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Contests – A comparison of timing and information structures

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Abstract

We study a model of imperfectly discriminating contests with two ex ante symmetric agents. We consider four institutional settings: Contestants move either sequentially or simultaneously and in addition their types are either public or private information. We find that an effort-maximizing designer of the contest prefers the sequential to the simultaneous setting from an ex ante perspective. Moreover, the sequential contest Pareto dominates the simultaneous one when the contestants' types are sufficiently negatively correlated. Regarding the information structure, the designer ex ante prefers private information while the contestants prefer public information.

JEL Classification Numbers: D72, C72
Keywords: sequential contests, asymmetric information, rent-seeking

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1 Introduction

Contests are situations in which agents spend resources – they make a bid or exert an effort – in order to win a prize. Contests exhibit various structures. In many settings contestants move one after another and those who move later observe the predecessor's action. Examples include sports contests (e.g., alpine skiing) and rent-seeking processes when contributions are publicly announced during the ongoing process. In other settings contestants act simultaneously or act sequentially but cannot observe the predecessor's action (e.g., when contributions are not publicly announced during a rent-seeking process).[1] In addition to the timing, contests differ in their information condition. Contestants may or may not know their rivals' abilities or valuations of the prize (i.e., their rivals' types). For example, valuations for an auctioned good or lobbying capabilities may be private information.

In this article we consider four institutional settings. The order of moves is either simultaneous or sequential and, additionally, the contestants' types are either public or private information (cf. Figure 1). In reality, the structure of a contest might be exogenously given. Yet, it might also result from institutional design where one important objective is to maximize the overall effort by the contestants, for instance in sports or promotion contests. The main purpose of this article is to identify which of the four settings induces the highest overall effort and is thus favored by the designer of the contest if he can choose the timing and information structure.

More precisely, we consider imperfectly discriminating contests in which the agent who exerts more effort does not win for sure. Most of the extensive literature on imperfectly discriminating contests (starting with Tullock 1980) focuses on simultaneous contests and symmetric agents or on public information on the agents' types.[2] While solutions to the public information

[1] Simultaneous contests are strategically equivalent to sequential contests in which no information on the predecessor's action is available if we keep everything else constant and abstract from discounting. We therefore subsume under simultaneous contests also sequential contests in which the predecessor's action is not observable.

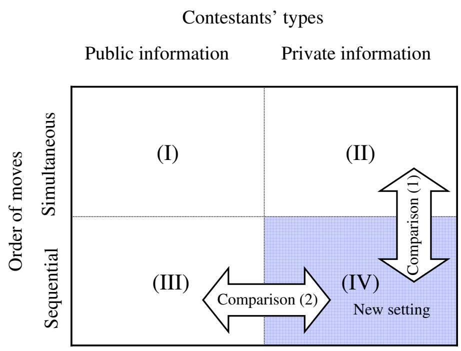
[2] For an early literature survey see, e.g., Nitzan (1994); for a recent one, see Konrad (2007).

settings and to simultaneous contests with private information, i.e., settings I, II, and III in Figure 1, are known (see later discussion), there is a dearth of scholarly work on sequential contests in which the agents' types are private information. This paper takes a step toward filling this gap. We provide an analytical solution to sequential two-player contests (in which the second mover observes the first mover's action) with private information on the contestants' types (i.e., their abilities or valuations of the prize). We consider ex ante symmetric agents where each agent can be of either high or low type so that asymmetric contests can arise. For this setting we derive the unique symmetric pure strategy perfect Bayesian equilibrium outcome in which all players make positive bids.

Having solved the sequential contest with private information, we can then compare the outcome with (1) a simultaneous contest with private information and with (2) a sequential contest with public information keeping in each case all other features of the contest fixed. The comparisons (1) and (2) are indicated in Figure 1. Both comparisons are new to the literature and complement Morgan's (2003) comparison of simultaneous and sequential contests given public information and Malueg and Yates' (2004) comparison of simultaneous contests with public and private information. To be able to draw conclusions as to which of the four institutional settings is favored and identify the driving forces behind the results, we additionally consider the known comparisons of settings I with II and I with III. In particular, we ask for which of the four settings the overall ex ante expected effort is highest (i.e., which setting a risk neutral effort-maximizing designer prefers). We also ask which setting risk neutral contestants prefer from an ex ante perspective (i.e., before knowing their type).

Overall, we conclude that a sequential contest with private information is favored by a designer who aims at maximizing the ex ante expected effort sum, while contestants ex ante prefer a sequential contest with public information. Although the overall effort in the sequential setting is higher (which is driven by the greater bid of the high type of the first mover), the contestants also prefer a sequential setting – at least if their types are sufficiently negatively correlated. This result is due to an *efficiency gain effect*: In the sequential contest the agent with the higher valuation is more likely to win than in

Figure 1: The four institutional settings and the new comparisons



the simultaneous contest. When comparing public with private information, no (further) efficiency gain effect is present and thus the agents' prefer (in contrast to the designer) the public information setting that induces lower expected efforts.

Finally, we apply our setting to an environment in which the designer wants to reach a fixed effort sum for the lowest possible offer of prize money. We show that in this case again the sequential setting with private information is optimal. Thus, we can conclude, for instance, that a designer may choose a sequential setting even if it is more costly because the contest takes longer.

The literature on sequential imperfectly discriminating contests starting with Dixit (1987) assumes either symmetric contestants or publicly known types. Linster (1993) compares simultaneous and sequential contests with two asymmetric agents and publicly known types[3] and shows that aggregate efforts in the sequential contest are greater only if the high type moves first.[4] Glazer and Hassin (2000) consider more than two symmetric players when comparing different sequences of moves. While in these studies the order of moves

[3]He briefly addresses one-sided asymmetric information but does not analyze any implications for the comparison.

[4]Similarly, Jost and Kräkel (2006) show that the designer of an asymmetric tournament may prefer a sequential setting if the first mover's type is sufficiently high.

is given exogenously, Baik and Shogren (1992) and Leininger (1993) endogenize the order of moves. Morgan (2003) extends these analyses by allowing for ex ante uncertainty. In contrast to our setting with private information, agents know each other's type when the contest begins. Considering a symmetric, independent distribution of types, he finds that an effort-maximizing designer and both contestants ex ante prefer a sequential contest. This result also holds for our distribution of types that allows for correlation between them.

There is a related literature on dynamic contests – in which agents simultaneously choose an effort in multiple rounds – that analyzes the effects of intermediate information.[5] Comparing settings in which agents observe or not the opponent's effort at an intermediate stage is similar to our comparison of sequential and simultaneous contests. In a dynamic setting, however, agents cannot commit to take an action only once and thus it becomes important *when* agents exert effort. The focus of this strand of literature is again on symmetric types or on public information (see Yildirim 2005; Romano and Yildirim 2005; and Aoyagi 2010).[6] An exception is Münster (2009) who allows for asymmetric information in a two-period game. He finds that intermediate information leads to lower expected efforts and is beneficial for both contestants. The results are driven by signaling issues that are not present in this paper: high ability contestants might put in little effort in the first period to make their opponents believe that they are of low ability.

There are only a few other studies that allow for asymmetric information, and none of them investigates sequential contests. Most closely related to our analysis is the aforementioned study by Malueg and Yates (2004). They analyze simultaneous contests with two-sided asymmetric information with a symmetric distribution of types that allows for correlation. We adopt this distributional assumption in our analysis. They find that the ex ante expected

[5] In such dynamic contests an agent faces the same opponent in each round. In elimination tournaments, in contrast, the losers of each round are eliminated (cf. Moldovanu and Sela 2006; Gradstein and Konrad 1999; or Rosen 1986).

[6] Similar games are studied in the industrial organization literature. For example, Saloner (1987) and Pal (1991) consider duopolies with two production periods. Harris and Vickers (1985, 1987) analyze dynamic patent races.

effort sum in the case of private and public information is the same and the effort-maximizing designer and the contestants are indifferent between both information conditions.[7] This stands in sharp contrast to our finding for sequential contests. Hurley and Shogren (1998a, b) consider simultaneous two-player contests with one- and two-sided asymmetric information. They approach the case of two-sided asymmetry numerically as it is analytically intractable in their setting. Whether the total effort is greater or smaller than in the case of complete information depends on the agents' perceptions of their relative abilities and the variance of the opponent's type.

The paper is structured as follows. Next, we present the basic model. In Section 2.1, we briefly review contests with public information. In Section 2.2, we focus on private information. We first consider simultaneous contests, before we introduce sequential contests with private information. In Section 3, we compare the four institutional settings. We conclude in Section 4. All proofs are in the Appendix if not indicated otherwise.

2 The model

We consider two risk neutral agents $i = 1, 2$ who compete for a prize by making non-negative effort expenditures x_i (or bids). Bids are investments that are not recovered (not even by the winner) and measured in the same units as the prize. Agent i's valuation of the prize (agent i's *type*) is $V_i \in \{V_L, V_H\}$, where $0 < V_L \leq V_H$. The prior probability distribution of the valuations is common knowledge and is given in Table 1.[8] An agent is with equal probability a high or low type. Moreover, both agents are symmetric from an ex ante perspective but their types can be different. The distribution allows for a correlation between types; the parameter $r \in [0, 1]$ is monotonically related to the correlation coefficient ρ of the valuations, where $\rho = 2(r - \frac{1}{2})$. More precisely, types are perfectly negatively (positively) correlated if $r = 0$ ($r = 1$), and independent from each other if $r = \frac{1}{2}$. The distribution of-

[7] Wärneryd (2003) shows a similar result for simultaneous common value contests: the expected effort sum is identical if both agents are uninformed or informed about the value.

[8] This distribution has also been used in Malueg and Yates (2004).

Table 1: Probability distribution of valuations

		V_1	
		V_L	V_H
V_2	V_L	$\frac{1}{2}r$	$\frac{1}{2}(1-r)$
	V_H	$\frac{1}{2}(1-r)$	$\frac{1}{2}r$

fers the whole range of correlation intensities, which will be important when we compare sequential with simultaneous contests given private information. While the symmetry of the distribution restricts its generality, it enables us to calculate explicit solutions for all four institutional settings that we consider.[9]

We assume that the probability that agent i wins the prize is given by the following symmetric, logit form contest success function (going back to Tullock 1980), where x_i denotes agent i's bid,[10]

$$\pi_i(x_1, x_2) = \begin{cases} \frac{x_i}{x_1+x_2} & \text{if } (x_1, x_2) \neq (0,0) \\ \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{1}$$

The contest success function offers a simple and tractable framework that is commonly used in the literature.[11] It implies positive, diminishing marginal returns to an agent's effort and that no one can win or lose with certainty.

[9]If we depart from the symmetry assumptions, the simultaneous contest with private information becomes analytically less tractable in our framework. See Hurley and Shogren (1998b) for a numerical analysis of simultaneous contests with private information when high and low types are not necessarily equally likely.

[10]We restrict to a discriminatory power of one. Otherwise the existence of pure strategy equilibria is not guaranteed and the explicit solution of sequential settings is difficult.

[11]Skaperdas (1996) and Clark and Riis (1998) provide an axiomatic foundation for the contest success function having this form.

The expected payoff of agent i given V_i and both agents' bids is then

$$\Psi_i(x_1, x_2, V_i) = \pi_i(x_1, x_2) \cdot V_i - x_i. \tag{2}$$

Note that the best reply of an agent to zero effort is not well-defined as the contest success function is discontinuous at $(x_1, x_2) = (0, 0)$. To solve this technical problem, we assume that a contestant can either choose an effort of zero or, if he wants to exert a positive effort, he has to exert at least some strictly positive (but arbitrarily small) amount of effort $\epsilon > 0$ (e.g., the smallest possible bid equals the smallest unit of money).[12] Then, the best reply to zero effort is ϵ.[13]

We consider four different settings: Contestants either make their bids simultaneously or sequentially and additionally the agents' types are either public or private information. The timing of the game is as follows. First, the risk neutral designer of the contest determines whether agents act simultaneously or sequentially, and in the latter case, which agent moves first. Then, the agents' types realize and each agent learns his type. Given public information, agents additionally learn the opponent's type. In a simultaneous contest, contestants then simultaneously make their bids. In a sequential contest, first agent 1 chooses his bid, then agent 2 observes the first mover's bid and makes his own bid. Finally, payoffs realize.

For some applications (e.g., sports contests) different abilities of agents are a more natural approach for modeling heterogeneity than different valuations of the prize. Note that our basic framework also captures heterogeneous abilities in terms of different effort costs. Instead of the expected payoff function $\Psi_i(x_1, x_2, V_i)$, we could consider $\widetilde{\Psi_i}(x_1, x_2, \tau_i) = \frac{x_i}{x_1+x_2} V - \tau_i x_i$, with $\tau_i \in \mathbb{R}^+$ and $V = V_i \cdot \tau_i$. Then, $\Psi_i(x_1, x_2, V_i) = \frac{1}{\tau_i} \widetilde{\Psi_i}(x_1, x_2, \tau_i)$ and, hence, equilibrium behavior is the same – irrespective of which notion of heterogeneity we apply – as behavior is invariant with respect to positive affine transformations of the expected utility function.

[12] An alternative approach are endogenous tie-breaking (or sharing) rules as proposed by Simon and Zame (1990) or Jackson et al. (2002).

[13] By bidding zero, the agent's expected payoff is $\frac{1}{2}V_i$; by bidding ϵ, it is $V_i - \epsilon$. For ϵ sufficiently small, we have $\frac{1}{2}V_i < V_i - \epsilon$.

2.1 Contests with public information

In this section, we briefly discuss the well-known case with publicly known valuations. Leininger (1993) derives the following equilibrium bids.[14]

Proposition 1 (Leininger 1993) *In the simultaneous contest with public information the unique pure strategy Nash-equilibrium is*

$$x_1^* = V_1 \frac{V_1 V_2}{(V_1 + V_2)^2} \equiv \omega V_1 \quad \textit{and} \quad x_2^* = V_2 \frac{V_1 V_2}{(V_1 + V_2)^2} \equiv \omega V_2. \tag{3}$$

Proposition 2 (Leininger 1993) *In the sequential contest with public information, where agent 1 is the first and agent 2 the second mover, the unique pure strategy subgame perfect equilibrium outcome is*

$$x_1^* = \frac{V_1}{2}\left(\frac{V_1}{2V_2}\right) \quad \textit{and} \quad x_2^* = \frac{V_1}{2}\left(1 - \frac{V_1}{2V_2}\right) \quad \textit{if } V_1 \leq 2V_2, \tag{4}$$

$$x_1^* = V_2 \quad \textit{and} \quad x_2^* = 0 \quad \textit{if } V_1 > 2V_2. \tag{5}$$

When contestants are homogeneous, they make identical bids irrespective of the order of moves. Moreover, these bids are the same in simultaneous and sequential contests. The reason is that given an equally strong opponent, no contestant is able to make a strategic gain by moving first. When contestants are heterogeneous, the contestant with the higher type bids more. In the simultaneous contest, both agents bid the same fraction $0 < \omega \equiv \frac{V_2 V_1}{(V_1+V_2)^2} < 1$ of their own valuation. In the sequential contest, in contrast, contestants do not bid the same fraction of their valuation due to the different strategic situation they face. Yet, the high type still bids more. More precisely, in the interior solution (i.e., $V_1 \leq 2V_2$) where the contestants' valuations are rather similar, $x_1^* \geq x_2^* \Leftrightarrow V_1 \geq V_2$; in the boundary solution (i.e., $V_1 > 2V_2$) where the first mover's valuation is larger, he can preempt the second mover who then spends zero effort. Moreover, for both settings, it can easily be verified that an agent's bid increases in the agent's own valuation (weakly so for the boundary solution). We refer to these bid increasing effects of higher

[14] The present setting corresponds to an unbiased contest in Leininger (1993) and to the setting in Morgan (2003), except for the distributional assumption, which, however, does not affect bids under public information.

valuations as *ability effects*.

In the simultaneous contest, bids increase (decrease) in the opponent's valuation if the opponent has a lower (higher) type as $\frac{\partial x_i^*}{\partial V_j} \geq (\leq)\ 0$ is equivalent to $V_i \geq (\leq)\ V_j$. Thus, bids increase in the opponent's valuation when the contest evens out (i.e., valuations become more similar). We call this effect the *competition intensity effect*. Competition intensity explains why the fraction that agents invest of their valuation in a symmetric contest, $\omega = \frac{1}{4}$, is larger than in an asymmetric contest, where $\omega = \frac{V_H V_L}{(V_H+V_L)^2} \leq \frac{1}{4}$. A crucial observation is that a sequential order of moves partly offsets the competition intensity effect. While it is present for the second mover,[15] it is not always present for the first mover: If $V_1 \leq 2V_2$, the first mover's effort falls in the second mover's valuation ($\frac{\partial x_1^*}{\partial V_2} = -\frac{V_1^2}{4V_2^2} < 0$), which cannot be explained by the contest becoming less close if $V_2 < V_1$. In particular, the high type of the first mover bids more if the follower is a low type than if he is a high type. The intuition is that if the first mover has the high(er) type (i.e., $V_2 < V_1$), he profits from committing to a high bid and this *commitment effect* offsets the competition intensity effect.

2.2 Contests with private information

We now consider contests in which valuations are private information. For simultaneous contests, Malueg and Yates (2004) derive the unique symmetric pure strategy Bayesian equilibrium. Equilibrium bids are as follows, where x_{it}^* (with $i = 1, 2$; $t = L, H$) denotes the bid of type t of agent i.[16]

Proposition 3 (Malueg and Yates 2004) *In the simultaneous contest with private information a unique symmetric pure strategy Bayesian equilibrium*

[15]If $V_1 < 2V_2$, $\frac{\partial x_2^*}{\partial V_1} = \frac{1}{2}(1 - \frac{V_1}{V_2})$; thus $\frac{\partial x_2^*}{\partial V_1} > 0 \Leftrightarrow V_1 > V_2$. If $V_1 > 2V_2$, i.e., in particular $V_1 > V_2$, the second mover's bid does not change with the first mover's valuation as long as $V_1 \geq 2V_2$ still holds. If $V_1 = 2V_2$ and the first mover's valuation falls (increases), the second mover's bid increases (decreases) as then $V_1 < (>) 2V_2$ and, at the same time, valuations become more (less) similar.

[16]Malueg and Yates (2004) consider a variable range of the discriminatory power and derive a condition for equilibrium existence. For a discriminatory power of one – as we consider – this condition is satisfied.

exists. In this equilibrium for $i = 1, 2$

$$x^*_{iL} = \left(\frac{r}{4} + (1-r)\frac{V_H V_L}{(V_H + V_L)^2}\right) V_L \equiv \lambda V_L \tag{6}$$

$$x^*_{iH} = \left(\frac{r}{4} + (1-r)\frac{V_H V_L}{(V_H + V_L)^2}\right) V_H \equiv \lambda V_H. \tag{7}$$

If $r = 1$ or $r = 0$, types are quasi-public information as they are perfectly correlated. Similarly, if $V_L = V_H$, valuations are common knowledge. In these three cases, equilibrium bids coincide with the corresponding bids under public information.[17] There are again ability effects: the bid of a high (low) type increases in his respective valuation (as $\frac{\partial x^*_{it}}{\partial V_t} = \frac{r}{4} + (1-r)\frac{2V_L V_H}{(V_L+V_H)^3} V_t > 0$ with $t = L, H,\ i = 1, 2$), and the bid of the high type is larger than the bid of a low type. As in the simultaneous contest with public information, agents invest a fraction, $0 < \lambda < 1$, of their own valuation.

To analyze the effect of 'more similar' types, we consider how bids vary with the degree of correlation since contestants do not know the opponent's type as long as the correlation of types is imperfect, $\rho \in \,]-1, 1[$, and $V_L < V_H$. An increase in ρ can be interpreted as a more even contest since it corresponds to a greater probability of the opponent having the same valuation. The invested fraction λ (and thus bids conditional on an agent's valuation) becomes larger, the more positively the agents' types are correlated.[18] Hence, we again have a competition intensity effect that drives behavior.

We now turn to the novel setting of sequential contests with private information. Note that all subsequent Propositions (4–10) are new contributions as they refer to this setting.[19] We consider symmetric perfect Bayesian equilibria in pure strategies. In equilibrium, the bid of each type of the first mover, x^*_{1t} $(t = L, H)$, maximizes this type's expected payoff given his beliefs about the second mover's type and anticipating the best response of each type of

[17]The case $r = 0$ is equivalent to $\rho = -1$ and thus corresponds to an asymmetric public information contest. In this case $\omega = \lambda = \frac{V_H V_L}{(V_H+V_L)^2}$. In contrast, $r = 1$ (i.e., $\rho = 1$) and $V_H = V_L$ correspond to a symmetric public information contest. In these cases $\omega = \lambda = \frac{1}{4}$.

[18]Using $\rho = 2(r - \frac{1}{2})$, we have $\frac{\partial \lambda}{\partial \rho} = \frac{1}{8} - \frac{V_H V_L}{2(V_H+V_L)^2}$. It follows that λ increases with ρ if $(V_H - V_L)^2 \geq 0$ which always holds.

[19]Nevertheless, we can partly build on results by Morgan (2003) and Malueg and Yates (2004) when comparing the four institutional settings in Section 3.

the second mover. Each type of the second mover maximizes his expected payoff contingent on the observed action of the first mover. We denote the bid of type $k = L, H$ of the second mover by x_{2k}. For the sake of brevity, we restrict ourselves to the interior solution of the equilibrium in which both types of the second mover make strictly positive bids. For the boundary solution see the previous version of this paper (Ludwig 2007).

Proposition 4 *Consider the sequential contest with private information with belief $r \in [0,1]$ of type $t = L, H$ of the first mover that the second mover has type $k = t$ and belief $1 - r$ that the second mover has type $k \neq t$, where $k = L, H$, and arbitrary beliefs of the second mover. In the interior solution of the unique symmetric pure strategy perfect Bayesian equilibrium outcome*

$$\begin{aligned} x_{1t}^{*} &= \alpha_t^2 \\ x_{2k}^{t*} &= \alpha_t \left(\sqrt{V_k} - \alpha_t\right) \end{aligned}$$

where $\alpha_L = \frac{V_L}{2}\left(r(\frac{1}{\sqrt{V_L}} - \frac{1}{\sqrt{V_H}}) + \frac{1}{\sqrt{V_H}}\right)$, $\alpha_H = \frac{V_H}{2}\left(r(\frac{1}{\sqrt{V_H}} - \frac{1}{\sqrt{V_L}}) + \frac{1}{\sqrt{V_L}}\right)$. *We have an interior solution if either (i) $V_L = V_H$ or if (ii) $V_L < V_H$ and $r > \frac{2V_L - V_H}{\sqrt{V_H}\left(\sqrt{V_L} - \sqrt{V_H}\right)} \equiv \widetilde{r}$.*

We can rewrite the condition $r > \widetilde{r}$ for an interior equilibrium outcome (if $V_L < V_H$) in terms of the correlation coefficient ρ using the relation $\rho = 2r - 1$:

$$\rho > \frac{4V_L - V_H - \sqrt{V_L}\sqrt{V_H}}{\sqrt{V_H}(\sqrt{V_L} - \sqrt{V_H})} \equiv \widetilde{\rho}. \tag{8}$$

Hence, for an interior solution valuations must have a sufficiently strong tendency to positive correlation – but $\widetilde{\rho}$ does not necessarily have to be positive. In other words, given an agent's valuation, the expected valuation of the opponent must be sufficiently similar. Compared to the condition for an interior solution in the case of public information, i.e., $V_H \leq 2V_L$, the condition under private information is less strict: It is satisfied if $V_H \leq 2V_L$ (as then $\widetilde{r} < 0$ and $\widetilde{\rho} < -1$, resp.) but $V_H \leq 2V_L$ need not hold.
Similar to the results with public information, sequential moves imply that contestants do not bid the same fraction of their valuations. Moreover, ability effects are again a driving force behind bidding behavior:

Proposition 5 *Let $V_L < V_H$ and $r \in [0, 1[$. In the sequential contest with private information*

(i) a contestant bids strictly more when he has the high valuation than when he has the low valuation and

(ii) the first mover bids strictly more (less) than the second mover when he is a high (low) type.

In contrast to public information, under private information bids of identical types differ as soon as valuations are not perfectly correlated or $V_L \neq V_H$ (cf. Proposition 5.ii). The reason is that the first mover is no longer sure which type the second mover has and adjusts his bid according to his belief about the second mover's type. In the limit, when information is quasi complete as there is perfect positive ($r = 1$) or negative correlation ($r = 0$) or the high and low valuation coincide, equilibrium bids equal the corresponding bids of the sequential contest with public information. The relation between bids and the correlation of valuations can be summarized as follows.

Proposition 6 *In the sequential contest with private information*

(i) the expected bid of both types of the second mover increases in ρ and

(ii) the bid of the low (high) first mover's type increases (decreases) in ρ.

A competition intensity effect explains the behavior of the second mover (Proposition 6.i). A stronger positive correlation between the types (i.e., an increase in ρ) implies that the contest evens out: given the own type, the probability that the opponent has the same type increases. The closer contest leads to more aggressive bidding. In particular, the second mover bids more if the first mover has the same type than if he has a different type. For the first mover (Proposition 6.ii), however, only the behavior of the low type can be explained by a competition intensity effect. The high type of the first mover, in contrast, bids less when the contest evens out. As in the sequential contest with public information, this is due to a commitment effect that offsets the competition intensity effect.

3 Comparison

We now compare the four institutional settings from an ex ante perspective using the equilibrium bids given in Propositions 1, 2, 3, and 4.

3.1 Sequential versus simultaneous contests

We now compare sequential with simultaneous contests given public information. For this case, we can build on the results by Morgan (2003) who considers the same framework apart from the distribution of types. He assumes ex ante symmetric but independently distributed types. However, the distribution does not influence equilibrium bids. Thus, bids are identical to the ones in Propositions 1 and 2.

Since efforts in the simultaneous and sequential contest with public information are identical when contestants have the same type (cf. Section 2.1), only the efforts when types are heterogeneous determine for which setting the ex ante expected effort sum is greater. Morgan shows that the ex ante expected effort sum is greater in the sequential contest under his distributional assumption. The result is driven by the first mover with a high valuation who commits to a greater bid in an asymmetric sequential contest than a high type in the corresponding simultaneous contest. The result immediately extends to the distribution that we assume as only the ex ante symmetry of agents matters for the result, which also holds true for our distribution. It follows that in our setting, a designer who wants to maximize the expected effort sum prefers a sequential contest under public information. Analogously, Morgan's result that the ex ante expected payoffs of the contestants are larger in the sequential setting extends to the setting considered here. Thus, we can conclude that the contestants also prefer the sequential contest (which hence Pareto dominates the simultaneous contest). This may be surprising at first glance since agents spend more effort from an ex ante perspective. The intuition behind the result is an *efficiency gain effect*: the agent with the greater valuation wins "more often" in the sequential setting since the gap between the bids of heterogeneous types is larger (and the high type bids more; cf. Section 2.1). This efficiency gain effect increases the surplus of both agents and outweighs the effect of greater expected efforts.

Comparing the ex ante expected effort sum in simultaneous and sequential contests given private information, we also find that it is greater in the sequential setting. Similar to the case of public information, the result is driven by the high type of the first mover who bids more in the sequential than in the simultaneous contest.[20]

Proposition 7 *Given private information, the risk neutral designer of the contest weakly prefers a sequential setting when he aims at maximizing the ex ante expected effort sum (strictly so if $\rho < 1$ and $V_L < V_H$).*

The result that sequential contests Pareto dominate simultaneous ones given public information can, however, only partly be extended to the case of private information. While the first mover still prefers the sequential contest, the second mover prefers the sequential contest only if the correlation of valuations is sufficiently "negative", in the sense that there is a sufficiently strong tendency to negative correlation (the correlation may be positive, though). The advantage of sequential contests is driven by an efficiency gain effect: The bid of the high type of the first mover increases, the smaller ("the more negative") ρ due to the commitment effect that offsets the competition intensity effect. In contrast, the expected bid of both types of the second mover decreases the smaller ρ (competition intensity effect). Hence, the high type of the first mover is more likely to win the smaller ρ. Moreover, the smaller ρ, the more important is the efficiency gain effect since it becomes more likely that valuations differ. If valuations differ, it matters which contestant wins in terms of allocative efficiency. When ρ is sufficiently "negative", and thus the efficiency gain effect sufficiently strong, both agents benefit. The intuition for the result that the first mover always benefits is a first mover advantage in the sequential contest with private information.

The overall ex ante expected payoff of the contestants is another criterion for the institutional choice, at least from the perspective of the contestants. Ex ante agents may not know whether they move first or second.[21] The overall

[20]The proof that the high type of the first mover exerts a greater effort in the sequential contest is not included but available from the author upon request.

[21]The designer fixes who moves first before types are known. Since agents are ex ante symmetric, we assume that each agent moves first with probability one-half.

expected payoff is larger in the sequential than in the simultaneous contest if ρ is sufficiently "negative". The resulting critical value for ρ is larger than the critical value for ρ up to which the second mover prefers the seqential setting since the first mover does always better in the sequential contest. The driving force behind the result is again the aforementioned efficiency gain effect. These observations are summarized in the following proposition.

Proposition 8 *In the sequential contest with private information*

(i) the first mover always receives a greater ex ante expected payoff than in the simultaneous contest;

(ii) the second mover receives the same ex ante expected payoff than in the simultaneous contest if $V_H = V_L$ or $\rho = 1$; when $V_L < V_H$ and $\rho < 1$, he receives a greater payoff than in the simultaneous contest if

$$\rho \leq \rho^s := \frac{\left(V_L^{\frac{3}{2}} - V_H^{\frac{3}{2}}\right)^2 - V_L V_H (V_H + V_L) - 10 V_H^{\frac{3}{2}} V_L^{\frac{3}{2}}}{(V_L + V_H)\left(V_H^2 + 4 V_H^{\frac{1}{2}} V_L^{\frac{1}{2}} (V_H + V_L) + V_L^2\right)} \text{ where } \rho^s \in \,]-1, 1[\quad ;$$

(iii) the overall expected payoff of the contestants is the same than in simultaneous contests if $V_H = V_L$ or $\rho = 1$; when $V_L < V_H$ and $\rho < 1$ the overall expected payoff is greater than in simultaneous contests if

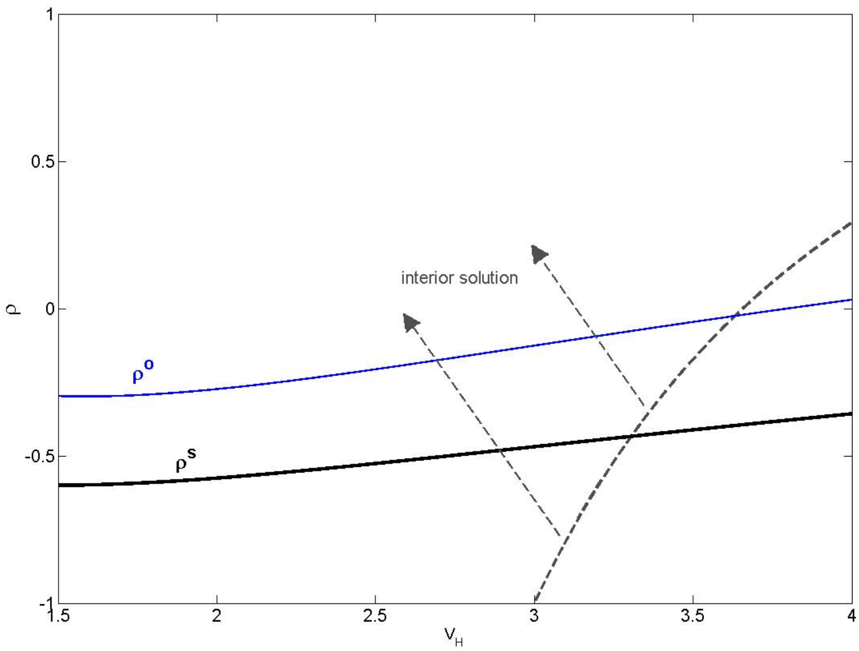
$$\rho \leq \rho^o := \frac{V_L^3 + V_H^3 + 2 V_H^{\frac{1}{2}} V_L^{\frac{1}{2}} (V_H - V_L)^2 - V_L V_H (V_L + V_H) - 4 V_L^{\frac{3}{2}} V_H^{\frac{3}{2}}}{V_L^3 + V_H^3 + V_H^{\frac{1}{2}} V_L^{\frac{1}{2}} (V_L + V_H)\left(2(V_H + V_L) + V_H^{\frac{1}{2}} V_L^{\frac{1}{2}}\right)} \text{ where } \rho^o \in \,]-1, 1[.$$

Results (ii) and (iii) are illustrated in Figure 2 for the case $V_L < V_H$ where additionally $V_L = 1.5$. The solid lines describe the critical values ρ^s and ρ^o. Below (above) these lines the sequential (simultaneous) contest leads to greater expected payoffs for the second mover and overall. The dashed line marks the threshold for the interior solution (i.e., $\rho > \widetilde{\rho}$).

3.2 Private versus public information

We now compare the effects of public and private information for a given order of moves. Consider first simultaneous contests. We have already seen that bids are driven by a competition intensity effect, i.e., agents bid more

Figure 2: Ex ante expected payoff comparison of sequential and simultaneous contests with private information if $V_L = 1.5 < V_H$



aggressively, the more similar are their types. When information is private, competition intensity is stronger relative to a public information contest with asymmetric types and is weaker relative to one with symmetric types as agents are uncertain about their opponent's type (as long as $\rho \in]-1, 1[$ and $V_L < V_H$). Therefore, in the case of private information, bids (conditional on an agent's type) lie in between the bids of symmetric and asymmetric public information contests and coincide for quasi-public information, i.e., $|\rho| = 1$ or $V_L = V_H$ (cf. Malueg and Yates 2004). Yet, the information condition does not really distort the bids: The expected bid of a high (low) type in the case of public information (taking the expectation over the opponent's type) equals the bid of a high (low) type in the case of private information as shown by Malueg and Yates (2004). It follows that for simultaneous contests the ex ante expected effort sum of contestants and their conditional probability of winning is the same under both information conditions. Thus, also the

contestants' ex ante expected payoffs under public and private information are identical. Hence, the designer who wants to maximize the expected effort sum as well as the contestants are indifferent between both information conditions.
In contrast to these results for simultaneous contests, we find that the information condition matters for a sequential setting.

Proposition 9 *Given a sequential contest, the risk neutral designer, who maximizes the ex ante expected effort sum, weakly prefers private to public information (strictly so if $\rho \in \,]-1, 1[$ and $V_L < V_H$).*

The intuition for the result is that with private information the commitment effect (that offsets the competition intensity effect) for the high type of the first mover is less strong as he does not know for sure if the second mover is a low type. This implies that the high type of the first mover makes a smaller bid than when he knows that the second mover is a low type. The low type of the second mover who observes a smaller bid of the high type of the first mover reacts with a larger bid than under public information, which can intuitively be explained by a stronger competition intensity. This increase in the second mover's effort more than compensates the decrease in the effort of the high type of the first mover.
Regarding the contestants' overall expected payoffs, it immediately follows that these are smaller in the case of private information as the expected effort costs are greater. Yet, also individually expected payoffs are smaller. While the first mover saves effort costs in the case of private information, his expected probability of winning decreases disproportionately as the second mover's expected effort increases to a larger extent. Similarly, while the second mover's winning probability increases in the case of private information, the increase does not compensate him for the increased effort costs.

Proposition 10 *Given a sequential contest, the contestants weakly prefer public information to private information (strictly so if $\rho \in \,]-1, 1[$ and $V_L < V_H$).*

Overall, we can conclude from the preceding analysis – taking into account that the sequential contest Pareto dominates the simultaneous contest under public information and the players' indifference between the information structures for simultaneous contests – that the designer who wants to maximize the ex ante expected effort sum and who can choose the timing and information structure favors a sequential contest with private information, while the contestants prefer a sequential contest with public information.

3.3 Minimizing the prize money for a fixed effort level

In many contests the prize for the winner can be thought of as money the designer spends, for instance in job-promotion tournaments. In this case the designer may aim at reaching a specific expected effort level at the lowest possible cost rather than simply maximizing the efforts. From the preceding analysis, we already know which contest structure leads to the highest overall expected effort for given prizes. We now ask with which contest structure the designer achieves the fixed expected effort level at the lowest cost. To analyze this issue, we consider that a reduction in the prize implies that both an agent's high and low valuation decrease. For simplicity we assume that the high and the low valuation decrease by the same percentage.

Proposition 11 *The same ex ante expected effort sum can be reached*

(i) at a lower prize money in sequential than in simultaneous contests and

(ii) at a weakly lower prize money under private than under public information (strictly lower given sequential contests).

Thus, the sequential contest with private information is the "cheapest" setting. Note that this result implies that the designer may prefer a sequential contest even if it were more expensive than a simultaneous contest because it takes longer to carry out a sequential contest.[22]

4 Conclusion

We extend the literature on imperfectly discriminating contests by considering a model of a sequential contest in which the players' types are private

[22]We abstract from explicitly considering such time costs by, e.g., a discount factor as we can think of simultaneous contests as being sequential contest where the second mover cannot observe the first mover's effort.

information and the second mover can observe the first mover's action. We then compare four institutional settings, in which the agents either move sequentially or simultaneously and their types are either private or public information. In particular, we show that for the case of private information the sequential contest is preferred to the simultaneous contest by a risk neutral, effort-maximizing designer and also by the first mover. If the correlation between the contestants' types is sufficiently negative, the second mover also prefers the sequential contest due to an efficiency gain effect. Hence, the sequential contest may Pareto dominate the simultaneous one given private information. Given a sequential structure, we find that the effort-maximizing designer prefers private information. Contestants, however, prefer public information. Overall, considering all four institutional settings, we find that an effort-maximizing designer of the contest favors a sequential contest with private information.

In reality, in particular the information structure might not be a real choice variable of the designer. Nevertheless, our results suggests that the designer may try to conceal the agents' types. For instance, in tendering procedures he may try to conceal the identities of the competitors. Similarly, if agents compete for a job or a promotion, he may prevent them from working closely together – physically as well as with regard to the assigned tasks. Even if concealing the types is costly for the designer, he may try to do so: we find that a private information contest is cheaper for the designer in the sense that he can reduce the prize money and still achieve the same expected effort.

Moreover, in reality, a sequential structure may prolong a contest and thus cause extra costs for the designer (e.g., a sequential marathon). However, a sequential structure can possibly outweigh the additional costs. We show that the sequential contest is cheaper in the aforementioned sense than the simultaneous contest.

Our comparison between sequential and simultaneous contests is equivalent to a comparison between sequential contests in which either the first mover's action is observable or it is not observable. Thus, our results also suggest that it can be profitable for the designer to provide the second mover with information about the opponent's action – even if the information provision is costly for the designer.

We use a simple and tractable framework to illustrate the advantage (from the designer's perspective) of sequential contests with private information. It might be interesting to generalize our analysis to other specifications of the contest success function, e.g., to non-constant returns to scale from effort, and to more general, e.g., asymmetric, distributions of types. The effects that we identify to drive effort choices and our results, however, will be present to some extent in other specifications of imperfectly discriminating contests.

References

Aoyagi, M. (2010), "Information Feedback in a Dynamic Tournament," *Games and Economic Behavior*, 70, 242–260.

Baik, K. H. and J. F. Shogren (1992), "Strategic Behavior in Contests: Comment," *American Economic Review* 82(1), 359-362.

Clark, D. and C. Riis (1998), "Contest Success Functions: An Extension," *Economic Theory* 11, 201-204.

Dixit, A. (1987), "Strategic Behavior in Contests," *American Economic Review* 77(5), 891-898.

Gradstein, M. and K. Konrad (1999), "Orchestrating Rent Seeking Contests," *Economic Journal* 109, 536-545.

Harris C. and J. Vickers (1985), "Perfect Equilibrium in a Model of a Race," *The Review of Economic Studies* 52 (2), 193-209.

Harris C. and J. Vickers (1987), "Racing with Uncertainty," *The Review of Economic Studies* 54 (1), 1-21.

Hurely, T. and J. Shogren (1998a), "Effort Levels in a Cournot Nash Contest with Asymmetric Information," *Journal of Public Economics* 69, 195-210.

Hurely, T. and J. Shogren (1998b), "Asymmetric Information Contests," *European Journal of Political Economy* 14, 645–665.

Jackson, M.O., Simon, L.K. Swinkels, J.M. and W.R. Zame (2002), "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica* 70 (5), 1711–1740.

Jost, P.-J. and M. Kräkel (2006), "Simultaneous- and Sequential-Move Tournaments with Heterogeneous Agents," *Schmalenbach Business Review*, 58, 306-331.

Konrad, K. A. (2007), "Strategy in Contests - An Introduction," WZB-Markets and Politics Working Paper No. SP II 2007-01.

Leininger, W. (1993), "More Efficient Rent-Seeking – A Münchhausen Solution," *Public Choice* 75, 43-64.

Linster, B. (1993), "Stackelberg Rent-Seeking," *Public Choice* 77 (2), 307-321.

Ludwig, S. (2007), "Essays on the Effects of Information on Incentives and on People's Awareness and Assessment of Biases," Ph.D. thesis, University of Bonn.

Malueg, D. and A. Yates (2004), "Rent Seeking with Private Values," *Public Choice* 119, 161-178.

Moldovanu, B. and A. Sela (2006), "Contest Architecture," *Journal of Economic Theory* 126 (1), 70-97.

Morgan, J. (2003), "Sequential Contests," *Public Choice* 116, 1-18.

Münster, J. (2009), "Repeated Contests with Asymmetric Information," *Journal of Public Economic Theory* 11 (1), 89-118.

Nitzan, S. (1994), "Modeling Rent Seeking Contests," *European Journal of Political Economy* **10**, 41-60.

Pal, D. (1991), "Cournot Duopoly with Two Production Periods and Cost Differentials," *Journal of Economic Theory* 55, 441-448.

Romano, R. and H. Yildirim (2005), "On the Endogeneity of Cournot-Nash and Stackelberg Equilibria: Games of Accumulation," *Journal of Economic Theory* 102, 73-107.

Rosen, S. (1986), "Prizes and Incentives in Elimination Tournaments," *American Economic Review* 76 (4), 701–715.

Saloner, G. (1987), "Cournot Duopoly with Two Production Periods," *Journal of Economic Theory* 42, 183-187.

Simon, L.K. and W.R. Zame (1990), "Discontinuous Games and Endogenous Sharing Rules," *Econometrica* 58 (4), 861–872.

Skaperdas, S. (1996), "Contest Success Functions," *Economic Theory* 7, 283-290.

Tullock, G. (1980), "Efficient Rent-Seeking," in J. Buchanan et al. (Eds.), Toward a theory of the rent-seeking society, 97-112. College Station: Texas A&M University Press.

Wärneryd, K. (2003), "Information in Conflicts," *Journal of Economic Theory* 110, 121136.

Yildirim, H. (2005), "Contests with Multiple Rounds," *Games and Economic Behavior* 51, 213-227.

Appendix

Proof of Proposition 4.

In equilibrium, the bid of each type of the first mover maximizes this type's expected payoff given his beliefs about the second mover's type and anticipating the best response of each type of the second mover. Conditional on the own type, the first mover believes that the second mover has the same type with probability r and the other type with probability $1 - r$. Each type of the second mover maximizes his expected payoff contingent on the observed action of the first mover. Note that the beliefs of the second mover about the first mover's type do not influence equilibrium outcomes: Irrespective of the first mover's type, the second mover's payoff and thus his best response depend only on the first mover's perfectly observable action (which can depend on the type). Thus, the second mover's beliefs can be arbitrary. Moreover, as the follower's action is independent of his beliefs, we can solve the game backward.

Suppose the bid of type $t = H, L$ of the first mover is positive, i.e., $x_{1t} > 0$. Then, type $k = H, L$ of the second mover maximizes

$$\max_{x_{2k} \geq 0} \frac{x_{2k}}{x_{2k} + x_{1t}} V_k - x_{2k}.$$

Since the maximization problem is concave in the second mover's bid, his best response is

$$x_{2k}(x_{1t}) = \max\{\sqrt{x_{1t}}\sqrt{V_k} - x_{1t}, 0\} \quad \text{for } x_{1t} > 0. \tag{9}$$

Hence, only if $x_{1t} < V_L$ do both types of the second mover make positive bids. If the first mover bids weakly more than the low valuation, at least one type of the second mover bids zero.

Turning to the first mover, suppose first that both types of the second mover make a positive bid irrespective of the first mover's type. It follows from (9) that the second mover's bids are positive if $x_{1t} < V_L$ for $t = L, H$. Then, the second mover's best response is $x_{2k}(x_{1t}) = \sqrt{x_{1t}}\sqrt{V_k} - x_{1t}$ and the maximization problem of type t of the first mover who believes that the second mover also has type t with probability r and with $1 - r$ the other type is

$$\max_{V_L > x_{1t} > 0} \left(\frac{\sqrt{x_{1t}}}{\sqrt{V_t}} r + \frac{\sqrt{x_{1t}}}{\sqrt{V_n}} (1 - r) \right) V_t - x_{1t} \quad \text{with } n = L, H \quad \text{and } t \neq n, \tag{10}$$

which is concave in x_{1t}. Thus, the first-order condition

$$x_{1t} = \frac{V_t^2}{4} \left(r \left(\frac{1}{\sqrt{V_t}} - \frac{1}{\sqrt{V_n}} \right) + \frac{1}{\sqrt{V_n}} \right)^2 \equiv \alpha_t^2 \quad \text{with } t \neq n \tag{11}$$

is sufficient as long as the constraints on x_{1t} are fulfilled. Obviously, α_t^2 is strictly positive but it remains to check whether $\alpha_t^2 < V_L$. The low type of the first mover never bids more than V_L as this yields a negative payoff and, for example, investing zero is better. He bids less since $\alpha_L^2 < V_L \Leftrightarrow (1 - r)\sqrt{V_L} < (2 - r)\sqrt{V_H}$ always holds as $r \in [0, 1]$ and $V_L \leq V_H$. Thus, $x_{1L}^* = \alpha_L^2$ and both types of the second mover make positive bids when the first mover is a low type. The corresponding equilibrium bids of type $k = L, H$ of the second mover are $x_{2k}^{L*} = \alpha_L(\sqrt{V_k} - \alpha_L)$, where the superscript L indicates the first mover's type.[23] The high type of the

[23] Although the second mover does not observe the leader's type, he can infer it from the bid as the first mover's types 'separate'. For $V_L < V_H$, $\alpha_H = \alpha_L$ cannot hold as

first mover may want to bid more than V_L, though. He bids less if

$$\alpha_H^2 < V_L \Leftrightarrow V_H \left(r \left(\frac{1}{\sqrt{V_H}} - \frac{1}{\sqrt{V_L}} \right) + \frac{1}{\sqrt{V_L}} \right) < 2\sqrt{V_L}. \tag{12}$$

If $V_L = V_H$, (12) is satisfied; the 'high' type does not want to bid more than his valuation V_L. For $V_H > V_L$ we can rewrite (12) as

$$r > \frac{2V_L - V_H}{\sqrt{V_H}\left(\sqrt{V_L} - \sqrt{V_H}\right)} \equiv \widetilde{r}. \tag{13}$$

Hence, provided that the first mover makes a strictly positive bid, both types of the second mover make a strictly positive bid if either $V_L = V_H$ or if $V_L < V_H$ and $r > \widetilde{r}$. More precisely, type $k = L, H$ of the follower bids $x_{2k}^{H*} = \alpha_H(\sqrt{V_k} - \alpha_H)$. It remains to show that the first mover wants to make a strictly positive bid. Suppose type $t = H, L$ of the first mover bids zero, i.e., $x_{1t} = 0$. Then the second mover could ensure winning by bidding $\epsilon > 0$ (arbitrarily small),[24] which is better than bidding zero and winning with probability 0.5. By bidding zero, the second mover's expected payoff is $\frac{1}{2}V_2$ which is smaller than his payoff when bidding ϵ and winning for sure, $V_2 - \epsilon$, if ϵ is sufficiently small. Thus, by bidding zero, the payoff of type t of the first mover is zero. If type t of the first mover, however, bids $\alpha_t^2 > 0$, his expected payoff given belief r that the second mover has the same type, $k = t$, and belief $1 - r$ that $k \neq t$ is

$$\Psi_t^{1st} \quad = \quad V_t \left(r \frac{\alpha_t}{\sqrt{V_t}} + (1 - r) \frac{\alpha_k}{V_k} \right) - \alpha_t^2 = \alpha_t^2 \;\text{ with } t \neq k, \tag{14}$$

which is strictly larger than zero. Hence, the first mover does not want to bid zero. Thus, we have an interior solution, i.e., both types of the second mover make positive bids, if either $V_L = V_H$ or if $V_L < V_H$ and $r > \widetilde{r}$. Otherwise, if $V_L < V_H$ and $r \leq \widetilde{r}$, we have a boundary solution in which at least the low type of the second mover bids zero. ■

Proof of Proposition 5.

Part (i) For the second mover we have to verify $\alpha_t(\sqrt{V_L} - \alpha_t) \leq \alpha_t(\sqrt{V_H} - \alpha_t)$ (where $t = L, H$). This holds as $V_H \geq V_L$ (with equality for $V_H = V_L$). For

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$\alpha_H = \alpha_L \Leftrightarrow V_L^{-\frac{1}{2}} V_H^{-\frac{1}{2}} \left(V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}} \right) + r \left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}} \right) = 0$, where the left hand side is larger than zero for $V_L < V_H$. Thus, $\alpha_H \neq \alpha_L$. For $V_L = V_H$ there are no different types and the first mover's bids coincide for 'both' types.

[24] Recall that if an agent wants to make a positive bid, he has to bid at least ϵ.

the first mover, we have to verify $\alpha_L \leq \alpha_H$. Plugging in α_L and α_H as given in Proposition 4 yields

$$\begin{aligned}\alpha_L \leq \alpha_H \quad &\Leftrightarrow \quad r(\frac{1}{\sqrt{V_L}} - \frac{1}{\sqrt{V_H}})\,(V_L + V_H) \leq \frac{V_H}{\sqrt{V_L}} - \frac{V_L}{\sqrt{V_H}} \\ &\Leftrightarrow \quad r(\sqrt{V_H} - \sqrt{V_L})\,(V_L + V_H) \leq (V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}}).\end{aligned}$$

This holds with equality if $V_L = V_H$. For $V_L < V_H$, we have

$$\alpha_L \leq \alpha_H \Leftrightarrow r\,(V_L + V_H) \leq (V_H + \sqrt{V_H}\sqrt{V_L} + V_L),$$

which always holds.

Part (ii) To show that the first mover exerts more effort than the second mover when he is a high type and less when he is a low type it suffices to check that he makes more (less) effort than the high (low) type of the second mover (as we know from part (i) that the high type makes a higher effort than the low type): $x^*_{1H} \geq x^{*H}_{2H}$ is equivalent to $2\alpha_H \geq \sqrt{V_H}$. This is equivalent to $(1-r)(\frac{\sqrt{V_H}}{\sqrt{V_L}} - 1) \geq 0$ which is fulfilled (for $V_L = V_H$ and $r = 1$ with equality). As $x^{*H}_{2H} \geq x^{*H}_{2L}$, we also have $x^*_{1H} \geq x^{*H}_{2L}$. $x^*_{1L} \leq x^{*L}_{2L}$ is equivalent to $2\alpha_L \leq \sqrt{V_L}$. This is equivalent to $(1-r)V_L^{\frac{1}{2}} \leq (1-r)V_H^{\frac{1}{2}}$, which always holds (with equality for $V_L = V_H$ and for $r = 1$). As $x^{*L}_{2H} \geq x^{*L}_{2L}$, we also have $x^*_{1L} \leq x^{*L}_{2H}$. ■

Proof of Proposition 6.

In order to show how bids (as given in Proposition 4) vary with the correlation coefficient ρ, we show how they vary with r. Since $r = \frac{1}{2}(\rho + 1)$, the signs of the derivatives of the bids with respect to r and to ρ are identical. Using $V_H \geq V_L > 0$ and $r \in [0, 1]$, we have the following results that establish the proposition.

For the low type of the first mover, $\frac{\partial x^*_{1L}}{\partial r} = \alpha_L V_L \left(\frac{1}{\sqrt{V_L}} - \frac{1}{\sqrt{V_H}}\right) \geq 0$ as $V_L \leq V_H$ and $\alpha_L \geq 0$ (which holds true as $V_L \leq V_H$). For the high type of the first mover, $\frac{\partial x^*_{1H}}{\partial r} = \alpha_H V_H \left(\frac{1}{\sqrt{V_H}} - \frac{1}{\sqrt{V_L}}\right) \leq 0$ as $V_L \leq V_H$ and $\alpha_H \geq 0$ (which holds true as $V_L \leq V_H$).

The expected bid of the low type of the second mover is $x^*_{2L} = r x^{*L}_{2L} + (1-r)x^{*H}_{2L}$. Hence, $\frac{\partial x^*_{2L}}{\partial r} = x^{*L}_{2L} - x^{*H}_{2L} + r\frac{\partial x^{*L}_{2L}}{\partial r} + (1-r)\frac{\partial x^{*H}_{2L}}{\partial r}$, where

$\frac{\partial x^{*L}_{2L}}{\partial r} = \frac{\partial \alpha_L}{\partial r}(\sqrt{V}_L - 2\alpha_L) = \frac{1}{2}(1-r)V_L V_H^{-1}\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^2 \geq 0,$

$\frac{\partial x^{*H}_{2L}}{\partial r} = \frac{\partial \alpha_H}{\partial r}(\sqrt{V}_L - 2\alpha_H) = \frac{1}{2}V_H^{\frac{1}{2}}V_L^{-1}\left(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}}\right)^2\left(V_L^{\frac{1}{2}} + (1-r)V_H^{\frac{1}{2}}\right) \geq 0,$ and

$x_{2L}^{*L} - x_{2L}^{*H} = (\alpha_H - \alpha_L)(\alpha_H + \alpha_L - V_L^{\frac{1}{2}})$. Plugging in α_H and α_L in the last bracket and rearranging yields
$x_{2L}^{*L} - x_{2L}^{*H} = \frac{1}{2}(\alpha_H - \alpha_L)V_H^{-\frac{1}{2}}V_L^{-\frac{1}{2}}\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)((1-r)(V_H - V_L) + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}) \geq 0$ as $\alpha_H \geq \alpha_L$ (cf. Proposition 5). Hence, $\frac{\partial x_{2L}^*}{\partial r} \geq 0$.
The expected bid of the high type of the second mover is $x_{2H}^* = rx_{2H}^{*H} + (1-r)x_{2H}^{*L}$. Hence, $\frac{\partial x_{2H}^*}{\partial r} = x_{2H}^{*H} - x_{2H}^{*L} + r\frac{\partial x_{2H}^{*H}}{\partial r} + (1-r)\frac{\partial x_{2H}^{*L}}{\partial r}$, where
$\frac{\partial x_{2H}^{*L}}{\partial r} = \frac{\partial \alpha_L}{\partial r}(\sqrt{V}_H - 2\alpha_L) = \frac{1}{2}V_L^{\frac{1}{2}}V_H^{-1}\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^2\left(V_H^{\frac{1}{2}} + (1-r)V_L^{\frac{1}{2}}\right) \geq 0$,
$\frac{\partial x_{2H}^{*H}}{\partial r} = \frac{\partial \alpha_H}{\partial r}(\sqrt{V}_H - 2\alpha_H) = \frac{1}{2}(1-r)V_H V_L^{-1}\left(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}}\right)^2 \geq 0$, and
$x_{2H}^{*H} - x_{2H}^{*L} = (\alpha_H - \alpha_L)(V_H^{\frac{1}{2}} - (\alpha_H + \alpha_L))$. Plugging in α_H and α_L in the last bracket and rearranging yields
$x_{2H}^{*H} - x_{2H}^{*L} = \frac{1}{2}(\alpha_H - \alpha_L)V_H^{-\frac{1}{2}}V_L^{-\frac{1}{2}}\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)\left(-(1-r)(V_H - V_L) + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}\right)$. If $V_H = V_L$, $x_{2H}^{*H} - x_{2H}^{*L} = 0$ and thus $\frac{\partial x_{2H}^*}{\partial r} = 0$. For $V_H > V_L$, $\alpha_H > \alpha_L$ (cf. Proof of Proposition 5), and hence $x_{2H}^{*H} - x_{2H}^{*L} \leq 0$ if $r \leq \frac{V_H - V_L - \sqrt{V}_H\sqrt{V}_L}{V_H - V_L} = r^*$. Can it be that $r \leq r^*$ but also $r > \widetilde{r} = \frac{2V_L - V_H}{\sqrt{V}_H(\sqrt{V}_L - \sqrt{V}_H)}$ (condition for the interior solution) and $r \in [0,1]$? $r^* > \widetilde{r} \Leftrightarrow 0 < V_L^{\frac{1}{2}}\left(-(V_H - V_L - \sqrt{V}_H\sqrt{V}_L) - (V_H - V_L)\right)$, which can be satisfied only if $V_H - V_L - \sqrt{V}_H\sqrt{V}_L < 0$. Yet, the latter implies $r^* < 0$ and thus $r \leq r^*$ is impossible for $r \in [0,1]$. Hence, $x_{2H}^{*H} - x_{2H}^{*L} \geq 0$ and thus $\frac{\partial x_{2H}^*}{\partial r} \geq 0$. ■

Proof of Proposition 7.

Consider the ex ante expected effort sum in the sequential contest with private information:

$$\begin{aligned}
\xi_{seq}^{priv} : &= \frac{1}{2}\Big[\alpha_L^2 + \alpha_H^2 + r\alpha_L\left(\sqrt{V_L} - \alpha_L\right) + (1-r)\alpha_H\left(\sqrt{V_L} - \alpha_H\right) \\
&\quad + r\alpha_H\left(\sqrt{V_H} - \alpha_H\right) + (1-r)\alpha_L\left(\sqrt{V_H} - \alpha_L\right)\Big] \\
&= \frac{1}{2}\left[r(\alpha_H - \alpha_L)\left(\sqrt{V_H} - \sqrt{V_H}\right) + \alpha_L\sqrt{V_H} + \alpha_H\sqrt{V_L}\right].
\end{aligned}$$

Substituting α_L and α_H (as given in Proposition 4) and rearranging yields

$$\xi_{seq}^{priv} = \frac{1}{4}\left[V_L + V_H + r(1-r)\left(\frac{1}{\sqrt{V_L V_H}}(V_L + V_H)\left(\sqrt{V_H} - \sqrt{V_L}\right)^2\right)\right]. \quad (15)$$

The ex ante expected effort sum in the simultaneous contest with private information is

$$\xi_{sim}^{priv} := \lambda(V_L + V_H) = \frac{1}{4(V_L + V_H)}\left[r(V_L - V_H)^2 + 4V_H V_L\right]. \quad (16)$$

Routine transformations yield that

$$\xi_{seq}^{priv} \geq \xi_{sim}^{priv} \Leftrightarrow (1-r)\left[(V_L - V_H)^2 + \frac{r}{\sqrt{V_L V_H}}(V_L + V_H)^2 \left(\sqrt{V_H} - \sqrt{V_L}\right)^2\right] \geq 0,$$

which is always satisfied (with equality for $r = 1$). Thus, we have for $r < 1$ (or equivalently $\rho < 1$) that $\xi_{seq}^{priv} \geq \xi_{sim}^{priv}$ (with equality for $V_L = V_H$). ■

Proof of Proposition 8.

The ex ante expected payoff, Ψ, of a contestant in the simultaneous contest with private information is the sum of his expected payoff when he is a low and a high type respectively, each with probability one half:

$$\begin{aligned} \Psi &= \frac{1}{2}\left(V_L\left(\frac{x_{iL}}{x_{iL}+x_{jL}}r + \frac{x_{iL}}{x_{iL}+x_{jH}}(1-r)\right) - x_{iL}\right) \\ &+ \frac{1}{2}\left(V_H\left(\frac{x_{iH}}{x_{iH}+x_{jH}}r + \frac{x_{iH}}{x_{iH}+x_{jL}}(1-r)\right) - x_{iH}\right). \end{aligned}$$

Plugging in equilibrium bids and simplifying yields

$$\Psi = \frac{1}{2}\left[\frac{V_L^2 - V_H V_L + V_H^2}{V_L + V_H} - \frac{3}{4}r\frac{(V_L - V_H)^2}{V_L + V_H}\right]. \tag{17}$$

The ex ante expected payoff of the first mover in the sequential contest with private information is $\Psi^{1st} := \frac{1}{2}\left(\Psi_H^{1st} + \Psi_L^{1st}\right)$, where $\Psi_H^{1st} = \alpha_H^2$ and $\Psi_L^{1st} = \alpha_L^2$ as given in (14). Hence,

$$\Psi^{1st} = \frac{1}{2}(\alpha_L^2 + \alpha_H^2). \tag{18}$$

The expected payoff, Ψ_k^{2nd}, of type $k = H, L$ of the second mover when the first mover has with probability r the same type, $t = k$, and with probability $1 - r$ a different type, $t \neq k$ is

$$\begin{aligned} \Psi_k^{2nd} &= V_k\left(r\left(\frac{\sqrt{V_k} - \alpha_k}{\sqrt{V_k}}\right) + (1-r)\left(\frac{\sqrt{V_k} - \alpha_t}{\sqrt{V_k}}\right)\right) \\ &\quad -r\left(\sqrt{V_k}\alpha_k - \alpha_k^2\right) - (1-r)\left(\sqrt{V_k}\alpha_t - \alpha_t^2\right) \\ &= r\left(\sqrt{V_k} - \alpha_k\right)^2 + (1-r)\left(\sqrt{V_k} - \alpha_t\right)^2, \quad \text{where } t \neq k. \end{aligned} \tag{19}$$

The ex ante expected payoff of the second mover, Ψ^{2nd}, then equals $\frac{1}{2}(\Psi_H^{2nd} + \Psi_L^{2nd})$:

$$\begin{aligned} \Psi^{2nd} &= \frac{1}{2}\Big[\left(\sqrt{V_L} - \alpha_H\right)^2 + \left(\sqrt{V_H} - \alpha_L\right)^2 + \\ &\quad r\left(\left(\sqrt{V_L} - \alpha_L\right)^2 - \left(\sqrt{V_L} - \alpha_H\right)^2 + \left(\sqrt{V_H} - \alpha_H\right)^2 - \left(\sqrt{V_H} - \alpha_L\right)^2\right)\Big] \\ &= \frac{1}{2}\left[\left(\sqrt{V_L} - \alpha_H\right)^2 + \left(\sqrt{V_H} - \alpha_L\right)^2 + 2r\left(\alpha_H - \alpha_L\right)\left(\sqrt{V_L} - \sqrt{V_H}\right)\right] \end{aligned} \tag{20}$$

Part (i). Routine transformations yield that $\Psi^{1st} \geq \Psi \Leftrightarrow z + ry + r^2 x \geq 0$, where

$$\begin{aligned}
z &= \left(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}}\right)^2 \left(V_L^{\frac{1}{2}} + V_H^{\frac{1}{2}}\right)^2 \left(V_H^2 - V_H V_L + V_L^2\right) \\
y &= -\left(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}}\right)^2 \left[V_L V_H (V_L + V_H) + \left(V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}}\right)^2 + 2V_L^{\frac{1}{2}} V_H^{\frac{1}{2}} \left(V_L^2 + V_H^2\right)\right. \\
&\quad \left. + V_H^3 + V_L^3\right] \\
x &= \left(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}}\right)^2 \left(V_H^2 + V_L^2\right)(V_H + V_L).
\end{aligned}$$

It can be seen that $x \geq 0$, $y \leq 0$, and $z \geq 0$, each with strict inequality for $V_L < V_H$. Hence for $V_L = V_H$ we have $x = y = z = 0$ and thus $\Psi^{1st} = \Psi$.
Let $V_H > V_L$ for the rest of the proof for the first mover, which implies $x > 0$, $z > 0$ and $y < 0$. Consider the function $f(r) = z + ry + r^2 x$ for $r \in [0, 1]$. $f(r) \geq 0 \Leftrightarrow \Psi^{1st} \geq \Psi$. As $z > 0$, and hence, $f(0) > 0$, we have (by continuity) for "sufficiently small" r that $\Psi^{1st} \geq \Psi$. f is decreasing in r for $r \leq \frac{-y}{2x} \equiv r^*$ and increasing for $r > r^*$. Depending on the sign of $f(1)$, on whether $r^* \gtrless 1$, and on the roots of f, we can determine the regions of r for which $\Psi^{1st} \geq \Psi$.
First, we check whether $r^* > 1$. This is equivalent to $-y > 2x$. Plugging in y and x and simplifying yields (for $V_H > V_L$) that $2V_L^{\frac{5}{2}} V_H^{\frac{1}{2}} + 2V_L^{\frac{1}{2}} V_H^{\frac{5}{2}} - 2V_L^{\frac{3}{2}} V_H^{\frac{3}{2}} - V_L^2 V_H - V_L V_H^2 > 0$. We can write this as

$$V_L^{\frac{1}{2}} V_H^{\frac{1}{2}} (V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}})^2 \left[(V_L + V_L^{\frac{1}{2}} V_H^{\frac{1}{2}} + V_H) + \left(V_L^{\frac{1}{2}} + V_H^{\frac{1}{2}}\right)^2\right] > 0,$$

which is obviously satisfied as $V_H > V_L$. Hence, $r^* > 1$.
Next, we check whether $f(1) = x + y + z$. It is straightforward to verify $x + y + z = 0$, hence $f(1) = 0$ and thus $r = 1$ is one of the roots. As $r^* > 1$, f decreasing in r for $r \leq r^*$ and increasing otherwise, and $f(1) = 0$, the second root has to be larger than $r^* > 1$. It follows that for all $r \in [0, 1]$ we have $r^2 x + ry + z \geq 0$ and hence, $\Psi^{1st} \geq \Psi$ (with equality for $r = 1$ and $V_L = V_H$ as shown before).

Part (ii). For the second mover, we receive (by plugging in α_L and α_H) that

$\Psi^{2nd} \geq \Psi \Leftrightarrow a + rb + r^2c \geq 0$ where

$$
\begin{aligned}
a &= \frac{1}{4V_H V_L}\left(V_H - V_L\right)^2\left(V_H^2 - V_H V_L + V_L^2\right), \\
b &= -\frac{(\sqrt{V_L} - \sqrt{V_H})^2}{4V_H V_L}\left[2\left(V_H^3 + V_L^3\right) + V_H V_L\left(V_H + V_L\right) + \right. \\
&\quad \left. 6\sqrt{V_L}\sqrt{V_H}\left(V_H V_L + V_H^2 + V_L^2\right)\right] \\
c &= \left(\frac{1}{\sqrt{V_L}} - \frac{1}{\sqrt{V_H}}\right)^2 \frac{(V_H + V_L)}{4}\left[V_H^{\frac{3}{2}}\left(\sqrt{V_H} + 4\sqrt{V_L}\right) + V_L^{\frac{3}{2}}\left(\sqrt{V_L} + 4\sqrt{V_H}\right)\right];
\end{aligned}
$$

we use for a that $\left(4V_H^2V_L^2 - 3\left(V_H^3V_L + V_L^3V_H\right) + V_H^4 + V_L^4\right)$ is equal to $\left(V_H - V_L\right)^2\left(V_H^2 - V_HV_L + V_L^2\right)$. It can be seen that $a \leq 0$, $b \leq 0$, and $c \geq 0$ (with equality if $V_L = V_H$). For $V_L = V_H$ we have $a = b = c = 0$ and hence $\Psi^{2nd} = \Psi$. Let $V_H > V_L$ for the rest of the proof. Consider $\widetilde{f}(r) = a + rb + r^2c$ for $r \in [0, 1]$. Note that $\widetilde{f}(r) \geq 0 \Leftrightarrow \Psi^{2nd} \geq \Psi$. Hence, since a is positive, we have (by continuity) for "sufficiently small" r that $\widetilde{f}(r) \geq 0$ and therefore $\Psi^{2nd} \geq \Psi$. $\widetilde{f}$ is decreasing in r for $r \leq -\frac{b}{2c} \equiv \widetilde{r}^*$ and increasing for $r > \widetilde{r}^*$. Depending on the sign of $\widetilde{f}(1)$, on whether $\widetilde{r}^* \gtrless 1$, and on the roots of $\widetilde{f}$, we can determine the regions of r for which $\Psi^{2nd} \geq \Psi$.

First, we check whether $\widetilde{r}^* \leq 1$. This is equivalent to $-b \leq 2c$. Plugging in b and c and simplifying yields (for $V_H > V_L$) that $V_L V_H \left(V_L^{\frac{1}{2}} + V_H^{\frac{1}{2}}\right)^2 + 2V_L^{\frac{1}{2}}V_H^{\frac{1}{2}}(V_L^2 + V_H^2) \geq 0$. Note that this is always fulfilled with strict inequality. Hence, $\widetilde{r}^* < 1$.

Next, we check whether $\widetilde{f}(1) = a + b + c \geq 0$. It is straightforward to verify $a + b + c = 0$. Hence, $\widetilde{f}(1) = 0$ and thus $r = 1$ is one of the roots ($\widetilde{r}_1$). As $\widetilde{r}^* < 1$, $\widetilde{f}$ decreasing in r for $r \leq \widetilde{r}^*$, and $\widetilde{f}(1) = 0$, the second root ($\widetilde{r}_2$) has to be smaller than $\widetilde{r}^* \leq 1$. Moreover, it follows that $\widetilde{r}_2$ is the critical value of r such that we have $\Psi^{2nd} \geq \Psi$ for $r \leq \widetilde{r}_2$ (as $\widetilde{f}(r) \geq 0$ for $r \leq \widetilde{r}_2$).

The roots of $\widetilde{f}(r)$ are $\widetilde{r}_{1,2} = \frac{1}{2c}\left[-b \pm \sqrt{b^2 - 4ac}\right]$. We know that the first root is equal to one. As $\frac{1}{2c}\left[-b + \sqrt{b^2 - 4ac}\right] > \frac{1}{2c}\left[b - \sqrt{b^2 - 4ac}\right]$, we have $\widetilde{r}_1 = \frac{1}{2c}\left[b + \sqrt{b^2 - 4as}\right] = 1$[25], and hence $\widetilde{r}_2 = \frac{1}{2c}\left[b - \sqrt{b^2 - 4ac}\right]$. Plugging in a, b, and c, and simplifying yields

$$
\widetilde{r}_2 = \frac{\left(V_L^{\frac{3}{2}} - V_H^{\frac{3}{2}}\right)^2 + 2V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H^2 + V_L^2)}{(V_L + V_H)\left(V_H^2 + 4V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H + V_L) + V_L^2\right)}. \tag{21}
$$

[25]It can be verified that $\widetilde{r}_1 = \frac{V_L^4 + V_H^4 - 3V_L^3V_H - 3V_L^3V_H + 4V_L^2V_H^2}{(V_H + V_L)\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^2\left(V_L^2 + V_H^2 + 4V_L^{\frac{3}{2}}V_H^{\frac{1}{2}} + 4V_L^{\frac{1}{2}}V_H^{\frac{3}{2}}\right)} = 1$.

Summarizing the results, it follows that $\Psi^{2nd} = \Psi$ for $V_L = V_H$ and for perfect positive correlation (i.e., $r = 1$ or equivalently $\rho = 1$). For $V_L < V_H$ and for $r < 1$, we have $\Psi^{2nd} \geq \Psi$ if and only if $r \leq \widetilde{r}_2$. Since $r = \frac{1}{2}(\rho + 1)$, we can rewrite the results in terms of ρ. For $\rho < 1$ and $V_L < V_H$ we have by (21) that $\Psi^{2nd} \geq \Psi$ if

$$\rho \leq \frac{2\left(\left(V_L^{\frac{3}{2}} - V_H^{\frac{3}{2}}\right)^2 + 2V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H^2 + V_L^2)\right)}{(V_L + V_H)\left(V_H^2 + 4V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H + V_L) + V_L^2\right)} - 1,$$

which is equivalent to

$$\rho \leq \frac{\left(V_L^{\frac{3}{2}} - V_H^{\frac{3}{2}}\right)^2 - V_H V_L(V_H + V_L) - 10V_H^{\frac{3}{2}}V_L^{\frac{3}{2}}}{(V_L + V_H)\left(V_H^2 + 4V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H + V_L) + V_L^2\right)} \equiv \rho^s.$$

Comparing the denominator and numerator of ρ^s, we see that ρ^s is strictly smaller than one. Moreover, it can be verified that $\rho^s > -1$ is equivalent to $2(V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}})^2 + 4V_L^{\frac{1}{2}}V_H^{\frac{1}{2}}(V_H^2 + V_L^2) > 0$, which is always satisfied as $V_L < V_H$.
Finally, we have to verify whether there exists any ρ such that $\widetilde{\rho} < \rho \leq \rho^s$ (since $\rho > \widetilde{\rho} = \frac{4V_L - V_H - \sqrt{V_L}\sqrt{V_H}}{\sqrt{V_H}(\sqrt{V_L} - \sqrt{V_H})}$ has to be satisfied for the interior solution). If $\widetilde{\rho} < \rho^s$, such ρ always exist. $\widetilde{\rho} < \rho^s$ is equivalent to

$$\left(\left(V_L^{\frac{3}{2}} - V_H^{\frac{3}{2}}\right)^2 - V_H V_L(V_H + V_L) - 10V_H^{\frac{3}{2}}V_L^{\frac{3}{2}}\right)\sqrt{V_H}(\sqrt{V_L} - \sqrt{V_H}) >$$

$$\left(4V_L - V_H - \sqrt{V_L}\sqrt{V_H}\right)(V_L + V_H)\left(V_H^2 + 4V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H + V_L) + V_L^2\right),$$

which can be shown to be equivalent to

$$14V_L^{\frac{5}{2}}V_H^{\frac{1}{2}}(V_H - V_L) + 6V_L^{\frac{1}{2}}V_H^2(V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}}) + 4V_L^{\frac{3}{2}}(V_H^{\frac{5}{2}} - V_L^{\frac{5}{2}}) + 2V_L V_H^3 + 14V_L^{\frac{5}{2}}V_H^{\frac{3}{2}} > 0.$$

This always holds as $V_H \geq V_L$.

Part *(iii)*. The ex ante expected payoff sum of the contestants in the simultaneous contest with private information is

$$W^{sim} := \frac{V_L^2 - V_H V_L + V_H^2}{V_L + V_H} - \frac{3}{4}r\frac{(V_L - V_H)^2}{V_L + V_H}.$$

The ex ante expected payoff sum in the sequential contest with private information is $W^{seq} = \Psi^{1st} + \Psi^{2nd}$, where Ψ^{1st} and Ψ^{2nd} are as given in (18) and (20)). The rest of the proof works exactly as the proof of part (ii). By standard transformations we can show that $W^{seq} \geq W^{sim} \Leftrightarrow u - rv + r^2 w \geq 0$ where

$$\begin{aligned}
u &= \frac{1}{V_L V_H}(V_L^2 - V_L V_H + V_H^2)(V_L - V_H)^2, \\
v &= \frac{1}{V_L V_H}\left[(V_L^2 - V_H^2)^2 + \left(V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}}\right)\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^3\left(V_H + 3V_L^{\frac{1}{2}}V_H^{\frac{1}{2}} + V_L\right)\right], \\
w &= 2(V_L + V_H)\frac{1}{V_L V_H}\left(\sqrt{V_L} - \sqrt{V_H}\right)^2\left(\frac{V_L^2}{2} + \frac{V_H^2}{2} + \sqrt{V_L}\sqrt{V_H}(V_L + V_H)\right);
\end{aligned}$$

we use for v that $\left(V_H^3 - V_L^3\right)(V_H - V_L) + 4V_L V_H\left(V_L^{\frac{3}{2}} - V_H^{\frac{3}{2}}\right)\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)$ is equal to $\left(V_H^{\frac{3}{2}} - V_L^{\frac{3}{2}}\right)\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^3\left(V_H + 3V_L^{\frac{1}{2}}V_H^{\frac{1}{2}} + V_L\right)$.

It can be seen that $u \geq 0$, $v \geq 0$, and $w \geq 0$ (strictly for $V_L < V_H$). Hence, for $V_L = V_H$ we have $W^{seq} = W^{sim}$ as $u = v = w = 0$.

Let $V_H > V_L$ for the rest of the proof. Consider the function $\hat{f}(r) = u - rv + r^2 w$ for $r \in [0, 1]$. $\hat{f}(r) \geq 0$ is equivalent to $W^{seq} \geq W^{sim}$. As $u > 0$ for $V_H > V_L$ and hence, $\hat{f}(0) > 0$, we have (by continuity) $W^{seq} \geq W^{sim}$ for r "sufficiently small". $\hat{f}$ is decreasing in r for $r \leq \frac{v}{2w} \equiv \hat{r}^*$ and increasing for $r > \hat{r}^*$. We can determine the regions of r for which $W^{seq} \geq W^{sim}$ if we know the roots of $\hat{f}$, the sign of $\hat{f}(1)$, and whether $\hat{r}^* \gtreqless 1$.

First, we check whether $\hat{r}^* \leq 1$. This is equivalent to $v \leq 2w$. Plugging in v and w and simplifying yields $10V_L^2 V_H^2 - V_L^3 V_H - V_L V_H^3 - 4V_L^{\frac{5}{2}} V_H^{\frac{3}{2}} - 4V_L^{\frac{3}{2}} V_H^{\frac{5}{2}} \leq 0$. We can write this as $V_L V_H\left[-(V_L - V_H)^2 - 4V_L^{\frac{1}{2}} V_H^{\frac{1}{2}}\left(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}}\right)^2\right] \leq 0$, which is always fulfilled, with strict inequality for $V_H > V_L$. Hence, $\hat{r}^* < 1$.

Next, we check whether $\hat{f}(1) = u - v + w \geq 0$. It is straightforward to verify $u - v + w = 0$. Hence, $\hat{f}(1) = 0$ and thus $r = 1$ is one of the roots ($\hat{r}_1$). This implies that for perfect positive correlation $W^{seq} = W^{sim}$.

As $\hat{r}^* < 1$ and $\hat{f}$ is decreasing in r for $r \leq \hat{r}^*$, and $\hat{f}(1) = 0$, the second root ($\hat{r}_2$) of $\hat{f}$ has to be smaller than $\hat{r}^*$. Moreover, it follows that $\hat{r}_2$ is the critical value of r such that we have $W^{seq} \geq W^{sim}$ for $r \leq \hat{r}_2$ (as $\hat{f}(r) \geq 0$ for $r \leq \hat{r}_2$).

The roots of $\hat{f}(r)$ are $\hat{r}_{1,2} = \frac{1}{2w}\left[v \pm \sqrt{v^2 - 4uw}\right]$. We know that the first root is equal to one. As $\frac{1}{2w}\left[v + \sqrt{v^2 - 4uw}\right] > \frac{1}{2w}\left[v - \sqrt{v^2 - 4uw}\right]$, we have that $\hat{r}_1 = \frac{1}{2w}\left[v + \sqrt{v^2 - 4uw}\right] = 1$[26], and hence $\hat{r}_2 = \frac{1}{2w}\left[v - \sqrt{v^2 - 4uw}\right]$. Plugging

[26]It can be verified that $\hat{r}_1 = \frac{V_L^3 + V_H^3 - 3V_L^2 V_H - 3V_L^2 V_H + 4V_L^{\frac{3}{2}} V_H^{\frac{3}{2}}}{\left(V_L^2 + V_H^2 + 2V_L^{\frac{3}{2}} V_H^{\frac{1}{2}} + 2V_L^{\frac{1}{2}} V_H^{\frac{3}{2}}\right)\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^2} = 1$.

in w, u and v, and simplifying yields

$$\hat{r}_2 = \frac{V_L^3 + V_H^3 + 2V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H^2 - V_LV_H + V_L^2)}{V_L^3 + V_H^3 + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_L + V_H)\left(2(V_H + V_L) + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}\right)}. \tag{22}$$

Summarizing the results, it follows that $W^{seq} = W^{sim}$ for $V_L = V_H$ and for perfect positive correlation (i.e., $r = 1$ or equivalently $\rho = 1$). For $V_L < V_H$ and for $r < 1$, we have $W^{seq} \geq W^{sim}$ if and only if $r \leq \hat{r}_2$.

Since $r = \frac{1}{2}(\rho + 1)$, we can rewrite the results in terms of ρ. For $\rho < 1$ and $V_L < V_H$, we receive by (22) that $W^{seq} \geq W^{sim}$ if

$$\rho \leq \frac{2(V_L^3 + V_H^3 + 2V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H^2 - V_LV_H + V_L^2))}{V_L^3 + V_H^3 + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_L + V_H)\left(2(V_H + V_L) + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}\right)} - 1,$$

which is equivalent to

$$\rho \leq \frac{V_L^3 + V_H^3 + 2V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_H - V_L)^2 - V_LV_H(V_L + V_H) - 4V_L^{\frac{3}{2}}V_H^{\frac{3}{2}}}{V_L^3 + V_H^3 + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}(V_L + V_H)\left(2(V_H + V_L) + V_H^{\frac{1}{2}}V_L^{\frac{1}{2}}\right)} \equiv \rho^o.$$

Comparing the denominator and numerator of ρ^o, we see that ρ^o is strictly smaller than one and larger than minus one.

Finally, we have to verify whether there exists any ρ such that $\widetilde{\rho} < \rho \leq \rho^o$ (since $\rho > \widetilde{\rho} = \frac{4V_L - V_H - \sqrt{V_L}\sqrt{V_H}}{\sqrt{V_H}(\sqrt{V_L} - \sqrt{V_H})}$ has to be satisfied for the interior solution). If $\widetilde{\rho} < \rho^o$, such ρ always exist. Since $\rho^o > \rho^s$ (as the nominator (denominator) of ρ^o is larger (smaller) than of ρ^s), $\widetilde{\rho} < \rho^o$ follows from the proof of part (ii). ■

Proof of Proposition 9.

The ex ante expected effort sum in sequential contests with public information is

$$\xi_{seq}^{pub} := \frac{1}{2}r\left(\frac{V_L}{2} + \frac{V_H}{2}\right) + \frac{1}{2}(1-r)\left(\frac{V_L}{2} + \frac{V_H}{2}\right) = \frac{V_L}{4} + \frac{V_H}{4}, \tag{23}$$

and with private information it is ξ_{seq}^{priv} as given in (15), which we can write as

$$\xi_{seq}^{priv} \quad = \quad \frac{1}{4}\left[V_H + V_L + r(1-r)\left(2\left(V_H^{\frac{1}{2}}V_L^{\frac{1}{2}} - V_L - V_H\right) + V_L^{-\frac{1}{2}}V_H^{-\frac{1}{2}}\left(V_H^2 + V_L^2\right)\right)\right].$$

It follows that $\xi_{seq}^{priv} \geq \xi_{seq}^{pub}$ is equivalent to

$$r(1-r)\left[2\left(V_H^{\frac{1}{2}}V_L^{\frac{1}{2}} - V_L - V_H\right) + V_L^{-\frac{1}{2}}V_H^{-\frac{1}{2}}\left(V_H^2 + V_L^2\right)\right] \geq 0. \tag{24}$$

Hence, for $r = 0$ and $r = 1$ the ex ante expected effort sum is identical in both information conditions. This is intuitive since in these cases we have public information. For $r \in (0,1)$ we can simplify (24) to

$$V_L^{-\frac{1}{2}} V_H^{-\frac{1}{2}} \left(V_H + V_L\right)^2 - 2\left(V_L + V_H\right) \geq 0 \Leftrightarrow \left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^2 \geq 0,$$

which is always fulfilled (with equality for $V_L = V_H$). ■

Proof of Proposition 10.

As we have seen in Section 2.1, the ex ante expected payoffs of the first mover, $\Psi^{1st/pub}$, and the second mover, $\Psi^{2nd/pub}$, in the sequential contest with public information are identical. The expected payoffs are given by

$$\Psi^{1st/pub} = \Psi^{2nd/pub} = \frac{1}{2}\left[r\left(\frac{V_L}{4} + \frac{V_H}{4}\right) + (1-r)\left(\frac{V_L^2}{4V_H} + \frac{V_H^2}{4V_L}\right)\right].$$

The ex ante expected payoff of the first and second mover in the sequential contest with private information are Ψ^{1st} and Ψ^{2nd}, respectively, as given in (18) and (20). Plugging in α_L and α_H in Ψ^{1st} yields

$$\Psi^{1st} = \frac{1}{8V_L V_H}\left[V_L^3 + V_H^3 + r^2(V_L^2 + V_H^2)(V_L^{\frac{1}{2}} - V_H^{\frac{1}{2}})^2 - 2r\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)\left(V_H^{\frac{5}{2}} - V_L^{\frac{5}{2}}\right)\right]$$

Then, $\Psi^{1st/pub} \geq \Psi^{1st} \Leftrightarrow r(1-r)\left(V_H^2 + V_L^2\right)\left(V_H^{\frac{1}{2}} - V_L^{\frac{1}{2}}\right)^2 \geq 0$, which always holds. For $r = 1$, $r = 0$, and $V_L = V_H$ we have $\Psi^{1st/pub} = \Psi^{1st}$ – which is intuitive as in these cases agents know each other's valuations. Thus, given a sequential order of moves, the first mover prefers public information.

For the second mover, we first plug in α_H and α_L in Ψ^{2nd}. Simplifying yields

$$\begin{aligned}
\Psi^{2nd} = & \frac{1}{2}\Bigg[\frac{V_H^2}{4V_L} - \frac{V_L^2}{4V_H} - \\
& r\left(2V_L^{\frac{1}{2}}V_H^{\frac{1}{2}} - 2\left(V_H + V_L\right) + \frac{1}{2}\left(\frac{V_H^2}{V_L} + \frac{V_L^2}{V_H}\right) + \frac{1}{2}\left(\frac{V_H^{\frac{3}{2}}}{V_L^{\frac{1}{2}}} + \frac{V_L^{\frac{3}{2}}}{V_H^{\frac{1}{2}}}\right)\right) \\
& + r^2\left(2V_L^{\frac{1}{2}}V_H^{\frac{1}{2}} - \frac{7}{4}\left(V_H + V_L\right) + \frac{1}{4}\left(\frac{V_H^2}{V_L} + \frac{V_L^2}{V_H}\right) + \frac{1}{2}\left(\frac{V_H^{\frac{3}{2}}}{V_L^{\frac{1}{2}}} + \frac{V_L^{\frac{3}{2}}}{V_H^{\frac{1}{2}}}\right)\right)\Bigg]
\end{aligned}$$

Then, $\Psi^{2nd/pub} \geq \Psi^{2nd}$ is equivalent to

$$(r^2 - r)\left(2V_L^{\frac{1}{2}}V_H^{\frac{1}{2}} - \frac{7}{4}\left(V_H + V_L\right) + \frac{1}{4}\left(\frac{V_H^2}{V_L} + \frac{V_L^2}{V_H}\right) + \frac{1}{2}\left(\frac{V_H^{\frac{3}{2}}}{V_L^{\frac{1}{2}}} + \frac{V_L^{\frac{3}{2}}}{V_H^{\frac{1}{2}}}\right)\right) \geq 0$$

Rearranging yields

$$\frac{-r(1-r)}{4V_LV_H}\left(V_H^{\frac{1}{2}}-V_L^{\frac{1}{2}}\right)\left[\left(V_H^2-V_L^2\right)\left(V_H^{\frac{1}{2}}+V_L^{\frac{1}{2}}\right)+\right.$$
$$\left.2\left(V_H^{\frac{1}{2}}-V_L^{\frac{1}{2}}\right)\left(V_HV_L^{\frac{1}{2}}\left(V_H^{\frac{1}{2}}-V_L^{\frac{1}{2}}\right)+V_H^{\frac{1}{2}}V_L^{\frac{3}{2}}\right)\right]\geq 0,$$

which is always satisfied (for $r=0$, $r=1$ as well as for $V_H=V_L$ with equality). ■

Proof of Proposition 11.

In the simultaneous contest with public information, the ex ante expected effort sum given valuations V_H and V_L is $\xi_{sim}^{pub}=\frac{1}{2}r(\frac{V_L}{2}+\frac{V_H}{2})+\frac{1}{2}(1-r)\frac{V_LV_H}{V_L+V_H}$. In the other three settings, the expected effort sums ξ_{sim}^{priv}, ξ_{seq}^{pub}, and ξ_{seq}^{priv} are as given in (15), (16), and (23). Let V denote the prize money the designer spends. Suppose that reducing V to $\widetilde{V}=\tau V$ where $\tau\leq 1$ implies valuations $\widetilde{V}_H=\tau V_H$ and $\widetilde{V}_L=\tau V_L$. Given $\widetilde{V}_H$ and $\widetilde{V}_L$, the ex ante expected effort sums become: $\widetilde{\xi}_{seq}^{priv}=\tau\xi_{seq}^{priv}$, $\widetilde{\xi}_{sim}^{priv}=\tau\xi_{sim}^{priv}$, $\widetilde{\xi}_{seq}^{pub}=\tau\xi_{seq}^{pub}$, and $\widetilde{\xi}_{sim}^{pub}=\tau\xi_{sim}^{pub}$. Then, $\widetilde{\xi}_{seq}^{pub}\geq\xi_{sim}^{pub}\Leftrightarrow\tau\geq\frac{\xi_{sim}^{pub}}{\xi_{seq}^{pub}}=\tau_1^*$ where $\tau_1^*\leq 1$ by Proposition 7. Thus, for $\tau\geq\tau_1^*$ the sequential contest yields a higher effort sum at a lower cost. Analogously, for the case of public information, the sequential contest yields a higher effort sum at a lower cost than the simultaneous contest if $\tau\geq\frac{\xi_{sim}^{priv}}{\xi_{seq}^{priv}}=\tau_2^*$ where $\tau_2^*\leq 1$ by Proposition 7. Comparing simultaneous contests with private and public information yields that the designer cannot reduce the prize in one setting and still reach at least the same effort level as in the other setting: Since $\xi_{sim}^{priv}=\xi_{sim}^{pub}$ (cf. Section 3), $\widetilde{\xi}_{sim}^{priv}\geq\xi_{sim}^{pub}$ (or $\widetilde{\xi}_{sim}^{pub}\geq\xi_{sim}^{priv}$) $\Leftrightarrow\tau\geq 1$. Finally, for $\tau\geq\frac{\xi_{seq}^{pub}}{\xi_{seq}^{priv}}=\tau_3^*$ where $\tau_3^*\leq 1$ by Proposition 9, sequential contests with private information yield a higher effort sum at a lower cost than if information is public. ■