilon+l-1}{2}} M_{D_k f} (-\Delta)^{-\frac{l}{2}} + (-\Delta)^{-\frac{l}{2}} M_{D_k f} (-\Delta)^{\frac{\epsilon+l-1}{2}} \right), \quad 1 \leq k \leq d.$$

Here, for $n = 0$, Y_k is assumed to be zero. Then

$$\|X_k\|_{\frac{d}{1-\epsilon},\infty}, \|Y_k\|_{\frac{d}{1-\epsilon},\infty} \leq \|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}}.$$

We are now in position to state and prove a useful representation formula for the commutators of interest.

Proposition 4.9. *Let $d \geq 2$ and let $\epsilon \in (-\frac{d}{2}, 1)$ (alternatively, let $d = 1$ and $\epsilon \in (0, 1)$). Choose $n \in \mathbb{Z}_+$ with $\epsilon \in [-2n-1, 2n+1)$. For every $f \in C_c^\infty(\mathbb{R}^d)$, we have*

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] = \sum_{k=1}^{d} T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_{\epsilon+2n}}(X_k)\Big) - \sum_{k=1}^{d} T^{\nabla}_{\psi_k}(Y_k).$$

Here, X_k and Y_k are given in Lemma 4.8, ϕ_ϵ is given in (6) and ψ_k is given in (7).

Proof. Fix $m \in \mathbb{N}$. Let $P_m = \chi_{(\frac{1}{m}, m)}(-\Delta)$. Consider the tuple $\mathbf{A} = \nabla \cdot P_m$ and $B = P_m M_f P_m$ on the Hilbert space $P_m(L_2(\mathbb{R}^d))$. Clearly, $\mathbf{A}$ is bounded and $|\mathbf{A}|$ is bounded from below. Also, $B \in \mathcal{L}_2$ (due to the fact that $M_f g(\nabla)$ belongs to $\mathcal{L}_2$ whenever $f, g \in L_2(\mathbb{R}^d)$). We now apply Lemma 4.6 (for $n \in \mathbb{N}$) or Lemma 4.5 (for $n = 0$). Abbreviating $\delta := \epsilon + 2n$ we obtain

$$\begin{aligned} T_m &:= (P_m(-\Delta)^{\frac{\epsilon}{2}}) \cdot M_f \cdot P_m - P_m \cdot M_f \cdot (P_m(-\Delta)^{\frac{\epsilon}{2}}) = [|\mathbf{A}|^\epsilon, B] \\ &= \sum_{k=1}^{d} T^{\mathbf{A}}_{\psi_k}\Big(T^{\mathbf{A}}_{\phi_\delta}\Big(|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k, B]|\mathbf{A}|^{\frac{\epsilon-1}{2}}\Big)\Big) \\ &\quad - \sum_{k=1}^{d} T^{\mathbf{A}}_{\psi_k}\Big(\sum_{l=1}^{n} |\mathbf{A}|^{\epsilon+l-1}[A_k, B]|\mathbf{A}|^{-l} + \sum_{l=1}^{n} |\mathbf{A}|^{-l}[A_k, B]|\mathbf{A}|^{\epsilon+l-1}\Big). \end{aligned}$$

Since

$$|\mathbf{A}|^{\frac{\epsilon-1}{2}}[A_k, B]|\mathbf{A}|^{\frac{\epsilon-1}{2}} = P_m X_k P_m,$$

$$\sum_{l=1}^{n} |\mathbf{A}|^{\epsilon+l-1}[A_k, B]|\mathbf{A}|^{-l} + \sum_{l=1}^{n} |\mathbf{A}|^{-l}[A_k, B]|\mathbf{A}|^{\epsilon+l-1} = P_m Y_k P_m,$$

we see that

$$T_m = \sum_{k=1}^{d} T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_\delta}\Big(P_m \cdot X_k \cdot P_m\Big)\Big) - \sum_{k=1}^{d} T^{\nabla}_{\psi_k}\Big(P_m \cdot Y_k \cdot P_m\Big).$$

We now want to pass to the limit $m \to \infty$. We have, by Lemma 4.8,

$$X_k, Y_k \in \mathcal{L}_{\frac{d}{1-\epsilon}, \infty} \subset \mathcal{L}_p, \quad \frac{d}{1-\epsilon} < p < \infty.$$

Since $P_m \uparrow 1$ strongly, it follows that

$$P_m \cdot X_k \cdot P_m \to X_k, \quad P_m \cdot Y_k \cdot P_m \to Y_k \qquad \text{in } \mathcal{L}_p.$$

The operators $T^{\nabla}_{\phi_\delta}$ and $T^{\nabla}_{\psi_k}$ are bounded in $\mathcal{L}_p$ by Lemma 4.2 and Lemma 4.4, respectively. It follows that

$$T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_\delta}\Big(P_m \cdot X_k \cdot P_m\Big)\Big) \to T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_\delta}(X_k)\Big), \qquad T^{\nabla}_{\psi_k}(P_m \cdot Y_k \cdot P_m) \to T^{\nabla}_{\psi_k}(Y_k)$$

in $\mathcal{L}_p$. We therefore have

$$\mathcal{L}_p \ni T_\infty := \lim_{m\to\infty} T_m = \sum_{k=1}^{d} T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_\delta}(X_k)\Big) - \sum_{k=1}^{d} T^{\nabla}_{\psi_k}(Y_k).$$

To complete the proof, it remains to notice that $T_\infty : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is the unique bounded extension of the operator $[(-\Delta)^{\frac{\epsilon}{2}}, M_f] : C_c^\infty(\mathbb{R}^d) \to (C_c^\infty(\mathbb{R}^d))'$ defined via Lemma 3.1. Indeed, by Lemma 3.2 we have for all $\phi \in C_c^\infty(\mathbb{R}^d)$

$$T_m\phi \to (-\Delta)^{\frac{\epsilon}{2}} M_f\phi - M_f(-\Delta)^{\frac{\epsilon}{2}}\phi = [(-\Delta)^{\frac{\epsilon}{2}}, M_f]\phi$$

in $(C_c^\infty(\mathbb{R}^d))'$. This implies

$$T_\infty\phi = [(-\Delta)^{\frac{\epsilon}{2}}, M_f]\phi, \quad \phi \in C_c^\infty(\mathbb{R}^d),$$

and therefore proves the claim. □

Remark 4.10. In the proof above, we used the folklore result that in any separable Banach ideal $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ and for any sequence of projections $P_m \uparrow 1$, we have $\|x - P_m x P_m\|_{\mathcal{E}} \to 0$ as $m \to \infty$. For a proof of a similar but more general fact, we refer to [14, Proposition 2.5]. In particular, this fact holds in any Schatten ideal $\mathcal{L}_p$, $1 \le p < \infty$, but fails in the non-separable ideals $\mathcal{L}_{p,\infty}$.

We can now derive the trace ideal inequality in Theorem 1.1 (i) and Theorem 1.2 (i) for smooth functions.

Lemma 4.11. *Let $\epsilon \in (-\frac{d}{2}, 1)$ with $\epsilon > 0$ when $d = 1$. For every $f \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\Big\|[(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big\|_{\frac{d}{1-\epsilon},\infty} \le c_{d,\epsilon}\|f\|_{\dot W^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}.$$

Proof. Choosing n as in Proposition 4.9 and applying its result, we find

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] = \sum_{k=1}^{d} T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_{\epsilon+2n}}(X_k)\Big) - \sum_{k=1}^{d} T^{\nabla}_{\psi_k}(Y_k).$$

By the triangle inequality, we obtain

$$\Big\|[(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big\|_{\frac{d}{1-\epsilon},\infty} \le$$

$$\leq c'_{d,\epsilon}\Big(\sum_{k=1}^{d}\Big\|T^{\nabla}_{\psi_k}\Big(T^{\nabla}_{\phi_{\epsilon+2n}}(X_k)\Big)\Big\|_{\frac{d}{1-\epsilon},\infty}+\sum_{k=1}^{d}\Big\|T^{\nabla}_{\psi_k}(Y_k)\Big\|_{\frac{d}{1-\epsilon},\infty}\Big)\leq$$

$$\leq c'_{d,\epsilon}\Big\|T^{\nabla}_{\psi_k}\Big\|_{\mathcal{L}_{\frac{d}{1-\epsilon},\infty}\to\mathcal{L}_{\frac{d}{1-\epsilon},\infty}}\cdot\Big(1+\Big\|T^{\nabla}_{\phi_{\epsilon+2n}}\Big\|_{\mathcal{L}_{\frac{d}{1-\epsilon},\infty}\to\mathcal{L}_{\frac{d}{1-\epsilon},\infty}}\Big)\cdot$$

$$\cdot\Big(\sum_{k=1}^{d}\|X_k\|_{\frac{d}{1-\epsilon},\infty}+\sum_{k=1}^{d}\|Y_k\|_{\frac{d}{1-\epsilon},\infty}\Big).$$

By Lemma 4.4, the first factor is finite (its value depends only on d and ϵ). By Lemma 4.2, the second factor is finite (its value depends only on d and ϵ). The assertion follows now from Lemma 4.8. □

Fact 4.12. *Let (A_n) be a sequence of bounded operators with $\sup_n \|A_n\|_\infty < \infty$ and let $A : C_c^\infty(\mathbb{R}^d) \to (C_c^\infty(\mathbb{R}^d))'$ be such that $A_n\phi \to A\phi$ in the sense of distributions for every $\phi \in C_c^\infty(\mathbb{R}^d)$. Then A extends to a bounded operator on $L_2(\mathbb{R}^d)$ and $A_n \to A$ strongly.*

Proof. Let $M := \sup_n \|A_n\|_\infty$. For $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ we have

$$|\langle\psi, A\phi\rangle| = \lim_{n\to\infty} |\langle\psi, A_n\phi\rangle| \leq M\|\psi\|_2\|\phi\|_2\,.$$

By density and the Riesz representation theorem, this implies $A\phi \in L_2(\mathbb{R}^d)$ with $\|A\phi\|_2 \leq M\|\phi\|_2$. Consequently, A extends to a bounded operator on $L_2(\mathbb{R}^d)$ and one easily verifies that $A_nf \to Af$ for any $f \in L_2(\mathbb{R}^d)$. □

Proof of Theorem 1.1 (i). The assertion follows from Lemma 4.11 by a simple density argument. Let us give the details.

Let $f \in \dot W^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$. It is well known (see, e.g., [33, Theorem 11.43]) that $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot W^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$. (Observe that [33, Theorem 11.43] requires that either $d \geq 2$ or else $p > 1$. In our case, if $d < 2$, then $d = 1$ and $p = \frac{1}{1-\epsilon}$. So, $p > 1$ means $\epsilon \in (0,1)$, satisfying the assumption.) Choose a sequence $\{f_n\}_{n\geq 0} \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $\dot W^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$. The proof of [33, Theorem 11.43] also shows that there is a sequence $(c_n) \subset \mathbb{C}$ such that $f_n - c_n \to f$ in the space $L_{\frac{d}{1-\epsilon},\mathrm{loc}}(\mathbb{R}^d)$. (Indeed, c_n can be chosen as the mean value of f over $\{x \in \mathbb{R}^d :\ n < |x| < 2n\}$.)

Denote, for brevity,

$$A_n := [(-\Delta)^{\frac{\epsilon}{2}}, M_{f_n}] = [(-\Delta)^{\frac{\epsilon}{2}}, M_{f_n-c_n}], \qquad A := [(-\Delta)^{\frac{\epsilon}{2}}, M_f].$$

Since $f_n - c_n \to f$ in $L_{1,\mathrm{loc}}(\mathbb{R}^d)$, it follows from Lemma 3.1 that $A_n\phi \to A\phi$ in $(C_c^\infty(\mathbb{R}^d))'$ for every $\phi \in C_c^\infty(\mathbb{R}^d)$. By Fact 4.12 and the Fatou property of the ideal $\mathcal{L}_{\frac{d}{1-\epsilon},\infty}$ (see, e.g., [45, Theorem 2.7 (d)]), we have

$$\|A\|_{\frac{d}{1-\epsilon},\infty} \leq \liminf_{n\to\infty} \|A_n\|_{\frac{d}{1-\epsilon},\infty}$$

Since, by Lemma 4.11,

$$\|A_n\|_{\frac{d}{1-\epsilon},\infty} \leq c_{d,\epsilon}\|f_n\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}}, \quad n \geq 0,$$

we obtain the assertion. □

Proof of Theorem 1.2 (i). The assertion follows from Lemma 4.11 by a simple density argument. Let us give the details.

Let $f \in \dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$. It is well known (see, e.g., [33, Theorem 11.43]) that $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$. Choose a sequence $(f_n) \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$.

Denote for brevity

$$A_n = [(-\Delta)^{\frac{\epsilon}{2}}, M_{f_n}], \quad A = [(-\Delta)^{\frac{\epsilon}{2}}, M_f].$$

It follows from (5) that

$$\|A_n - A\|_\infty \leq c^{(1)}_{d,\epsilon}\|f_n - f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}}.$$

Recall from Lemma 4.11 that

$$\|A_n\|_{\frac{d}{1-\epsilon},\infty} \leq c_{d,\epsilon}\|f_n\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}}, \quad n \geq 0.$$

Using the Fatou property of the ideal $\mathcal{L}_{\frac{d}{1-\epsilon},\infty}$ (see, e.g., [45, Theorem 2.7 (d)]), we obtain

$$\|A\|_{\frac{d}{1-\epsilon},\infty} \leq \limsup_{n\to\infty}\|A_n\|_{\frac{d}{1-\epsilon},\infty} \leq c_{d,\epsilon}\limsup_{n\to\infty}\|f_n\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}} = c_{d,\epsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}},$$

as claimed. □

5. Approximate expression for commutator

In this section, we prove an approximation results, which provides the leading term of the commutator $[(1-\Delta)^{\frac{\epsilon}{2}}, M_f]$. This approximation is needed in the proof of Theorem 1.1 (ii) and Theorem 1.2 (ii).

Theorem 5.1. *Let $d \geq 2$ and $\epsilon \in (-\frac{d}{2}, 1)$ (alternatively, let $d = 1$ and $\epsilon \in (0,1)$). For every $f \in C_c^\infty(\mathbb{R}^d)$ we have*

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] \in -\tfrac{\epsilon}{2}[\Delta, M_f](1-\Delta)^{\frac{\epsilon}{2}-1} + (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

The rest of this section is devoted to the proof of this theorem.

Let (Ω, ν) be a measure space and H a complex, separable Hilbert space. Recall that a function $f : (\Omega, \nu) \to \mathcal{B}(H)$ is called *measurable in the weak operator topology* if for every pair of vectors $\xi, \eta \in H$ the function $s \mapsto \langle f(s)\xi, \eta\rangle$ is measurable. A function $f : (\Omega, \nu) \to \mathcal{B}(H)$ is said to be integrable in the weak operator topology if it is measurable in the weak operator topology and $\int_\Omega \|f(s)\|_\infty ds < \infty$; for details see, e.g., [18, Subsection 2.7]. In this case, $\int_\Omega f(s)ds$ determines uniquely an operator in $\mathcal{B}(H)$ called the weak integral. If the function $F : (0, \infty) \to \mathcal{L}_{p,\infty}$, $1 < p < \infty$, is continuous in the weak operator topology, and $\int_0^\infty \|F(\lambda)\|_{p,\infty} d\lambda < \infty$, then its weak integral exists and belongs to $\mathcal{L}_{p,\infty}$. Furthermore, we record the following fact, established in [49, Proposition 2.3.2].

Fact 5.2. *For every $1 < p < \infty$ there is a constant c_p such that, if $F : (0, \infty) \to \mathcal{L}_{p,\infty}$ is continuous in the weak operator topology, then*

$$\Big\| \int_0^\infty F(\lambda) d\lambda \Big\|_{p,\infty} \leq c_p \int_0^\infty \|F(\lambda)\|_{p,\infty} d\lambda.$$

Fact 5.3. *Let $f \in C_c^\infty(\mathbb{R}^d)$. For every $n \geq 2$, we have*

$$[\frac{1}{1+\lambda-\Delta}, M_f] = \sum_{k=1}^{n-1} A_{k,f} \frac{1}{(1+\lambda-\Delta)^{k+1}} + \frac{1}{1+\lambda-\Delta} A_{n,f} \frac{1}{(1+\lambda-\Delta)^n}.$$

Here, we use the inductive notation

$$A_{0,f} = M_f, \quad A_{k+1,f} = [\Delta, A_{k,f}], \quad k \geq 0.$$

Lemma 5.4. *Let $\epsilon \in (0, 1)$. For $f \in C_c^\infty(\mathbb{R}^d)$ we have*

$$\int_0^\infty \frac{1}{1+\lambda-\Delta} A_{2,f} \frac{1}{(1+\lambda-\Delta)^2} \cdot \lambda^{\frac{\epsilon}{2}} d\lambda \in \mathcal{L}_{d,\infty}.$$

Proof. We first assume $d \geq 2$ and bound

$$\begin{aligned}&\Big\| \frac{1}{1+\lambda-\Delta} A_{2,f} \frac{1}{(1+\lambda-\Delta)^2} \Big\|_{d,\infty} \\ &\leq \Big\| \frac{1}{1+\lambda-\Delta} \Big\|_\infty \Big\| A_{2,f}(1-\Delta)^{-\frac{3}{2}} \Big\|_{d,\infty} \Big\| \frac{(1-\Delta)^{\frac{3}{2}}}{(1+\lambda-\Delta)^2} \Big\|_\infty.\end{aligned}$$

One easily finds that

$$\Big\| \frac{1}{1+\lambda-\Delta} \Big\|_\infty \leq \frac{1}{1+\lambda}, \quad \Big\| \frac{(1-\Delta)^{\frac{3}{2}}}{(1+\lambda-\Delta)^2} \Big\|_\infty \leq \frac{1}{(1+\lambda)^{\frac{1}{2}}}.$$

Moreover, $A_{2,f}$ is a second order differential operator with bounded, compactly supported coefficients. Explicitly,

$$A_{2,f} = 4\sum_{j,k} M_{\partial_j\partial_k f}\partial_j\partial_k + 4\sum_j M_{\partial_j\Delta f}\partial_j + M_{\Delta^2 f}\,.$$

We claim that by Cwikel's estimate (Theorem 2.4 (i) for $d \geq 3$ and Theorem 2.4 (ii) for $d = 2$) we have $A_{2,f}(1-\Delta)^{-\frac{3}{2}} \in \mathcal{L}_{d,\infty}$. To see this, we write $\partial_j\partial_k(1-\Delta)^{-\frac{3}{2}}$ as $(1-\Delta)^{-\frac{1}{2}}$ times the bounded operator $\partial_j\partial_k(1-\Delta)^{-1}$. The operator $M_{\partial_j\partial_k f}(1-\Delta)^{-\frac{1}{2}}$ belongs to the claimed trace ideal by Cwikel's bound. The other terms in the expression for $A_{2,f}$ can be handled similarly (and, in fact, enjoy better trace ideal properties than $\mathcal{L}_{d,\infty}$). Thus, we have shown that

$$\Big\|\frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\Big\|_{d,\infty} \leq \frac{c_{n,f}}{(1+\lambda)^{\frac{3}{2}}}.$$

Hence, the integrand is absolutely integrable in $\mathcal{L}_{d,\infty}$ and the assertion of the lemma for $d \geq 2$ follows from Fact 5.2.

The proof for $d = 1$ is a variation of this argument. We bound

$$\begin{aligned}&\Big\|\frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\Big\|_1\\ &\leq \Big\|\frac{1}{1+\lambda-\Delta}\Big\|_\infty \Big\|A_{2,f}(1-\Delta)^{\frac{\epsilon}{2}-2}\Big\|_1 \Big\|\frac{(1-\Delta)^{2-\frac{\epsilon}{2}}}{(1+\lambda-\Delta)^2}\Big\|_\infty.\end{aligned}$$

One easily finds that

$$\Big\|\frac{1}{1+\lambda-\Delta}\Big\|_\infty \leq \frac{1}{1+\lambda}, \quad \Big\|\frac{(1-\Delta)^{2-\frac{\epsilon}{2}}}{(1+\lambda-\Delta)^2}\Big\|_\infty \leq \frac{1}{(1+\lambda)^{\frac{\epsilon}{2}}}.$$

Furthermore, we have $A_{2,f}(1-\Delta)^{\frac{\epsilon}{2}-2} \in \mathcal{L}_1$ using Theorem 2.3. We, therefore, established the inequality

$$\Big\|\frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\Big\|_1 \leq \frac{c_{n,f}}{(1+\lambda)^{1+\frac{\epsilon}{2}}}.$$

The integrand is absolutely integrable in $\mathcal{L}_1$. Using Fact 5.2 (with $\mathcal{L}_1$ instead of $\mathcal{L}_{d,\infty}$), we infer that the integral in the assertion belongs to $\mathcal{L}_1$ and, therefore, to $\mathcal{L}_{1,\infty}$. □

Proof of Theorem 5.1 for $0 < \epsilon < 1$. Set $\delta = 2-\epsilon \in (1,2)$. Using the functional calculus, we write (see, e.g., [8, Theorem 1, p.232])

$$(1-\Delta)^{\frac{\epsilon}{2}} = \frac{\sin(\frac{\pi\epsilon}{2})}{\pi}\int_0^\infty \frac{1-\Delta}{1+\lambda-\Delta}\lambda^{-\frac{\delta}{2}}d\lambda.$$

Therefore,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] = \frac{\sin(\frac{\pi\epsilon}{2})}{\pi}\int\limits_0^\infty [\frac{1-\Delta}{1+\lambda-\Delta}, M_f]\lambda^{-\frac{\delta}{2}}d\lambda.$$

It is immediate that

$$[\frac{1-\Delta}{1+\lambda-\Delta}, M_f] = [1-\frac{\lambda}{1+\lambda-\Delta}, M_f] = -[\frac{1}{1+\lambda-\Delta}, M_f]\cdot\lambda.$$

Hence,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] = -\frac{\sin(\frac{\pi\epsilon}{2})}{\pi}\int\limits_0^\infty [\frac{1}{1+\lambda-\Delta}, M_f]\lambda^{\frac{\epsilon}{2}}d\lambda.$$

Using Fact 5.3 with $n=2$, we write

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] = -A_{1,f}\cdot\frac{\sin(\frac{\pi\epsilon}{2})}{\pi}\int\limits_0^\infty \frac{\lambda^{\frac{\epsilon}{2}}d\lambda}{(1+\lambda-\Delta)^2} -$$

$$-\frac{\sin(\frac{\pi\epsilon}{2})}{\pi}\int\limits_0^\infty \frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\lambda^{\frac{\epsilon}{2}}d\lambda.$$

It follows from Lemma 5.4 that

$$\int\limits_0^\infty \frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\lambda^{\frac{\epsilon}{2}}d\lambda \in \mathcal{L}_{d,\infty}\subset(\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

Again by the functional calculus, we have

$$\int\limits_0^\infty \frac{\lambda^{\frac{\epsilon}{2}}d\lambda}{(1+\lambda-\Delta)^2} = (1-\Delta)^{\frac{\epsilon}{2}-1}B(1+\tfrac{\epsilon}{2}, 1-\tfrac{\epsilon}{2}) = \frac{\frac{\pi\epsilon}{2}}{\sin(\frac{\pi\epsilon}{2})}(1-\Delta)^{\frac{\epsilon}{2}-1},$$

where $B(\cdot,\cdot)$ denotes the beta function. This completes the proof. □

Lemma 5.5. *Let $\alpha,\beta>0$ and $\delta\in(0,2)$. Suppose $p>1$ is such that $p\geq\frac{d}{\alpha+\beta+2}$. We have*

$$(1-\Delta)^{-\frac{\alpha}{2}}\cdot\int\limits_0^\infty \frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\lambda^{-\frac{\delta}{2}}d\lambda\cdot(1-\Delta)^{-\frac{\beta}{2}}\in\mathcal{L}_{p,\infty}.$$

Proof. We claim that

$$(1-\Delta)^{-\frac{\alpha}{2}}M_h(1-\Delta)^{-\frac{\beta}{2}-1}\in\mathcal{L}_{\frac{d}{\alpha+\beta+2},\infty},\quad h\in C_c(\mathbb{R}^d). \tag{8}$$

By complex interpolation [45, Theorem 2.9], it suffices to prove that

$$M_h(1-\Delta)^{-\frac{\alpha+\beta+2}{2}} \in \mathcal{L}_{\frac{d}{\alpha+\beta+2},\infty}, \quad h \in C_c(\mathbb{R}^d).$$

The latter inclusion follows from Theorem 2.4, thus proving (8).

We now note that

$$(1-\Delta)^{-\frac{\alpha}{2}} A_{2,f} (1-\Delta)^{-\frac{\beta}{2}-2} \in \mathcal{L}_{\frac{d}{\alpha+\beta+2},\infty}.$$

Indeed, $A_{2,f}$ is a differential operator of second order with smooth, compactly supported coefficients. Hence, the inclusion follows from (8).

We now argue similarly as in the proof of Lemma 5.4. We bound

$$\Big\|(1-\Delta)^{-\frac{\alpha}{2}} \cdot \frac{1}{1+\lambda-\Delta} A_{2,f} \frac{1}{(1+\lambda-\Delta)^2} \cdot (1-\Delta)^{-\frac{\beta}{2}}\Big\|_{\frac{d}{\alpha+\beta+2},\infty} \leq$$
$$\leq \frac{1}{1+\lambda}\Big\|(1-\Delta)^{-\frac{\alpha}{2}} A_{2,f} (1-\Delta)^{-\frac{\beta}{2}-2}\Big\|_{\frac{d}{\alpha+\beta+2},\infty} \leq \frac{c_{\alpha,\beta,f}}{1+\lambda}.$$

Hence, the integrand is absolutely integrable in $\mathcal{L}_{\frac{d}{\alpha+\beta+2},\infty}$. Consequently, it is absolutely integrable in $\mathcal{L}_{p,\infty}$ and the assertion follows from Fact 5.2. □

Proof of Theorem 5.1 for $-\frac{d}{2} < \epsilon < 0$, $d \geq 2$. Choose $0 < \delta < 1$ and $m \in \mathbb{N}$ such that $\epsilon = -m\delta$. Using Leibniz's rule, we write

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] = [(1-\Delta)^{-\frac{m\delta}{2}}, M_f] =$$
$$= \sum_{l=0}^{m-1} (1-\Delta)^{-\frac{l\delta}{2}} [(1-\Delta)^{-\frac{\delta}{2}}, M_f] (1-\Delta)^{\frac{(l+1-m)\delta}{2}}.$$

Using the functional calculus, we write

$$(1-\Delta)^{-\frac{\delta}{2}} = \frac{\sin(\frac{\pi\delta}{2})}{\pi} \int_0^\infty \frac{\lambda^{-\frac{\delta}{2}}\,d\lambda}{1+\lambda-\Delta},$$

where the integrand is a norm-continuous function of λ and the integral converges in operator norm.

We have

$$[(1-\Delta)^{-\frac{\delta}{2}}, M_f] = \frac{\sin(\frac{\pi\delta}{2})}{\pi} \int_0^\infty [\frac{1}{1+\lambda-\Delta}, M_f] \cdot \lambda^{-\frac{\delta}{2}} d\lambda.$$

Using Fact 5.3 with $n = 2$, we write

$$[(1-\Delta)^{-\frac{\delta}{2}}, M_f] = A_{1,f}\cdot\frac{\sin(\frac{\pi\delta}{2})}{\pi}\int_0^\infty \frac{\lambda^{-\frac{\delta}{2}}d\lambda}{(1+\lambda-\Delta)^2}$$

$$+\frac{\sin(\frac{\pi\delta}{2})}{\pi}\int_0^\infty \frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\lambda^{-\frac{\delta}{2}}d\lambda.$$

Thus,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] =$$

$$=\sum_{l=0}^{m-1}(1-\Delta)^{-\frac{l\delta}{2}}A_{1,f}\cdot\frac{\sin(\frac{\pi\delta}{2})}{\pi}\int_0^\infty\frac{\lambda^{-\frac{\delta}{2}}d\lambda}{(1+\lambda-\Delta)^2}\cdot(1-\Delta)^{\frac{(l+1-m)\delta}{2}}+$$

$$+\sum_{l=0}^{m-1}\frac{\sin(\frac{\pi\delta}{2})}{\pi}(1-\Delta)^{-\frac{l\delta}{2}}\cdot\int_0^\infty\frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\lambda^{-\frac{\delta}{2}}d\lambda\cdot(1-\Delta)^{\frac{(l+1-m)\delta}{2}}.$$

Choose $p>1$ such that $\frac{d}{2-\epsilon-\delta}\le p<\frac{d}{1-\epsilon}$. (Note that $\frac{d}{1-\epsilon}>1$ since $\epsilon>-\frac{d}{2}$ and $d\ge 2$.) It follows from Lemma 5.5 that

$$(1-\Delta)^{-\frac{l\delta}{2}}\cdot\int_0^\infty\frac{1}{1+\lambda-\Delta}A_{2,f}\frac{1}{(1+\lambda-\Delta)^2}\lambda^{-\frac{\delta}{2}}d\lambda\cdot(1-\Delta)^{\frac{(l+1-m)\delta}{2}}\in\mathcal{L}_{p,\infty}.$$

Thus,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f]\in$$

$$\in\sum_{l=0}^{m-1}(1-\Delta)^{-\frac{l\delta}{2}}A_{1,f}\cdot\frac{\sin(\frac{\pi\delta}{2})}{\pi}\int_0^\infty\frac{\lambda^{-\frac{\delta}{2}}d\lambda}{(1+\lambda-\Delta)^2}\cdot(1-\Delta)^{\frac{(l+1-m)\delta}{2}}+(\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

Again by the functional calculus, we have

$$\int_0^\infty\frac{\lambda^{-\frac{\delta}{2}}d\lambda}{(1+\lambda-\Delta)^2}=(1-\Delta)^{-\frac{\delta}{2}-1}\cdot B(\tfrac{\delta}{2}+1,1-\tfrac{\delta}{2})=\frac{\frac{\pi\delta}{2}}{\sin(\frac{\pi\delta}{2})}(1-\Delta)^{-\frac{\delta}{2}-1}.$$

Thus,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f]\in\tfrac{\delta}{2}\sum_{l=0}^{m-1}(1-\Delta)^{-\frac{l\delta}{2}}A_{1,f}(1-\Delta)^{\frac{(l-m)\delta}{2}-1}+(\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

With the help of Cwikel estimates (Theorem 2.4) it is easy to see that

$$[(1-\Delta)^{-\frac{l\delta}{2}},A_{1,f}](1-\Delta)^{\frac{(l-m)\delta}{2}-1}\in(\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0,\quad 0\le l<m.$$

Thus,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] \in \frac{\delta}{2}\sum_{l=0}^{m-1} A_{1,f}(1-\Delta)^{-\frac{l\delta}{2}}(1-\Delta)^{\frac{(l-m)\delta}{2}-1} + (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

Since all summands on the right hand side are equal, the assertion follows. □

6. Spectral asymptotics for pseudo-differential operators

Our goal in this section is to extend the main result in [24].

Let Π be the C^*-subalgebra of $\mathcal{B}(L_2(\mathbb{R}^d))$ generated by the algebras

$$\{M_f: \ f \in \mathbb{C} + C_0(\mathbb{R}^d)\} \qquad \text{and} \qquad \{g(\nabla(-\Delta)^{-\frac{1}{2}}): \ g \in C(\mathbb{S}^{d-1})\}.$$

According to [48] (where a much stronger result is given in Theorem 1.2) or [39] (where a very general result is given in Theorem 3.3 and examplified on p. 284), there is a $*$-homomorphism

$$\mathrm{sym}: \Pi \to C(\mathbb{S}^{d-1}, \mathbb{C} + C_0(\mathbb{R}^d))$$

such that, for all $f \in \mathbb{C} + C_0(\mathbb{R}^d)$ and $g \in C(\mathbb{S}^{d-1})$,

$$\mathrm{sym}(M_f) = f \otimes 1 \qquad \text{and} \qquad \mathrm{sym}(g(\nabla(-\Delta)^{-\frac{1}{2}})) = 1 \otimes g\,.$$

We say that $T \in \mathcal{B}(L_2(\mathbb{R}^d))$ is compactly supported from the right if there is a $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $T = TM_\phi$. We say that $T \in \mathcal{B}(L_2(\mathbb{R}^d))$ is compactly supported if there is a $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $T = M_\phi T M_\phi$.

Theorem 6.1. *Let $d \geq 2$ and $p > 0$. If $T \in \Pi$ is compactly supported from the right, then*

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu\Big(t, T(1-\Delta)^{-\frac{d}{2p}}\Big) = d^{-\frac{1}{p}}(2\pi)^{-\frac{d}{p}} \|\mathrm{sym}(T)\|_{L_p(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

This theorem with $p = d$ appears in [24]. The above more general assertion can be obtained either by following the same steps as in that paper or, as we shall show here, as a consequence of the results proved there. The corresponding result for $d = 1$ is essentially the well-known Weyl asymptotic and will be discussed separately in Subsection 6.3.

6.1. An abstract result on spectral asymptotics

Our goal in this subsection is to prove the following result, which allows us to reduce spectral asymptotics for the product of powers to the power of a product. The parameter d in this subsection is an arbitrary real number, not necessarily an integer (although it will be in the application to the proof of Theorem 6.1).

Proposition 6.2. *Let $d > p > 1$. Let $0 \leq A, B \in \mathcal{B}(H)$. Suppose $BA^{\frac{1}{2}} \in \mathcal{L}_{d,\infty}$, $B^pA^p \in \mathcal{L}_{\frac{d}{p},\infty}$ and $[B^p, A^{\frac{1}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0$. Then*

$$B^pA^p - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^p \in (\mathcal{L}_{\frac{d}{p},\infty})_0.$$

The first step in the proof of this proposition is an integral representation for the difference on the left side of the proposition. This representation formula is a special case of [49, Theorem 5.2.1], with a predecessor in [18].

Lemma 6.3. *Let $0 \leq A, B \in \mathcal{B}(H)$ and let $Y = A^{\frac{1}{2}}BA^{\frac{1}{2}}$. For $p > 1$, define the mapping $T_p : \mathbb{R} \to \mathcal{B}(H)$ by,*

$$T_p(s) := \begin{cases} B^{p-1}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}}] + [BA^{\frac{1}{2}}, A^{\frac{1}{2}}]Y^{p-1} & \text{if } s = 0\,, \\ B^{p-1+is}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}+is}]Y^{-is} + B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{p-1-is} & \text{if } s \neq 0. \end{cases}$$

We also define the function $g_p \in \mathcal{S}(\mathbb{R})$ by setting

$$g_p(t) := \begin{cases} 1 - \frac{p}{2} & \text{if } t = 0\,, \\ 1 - \frac{e^{\frac{p}{2}t} - e^{-\frac{p}{2}t}}{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})(e^{(\frac{p-1}{2})t} + e^{-(\frac{p-1}{2})t})} & \text{if } t \neq 0\,. \end{cases}$$

Then the mapping $T_p : \mathbb{R} \to \mathcal{B}(H)$ is continuous in the weak operator topology and we have

$$B^pA^p - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^p = T_p(0) - \int_{\mathbb{R}} T_p(s)\widehat{g}_p(s)ds.$$

Lemma 6.4. *Let $d > 0$ and $p > 1$. Let $0 \leq A, B \in \mathcal{B}(H)$ and let T_p be as in Lemma 6.3. Suppose $B^pA^p \in \mathcal{L}_{\frac{d}{p},\infty}$. We have*

$$\sup_{s \in \mathbb{R}} \|T_p(s)\|_{\frac{d}{p},\infty} < \infty.$$

Proof. We use the notion of logarithmic submajorization discussed, for instance in [37, Definition 2.3.10]. For $x, y \in \mathcal{B}(H)$ one write $x \prec\prec_{\log} y$ if $\prod_{k=1}^n \mu(k, x) \leq \prod_{k=1}^n \mu(k, y)$ for all $n \geq 0$. In this notation the Araki–Lieb–Thirring inequality (see [1]) states that

$$X^rY^r \prec\prec_{\log} |XY|^r, \quad 0 < r \leq 1.$$

Using this inequality with $X = B^p$, $Y = A^p$ and taking into account that every quasi-Banach ideal is closed with respect to the logarithmic submajorization (see [37, Proposition 2.4.18]), we obtain from the assumption $B^pA^p \in \mathcal{L}_{\frac{d}{p},\infty}$ the inclusions

$$B^{p-1}A^{p-1} \in \mathcal{L}_{\frac{d}{p-1},\infty}, \quad BA \in \mathcal{L}_{d,\infty}, \quad B^{\frac{1}{2}}A^{\frac{1}{2}} \in \mathcal{L}_{2d,\infty}.$$

It follows from (2) that

$$\|x+y\|_{r,\infty} \le 2^{\frac{1}{r}}(\|x\|_{r,\infty} + \|y\|_{r,\infty}) \qquad \text{for all } 0 < r < \infty.$$

Applying this quasi-triangle inequality, we have with $Y := A^{\frac{1}{2}} B A^{\frac{1}{2}}$

$$\|T_p(s)\|_{\frac{d}{p},\infty} \le 2^{\frac{p}{d}} \|B^{p-1}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}+is}]\|_{\frac{d}{p},\infty} + 2^{\frac{p}{d}} \|[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{p-1}\|_{\frac{d}{p},\infty}.$$

Again using the quasi-triangle inequality, we obtain

$$\|B^{p-1}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}+is}]\|_{\frac{d}{p},\infty} \le 2^{\frac{p}{d}} \|B^p A^p\|_{\frac{d}{p},\infty} + 2^{\frac{p}{d}} \|B^{p-1} A^{p-\frac{1}{2}+is} B A^{\frac{1}{2}}\|_{\frac{d}{p},\infty},$$

$$\|[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{p-1}\|_{\frac{d}{p},\infty} \le 2^{\frac{p}{d}} \|BA^{1+is} Y^{p-1}\|_{\frac{d}{p},\infty} + 2^{\frac{p}{d}} \|A^{\frac{1}{2}} B A^{\frac{1}{2}} Y^{p-1}\|_{\frac{d}{p},\infty}.$$

Using Hölder's inequality, we obtain

$$\|B^{p-1} A^{p-\frac{1}{2}+is} B A^{\frac{1}{2}}\|_{\frac{d}{p},\infty} \le 2^{\frac{p}{d}} \|B^{p-1} A^{p-1}\|_{\frac{d}{p-1},\infty} \|A^{\frac{1}{2}} B A^{\frac{1}{2}}\|_{d,\infty},$$

$$\|BA^{1+is} Y^{p-1}\|_{\frac{d}{p},\infty} \le 2^{\frac{p}{d}} \|BA\|_{d,\infty} \|Y\|_{d,\infty}^{p-1}.$$

Clearly,

$$\|A^{\frac{1}{2}} B A^{\frac{1}{2}} Y^{p-1}\|_{\frac{d}{p},\infty} = \|Y\|_{d,\infty}^{p} = \|B^{\frac{1}{2}} A^{\frac{1}{2}}\|_{2d,\infty}^{2p}.$$

Combining these estimates, we complete the proof. □

Lemma 6.5. *Let* $1 < r < \infty$. *Let* $0 \le X, Y \in \mathcal{B}(H)$. *If* $[X, Y] \in (\mathcal{L}_{r,\infty})_0$, *then also* $[X, Y^z] \in (\mathcal{L}_{r,\infty})_0$ *for every* $z \in \mathbb{C}$ *with* $\Re(z) \ge 1$.

Proof. Without loss of generality, $\|Y\|_\infty = 1$. Let f_z be a Lipschitz function on $\mathbb{R}$ such that $f_z(t) = t^z$ for $t \in [0, 1]$. We have

$$[X, Y^z] = [X, f_z(Y)] = T^Y_{f_z^{[1]}}([X, Y]).$$

The main result of [43] yields that $T^Y_{f_z^{[1]}} : (\mathcal{L}_{r,\infty})_0 \to (\mathcal{L}_{r,\infty})_0$, and this is enough to complete the proof. □

Lemma 6.6. *Let* $d > p > 1$. *Let* $0 \le A, B \in \mathcal{B}(H)$ *and let* T_p *be as in Lemma 6.3. Suppose* $BA^{\frac{1}{2}} \in \mathcal{L}_{d,\infty}$ *and* $[B^p, A^{\frac{1}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0$. *We have*

$$T_p(s) \in (\mathcal{L}_{\frac{d}{p},\infty})_0, \quad s \in \mathbb{R}.$$

Proof. Since $[B^p, A^{\frac{1}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0$, it follows from [25, Corollary 7.1] that

$$[B^{p-1}, A^{\frac{1}{2}}] \in (\mathcal{L}_{\frac{d}{p-1},\infty})_0, \quad [B, A^{\frac{1}{2}}] \in (\mathcal{L}_{d,\infty})_0.$$

By the Leibniz rule, we have

$$B^{p-1}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}+is}] = X_1X_2 - X_3X_4,$$
$$X_1 = [B^p, A^{p-\frac{1}{2}+is}], \quad X_2 = A^{\frac{1}{2}}, \quad X_3 = [B^{p-1}, A^{p-\frac{1}{2}+is}], \quad X_4 = BA^{\frac{1}{2}}.$$

Applying Lemma 6.5 to $X = B^p$ (respectively, $X = B^{p-1}$), $Y = A^{\frac{1}{2}}$, $r = \frac{d}{p}$ (respectively, $r = \frac{d}{p-1}$) and $z = 2p - 1 + 2is$, we obtain

$$X_1 \in (\mathcal{L}_{\frac{d}{p},\infty})_0\,, \quad X_3 \in (\mathcal{L}_{\frac{d}{p-1},\infty})_0\,.$$

Since also X_2 is bounded and $X_4 \in \mathcal{L}_{d,\infty}$, it follows from the inclusions above and from Hölder's inequality that

$$B^{p-1}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}+is}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0.$$

Thus,

$$B^{p-1+is}[BA^{\frac{1}{2}}, A^{p-\frac{1}{2}+is}]Y^{-is} \in (\mathcal{L}_{\frac{d}{p},\infty})_0. \tag{9}$$

We have

$$B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{p-1-is} = B^{is} \cdot [B, A^{\frac{1}{2}+is}] \cdot A^{\frac{1}{2}}Y^{-is} \cdot Y^{p-1}.$$

Recall that $[B, A^{\frac{1}{2}}] \in (\mathcal{L}_{d,\infty})_0$. Applying Lemma 6.5 with $X = B$, $Y = A^{\frac{1}{2}}$, $r = d$ and $z = 1 + 2is$, we obtain

$$[B, A^{\frac{1}{2}+is}] \in (\mathcal{L}_{d,\infty})_0.$$

By assumption, $BA^{\frac{1}{2}} \in \mathcal{L}_{d,\infty}$. Thus, $Y \in \mathcal{L}_{d,\infty}$ and $Y^{p-1} \in \mathcal{L}_{\frac{d}{p-1},\infty}$. Hence,

$$B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{p-1-is} \in (\mathcal{L}_{\frac{d}{p},\infty})_0. \tag{10}$$

The assertion follows now by combining (9) and (10). □

We are finally in position to prove the main result of this subsection.

Proof of Proposition 6.2. By Lemmas 6.4 and 6.6, the mapping

$$s \to T_p(s)\widehat{g_p}(s), \quad s \in \mathbb{R},$$

is absolutely integrable in $(\mathcal{L}_{\frac{d}{p},\infty})_0$. Since $d > p$, it follows that the latter mapping is Bochner integrable in $(\mathcal{L}_{\frac{d}{p},\infty})_0$. In particular, its integral belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$. □

6.2. Proof of Theorem 6.1

We now combine the result from [24] with Proposition 6.2 and obtain the following preliminary version of Theorem 6.1.

Lemma 6.7. *Let* $1 < p < d$. *If* $0 \leq T \in \Pi$ *is compactly supported, then one has*

$$\lim_{t\to\infty} t^{\frac{p}{d}}\mu(t, T^p(1-\Delta)^{-\frac{p}{2}}) = \Big(d^{-\frac{1}{d}}(2\pi)^{-1}\|\mathrm{sym}(T)\|_{L_d(\mathbb{R}^d\times\mathbb{S}^{d-1})}\Big)^p.$$

Proof. Set $A = T$ and $B = (1-\Delta)^{-\frac{1}{2}}$. We want to apply Proposition 6.2.

Let us show that the conditions of the Proposition 6.2 are satisfied. Indeed, the inclusion $[B^p, A^{\frac{1}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0$ is established in Theorem A.1 for every $p > 0$. Recall that T is compactly supported and choose $\phi \in C_c(\mathbb{R}^d)$ such that $T = M_\phi T$. Then, by Theorem 2.4,

$$BA^{\frac{1}{2}} = (1-\Delta)^{-\frac{1}{2}}M_\phi \cdot T^{\frac{1}{2}} \in \mathcal{L}_{d,\infty}, \quad B^pA^p = (1-\Delta)^{-\frac{p}{2}}M_\phi \cdot T^{\frac{p}{2}} \in \mathcal{L}_{\frac{d}{p},\infty}.$$

By Proposition 6.2, we have

$$B^pA^p - \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^p \in (\mathcal{L}_{\frac{d}{p},\infty})_0.$$

By [24, Theorem 1.5], we have

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, AB) = c_T, \quad c_T := d^{-\frac{1}{d}}(2\pi)^{-1}\|\mathrm{sym}(T)\|_{L_d(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

By Theorem A.1, we have $[B, A^{\frac{1}{2}}] \in (\mathcal{L}_{d,\infty})_0$. By a standard result on spectral asymptotics (see, e.g., [24, Lemma 3.1]) this implies that

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, A^{\frac{1}{2}}BA^{\frac{1}{2}}) = c_T.$$

In other words, we have

$$\lim_{t\to\infty} t^{\frac{p}{d}}\mu\Big(t, \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^p\Big) = c_T^p.$$

Again using [24, Lemma 3.1], we conclude that

$$\lim_{t\to\infty} t^{\frac{p}{d}}\mu(t, B^pA^p) = c_T^p.$$

This is exactly our assertion. □

Lemma 6.8. *Let $d \geq 2$ and $p > 0$. The following conditions are equivalent:*

(i) *for every nonnegative and compactly supported $T \in \Pi$ one has*

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu(t, T(1-\Delta)^{-\frac{d}{2p}}) = d^{-\frac{1}{p}} (2\pi)^{-\frac{d}{p}} \|\mathrm{sym}(T)\|_{L_p(\mathbb{R}^d\times\mathbb{S}^{d-1})}$$

(ii) *for every nonnegative and compactly supported $T \in \Pi$ one has*

$$\lim_{t\to\infty} t^{\frac{2}{p}} \mu(t, T(1-\Delta)^{-\frac{d}{p}}) = d^{-\frac{2}{p}} (2\pi)^{-\frac{2d}{p}} \|\mathrm{sym}(T)\|_{L_{\frac{p}{2}}(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

Proof. We use the fact that the set of nonnegative, compactly supported elements of Π is invariant under taking powers.

Renaming T into T^2, we rewrite (ii) as follows: for every nonnegative and compactly supported $T \in \Pi$ one has

$$\lim_{t\to\infty} t^{\frac{2}{p}} \mu(t, T^2(1-\Delta)^{-\frac{d}{p}}) = d^{-\frac{2}{p}} (2\pi)^{-\frac{2d}{p}} \|\mathrm{sym}(T)\|^2_{L_p(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

It follows from Theorem A.1 (which we proved for $d \geq 2$) and simple arithmetic that

$$T^2(1-\Delta)^{-\frac{d}{p}} - (1-\Delta)^{-\frac{d}{2p}} T^2 (1-\Delta)^{-\frac{d}{2p}} \in (\mathcal{L}_{\frac{p}{2},\infty})_0.$$

Using a standard result about spectral asymptotics (see, e.g., [24, Lemma 3.1]), we rewrite (ii) as follows:

$$\lim_{t\to\infty} t^{\frac{2}{p}} \mu^2(t, T(1-\Delta)^{-\frac{d}{2p}}) = d^{-\frac{2}{p}} (2\pi)^{-\frac{2d}{p}} \|\mathrm{sym}(T)\|^2_{L_p(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

This is clearly equivalent to (i). □

Proof of Theorem 6.1. Let us first prove the assertion for $1 < p \leq d$. For $p = d$, the assertion is given by Theorem 1.5 in [24]. Hence, we may assume that $1 < p < d$. Applying Lemma 6.7 to $T^{\frac{1}{p}}$ (which belongs to Π and is compactly supported), we conclude that

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{p}{d}} \mu(t, T(1-\Delta)^{-\frac{p}{2}}) &= d^{-\frac{p}{d}} (2\pi)^{-p} \|\mathrm{sym}(T^{\frac{1}{p}})\|^p_{L_d(\mathbb{R}^d\times\mathbb{S}^{d-1})} \\ &= d^{-\frac{p}{d}} (2\pi)^{-p} \|\mathrm{sym}(T)\|_{L_{\frac{d}{p}}(\mathbb{R}^d\times\mathbb{S}^{d-1})}\end{aligned}$$

Renaming p into $\frac{d}{p}$ and noting that as p runs through $(1, d)$, $\frac{d}{p}$ runs through the same interval, we conclude the assertion for $1 < p < d$ (and, hence, for $1 < p \leq d$).

Let us now prove the assertion in full generality. Fix $p > 0$. Choose $n \in \mathbb{Z}$ such that $q = 2^n p \in (1, d]$. By Lemma 6.7, for every nonnegative and compactly supported $T \in \Pi$ we have

$$\lim_{t\to\infty} t^{\frac{1}{q}} \mu(t, T(1-\Delta)^{-\frac{d}{2q}}) = d^{-\frac{1}{q}} (2\pi)^{-\frac{d}{q}} \|\mathrm{sym}(T)\|_{L_q(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

By repeated use of Lemma 6.8, we deduce that for every nonnegative and compactly supported $T \in \Pi$ we have

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu(t, T(1-\Delta)^{-\frac{d}{2p}}) = d^{-\frac{1}{p}}(2\pi)^{-\frac{d}{p}} \|\mathrm{sym}(T)\|_{L_p(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

This proves the assertion for every nonnegative and compactly supported $T \in \Pi$.

Consider now the general case. Let $T \in \Pi$ be compactly supported from the right. Clearly, $|T| \in \Pi$ is nonnegative and compactly supported. By the preceding paragraph, we have

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu(t, |T|(1-\Delta)^{-\frac{d}{2p}}) = d^{-\frac{1}{p}}(2\pi)^{-\frac{d}{p}} \|\mathrm{sym}(|T|)\|_{L_p(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

Observing that for any operators $A, B \in \mathcal{B}(H)$ we have $\mu(AB) = \mu(|AB|)$ and $|AB| = \big||A|B\big|$, we write

$$\mu\Big(T(1-\Delta)^{-\frac{d}{2p}}\Big) = \mu\Big(|T|(1-\Delta)^{-\frac{d}{2p}}\Big), \quad \mathrm{sym}(|T|) = |\mathrm{sym}(T)|.$$

This proves the assertion in general case. $\square$

6.3. Asymptotics for $d = 1$

For $d = 1$, the sphere $\mathbb{S}^{d-1}$ is just a two-point set. Hence, the algebra Π consists of two 'copies' of the first algebra in the definition of Π. So, the result analogous to Theorem 6.1 should be stated only for multiplication operators. This is the well-known Weyl asymptotic formula. For the convenience of the reader we explain how this can be deduced from the results of Birman–Solomyak in [5].

Theorem 6.9. *Let $d = 1$ and $p > 0$. If $f \in C_c(\mathbb{R})$, then*

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu(t, M_f(1-\Delta)^{-\frac{1}{2p}}) = \pi^{-\frac{1}{p}} \|f\|_p.$$

The ingredient from [5] that we use is the following.

Lemma 6.10. *Let $d = 1$ and let $p > 1$. If $f \in C_c(\mathbb{R})$, then*

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu(t, M_{\bar f}(-\Delta)^{-\frac{1}{2p}} M_f) = \pi^{-\frac{1}{p}} \|f\|_{2p}^2.$$

Proof. This is a very special case of [5, Theorem 1]. Indeed, take $m = 1$ and $\alpha = \frac{1}{p} - 1 \in (-1, 0)$. Set $\theta(t) = |t|^\alpha$, $\Phi(t, t') = 1$ and $a(t) = f(t)$ for every $t \in \mathbb{R}$. Set $E = \mathrm{supp}(f)$.

Note that in [5] a non-standard notion of Fourier transform is used. However, for a homogeneous function θ of degree $\alpha \in (-1, 0)$, this notion coincides with the usual notion of Fourier transform. In particular, the operator T featuring in [5, (2.4)] is given by a constant times $M_{\bar f}(-\Delta)^{-\frac{1}{2p}} M_f$. $\square$

Lemma 6.11. *Let $d = 1$ and $p > 0$. The following conditions are equivalent:*

(i) *for every $0 \leq f \in C_c(\mathbb{R})$ one has*

$$\lim_{t\to\infty} t^{\frac{1}{p}} \mu(t, M_f(1-\Delta)^{-\frac{1}{2p}}) = \pi^{-\frac{1}{p}} \|f\|_p$$

(ii) *for every $0 \leq f \in C_c(\mathbb{R})$ one has*

$$\lim_{t\to\infty} t^{\frac{2}{p}} \mu(t, M_f(1-\Delta)^{-\frac{1}{p}}) = \pi^{-\frac{2}{p}} \|f\|_{\frac{p}{2}}.$$

Proof. The proof follows that of Lemma 6.8 *mutatis mutandi.* Instead of Theorem A.1, which we stated under the assumption $d \geq 2$, we apply Lemma A.3, whose proof remains valid for $d = 1$. $\square$

Proof of Theorem 6.9. First, take $p > 2$. Applying Lemma 6.10 to $\frac{p}{2}$ and taking into account that

$$\mu(M_{\bar{f}}(-\Delta)^{-\frac{1}{p}} M_f) = \mu^2(M_f(-\Delta)^{-\frac{1}{2p}}),$$

we arrive at

$$\lim_{t\to\infty} t^{\frac{2}{p}} \mu^2(t, M_f(-\Delta)^{-\frac{1}{2p}}) = \pi^{-\frac{2}{p}} \|f\|_p^2.$$

Noting that, by Theorem 2.3,

$$M_f(-\Delta)^{-\frac{1}{2p}} - M_f(1-\Delta)^{-\frac{1}{2p}} \in (\mathcal{L}_{p,\infty})_0,$$

and using a standard result about spectral asymptotics (see, e.g., [24, Lemma 3.1]), we infer the assertion for $p > 2$.

The assertion for $p \leq 2$ follows by induction in the same way as in the proof of Theorem 6.1, using Lemma 6.11 instead of Lemma 6.8. $\square$

7. Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii)

Parts (ii) of Theorems 1.1 and 1.2 state spectral asymptotics for $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ under the sole assumption that the leading term is finite. We begin by proving these asymptotics in the smooth case.

Lemma 7.1. *Let $d \geq 2$ and $\epsilon \in (-\frac{d}{2}, 1)$ (alternatively, let $d = 1$ and $\epsilon \in (0, 1)$). If $f \in C_c^\infty(\mathbb{R}^d)$, then*

$$\lim_{t\to\infty} t^{\frac{1-\epsilon}{d}} \mu\Big(t, [(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big) = \kappa_{d,\epsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)}$$

with $\kappa_{d,\epsilon}$ *given by* (1).

Proof. Set

$$T := \epsilon \sum_{k=1}^{d} \frac{D_k}{(-\Delta)^{\frac{1}{2}}} M_{D_k f}$$

and note that $T \in \Pi$ and that T is compactly supported from the right.

We claim that

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_f] - T(1-\Delta)^{-\frac{1-\epsilon}{2}} \in (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0. \tag{11}$$

To prove this, we note that by Theorem 2.3

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] - [(-\Delta)^{\frac{\epsilon}{2}}, M_f] =$$
$$= \big((1-\Delta)^{\frac{\epsilon}{2}} - (-\Delta)^{\frac{\epsilon}{2}}\big)M_f - M_f\big((1-\Delta)^{\frac{\epsilon}{2}} - (-\Delta)^{\frac{\epsilon}{2}}\big) \in \mathcal{L}_{\frac{d}{1-\epsilon}} \subset (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

Here we use the assumption $f \in C_c^\infty(\mathbb{R}^d)$ together with the fact that the function $\xi \mapsto (1+|\xi|^2)^{\frac{\epsilon}{2}} - |\xi|^\epsilon$ belongs to $L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)$ if $\frac{d}{1-\epsilon} \geq 2$ and to $\ell_{\frac{d}{1-\epsilon}}(L_2)(\mathbb{R}^d)$ if $\frac{d}{1-\epsilon} < 2$. Next, by Theorem 5.1,

$$[(1-\Delta)^{\frac{\epsilon}{2}}, M_f] + \tfrac{\epsilon}{2}[\Delta, M_f](1-\Delta)^{\frac{\epsilon}{2}-1} \in (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

Finally, we write

$$-[\Delta, M_f] = 2\sum_{k=1}^{d} M_{D_k f} D_k - M_{\Delta f}\,.$$

For the second term we use Theorem 2.3 similarly as before and find

$$M_{\Delta f}(1-\Delta)^{\frac{\epsilon}{2}-1} \in \mathcal{L}_{\frac{d}{1-\epsilon}} \subset (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

For the first term we again use Theorem 2.3 and find

$$\epsilon\sum_{k=1}^{d} M_{D_k f} D_k (1-\Delta)^{\frac{\epsilon}{2}-1} - T(1-\Delta)^{-\frac{1-\epsilon}{2}} \in \mathcal{L}_{\frac{d}{1-\epsilon}} \subset (\mathcal{L}_{\frac{d}{1-\epsilon},\infty})_0.$$

This completes the proof of (11).

Consider the case $d \geq 2$. The inclusion (11), together with simple limiting arguments (see, e.g., [24, Lemma 3.1]) and Theorem 6.1, applied with $p = \frac{d}{1-\epsilon}$, implies

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{1-\epsilon}{d}}\mu\Big(t,[(-\Delta)^{\frac{\epsilon}{2}},M_f]\Big) &= \lim_{t\to\infty} t^{\frac{1-\epsilon}{d}}\mu\Big(t,T(1-\Delta)^{-\frac{1-\epsilon}{2}}\Big)\\ &= d^{-\frac{1-\epsilon}{d}}(2\pi)^{-1+\epsilon}\|\mathrm{sym}(T)\|_{L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d\times\mathbb{S}^{d-1})}.\end{aligned}$$

Finally, we compute, similarly as in [24, Lemma 8.4],

$$\begin{aligned}\|\mathrm{sym}(T)\|^{\frac{d}{1-\epsilon}}_{L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d\times\mathbb{S}^{d-1})} &= |\epsilon|^{\frac{d}{1-\epsilon}}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}|\omega\cdot\nabla f(x)|^{\frac{d}{1-\epsilon}}\,d\omega\,dx\\ &= |\epsilon|^{\frac{d}{1-\epsilon}}\int_{\mathbb{R}^d}|\nabla f(x)|^{\frac{d}{1-\epsilon}}\,dx\int_{\mathbb{S}^{d-1}}|\omega_d|^{\frac{d}{1-\epsilon}}\,d\omega\,.\end{aligned}$$

This completes the proof for $d\geq 2$.

Consider the case $d=1$. The inclusion (11) reads as

$$[(-\Delta)^{\frac{\epsilon}{2}},M_f]\in -\epsilon M_{Df}(1-\Delta)^{\frac{\epsilon-1}{2}}\cdot\mathrm{sgn}(D)+(\mathcal{L}_{\frac{1}{1-\epsilon},\infty})_0.$$

By a simple limiting argument (see, e.g., [24, Lemma 3.1]), this reduces the assertion for $d=1$ to the corresponding assertion about the operator $\epsilon M_{Df}(1-\Delta)^{\frac{\epsilon-1}{2}}\cdot\mathrm{sgn}(D)$. Since $\mathrm{sgn}(D)$ is unitary, we have

$$\mu(M_{Df}(1-\Delta)^{\frac{\epsilon-1}{2}}\cdot\mathrm{sgn}(D))=\mu(M_{Df}(1-\Delta)^{\frac{\epsilon-1}{2}})\,.$$

The assertion for $d=1$ follows now from Theorem 6.9 (applied with $p=\frac{1}{1-\epsilon}$). □

Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii). The assertion follows from Lemma 7.1 and the universal bounds in parts (i) by means a simple approximation argument (see, e.g., [24, Lemma 3.2]). □

In the proof of part (iii) we need a lower bound with localization functions.

Lemma 7.2. *Let $d\geq 2$ and $\epsilon\in(-\frac{d}{2},1)$ (alternatively, let $d=1$ and $\epsilon\in(0,1)$). If $f\in C^\infty(\mathbb{R}^d)$ and $\chi\in C_c(\mathbb{R}^d)$, then*

$$\lim_{t\to\infty} t^{\frac{1-\epsilon}{d}}\mu\Big(t,M_{\bar\chi}[(-\Delta)^{\frac{\epsilon}{2}},M_f]M_\chi\Big)=\kappa_{d,\epsilon}\left(\int_{\mathbb{R}^d}|\chi|^{\frac{2d}{1-\epsilon}}|\nabla f|^{\frac{d}{1-\epsilon}}\right)^{\frac{1-\epsilon}{d}}$$

with $\kappa_{d,\epsilon}$ given by (1).

Proof. Let $\phi\in C_c^\infty(\mathbb{R}^d)$ be real-valued with $\chi\phi=\chi$. Then

$$M_{\bar\chi}[(-\Delta)^{\frac{\epsilon}{2}},M_f]M_\chi=M_{\bar\chi}[(-\Delta)^{\frac{\epsilon}{2}},M_{f\cdot\phi}]M_\chi.$$

Hence, we may assume without loss of generality that $f \in C_c^\infty(\mathbb{R}^d)$. The rest of the argument follows that in Corollary 7.1 *mutatis mutandi.* $\square$

The following bound is an immediate consequence of Lemma 7.2.

Lemma 7.3. *Let $d \geq 2$ and let $\epsilon \in (-\frac{d}{2}, 1)$ (alternatively, let $d = 1$ and $\epsilon \in (0,1)$). There is a constant $c_{d,\epsilon} > 0$ such that for all $f \in C^\infty(\mathbb{R}^d)$ and $\chi \in C_c(\mathbb{R}^d)$,*

$$\|M_{\bar{\chi}}[(-\Delta)^{\frac{\epsilon}{2}}, M_f]M_\chi\|_{\frac{d}{1-\epsilon},\infty} \geq c_{d,\epsilon} \left(\int_{\mathbb{R}^d} |\chi|^{\frac{2d}{1-\epsilon}} |\nabla f|^{\frac{d}{1-\epsilon}} \right)^{\frac{1-\epsilon}{d}}.$$

8. Proof of Theorems 1.1 and 1.2 (iii)

In this section, T_t, $t \in \mathbb{R}^d$, denotes the translation operator,

$$(T_t f)(s) = f(s+t), \ s \in \mathbb{R}^d.$$

Lemma 8.1. *Let $\epsilon \in (-\frac{d}{2}, 0)$ and let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. If $h, \Phi \in C_c(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \Phi(t)\langle T_{-t}f, h\rangle dt = \langle f * \Phi, h\rangle.$$

Proof. Let h be supported in $B(0, r_1)$ and let Φ be supported in $B(r_2)$. We have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |h(s)||\Phi(t)||f(s-t)|\, dsdt \leq \|h\|_\infty \|\Phi\|_\infty \int_{\substack{|s| \leq r_1 \\ |t| \leq r_2}} |f(s-t)|\, dsdt =$$

$$= \|h\|_\infty \|\Phi\|_\infty \int_{\substack{|u| \leq r_1 \\ |u+v| \leq r_2}} |f(v)|\, dudv \leq \|h\|_\infty \|\Phi\|_\infty \int_{\substack{|u| \leq r_1 \\ |v| \leq r_1 + r_2}} |f(v)|\, dudv =$$

$$= c_d r_1^2 \|h\|_\infty \|\Phi\|_\infty \int_{|v| \leq r_1 + r_2} |f(v)|\, dv < \infty.$$

By Fubini theorem, we have

$$\int_{\mathbb{R}^d} \Phi(t)\Big(\int_{\mathbb{R}^d} f(s-t)\overline{h(s)}\, ds\Big)dt = \int_{\mathbb{R}^d} \overline{h(s)}\Big(\int_{\mathbb{R}^d} f(s-t)\Phi(t)dt\Big)ds.$$

This completes the proof. $\square$

In the following lemma we use the notion of submajorization, which is discussed in detail for instance in [37, Section 2.3].

Lemma 8.2. *Let* $\epsilon \in (-\frac{d}{2}, 1]$ *and let* $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ *be such that* $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ *extends to a bounded operator on* $L_2(\mathbb{R}^d)$*. If* $\Phi \in C_c(\mathbb{R}^d)$ *is nonnegative with* $\|\Phi\|_1 = 1$*, then* $[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]$ *also extends to a bounded operator on* $L_2(\mathbb{R}^d)$ *and*

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}] \prec\prec [(-\Delta)^{\frac{\epsilon}{2}}, M_f].$$

Moreover, if $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ *is compact, then so is* $[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]$*.*

Proof. Let $A : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ be the extension of $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$. We consider the weak integral

$$B = \int_{\mathbb{R}^d} \Phi(t) T_{-t} A T_t dt.$$

Clearly, B is a bounded operator and we have $B \prec\prec A$; see, for instance, [35, Lemma 18 and its proof].

We now claim that

$$\langle B\phi, \psi\rangle = \langle [(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]\phi, \psi\rangle, \quad \phi, \psi \in C_c^\infty(\mathbb{R}^d). \tag{12}$$

By definition,

$$LHS = \int_{\mathbb{R}^d} \Phi(t) \langle T_{-t} A T_t \phi, \psi\rangle dt = \int_{\mathbb{R}^d} \Phi(t) \langle [(-\Delta)^{\frac{\epsilon}{2}}, M_{T_{-t}f}]\phi, \psi\rangle dt.$$

By Lemma 3.1, we have

$$\begin{aligned}\langle [(-\Delta)^{\frac{\epsilon}{2}}, M_{T_{-t}f}]\phi, \psi\rangle &= \langle T_{-t}f \cdot \phi, (-\Delta)^{\frac{\epsilon}{2}}\psi\rangle - \langle (-\Delta)^{\frac{\epsilon}{2}}\phi, \overline{T_{-t}f} \cdot \psi\rangle = \\ &= \langle T_{-t}f, \bar{\phi} \cdot (-\Delta)^{\frac{\epsilon}{2}}\psi\rangle - \langle \bar{\psi} \cdot (-\Delta)^{\frac{\epsilon}{2}}\phi, \overline{T_{-t}f}\rangle = \langle T_{-t}f, h\rangle,\end{aligned}$$

where

$$h = \bar{\phi} \cdot (-\Delta)^{\frac{\epsilon}{2}}\psi - \psi \cdot \overline{(-\Delta)^{\frac{\epsilon}{2}}\phi} \in C_c(\mathbb{R}^d).$$

Meanwhile, it follows from Lemma 3.1 that

$$RHS = \langle (f * \Phi) \cdot \phi, (-\Delta)^{\frac{\epsilon}{2}}\psi\rangle - \langle (-\Delta)^{\frac{\epsilon}{2}}\phi, \overline{(f * \Phi)} \cdot \psi\rangle = \langle f * \Phi, h\rangle.$$

The claim (12) follows now from Lemma 8.1.

By (12), we have

$$B\phi = [(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]\phi, \quad \phi \in C_c^\infty(\mathbb{R}^d).$$

Therefore, $[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]$ extends to a bounded operator on $L_2(\mathbb{R}^d)$ and its extension is exactly B.

The last statement concerning compactness follows from the general fact that any bounded operator that is submajorized by a compact one is compact. The latter fact follows immediately from the definition of submajorization via the singular value function, together with the simple fact that a bounded operator A is compact if and only if $N^{-1}\sum_{n=1}^N \mu(n,A) \to 0$ as $N \to \infty$. □

Lemma 8.3. *Let* $1 < p \le \infty$ *and let* $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ *be such that*

$$\|\nabla(f * \Phi)\|_{L_p(\mathbb{R}^d,\mathbb{C}^d)} \le 1$$

for every nonnegative $\Phi \in C_c^\infty(\mathbb{R}^d)$ *with* $\|\Phi\|_1 = 1$*. Then* $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ *and*

$$\|f\|_{\dot{W}_p^1(\mathbb{R}^d)} \le 1.$$

Proof. Let $0 \le \Phi \in C_c^\infty(\mathbb{R}^d)$ with $\|\Phi\|_1 = 1$ and set

$$\Phi_n(t) := n^d \Phi(nt), \quad t \in \mathbb{R}^d, \quad n \in \mathbb{N}.$$

By assumption, we have

$$\|\nabla(f * \Phi_n)\|_{L_p(\mathbb{R}^d,\mathbb{C}^d)} \le 1.$$

Since $1 < p \le \infty$, $L_p(\mathbb{R}^d,\mathbb{C}^d)$ is the Banach dual of a separable space, namely of $L_{p'}(\mathbb{R}^d,\mathbb{C}^d)$. Therefore (see, e.g., [53, Proposition 4.49]) there is a subsequence $(\nabla(f * \Phi_{n_k}))$ and a $G \in L_p(\mathbb{R}^d,\mathbb{C}^d)$ such that $\nabla(f * \Phi_{n_k}) \to G$ in weak*-topology on L_p. In particular, $\nabla(f * \Phi_{n_k}) \to G$ in the sense of distributions. However, $\nabla(f * \Phi_{n_k}) \to \nabla f$ in the sense of distributions. By uniqueness of the limit, we have $G = \nabla f$ and, therefore, $\nabla f \in L_p(\mathbb{R}^d,\mathbb{C}^d)$ and

$$\|\nabla f\|_{L_p(\mathbb{R}^d,\mathbb{C}^d)} \le 1,$$

as claimed. □

Proof of Theorems 1.1 and 1.2 (iii). Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and assume $[(-\Delta)^{\frac{\epsilon}{2}}, M_f]$ extends to a bounded operator on $L_2(\mathbb{R}^d)$ that belongs to $\mathcal{L}_{\frac{d}{1-\epsilon},\infty}$.

Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be nonnegative with $\|\Phi\|_1 = 1$. By Lemma 8.2, we have

$$[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}] \prec\prec [(-\Delta)^{\frac{\epsilon}{2}}, M_f].$$

Therefore, if $\chi \in C_c^\infty(\mathbb{R}^d)$ satisfies $\|\chi\|_\infty = 1$, we have

$$\Big\|M_{\overline{\chi}}[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]M_\chi\Big\|_{\frac{d}{1-\epsilon},\infty} \leq c^{(1)}_{d,\epsilon}\Big\|[(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big\|_{\frac{d}{1-\epsilon},\infty}.$$

Meanwhile, since $\chi \in C_c^\infty(\mathbb{R}^d)$ and $f * \Phi \in C^\infty(\mathbb{R}^d)$, Lemma 7.3 implies that

$$\||\chi|^2 \cdot |\nabla(f*\Phi)|\|_{L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)} \leq c^{(2)}_{d,\epsilon}\Big\|M_{\bar{\chi}}[(-\Delta)^{\frac{\epsilon}{2}}, M_{f*\Phi}]M_\chi\Big\|_{\frac{d}{1-\epsilon},\infty}.$$

Combining these bounds, we find

$$\||\chi|^2 \cdot |\nabla(f*\Phi)|\|_{L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d)} \leq c^{(1)}_{d,\epsilon}c^{(2)}_{d,\epsilon}\Big\|[(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big\|_{\frac{d}{1-\epsilon},\infty}.$$

Taking the supremum over $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\chi\|_\infty = 1$, we obtain

$$\|\nabla(f*\Phi)\|_{L_{\frac{d}{1-\epsilon}}(\mathbb{R}^d,\mathbb{C}^d)} \leq c^{(1)}_{d,\epsilon}c^{(2)}_{d,\epsilon}\Big\|[(-\Delta)^{\frac{\epsilon}{2}}, M_f]\Big\|_{\frac{d}{1-\epsilon},\infty}.$$

The assertion follows from Lemma 8.3. □

9. Proof of Theorem 1.3

Proof of Theorem 1.3 (i). The first part follows from [9, Theorem 2]. Indeed, note that for sufficiently nice functions ϕ,

$$(-\Delta)^{\frac{1}{2}}\phi(t) = c_d \int\limits_{\mathbb{R}^d} \frac{\phi(s)-\phi(t)}{|t-s|^{d+1}}ds,$$

where the integral is understood in the principal value sense. Therefore $[(-\Delta)^{\frac{1}{2}}, M_f]$ has integral kernel

$$c_d\frac{f(t)-f(s)}{|t-s|^{d+1}},$$

and this is precisely the setting of [9, Theorem 2]. □

For the proofs of (ii) and (iii), we use the following notation

$$e_\xi(t) := e^{i\xi\cdot t}, \quad t, \xi, \in \mathbb{R}^d.$$

and the following auxiliary results.

Lemma 9.1. *Let $\phi, \psi, f \in \mathcal{S}(\mathbb{R}^d)$. We have*

$$\langle \psi, f\phi\rangle = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} \widehat{f}(\eta_1-\eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2)d\eta_1\, d\eta_2,$$

$$\langle \psi, [(-\Delta)^{\frac{1}{2}}, M_f]\phi\rangle = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} (|\eta_1|-|\eta_2|)\widehat{f}(\eta_1-\eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2)d\eta_1\, d\eta_2.$$

Here $\widehat{f}(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-i\xi\cdot x}f(x)dx$ denotes the Fourier transform.

Proof. The first assertion is standard and the proof is omitted.

To prove the second assertion, suppose first that $\hat{\phi}$ and $\hat{\psi}$ vanish near 0. It follows that $(-\Delta)^{\frac{1}{2}}\phi$ and $(-\Delta)^{\frac{1}{2}}\psi$ are Schwartz functions. Note that

$$\widehat{(-\Delta)^{\frac{1}{2}}\psi}(\eta_1) = |\eta_1|\hat{\psi}(\eta_1), \quad \widehat{(-\Delta)^{\frac{1}{2}}\phi}(\eta_2) = |\eta_2|\hat{\phi}(\eta_2), \quad \eta_1,\eta_2\in\mathbb{R}^d.$$

We have

$$\langle \psi, [(-\Delta)^{\frac{1}{2}}, M_f]\phi\rangle = \langle (-\Delta)^{\frac{1}{2}}\psi, f\phi\rangle - \langle \psi, f((-\Delta)^{\frac{1}{2}}\phi)\rangle.$$

Using the first assertion, we write

$$\langle (-\Delta)^{\frac{1}{2}}\psi, f\phi\rangle = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |\eta_1|\hat{f}(\eta_1-\eta_2)\overline{\hat{\psi}(\eta_1)}\hat{\phi}(\eta_2)d\eta_1\, d\eta_2,$$

$$\langle \psi, f((-\Delta)^{\frac{1}{2}}\phi)\rangle = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |\eta_2|\hat{f}(\eta_1-\eta_2)\overline{\hat{\psi}(\eta_1)}\hat{\phi}(\eta_2)d\eta_1\, d\eta_2.$$

Combining these equalities, we obtain the second assertion for the case when $\hat{\phi}$ and $\hat{\psi}$ vanish near 0.

However,

$$\Bigg|\int\limits_{\mathbb{R}^d\times\mathbb{R}^d} (|\eta_1|-|\eta_2|)\hat{f}(\eta_1-\eta_2)\overline{\hat{\psi}(\eta_1)}\hat{\phi}(\eta_2)d\eta_1\, d\eta_2\Bigg| \leq \|\hat{f}'\|_\infty\|\hat{\phi}\|_1\|\hat{\psi}\|_1.$$

So, the right hand side in the second assertion is a continuous functional of $\hat{\psi}$ in L_1-norm. One can also see that so is the left hand side. Thus, we can remove the restriction on ϕ and ψ. $\square$

Lemma 9.2. *Let $f\in C_c^\infty(\mathbb{R}^d)$ and let $\omega\in\mathbb{S}^{d-1}$. We have*

$$M_{e_{-n\omega}}[(-\Delta)^{\frac{1}{2}}, M_f]M_{e_{n\omega}} \to M_{\omega\cdot\nabla f}, \quad n\to\infty,$$

in the weak operator topology.

Proof. Let $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$. Note that

$$\widehat{e_{n\omega}\phi}(\eta_1) = \hat{\phi}(\eta_1 - n\omega), \quad \widehat{e_{n\omega}\psi}(\eta_2) = \hat{\psi}(\eta_2 - n\omega).$$

Applying the second assertion of Lemma 9.1 to the functions $e_{n\omega}\phi, e_{n\omega\psi}, f \in C_c^\infty(\mathbb{R}^d)$, we arrive at

$$\begin{aligned}\langle \psi, \big(M_{e_{-n\omega}}[(-\Delta)^{\frac{1}{2}}, M_f]M_{e_{n\omega}}\big)\phi\rangle = \langle e_{n\omega}\psi, [(-\Delta)^{\frac{1}{2}}, M_f](e_{n\omega}\phi)\rangle = \\ = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} (|\eta_1| - |\eta_2|)\widehat{f}(\eta_1 - \eta_2)\overline{\widehat{\psi}(\eta_1 - n\omega)}\widehat{\phi}(\eta_2 - n\omega)d\eta_1\, d\eta_2.\end{aligned}$$

Replacing η_1 with $\eta_1 + n\omega$ and η_2 with $\eta_2 + n\omega$, we obtain

$$\begin{aligned}\langle \psi, \big(M_{e_{-n\omega}}[(-\Delta)^{\frac{1}{2}}, M_f]M_{e_{n\omega}}\big)\phi\rangle = \\ = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} (|\eta_1 + n\omega| - |\eta_2 + n\omega|)\widehat{f}(\eta_1 - \eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2)d\eta_1\, d\eta_2.\end{aligned}$$

Set

$$\begin{aligned}F_n(\eta_1,\eta_2) &:= (|\eta_1 + n\omega| - |\eta_2 + n\omega|)\widehat{f}(\eta_1 - \eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2), \quad \eta_1,\eta_2 \in \mathbb{R}^d,\\ F(\eta_1,\eta_2) &:= (|\eta_1 - \eta_2|)\widehat{f}(\eta_1 - \eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2), \quad \eta_1,\eta_2 \in \mathbb{R}^d,\\ F_\infty(\eta_1,\eta_2) &:= (\omega\cdot(\eta_1 - \eta_2))\widehat{f}(\eta_1 - \eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2), \quad \eta_1,\eta_2 \in \mathbb{R}^d.\end{aligned}$$

It follows from the triangle inequality that $|F_n| \le F$ on $\mathbb{R}^d \times \mathbb{R}^d$. We have

$$\lim_{n\to\infty} |\eta_1 + n\omega| - |\eta_2 + n\omega| \to \omega\cdot(\eta_1 - \eta_2), \quad \eta_1,\eta_2 \in \mathbb{R}^d.$$

Thus, $F_n \to F_\infty$ pointwise. Moreover, since

$$|F(\eta_1,\eta_2)| \le \|\widehat{(-\Delta)^{\frac{1}{2}}f}\|_\infty |\hat{\psi}|(\eta_1)|\hat{\phi}|(\eta_2), \quad \eta_1,\eta_2 \in \mathbb{R},$$

it follows that F is integrable on $\mathbb{R}^d \times \mathbb{R}^d$. By the Dominated Convergence Theorem, we have

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}^d} F_n(\eta_1,\eta_2)d\eta_1 d\eta_2 \to \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} F(\eta_1,\eta_2)d\eta_1 d\eta_2, \quad n\to\infty.$$

Thus,

$$\lim_{n\to\infty} \langle \psi, \big(M_{e_{-n\omega}}[(-\Delta)^{\frac{1}{2}}, M_f]M_{e_{n\omega}}\big)\phi\rangle =$$

$$= (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} (\omega\cdot(\eta_1-\eta_2))\widehat{f}(\eta_1-\eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2)d\eta_1 d\eta_2 =$$

$$= (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} \widehat{\omega\cdot\nabla f}(\eta_1-\eta_2)\overline{\widehat{\psi}(\eta_1)}\widehat{\phi}(\eta_2)d\eta_1 d\eta_2 = \langle\psi,(\omega\cdot\nabla f)\phi\rangle.$$

Since our sequence of operators is bounded in the uniform norm, the assertion follows. □

Lemma 9.3. *Let $f\in C^\infty(\mathbb{R}^d)$ be such that $[(-\Delta)^{\frac{1}{2}},M_f]$ is bounded. We have $f\in\dot{W}^1_\infty(\mathbb{R}^d)$ and*

$$\|f\|_{\dot{W}^1_\infty(\mathbb{R}^d)}\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty.$$

Proof. Let $\chi\in C_c^\infty(\mathbb{R}^d)$ with $\|\chi\|_\infty=1$ and choose $\theta\in C_c^\infty(\mathbb{R}^d)$ with $\theta\chi=\chi$. It is immediate that $f\theta\in C_c^\infty(\mathbb{R}^d)$ and

$$M_{\bar{\chi}}[(-\Delta)^{\frac{1}{2}},M_{f\theta}]M_\chi=M_{\bar{\chi}}[(-\Delta)^{\frac{1}{2}},M_f]M_\chi.$$

Therefore,

$$\|M_{\bar{\chi}}[(-\Delta)^{\frac{1}{2}},M_{f\theta}]M_\chi\|_\infty\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty$$

and, for every $n\in\mathbb{N}$ and for every $\omega\in\mathbb{S}^{d-1}$,

$$\|M_{\bar{\chi}}\cdot M_{e_{-n\omega}}[(-\Delta)^{\frac{1}{2}},M_{f\theta}]M_{e_{n\omega}}\cdot M_\chi\|_\infty\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty.$$

Letting $n\to\infty$, we deduce from Lemma 9.2 that for every $\omega\in\mathbb{S}^{d-1}$

$$\|M_{\bar{\chi}}\cdot M_{\omega\cdot\nabla(f\theta)}\cdot M_\chi\|_\infty\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty.$$

In other words, for every $\omega\in\mathbb{S}^{d-1}$ we have

$$\||\chi|^2(\omega\cdot\nabla(f\theta))\|_{L_\infty(\mathbb{R}^d)}\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty.$$

We have $|\chi|^2\nabla\theta=0$ and therefore $|\chi|^2(\omega\cdot\nabla(f\theta))=|\chi|^2(\omega\nabla f)$. Thus, we have shown that for every $\omega\in\mathbb{S}^{d-1}$ we have

$$\||\chi|^2(\omega\cdot\nabla f)\|_{L_\infty(\mathbb{R}^d}\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty$$

Taking the supremum over all $\chi\in C_c^\infty(\mathbb{R}^d)$ with $\|\chi\|_\infty=1$, we obtain

$$\|\omega\cdot\nabla f\|_{L_\infty(\mathbb{R}^d)}\le\|[(-\Delta)^{\frac{1}{2}},M_f]\|_\infty$$

for every $\omega\in\mathbb{S}^{d-1}$. Since ∇f is continuous, we deduce that

$$|\omega \cdot (\nabla f)(x)| \leq \|[(-\Delta)^{\frac{1}{2}}, M_f]\|_\infty, \qquad \text{for all } \omega \in \mathbb{S}^{d-1} \text{ and all } x \in \mathbb{R}^d.$$

For given $x \in \mathbb{R}^d$ with $\nabla f(x) \neq 0$ we pick $\omega = \nabla f(x)/|\nabla f(x)|$ and obtain

$$|(\nabla f)(x)| \leq \|[(-\Delta)^{\frac{1}{2}}, M_f]\|_\infty, \qquad x \in \mathbb{R}^d,$$

which is the claimed inequality. □

Proof of Theorem 1.3 (ii). Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and assume the operator $[(-\Delta)^{\frac{1}{2}}, M_f]$ is bounded. Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be nonnegative with $\|\Phi\|_1 = 1$. By Lemma 8.2, we have

$$\|[(-\Delta)^{\frac{1}{2}}, M_{f*\Phi}]\|_\infty \leq \|[(-\Delta)^{\frac{1}{2}}, M_f]\|_\infty.$$

Since $f * \Phi \in C^\infty(\mathbb{R}^d)$, it follows from Lemma 9.3 that

$$\|f * \Phi\|_{\dot{W}^1_\infty(\mathbb{R}^d)} \leq \|[(-\Delta)^{\frac{1}{2}}, M_f]\|_\infty.$$

The assertion follows now from Lemma 8.3. □

Lemma 9.4. *Let $f, \chi \in C_c^\infty(\mathbb{R}^d)$. If $M_{\bar{\chi}}[(-\Delta)^{\frac{1}{2}}, M_f]M_\chi$ extends to a compact operator, then $\chi \nabla f = 0$.*

Proof. For every $\psi, \phi \in C_c^\infty(\mathbb{R}^d)$ and for every $\omega \in \mathbb{S}^{d-1}$, we have $e_{n\omega}\psi, e_{n\omega}\phi \to 0$ weakly to zero in $L_2(\mathbb{R}^d)$ as $n \to \infty$. Therefore, the compactness of the operator implies

$$\langle e_{n\omega}\chi\psi, [(-\Delta)^{\frac{1}{2}}, M_f] e_{n\omega}\chi\phi\rangle \to 0, \quad n \to \infty.$$

In other words,

$$\langle \chi\psi, (M_{e_{-n\omega}}[(-\Delta)^{\frac{1}{2}}, M_f]M_{e_{n\omega}})(\chi\phi)\rangle \to 0, \quad n \to \infty.$$

By Lemma 9.2, we deduce that for every $\omega \in \mathbb{S}^{d-1}$

$$\langle \chi\psi, M_{\omega\cdot\nabla f}(\chi\phi)\rangle = 0.$$

In particular,

$$\int_{\mathbb{R}^d} |\chi|^2 \bar{\psi}\phi \partial_k f = 0, \quad 1 \leq k \leq d.$$

Taking $\bar{\psi}\phi$ to be (approximately) the sign of $\partial_k f$, we conclude that

$$\int_{\mathbb{R}^d} |\chi|^2 |\partial_k f| = 0, \quad 1 \leq k \leq d.$$

This completes the proof. □

Proof of Theorem 1.3 (iii). Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and assume that $[(-\Delta)^{\frac{1}{2}}, M_f]$ extends to a compact operator.

Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be nonnegative with $\|\Phi\|_1 = 1$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ and choose $\theta \in C_c^\infty(\mathbb{R}^d)$ with $\theta\chi = \chi$.

By Lemma 8.2, $[(-\Delta)^{\frac{1}{2}}, M_{\Phi * f}]$ is compact. Therefore,

$$M_{\bar{\chi}}[(-\Delta)^{\frac{1}{2}}, M_{(\Phi * f)\theta}]M_\chi = M_{\bar{\chi}}[(-\Delta)^{\frac{1}{2}}, M_{\Phi * f}]M_\chi$$

is also compact. Clearly, $(\Phi * f)\theta \in C_c^\infty(\mathbb{R}^d)$. It follows from Lemma 9.4 that

$$\chi \cdot \nabla((f * \Phi)\theta) = 0.$$

We have $\chi\nabla\theta = 0$ and therefore

$$\chi \cdot \nabla(f * \Phi) = 0.$$

Since $\chi \in C_c^\infty(\mathbb{R}^d)$ is arbitrary, it follows that

$$(\nabla f) * \Phi = \nabla(f * \Phi) = 0.$$

Since Φ is arbitrary, it follows that $\nabla f = 0$. This completes the proof. □

Appendix A. Commutator estimate

Throughout this appendix we assume $d \geq 2$. Our goal is to prove the following commutator bound. We recall that the algebra Π is defined at the beginning of Section 6.

Theorem A.1. *If $T \in \Pi$ is compactly supported, then*

$$[T, (1-\Delta)^{-\frac{p}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0, \quad p > 0.$$

The following lemma is a standard result, but for the convenience of the reader we provide a proof.

Lemma A.2. *If $f \in C_c^\infty(\mathbb{R}^d)$, then*

$$[M_f, (1-\Delta)^{-\frac{p}{2}}] \in \mathcal{L}_{\frac{d}{p+1},\infty}, \quad p > 0.$$

Proof. We write $f = f_1 f_2$, $f_1, f_2 \in C_c^\infty(\mathbb{R}^d)$. By the Leibniz rule, we have

$$[M_f, (1-\Delta)^{-\frac{p}{2}}] = [M_{f_1}, (1-\Delta)^{-\frac{p}{2}}] \cdot M_{f_2} + M_{f_1} \cdot [M_{f_2}, (1-\Delta)^{-\frac{p}{2}}] = X_1Y_1 + Y_2X_2,$$

where we denote

$$X_1 = [M_{f_1}, (1-\Delta)^{-\frac{p}{2}}](1-\Delta)^{\frac{p+1}{2}}, \quad X_2 = (1-\Delta)^{\frac{p+1}{2}}[M_{f_2}, (1-\Delta)^{-\frac{p}{2}}],$$
$$Y_1 = (1-\Delta)^{-\frac{p+1}{2}} M_{f_2}, \quad Y_2 = M_{f_1}(1-\Delta)^{-\frac{p+1}{2}}.$$

Clearly, X_1 and X_2 are pseudodifferential operators of order 0. Hence, they are bounded. By Theorem 2.4 we have $Y_1, Y_2 \in \mathcal{L}_{\frac{d}{p+1},\infty}$. This completes the proof. □

Using standard arguments we can now weaken the regularity on f required in Lemma A.2.

Lemma A.3. *If $f \in C_c(\mathbb{R}^d)$, then*

$$[M_f, (1-\Delta)^{-\frac{p}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0, \quad p > 0.$$

Proof. Assume for definiteness that f is supported on $[-1,1]^d$. Choose a sequence $\{f_n\}_{n\geq 0} \subset C_c^\infty(\mathbb{R}^d)$ supported in $[-1,1]^d$ such that $f_n \to f$ in the uniform norm. Using quasi-triangle and Hölder inequalities, we write

$$\|[M_{f_n-f}, (1-\Delta)^{-\frac{p}{2}}]\|_{\frac{d}{p},\infty} \leq 2^{1+\frac{p}{d}} \|f_n - f\|_\infty \|M_{\chi_{[-1,1]^d}}(1-\Delta)^{-\frac{p}{2}}]\|_{\frac{d}{p},\infty}.$$

The second factor on the right hand side is finite by Theorem 2.4. Therefore,

$$[M_{f_n}, (1-\Delta)^{-\frac{p}{2}}] \to [M_f, (1-\Delta)^{-\frac{p}{2}}]$$

in $\mathcal{L}_{\frac{d}{p},\infty}$. Since the sequence on the left hand side belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$ by Lemma A.2, the assertion follows. □

The following result is a special case of [38, Theorem 1.6] (with $\alpha = 1$ and $\beta = p$).

Lemma A.4. *If $f \in C_c^\infty(\mathbb{R}^d)$, then*

$$[(1-\Delta)^{\frac{1}{2}}, M_f](1-\Delta)^{-\frac{p}{2}} \in \mathcal{L}_{\frac{d}{p},\infty}, \quad p > 0.$$

The following lemma is a technical precursor to Lemma A.6.

Lemma A.5. *If $f \in C_c^\infty(\mathbb{R}^d)$, then*

$$[D_k(-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] \in \mathcal{L}_{\frac{d}{p+1},\infty}, \quad p > 0, \quad k \in \{1, \cdots, d\}.$$

Proof. Set

$$h_k(s) = \frac{s_k}{|s|} - \frac{s_k}{(1+|s|^2)^{\frac{1}{2}}}, \quad s \in \mathbb{R}^d.$$

It is immediate that

$$[D_k(-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] - [D_k(1-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] =$$
$$= h_k(\nabla) M_f(1-\Delta)^{-\frac{p}{2}} - M_f(1-\Delta)^{-\frac{p}{2}} h_k(\nabla) =$$
$$= h_k(\nabla)(1-\Delta)\cdot(1-\Delta)^{-1} M_f(1-\Delta)^{-\frac{p}{2}} - M_f(1-\Delta)^{-\frac{p+2}{2}}\cdot(1-\Delta)h_k(\nabla).$$

Using Theorem 2.4 as in the proof of Lemma A.2, we obtain

$$(1-\Delta)^{-1} M_f(1-\Delta)^{-\frac{p}{2}}, \quad M_f(1-\Delta)^{-\frac{p+2}{2}} \in \mathcal{L}_{\frac{d}{p+2},\infty}.$$

Since $h_k(\nabla)\cdot(1-\Delta)$ is bounded, it follows from (8) that

$$[D_k(-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] - [D_k(1-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] \in \mathcal{L}_{\frac{d}{p+2},\infty}. \tag{13}$$

Next,

$$[D_k(1-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] = [D_k(1-\Delta)^{-\frac{1}{2}}, M_f]\cdot(1-\Delta)^{-\frac{p}{2}} =$$
$$= M_{D_k f}(1-\Delta)^{-\frac{p+1}{2}} - D_k(1-\Delta)^{-\frac{1}{2}}\cdot[(1-\Delta)^{\frac{1}{2}}, M_f](1-\Delta)^{-\frac{p+1}{2}}.$$

Applying Theorem 2.4 to the first term and Theorem A.4 (applied with $p+1$ instead of p) to the second term, we deduce that

$$[D_k(-\Delta)^{-\frac{1}{2}}, M_f(1-\Delta)^{-\frac{p}{2}}] \in \mathcal{L}_{\frac{d}{p+1},\infty}. \tag{14}$$

Combining (13) and (14), we complete the proof. □

The next lemma is the crucial step in proving Theorem A.1.

Lemma A.6. *If $f \in C_c(\mathbb{R}^d)$ and if $g \in C(\mathbb{S}^{d-1})$, then*

$$[g(\nabla(-\Delta)^{-\frac{1}{2}}), M_f(1-\Delta)^{-\frac{p}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0, \quad p > 0.$$

Proof. If $g \in C(\mathbb{S}^{d-1})$ is a monomial, then Lemma A.5 and the Leibniz rule yield

$$[g(\nabla(-\Delta)^{-\frac{1}{2}}), M_f(1-\Delta)^{-\frac{p}{2}}] \in \mathcal{L}_{\frac{d}{p+1},\infty}.$$

If $g \in C(\mathbb{S}^{d-1})$ is a polynomial, then we obtain by linearity

$$[g(\nabla(-\Delta)^{-\frac{1}{2}}), M_f(1-\Delta)^{-\frac{p}{2}}] \in \mathcal{L}_{\frac{d}{p+1},\infty} \subset (\mathcal{L}_{\frac{d}{p},\infty})_0.$$

Consider now the general case. Fix $g \in C(\mathbb{S}^{d-1})$ and choose a sequence of polynomials $\{g_n\}_{n\geq 0} \in C(\mathbb{S}^{d-1})$ such that $g_n \to g$ in the uniform norm. Using quasi-triangle and Hölder inequalities, we write

$$\|[(g_n-g)(\nabla(-\Delta)^{-\frac{1}{2}}),M_f(1-\Delta)^{-\frac{p}{2}}]\|_{\frac{d}{p},\infty}\leq 2^{1+\frac{p}{d}}\|g_n-g\|_\infty\|M_f(1-\Delta)^{-\frac{p}{2}}\|_{\frac{d}{p},\infty}.$$

The second factor on the right hand side is finite by Theorem 2.4. Therefore,

$$[g_n(\nabla(-\Delta)^{-\frac{1}{2}}),M_f(1-\Delta)^{-\frac{p}{2}}]\to[g(\nabla(-\Delta)^{-\frac{1}{2}}),M_f(1-\Delta)^{-\frac{p}{2}}]$$

in $\mathcal{L}_{\frac{d}{p},\infty}$. Since the sequence on the left is in $(\mathcal{L}_{\frac{d}{p},\infty})_0$, the assertion follows. □

Next, we derive an approximation result for operators $T\in\Pi$.

Lemma A.7. *Let $T\in\Pi$. There are $\{f_{n,l}\}_{n,l\geq 1}\subset(C_0+\mathbb{C})(\mathbb{R}^d)$, $\{g_{n,l}\}_{n,l\geq 1}\subset C(\mathbb{S}^{d-1})$ and $\{S_n\}_{n\geq 0}\subset\mathcal{K}(L_2(\mathbb{R}^d))$ such that, with the limit in the uniform norm,*

$$T=\lim_{n\to\infty}T_n,\quad T_n:=\sum_{l=1}^{l_n}M_{f_{n,l}}g_{n,l}(\nabla(-\Delta)^{-\frac{1}{2}})+S_n.$$

Proof. If $T\in\Pi$, then we can find a sequence $\{T_n\}_{n\geq 0}\subset\Pi$ such that $T_n\to T$ in the uniform norm and such that

$$T_n=\sum_{l=1}^{l_n}\prod_{k=1}^{k_n}M_{f_{n,k,l}}g_{n,k,l}(\nabla(-\Delta)^{-\frac{1}{2}}).$$

Here, $f_{n,k,l}\in(C_0+\mathbb{C})(\mathbb{R}^d)$ and $g_{n,k,l}\in C(\mathbb{S}^{d-1})$ for every n, every k and every l.

By [31, Lemma 5.8], the operator

$$\prod_{k=1}^{k_n}M_{f_{n,k,l}}g_{n,k,l}(\nabla(-\Delta)^{-\frac{1}{2}})-M_{\prod_{k=1}^{k_n}f_{n,k,l}}(\prod_{k=1}^{k_n}g_{n,k,l})(\nabla(-\Delta)^{-\frac{1}{2}})$$

is compact. Setting

$$f_{n,l}=\prod_{k=1}^{k_n}f_{n,k,l},\quad g_{n,l}=\prod_{k=1}^{k_n}g_{n,k,l},\quad 1\leq l\leq l_n$$

the assertion follows. □

We are now in position to prove the main result of this appendix.

Proof of Theorem A.1. Let $T\in\Pi$ and let $T_n\in\Pi$ be as in Lemma A.7. Let $\phi\in C_c^\infty(\mathbb{R}^d)$. By the Leibniz rule, we have

$$[T_n,M_\phi(1-\Delta)^{-\frac{p}{2}}]=\sum_{l=1}^{l_n}[M_{f_{n,l}},M_\phi(1-\Delta)^{-\frac{p}{2}}]\cdot g_{n,l}(\nabla(-\Delta)^{-\frac{1}{2}})+$$

$$+\sum_{l=1}^{l_n} M_{f_{n,l}} \cdot [g_{n,l}(\nabla(-\Delta)^{-\frac{1}{2}}), M_\phi(1-\Delta)^{-\frac{p}{2}}]+$$
$$+[S_n, M_\phi(1-\Delta)^{-\frac{p}{2}}]. \quad (15)$$

The commutators on the right hand side of (15) belong to $(\mathcal{L}_{\frac{d}{p},\infty})_0$. Indeed,

$$[M_{f_{n,l}}, M_\phi(1-\Delta)^{-\frac{p}{2}}] = [M_{f_{n,l}\phi}, (1-\Delta)^{-\frac{p}{2}}] - [M_\phi, (1-\Delta)^{-\frac{p}{2}}] \cdot M_{f_{n,l}}.$$

Hence, the first commutator on the right hand side of (15) belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$ by Lemma A.3. The second commutator on the right hand side of (15) belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$ by Lemma A.6. The third commutator on the right hand side of (15) belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$ since $M_\phi(1-\Delta)^{-\frac{p}{2}} \in \mathcal{L}_{\frac{d}{p},\infty}$ and since S_n is compact. Therefore,

$$[T_n, M_\phi(1-\Delta)^{-\frac{p}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0.$$

Since $T_n \to T$ in the uniform norm and since $M_\phi(1-\Delta)^{-\frac{p}{2}} \in \mathcal{L}_{\frac{d}{p},\infty}$ by Theorem 2.4, it follows that

$$[T_n, M_\phi(1-\Delta)^{-\frac{p}{2}}] \to [T, M_\phi(1-\Delta)^{-\frac{p}{2}}],$$

where the limit is taken in $\mathcal{L}_{\frac{d}{p},\infty}$. Since the sequence on the left hand side belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$, it follows that

$$[T, M_\phi(1-\Delta)^{-\frac{p}{2}}] \in (\mathcal{L}_{\frac{d}{p},\infty})_0.$$

Suppose now that $T \in \Pi$ is compactly supported, that is, $T = M_\phi T = T M_\phi$ for some $\phi \in C_c^\infty(\mathbb{R}^d)$. We have

$$[T, (1-\Delta)^{-\frac{p}{2}}] = T(1-\Delta)^{-\frac{p}{2}} - (1-\Delta)^{-\frac{p}{2}}T =$$
$$= TM_\phi(1-\Delta)^{-\frac{p}{2}} - (1-\Delta)^{-\frac{p}{2}}M_\phi T = [T, M_\phi(1-\Delta)^{-\frac{p}{2}}] + [M_\phi, (1-\Delta)^{-\frac{p}{2}}] \cdot T.$$

The first term belongs to $(\mathcal{L}_{\frac{d}{p},\infty})_0$ as we have just shown above, and the second one does by Lemma A.2. This completes the proof. □

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