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Synthesis of Controllers for Partially-Observable Systems: A Data-Driven Approach ⋆

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Abstract: This paper is concerned with the formal synthesis of safety controllers for partially-observable continuous-time polynomial-type systems with unknown dynamics. Given a continuous-time polynomial-type estimator with a *partially-unknown* dynamic and a *known* upper bound on the estimation accuracy, we propose a data-driven approach to compute a polynomial-type controller ensuring safety of the system. The proposed framework is based on a notion of so-called *control barrier functions* and only requires a single output trajectory collected from the system and a single state trajectory collected from its estimator. We show the application of our technique by synthesizing a safety controller for a partially-observable jet engine with unknown dynamics.

Keywords: Polynomial-type systems, Partial-information, Safety property, Data-driven controller synthesis, Control barrier functions.

1. INTRODUCTION

Synthesizing controllers that enforce safety specifications has gained significant attentions in the last decade. In this regard, formal methods have emerged as a promising and reliable approach to synthesize safety controllers for complex dynamical systems. When synthesizing controllers, assuming that the values of all of the system's states are known is often an unrealistic assumption, and in many real world applications, full state information is not always accessible. This limitation led to new challenges in the synthesis of controllers for systems with partial or incomplete information. Motivated by these challenges, the works in (Jahanshahi et al., 2020b,a; Clark, 2021, 2019) provide an approach based on notions of *barrier functions* to synthesize safety controllers, where proper state estimators are used in order to compute a controller that makes the partially-observable stochastic control system safe. The results in (Jahanshahi et al., 2020b) and (Jahanshahi et al., 2020a) provide a lower bound on the probability that the trajectories of the system remain safe over a finite-time horizon. The proposed approaches in (Clark, 2021) and (Clark, 2019) provide infinite-time horizon guarantees for the safety of the system with probability 1 while assuming a prior knowledge of control barrier functions and considering an unbounded input set. The problem of synthesizing controllers for partially-observable Markov decision processes (POMDPs) using barrier functions has also been studied in (Ahmadi et al., 2019) and (Ahmadi et al., 2020).

The proposed methods in the above-mentioned literature require precise models of dynamical systems. However, closed-form mathematical models of many physical systems are either unavailable or too complicated to be of any use. Therefore, it is not possible to analyze or synthesize controllers for complex systems with unknown models using model-based methodologies. Since obtaining precise models for complex systems is typically a tedious and costly task (Hou and Wang, 2013), data-driven approaches are becoming increasingly popular when dealing with systems with unknown dynamics. To this end, over the past few years, several studies have investigated data-driven controller synthesis for systems with complete state information. When the system model is unavailable, (Fraile et al., 2020) offers an approach to synthesis controllers for single-input, single-output feedback linearizable systems. The work in (Coulson et al., 2019) examines a data-enabled predictive control technique for linear stochastic systems for which the model is unavailable and the controller is derived from noisy input-output data. Given that an upper bound on the dimension of the system is available, (Coulson et al., 2021) presents a data-driven model predictive control scheme solely based on initially measured input-output data. By collecting input-output data over a finite time horizon, (Guo et al., 2020) proposes a methodology to compute control laws for nonlinear polynomial-type systems. Using the so-called *behavioural framework*, which is a data-driven method proposed in (Willems and Polderman, 1997), state and output feedback stabilization and linear quadratic regulation (LQR) problems are studied in (De Persis and Tesi, 2019). Based on the same behavioural idea, the problem is extended to stabilizing polynomial-type systems (Esfahani

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et al., 2014), switched linear systems (Rotulo et al., 2022), and linear time-varying systems (Nortmann and Mylvaganam, 2020). In addition, barrier-based data-driven techniques, in which barrier functions are constructed directly from data, have also been investigated recently. In this respect, the work in (Salamati and Zamani, 2022) offers a data-driven verification strategy via barrier functions for stochastic systems with unknown dynamics as well as a probabilistic confidence over the verification. The extension of (Salamati and Zamani, 2022) from verification to synthesis of safety controllers is proposed in (Salamati et al., 2021). Under a certain rank condition, (Nejati et al., 2022) provides a data-driven controller synthesis methodology for continuous-time nonlinear polynomial-type systems based on a single trajectory acquired from the system. However, to the best of our knowledge, there is no work on the synthesis of safety controllers for unknown systems with partial state information. This work is the first to provide a data-driven methodology to synthesize controllers making a partially-observable system safe.

The main contribution of this work is to provide a data-driven framework for the synthesis of safety controllers for partially-observable continuous-time polynomial-type systems with unknown models. Given an appropriate estimator with a known estimation accuracy, we provide sufficient conditions for so-called *control barrier functions* under which the safety of the unknown system can be guaranteed. The control barrier function and its corresponding polynomial-type safety controller are constructed purely from data. Under a certain rank condition, which is linked to the condition of *persistency of excitation* (Willems et al., 2005), only a single state trajectory from the estimator and a single input-output trajectory from the system over a finite time horizon are needed in our setting. We illustrated our proposed results on a partially-observable jet engine example.

The remainder of the paper is organized as follows. Section 2 contains the definition of polynomial-type systems, mathematical notations, and a description of the problem. Control barrier functions are formally defined in Section 3. We outline our data-driven methodology in Section 4 for synthesizing safety controllers. Section 5 is dedicated to the computation of control barrier functions. Finally, in Section 6, we use a jet engine example with an unknown model to show the effectiveness of our results.

2. PARTIALLY-OBSERVABLE CONTINUOUS-TIME POLYNOMIAL-TYPE SYSTEMS

2.1 Notations

The sets of positive integers, non-negative integers, real numbers, non-negative real numbers, and positive real numbers are denoted by $\mathbb{N}, \mathbb{N}_{\geq 0}, \mathbb{R}, \mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$, respectively. We use $\mathbb{R}^n$ to denote an n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ to denote the space of real matrices with n rows and m columns. Given N vectors $x_i \in \mathbb{R}^{n_i}, n_i \in \mathbb{N}$, and $i \in \{1,\dots,N\}$, we use $[x_1;\dots;x_n]$ and $[x_1,\dots,x_n]$ to denote the corresponding column and row vectors, respectively, with dimension $\sum_i n_i$. The notation $\|x\|$ is used to indicate the infinity norm of a vector $x \in \mathbb{R}^n$. For a set X, we denote its ϵ-inflated version by X^ϵ, with $\epsilon \in \mathbb{R}_{>0}$, and define it as $X^\epsilon := \{\hat{x} \in X \mid \exists x \in X, \|\hat{x}-x\| \leq \epsilon\}$. We denote by $\mathbb{I}_n$ the identity matrix in $\mathbb{R}^{n\times n}$. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite, denoted by $P \succ 0$, if all its eigenvalues are positive.

2.2 Partially-Observable Continuous-Time Polynomial-Type Systems

In this work, we consider partially-observable continuous-time polynomial-type systems as formalized in the following definition.

Definition 2.1. A partially-observable continuous-time polynomial-type system (PO-ct-PS) is described by

$$\mathcal{S}: \begin{cases} \dot{x} = A\mathcal{M}(x) + Bu, \\ y = C\mathcal{M}(x), \end{cases} \tag{2.1}$$

where $A \in \mathbb{R}^{n\times N}, B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times N}$. The vector function $\mathcal{M}(x) \in \mathbb{R}^N$ contains monomials in state $x \in X$, with $X \subset \mathbb{R}^n$ being the state set. Furthermore, $u \in U$ is the control input with input set $U \subset \mathbb{R}^m$, and $y \in Y$ is the output with output set $Y \subset \mathbb{R}^p$. Here, we employ notation $x_{x_0\upsilon}$ to denote the trajectory of $\mathcal{S}$ starting from an initial state $x_0 = x(0)$, under an input υ, and $x_{x_0\upsilon}(t)$ denotes the value of this trajectory at time $t \in \mathbb{R}_{\geq 0}$.

For system $\mathcal{S}$ as in (2.1), we assume matrices A, B, and C are unknown and we employ the term *unknown model* to refer to this type of system. Furthermore, we raise the following assumption on the existence of an estimator that can estimate the states of $\mathcal{S}$ with an upper bound on the estimation accuracy.

Assumption 1. Consider a PO-ct-PS $\mathcal{S}$ as in (2.1). States of $\mathcal{S}$ can be estimated by a proper estimator $\widehat{\mathcal{S}}$ represented as:

$$\widehat{\mathcal{S}}: \begin{cases} \dot{\hat{x}} = A\mathcal{M}(\hat{x}) + Bu + K\big(C\mathcal{M}(x) - C\mathcal{M}(\hat{x})\big), \\ \hat{y} = C\mathcal{M}(\hat{x}), \end{cases} \tag{2.2}$$

with $\hat{x} \in \widehat{X}$ and $\hat{y} \in \widehat{Y}$, where $\widehat{X} \subset \mathbb{R}^n$ and $\widehat{Y} \subset \mathbb{R}^p$ are the estimator's state and output set, respectively. Furthermore, $X \subseteq \widehat{X}$ and $Y \subseteq \widehat{Y}$. The matrix $K \in \mathbb{R}^{n\times p}$ is the known estimator gain, and A, B, and C are the unknown matrices as in $\mathcal{S}$. Moreover, in this paper, we consider a guaranteed upper bound on the estimation accuracy as:

$$\|x(t) - \hat{x}(t)\| \leq \epsilon, \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{2.3}$$

where $\epsilon \in \mathbb{R}_{>0}$ is a known constant.

Now we can formally define the main synthesis problem that we are interested in solving in this paper.

Problem 2.2. Consider a PO-ct-PS $\mathcal{S}$ as in (2.1) with unknown matrices A, B, and C, its estimator $\widehat{\mathcal{S}}$ as in (2.2) with the estimation accuracy ϵ as in (2.3). Let $X_1, X_2 \subset X$ represent initial and unsafe sets for $\mathcal{S}$, respectively. Synthesize a polynomial-type safety controller using which the trajectories of $\mathcal{S}$ starting from initial set X_1 never reach the unsafe set X_2.

To synthesize a controller for Problem 2.2, we utilize a notion of *control barrier functions*, introduced in the next section.

3. CONTROL BARRIER FUNCTIONS

In this section, we define a notion of control barrier functions (CBFs), adopted from (Prajna et al., 2007), as formalized in the following definition.

Definition 3.1. Consider a PO-ct-PS $\mathcal{S}$ as in (2.1), its estimator $\widehat{\mathcal{S}}$ as in (2.2) together with an estimation accuracy ϵ as in (2.3), and $X_1, X_2 \subset X \subseteq \widehat{X}$ as initial and unsafe sets of $\mathcal{S}$, respectively. Let us define $X_2^\epsilon \subset \widehat{X}$ as an ϵ-inflated version of X_2. A function $\mathcal{B} : \widehat{X} \to \mathbb{R}$ is called a control barrier function for $\widehat{\mathcal{S}}$ if there exists constants $\lambda_1, \lambda_2 \in \mathbb{R}$, with $\lambda_1 < \lambda_2$, such that

- $\forall \hat{x} \in X_1$,
$$\mathcal{B}(\hat{x}) \leq \lambda_1, \tag{3.1}$$
- $\forall \hat{x} \in X_2^\epsilon$,
$$\mathcal{B}(\hat{x}) \geq \lambda_2, \tag{3.2}$$
- $\forall \hat{x} \in \widehat{X}, \exists u \in U$, such that $\forall x \in X$,
$$\mathsf{L}\mathcal{B}(x, \hat{x}, u) \leq 0, \tag{3.3}$$

where $\mathsf{L}\mathcal{B}$ is the Lie derivative of $\mathcal{B}$ with respect to the dynamic of the estimator, which is defined as

$$\mathsf{L}\mathcal{B}(x, \hat{x}, u) := \frac{\partial \mathcal{B}(\hat{x})}{\partial \hat{x}} \Big(A\mathcal{M}(\hat{x}) + Bu + K\big(C\mathcal{M}(x) - C\mathcal{M}(\hat{x})\big)\Big). \tag{3.4}$$

The above definition implicitly associates a controller to a CBF according to the existential quantifier over the input for any $\hat{x} \in \widehat{X}$.

Remark 3.2. Note that X_1 and X_2^ϵ should not intersect in order to enforce the safety property in Definition 3.1. This condition is implicitly enforced by imposing $\lambda_1 < \lambda_2$.

The next theorem shows how CBFs can be used in order to make sure that the unknown PO-ct-PS $\mathcal{S}$ in (2.1) is safe in the sense that its trajectories starting from X_1 never reach X_2.

Theorem 3.3. Let $\mathcal{S}$ be a PO-ct-PS as in (2.1) and $\widehat{\mathcal{S}}$ be its corresponding estimator as in (2.2) with the estimation accuracy ϵ as in (2.3). Suppose $\mathcal{B}$ is a CBF for $\widehat{\mathcal{S}}$ as in Definition 3.1. Then, one gets $x_{x_0 v}(t) \notin X_2$ for any $x_0 \in X_1$ and any $t \in \mathbb{R}_{\geq 0}$, where the control input u is chosen in such a way that (3.3) holds.

In the next section, we propose a data-driven approach in order to construct control barrier functions for unknown PO-ct-PSs as in (2.1).

4. DATA-DRIVEN CONTROLLER SYNTHESIS VIA CBFS

We now provide our data-driven approach in order to synthesize safety controllers for the unknown PO-ct-PS $\mathcal{S}$ in (2.1) using its estimator $\widehat{\mathcal{S}}$ in (2.2). To do so, we first collect input-output data from the unknown PO-ct-PS $\mathcal{S}$ and its estimator $\widehat{\mathcal{S}}$ over the time interval $[t_0, t_0 + (T-1)\Delta_t]$, where Δ_t is the sampling time, and $T \in \mathbb{N}_{>0}$ is the number of collected samples. Then, using the collected data from $\mathcal{S}$, we collect input-output data from the estimator $\widehat{\mathcal{S}}$. The collected samples are denoted as follows:

$$\begin{aligned}
\mathcal{U}_{0,T} &:= [u(t_0), u(t_0 + \Delta_t), \cdots, u(t_0 + (T-1)\Delta_t)], \\
\mathcal{Y}_{0,T} &:= [y(t_0), y(t_0 + \Delta_t), \cdots, y(t_0 + (T-1)\Delta_t)], \\
\widehat{\mathcal{X}}_{0,T} &:= [\hat{x}(t_0), \hat{x}(t_0 + \Delta_t), \cdots, \hat{x}(t_0 + (T-1)\Delta_t)], \\
\widehat{\mathcal{X}}_{1,T} &:= \big[\dot{\hat{x}}(t_0), \dot{\hat{x}}(t_0 + \Delta_t), \cdots, \dot{\hat{x}}(t_0 + (T-1)\Delta_t)\big].
\end{aligned} \tag{4.1}$$

Remark 4.1. Observe that in order to construct $\widehat{\mathcal{X}}_{1,T}$, one needs to have access to the derivatives of the states of the estimator at sampling times. Since this data is generally not available via measurements, proposed results in the relevant literature can be utilized in order to approximate derivatives using some filters (cf. (Padoan and Astolfi, 2015; Garnier et al., 2008, 2003)). Although a small numerical error gets introduced from approximating the derivatives, we do not consider this error in our analysis.

Next, we use the results of (Guo et al., 2020) in order to provide a data-based representation of the closed-loop estimator $\widehat{\mathcal{S}}$ in (2.2) using a polynomial-type safety controller of the form $u = Z(\hat{x})\mathcal{M}(\hat{x})$, where the matrix polynomial $Z(\hat{x})$ is to be synthesized.

Lemma 4.2. Let $F(\hat{x})$ be a $(T \times N)$ matrix polynomial such that $\mathbb{I}_N = \widehat{\mathcal{M}}_{0,T} F(\hat{x})$, where $\widehat{\mathcal{M}}_{0,T}$ is an $(N \times T)$ full row-rank matrix constructed from the vector $\mathcal{M}(\hat{x})$ and samples $\widehat{\mathcal{X}}_{0,T}$ as follows

$$\widehat{\mathcal{M}}_{0,T} = \big[\mathcal{M}\big(\hat{x}(t_0)\big), \cdots, \mathcal{M}\big(\hat{x}(t_0 + (T-1)\Delta_t)\big)\big].$$

If the input is set to be as $u = Z(\hat{x})\mathcal{M}(\hat{x}) = \mathcal{U}_{0,T} F(\hat{x})\mathcal{M}(\hat{x})$, then the data-based representation of the closed loop estimator $\dot{\hat{x}} = A\mathcal{M}(\hat{x}) + Bu + K\big(C\mathcal{M}(x) - C\mathcal{M}(\hat{x})\big)$ is as follows:

$$\dot{\hat{x}} = \big(\widehat{\mathcal{X}}_{1,T} - K\mathcal{Y}_{0,T}\big)F(\hat{x})\mathcal{M}(\hat{x}) + K\mathcal{Y}_{0,T}F(x)\mathcal{M}(x),$$

or equivalently,

$$A + BZ(\hat{x}) - KC = (\widehat{\mathcal{X}}_{1,T} - K\mathcal{Y}_{0,T})F(\hat{x}),$$

and

$$KC = K\mathcal{Y}_{0,T}F(x).$$

Remark 4.3. Note that the number of samples T should be at least N in order to insure $\widehat{\mathcal{M}}_{0,T}$ to have full row rank.

The following theorem, inspired by (Nejati et al., 2022, Theorem 8), shows the usefulness of CBFs in order to solve Problem 2.2. To do so, we construct the CBF from data and use the data-based representation in Lemma 4.2 in order to synthesize the controller gain $Z(\hat{x})$, such that $u = Z(\hat{x})\mathcal{M}(\hat{x})$ makes the unknown PO-ct-PS (2.1) safe.

Theorem 4.4. Let $\mathcal{S}$ be a PO-ct-PS in (2.1) and $\widehat{\mathcal{S}}$ be its estimator in (2.2) together with an estimation accuracy ϵ as in (2.3). Suppose there exists a matrix polynomial $H(\hat{x}) \in \mathbb{R}^{T \times N}$ such that $\widehat{\mathcal{M}}_{0,T} H(\hat{x}) = P^{-1}, \forall \hat{x} \in \widehat{X}$, with $P \succ 0$. If conditions (4.2)-(4.4) are satisfied, then $\mathcal{B}(\hat{x}) = \mathcal{M}(\hat{x})^\top [\widehat{\mathcal{M}}_{0,T} H(\hat{x})]^{-1} \mathcal{M}(\hat{x})$ is a CBF and $u = \mathcal{U}_{0,T} H(\hat{x}) \big(\widehat{\mathcal{M}}_{0,T} H(\hat{x})\big)^{-1} \mathcal{M}(\hat{x})$ is its corresponding safety controller which makes the unknown PO-ct-PS $\mathcal{S}$ safe:

- $\forall \hat{x} \in X_1$,

$$\mathcal{M}(\hat{x})^\top[\widehat{\mathcal{M}}_{0,T}H(\hat{x})]^{-1}\mathcal{M}(\hat{x}) \le \lambda_1, \quad (4.2)$$

- $\forall \hat{x} \in X_2^\epsilon$,

$$\mathcal{M}(\hat{x})^\top[\widehat{\mathcal{M}}_{0,T}H(\hat{x})]^{-1}\mathcal{M}(\hat{x}) \ge \lambda_2, \quad (4.3)$$

- $\forall \hat{x} \in \widehat{X}$,

$$\begin{aligned}&\mathcal{M}(\hat{x})^\top P\Big[\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}}(\widehat{\mathcal{X}}_{1,T} - K\mathcal{Y}_{0,T})H(\hat{x})\\ &+ H(\hat{x})^\top(\widehat{\mathcal{X}}_{1,T} - K\mathcal{Y}_{0,T})^\top(\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}})^\top\Big]P\mathcal{M}(\hat{x})\\ &+ \mathcal{M}(\hat{x})^\top P\Big[\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}}K\mathcal{Y}_{0,T}H(x)\Big]P\mathcal{M}(x)\\ &+ \mathcal{M}(x)^\top P\Big[H(x)^\top(K\mathcal{Y}_{0,T})^\top(\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}})^\top\Big]P\mathcal{M}(\hat{x}) \le 0,\end{aligned} \quad (4.4)$$

where $\lambda_1 < \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$. In the next section, we discuss the computation of CBFs.

5. COMPUTATION OF CBFS

In this section, we provide a systematic approach to implement Theorem 4.4 and search for CBFs and their corresponding controllers. The proposed method is based on a sum-of-square (SOS) optimization problem (Parrilo, 2003). To do so, we consider the state set of the system and the estimator X, $\widehat{X}$, the initial set X_1, and the unsafe set X_2^ϵ as

$$X = \bigcup_{i=1}^{n_x} X_i, X_i := \{x \in \mathbb{R}^n \mid g_{ij}(x) \ge 0, j = 1, \ldots, \ell\}, \quad (5.1)$$

$$\widehat{X} = \bigcup_{i=1}^{n_{\hat{x}}} \widehat{X}_i, \widehat{X}_i := \{\hat{x} \in \mathbb{R}^n \mid \hat{g}_{ij}(\hat{x}) \ge 0, j = 1, \ldots, \hat{\ell}\}, \quad (5.2)$$

$$X_1 = \bigcup_{i=1}^{n_{x_1}} X_{1i}, X_{1i} := \{\hat{x} \in \mathbb{R}^n \mid g_{ij}^1(\hat{x}) \ge 0, j = 1, \ldots, \ell_1\}, \quad (5.3)$$

$$X_2^\epsilon = \bigcup_{i=1}^{n_{x_2}} X_{2i}^\epsilon, X_{2i}^\epsilon := \{\hat{x} \in \mathbb{R}^n \mid g_{ij}^2(\hat{x}) \ge 0, j = 1, \ldots, \ell_2\}, \quad (5.4)$$

where $n_x, n_{\hat{x}}, n_{x_1}$, and n_{x_2} are the number of regions in $X, \widehat{X}, X_1$, and X_2^ϵ, respectively. Furthermore, $g_{ij}, \hat{g}_{ij}, g_{ij}^1$, and g_{ij}^2 are polynomial functions, with $\ell, \hat{\ell}, \ell_1$, and ℓ_2 being the number of polynomials required to characterize each region. The input set U is defined as

$$U := \{u \in \mathbb{R}^m \mid b_j^\top u \le 1, \text{ with } j = 1, \ldots, \ell_u\}, \quad (5.5)$$

where $b_j \in \mathbb{R}^m$ are some constant vectors. We now present the SOS formulations in the following corollary.

Corollary 5.1. Consider a PO-ct-PS $\mathcal{S}$ in (2.1) and its estimator $\widehat{\mathcal{S}}$ in (2.2) together with an estimation accuracy ϵ as in (2.3). Let $X, \widehat{X}, X_1$, and X_2^ϵ be as in (5.1)-(5.4), respectively, the input set U be as in (5.5), and data $\mathcal{U}_{0,T}, \mathcal{Y}_{0,T}, \widehat{\mathcal{X}}_{1,T}$, and $\widehat{\mathcal{M}}_{0,T}$ be as in (4.1) and in Lemma 4.2, respectively. If there exist a positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix polynomial $H(\hat{x}) \in \mathbb{R}^{T\times N}$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, with $\lambda_1 < \lambda_2$, such that the following expressions are sum-of-squares (SOS) polynomials

$$-\mathcal{M}(\hat{x})^\top P\mathcal{M}(\hat{x}) - \sum_{j=1}^{\ell_1} h_{ij}^1(\hat{x})g_{ij}^1(\hat{x}) + \lambda_1, \forall i \in \{1,\ldots,n_{x_1}\}, \quad (5.6)$$

$$\mathcal{M}(\hat{x})^\top P\mathcal{M}(\hat{x}) - \sum_{j=1}^{\ell_2} h_{ij}^2(\hat{x})g_{ij}^2(\hat{x}) - \lambda_2, \forall i \in \{1,\ldots,n_{x_2}\}, \quad (5.7)$$

$$\begin{aligned}&- \mathcal{M}(\hat{x})^\top P\Big[\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}}(\widehat{\mathcal{X}}_{1,T} - K\mathcal{Y}_{0,T})H(\hat{x})\\ &- H(\hat{x})^\top(\widehat{\mathcal{X}}_{1,T} - K\mathcal{Y}_{0,T})^\top(\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}})^\top\Big]P\mathcal{M}(\hat{x})\\ &- \mathcal{M}(\hat{x})^\top P\Big[\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}}K\mathcal{Y}_{0,T}H(x)\Big]P\mathcal{M}(x)\\ &- \mathcal{M}(x)^\top P\Big[H(x)^\top(K\mathcal{Y}_{0,T})^\top(\frac{\partial \mathcal{M}(\hat{x})}{\partial \hat{x}})^\top\Big]P\mathcal{M}(\hat{x})\\ &- \sum_{j=1}^{\hat{\ell}} \hat{h}_{\hat{i}j}(\hat{x})\hat{g}_{\hat{i}j}(\hat{x}) - \sum_{j=1}^{\ell} h_{ij}(x)g_{ij}(x),\\ &\forall i \in \{1, \ldots, n_x\}, \forall \hat{i} \in \{1, \ldots, n_{\hat{x}}\},\end{aligned} \quad (5.8)$$

$$\begin{aligned}&1 - b_j^\top \mathcal{U}_{0,T}H(\hat{x})P\mathcal{M}(\hat{x}) - h_u(\hat{x})\Big(\lambda_1 - \mathcal{M}(\hat{x})^\top P\mathcal{M}(\hat{x})\Big),\\ &\forall j \in \{1, \ldots \ell_u\},\end{aligned} \quad (5.9)$$

with $h_{ij}^1(\hat{x}), h_{ij}^2(\hat{x}), \hat{h}_{\hat{i}j}(\hat{x}), h_{ij}(x)$, and $h_u(\hat{x})$ being SOS polynomials of appropriate dimensions, then $\mathcal{B}(\hat{x}) = \mathcal{M}(\hat{x})^\top P\mathcal{M}(\hat{x})$ is a CBF for $\widehat{\mathcal{S}}$, and $u = \mathcal{U}_{0,T}H(\hat{x})P\mathcal{M}(\hat{x})$ is a safety controller for $\mathcal{S}$.

Remark 5.2. If one wishes to accommodate the error term coming from the calculation of derivatives, as discussed in Remark 4.1, one can add an extra positive term to $\widehat{\mathcal{X}}_{1,T}$ in (5.8) such that condition (5.8) is satisfied for that positive error term.

Remark 5.3. Note that in order to search for the matrix polynomial $H(\cdot)$ and matrix P fulfilling conditions (5.6)-(5.9), one can employ existing software tools in the relevant literature such as SOSTOOLS (Prajna et al., 2002), in conjunction with a semidefinite programming solver, such as SeDuMi (Sturm, 1999).

Remark 5.4. Observe that in condition (5.9) there exists a bilinearity between decision matrices P and $H(\cdot)$. In order to tackle this bilinear matrix inequality (BMI), one can first acquire a candidate for P derived from (5.6) and (5.7), and then attempt to obtain an appropriate candidate for $H(\cdot)$ based on (5.8) and (5.9). Another approach to resolve this problem is to utilize the proposed method in (Hassibi et al., 1999) in order to locally solve the BMI by linearizing it via a first-order perturbation approximation. Then, the problem reduces to solving the linearized version.

Remark 5.5. Note that we provide an approach that is *sound* but not *complete* in solving the synthesis problem. This means if one fails to find matrices P and $H(\cdot)$, then a safety controller may or may not exist.

6. CASE STUDY

Here, we consider a nonlinear Moore-Greitzer jet engine model in no-stall mode (Krstic and Kokotovic, 1995) given by:

$$\mathcal{S}: \begin{cases} \dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3, \\ \dot{x}_2 = x_1 - u, \\ y = x_2, \end{cases} \tag{6.1}$$

where $x = [x_1; x_2], x_1 = \Phi - 1, x_2 = \Psi - \phi - 2$, Φ is the mass flow, Ψ is the pressure rise, and ϕ is a constant. System $\mathcal{S}$ in (6.1) is in the form of the PO-ct-PS in (2.1), with

$$A = \begin{bmatrix} 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 & 0 & 0 \end{bmatrix}, \mathcal{M}(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_1^3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, C = [0 \ 1 \ 0 \ 0].$$

We assume that matrices A, B, and C are all unknown and treat the system as a black-box. Here, we consider the state set $X = [-5, 5] \times [-5, 5]$, the initial set $X_1 = [-1, 1] \times [-1, 1]$, the unsafe set $X_2 = [-4.7, 4.7] \times [2, 4.7]$, and the input set $U = [-5, 5]$. Here, we consider a partially-unknown estimator as in (2.2) with unknown A, B, and C matrices and a known gain matrix as $K = [0.06738; 0.09959]$. Note that we are not providing the design procedure of the estimator since it is out of the scope of this paper. Furthermore, we compute the estimator's accuracy empirically using the results of (Marchi et al., 2021) and a sufficiently large amount of data. Now with the estimator's state set as $\widehat{X} = X$ and an estimation accuracy as $\epsilon = 0.3$, we illustrate the results in Theorem 4.4. To do so, we collect data in the form of (4.1), with a sampling time of $\Delta_t = 0.01s$, and the number of samples as $T = 10$. With the help of Corollary 5.1, we obtain

$$P = \begin{bmatrix} 1.212 & 0 & 0 & 0 \\ 0 & 141.5 & -1.067 & 0 \\ 0 & -1.067 & 2.511 & 0 \\ 0 & 0 & 0 & 2.172 \times 10^{-9} \end{bmatrix},$$

with $\lambda_1 = 150, \lambda_2 = 400$, and the safety controller as follows:

$$\begin{aligned} u =& 0.0011\hat{x}_1^3\hat{x}_2 - 0.0211\hat{x}_1^3 + 0.00011\hat{x}_1^2\hat{x}_2^2 - 0.0007\hat{x}_1^2\hat{x}_2 \\ &- 0.0610\hat{x}_1\hat{x}_2^2 - 0.0006\hat{x}_1\hat{x}_2 + 0.0943\hat{x}_1 \\ &- 0.0075\hat{x}_2^3 + 0.0083\hat{x}_2. \end{aligned} \tag{6.2}$$

For the simulation results, we initialized the system and the estimator with 100 random initial states from the initial state set and simulated the closed-loop system under the controller (6.2). The input and state trajectories of the system are illustrated in Figure 1 and Figure 2, respectively.

7. CONCLUSION

In this work, we established a data-driven method for the synthesis of safety controllers for partially-observable continuous-time polynomial-type systems with unknown models. Given a partially-unknown polynomial-type estimator with an upper bound on the estimation accuracy, control barrier functions were utilized in order to synthesize safety controllers. The controller associated with the control barrier function (if existing) makes the system safe. Our proposed framework only requires a single state trajectory collected from the estimator and a single output trajectory of the system, given that a specified rank condition is met. Finally, we used a case study to demonstrate the effectiveness of our proposed results.

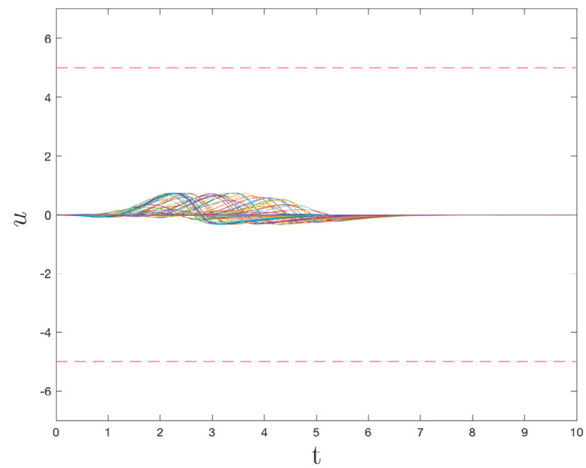

Fig. 1. Input trajectories of the system starting from different initial conditions.

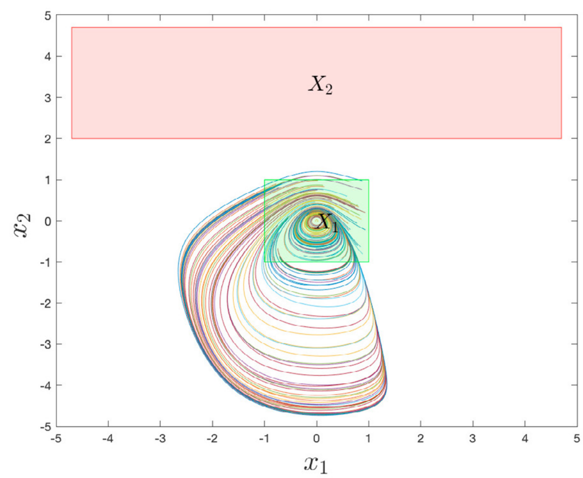

Fig. 2. A few closed-loop state trajectories starting from different initial conditions in X_1 under controller (6.2).

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REFERENCES

Ahmadi, M., Singletary, A., Burdick, J.W., and Ames, A.D. (2019). Safe policy synthesis in multi-agent pomdps via discrete-time barrier functions. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 4797–4803. IEEE.

Ahmadi, M., Singletary, A., Burdick, J.W., and Ames, A.D. (2020). Barrier functions for multiagent-pomdps with dtl specifications. In *2020 59th IEEE Conference on Decision and Control (CDC)*, 1380–1385. IEEE.

Clark, A. (2019). Control barrier functions for complete and incomplete information stochastic systems. In

2019 American Control Conference (ACC), 2928–2935. IEEE.

Clark, A. (2021). Control barrier functions for stochastic systems. *Automatica*, 130, 109688.

Coulson, J., Lygeros, J., and Dörfler, F. (2019). Regularized and distributionally robust data-enabled predictive control. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2696–2701. IEEE.

Coulson, J., Lygeros, J., and Dorfler, F. (2021). Distributionally robust chance constrained data-enabled predictive control. *IEEE Transactions on Automatic Control*.

De Persis, C. and Tesi, P. (2019). Formulas for data-driven control: Stabilization, optimality, and robustness. *IEEE Transactions on Automatic Control*, 65(3), 909–924.

Esfahani, P.M., Sutter, T., and Lygeros, J. (2014). Performance bounds for the scenario approach and an extension to a class of non-convex programs. *IEEE Transactions on Automatic Control*, 60(1), 46–58.

Fraile, L., Marchi, M., and Tabuada, P. (2020). Data-driven stabilization of siso feedback linearizable systems. *arXiv preprint arXiv:2003.14240*.

Garnier, H., Mensler, M., and Richard, A. (2003). Continuous-time model identification from sampled data: implementation issues and performance evaluation. *International journal of Control*, 76(13), 1337–1357.

Garnier, H., Wang, L., and Young, P.C. (2008). Direct identification of continuous-time models from sampled data: Issues, basic solutions and relevance. In *Identification of continuous-time models from sampled data*, 1–29. Springer.

Guo, M., De Persis, C., and Tesi, P. (2020). Learning control for polynomial systems using sum of squares relaxations. In *2020 59th IEEE Conference on Decision and Control (CDC)*, 2436–2441. IEEE.

Hassibi, A., How, J., and Boyd, S. (1999). A path-following method for solving bmi problems in control. In *Proceedings of the 1999 American control conference (Cat. No. 99CH36251)*, volume 2, 1385–1389. IEEE.

Hou, Z.S. and Wang, Z. (2013). From model-based control to data-driven control: Survey, classification and perspective. *Information Sciences*, 235, 3–35.

Jahanshahi, N., Jagtap, P., and Zamani, M. (2020a). Synthesis of partially observed jump-diffusion systems via control barrier functions. *IEEE Control Systems Letters*, 5(1), 253–258.

Jahanshahi, N., Jagtap, P., and Zamani, M. (2020b). Synthesis of stochastic systems with partial information via control barrier functions. *IFAC-PapersOnLine*, 53(2), 2441–2446.

Krstic, M. and Kokotovic, P.V. (1995). Lean backstepping design for a jet engine compressor model. In *Proceedings of International Conference on Control Applications*, 1047–1052. IEEE.

Marchi, M., Bunton, J., Gharesifard, B., and Tabuada, P. (2021). Safety and stability guarantees for control loops with deep learning perception. *IEEE Control Systems Letters*, 6, 1286–1291.

Nejati, A., Zhong, B., Caccamo, M., and Zamani, M. (2022). Data-driven controller synthesis of unknown nonlinear polynomial systems via control barrier certificates. In *Learning for Dynamics and Control Conference*, 763–776. PMLR.

Nortmann, B. and Mylvaganam, T. (2020). Data-driven control of linear time-varying systems. In *2020 59th IEEE Conference on Decision and Control (CDC)*, 3939–3944. IEEE.

Padoan, A. and Astolfi, A. (2015). Towards deterministic subspace identification for autonomous nonlinear systems. In *2015 54th IEEE Conference on Decision and Control (CDC)*, 127–132. IEEE.

Parrilo, P.A. (2003). Semidefinite programming relaxations for semialgebraic problems. *Mathematical programming*, 96(2), 293–320.

Prajna, S., Jadbabaie, A., and Pappas, G.J. (2007). A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE Transactions on Automatic Control*, 52(8), 1415–1428.

Prajna, S., Papachristodoulou, A., and Parrilo, P.A. (2002). Introducing sostools: A general purpose sum of squares programming solver. In *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, volume 1, 741–746. IEEE.

Rotulo, M., De Persis, C., and Tesi, P. (2022). Online learning of data-driven controllers for unknown switched linear systems. *Automatica*, 145, 110519.

Salamati, A., Lavaei, A., Soudjani, S., and Zamani, M. (2021). Data-driven verification and synthesis of stochastic systems through barrier certificates. *arXiv preprint arXiv:2111.10330*.

Salamati, A. and Zamani, M. (2022). Data-driven safety verification of stochastic systems via barrier certificates: A wait-and-judge approach. In *Learning for Dynamics and Control Conference*, 441–452. PMLR.

Sturm, J.F. (1999). Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization methods and software*, 11(1-4), 625–653.

Willems, J.C. and Polderman, J.W. (1997). *Introduction to mathematical systems theory: a behavioral approach*, volume 26. Springer Science & Business Media.

Willems, J.C., Rapisarda, P., Markovsky, I., and De Moor, B.L. (2005). A note on persistency of excitation. *Systems & Control Letters*, 54(4), 325–329.