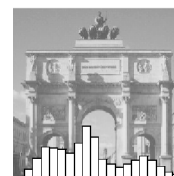




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# Likelihood-based Imprecise Regression

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## Abstract

We introduce a new approach to regression with imprecisely observed data, combining likelihood inference with ideas from imprecise probability theory, and thereby taking different kinds of uncertainty into account. The approach is very general and applicable to various kinds of imprecise data, not only to intervals.

In the present paper, we propose a regression method based on this approach, where no parametric distributional assumption is needed and interval estimates of quantiles of the error distribution are used to identify plausible descriptions of the relationship of interest. Therefore, the proposed regression method is very robust.

We apply our robust regression method to an interesting question in the social sciences. The analysis, based on survey data, yields a relatively imprecise result, reflecting the high amount of uncertainty inherent in the analyzed data set.

**Keywords:** imprecise data, likelihood inference, imprecise probability, complex uncertainty, robust regression, quantile estimation

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## 1. Introduction

Data are often available only with limited precision. That is, they contain only the information that the values of interest lie in certain subsets of the observation space. For example, technical measuring instruments usually provide a precise value and an assessment of the measurement uncertainty, which translates the measurement into an interval of possible values. However, only few general methods for analyzing the relationships between imprecisely observed variables have been proposed so far. These general approaches fall mainly in two categories. One of them consists of approaches suggesting to apply standard regression methods to all possible precise data compatible with the observations, and to consider the range of outcomes as the imprecise result [16, 2, 25, 17]. The approaches in the second category consist in representing the imprecise observations by few precise values (for example, intervals by center and width), and in applying standard regression methods to those values [14, 13, 5, 23, 15, 6].

In the present paper, we follow another line of approach and suggest a Likelihood-based Imprecise Regression (LIR) analysis directly applicable to the imprecise data. LIR combines likelihood inference with ideas from imprecise probability theory, allowing to take into account different kinds of uncertainty. This complex uncertainty is reflected in the imprecise results of the LIR analysis, consisting of all regression functions that cannot be excluded on the basis of the likelihood inference. In the present paper, which is an extended and refined version of [9], we focus on the case without distributional assumptions. The minimization of the quantiles of the residuals leads to a very robust regression method. We describe the details of the regression method in Section 3, which is based on the general methodology for likelihood inference with imprecise data introduced in Section 2.

In addition to the theoretical results, in Section 4 we apply the method to analyze an interesting question in the social sciences. We investigate the relationship between age and income on the basis of survey data. The source of data used in this paper is “Allgemeine Bevölkerungsumfrage der Sozialwissenschaften (ALLBUS) — German General Social Survey” of 2008. The data is provided by GESIS — Leibniz Institute for the Social Sciences.

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## 2. Imprecise Data

Before considering the specific problem of regression with imprecisely observed variables, in the present section we derive some general results about likelihood inference with imprecise data. Let  $V_1, \dots, V_n$  be  $n$  random objects taking values in a set  $\mathcal{V}$ , and let  $V_1^*, \dots, V_n^*$  be  $n$  random sets taking values in a set  $\mathcal{V}^* \subseteq 2^{\mathcal{V}}$ , such that the events  $V_i \in V_i^*$  are measurable. We are actually interested in the data  $V_i$ , but we can only observe the imprecise data  $V_i^*$ . The connection between precise and imprecise data is established by the following assumptions about the probability measures considered as models of the situation.

For each  $\varepsilon \in [0, 1]$ , let  $\mathcal{P}_\varepsilon$  be the set of all probability measures  $P$  such that the  $n$  random objects  $(V_1, V_1^*), \dots, (V_n, V_n^*)$  are independent and identically distributed and satisfy

$$P(V_i \in V_i^*) \geq 1 - \varepsilon \quad (1)$$

(where, as usual, probability measures and random objects are defined on an underlying measurable space). We assume that the precise and imprecise data can be modeled by a probability measure  $P$  included in a particular set  $\mathcal{P} \subseteq \mathcal{P}_\varepsilon$ , for some  $\varepsilon \in [0, 1]$ . Each  $P \in \mathcal{P}$  can be identified with a particular joint distribution for  $V_i$  and  $V_i^*$  (that is, the precise and imprecise data, respectively) satisfying condition (1). In particular,  $\mathcal{P} = \mathcal{P}_\varepsilon$  corresponds to the fully nonparametric assumption that any joint distribution for  $V_i$  and  $V_i^*$  satisfying condition (1) is a possible model of the situation (this is the assumption we consider in Sections 3 and 4). The usual choice for the value of  $\varepsilon$  is 0 (see for example [11, 32]), which corresponds to an assumption of correctness of the imprecise data:  $V_i^* = A$  implies  $V_i \in A$  (a.s.). However, this assumption is often too strong: some imprecise data can be incorrect, in the sense that  $V_i^* = A$ , but  $V_i \notin A$ . This is for example the case, when the imprecise data represent the classification of the precise data into categories, and some observations are misclassified. By choosing a positive value for  $\varepsilon$ , we allow each imprecise observation to be incorrect with probability at most  $\varepsilon$ .

The set  $\mathcal{V}^*$  describes which imprecise data  $V_i^* = A$  are considered as possible. As extreme cases we have the actually precise data (when  $A$  is a singleton) and the missing data (when  $A = \mathcal{V}$ ). In general, the fully nonparametric assumption  $\mathcal{P} = \mathcal{P}_\varepsilon$  does not exclude informative coarsening (see for example [37]): parametric models or uninformative coarsening can be imposed by a stronger assumption  $\mathcal{P} \subset \mathcal{P}_\varepsilon$ . However, it is important to note that the set  $\mathcal{P}_\varepsilon$  depends strongly on the choice of  $\mathcal{V}^*$ . For example, when  $\varepsilon = 0$ , the choice of a set  $\mathcal{V}^*$  such that its elements build a partition of  $\mathcal{V}$  implies the assumption that the coarsening is deterministic and uninformative, because each possible precise data value is contained in exactly one possible imprecise observation  $A \in \mathcal{V}^*$ .

**Example 1.** Let  $\mathcal{V} = \{0, 1\}$  and  $\mathcal{V}^* = 2^{\mathcal{V}}$ , and assume  $\mathcal{P} = \mathcal{P}_\varepsilon$  for some  $\varepsilon \in [0, 1]$ . In this case, each (unobserved) variable  $V_i$  assumes either the value 0 or 1, but we observe only the imprecise data  $V_i^* = A$ , with  $A \subseteq \{0, 1\}$ . When  $A = \{0\}$  or  $A = \{1\}$ , the observation is actually precise (but possibly incorrect): we know that  $V_i = 0$  or  $V_i = 1$ , respectively, with probability at least  $1 - \varepsilon$ . When  $A = \mathcal{V}$ , the data  $V_i$  is in fact missing: we did not learn anything about it by observing  $V_i^* = \{0, 1\}$ . Finally, when  $A = \emptyset$ , the imprecise observation does not tell us anything about  $V_i$ , because it is certainly incorrect; therefore, condition (1) implies  $P(V_i^* = \emptyset) \leq \varepsilon$ .

### 2.1. Complex Uncertainty

In general, we are uncertain about which of the probability measures in  $\mathcal{P}$  is the best model of the reality under consideration. Our uncertainty is composed of two parts. On the one hand, we are uncertain about the distribution of the imprecise data  $V_i^*$ : this uncertainty decreases when we observe more and more (imprecise) data; we call it statistical uncertainty. On the other hand, even if we (asymptotically) knew the distribution of the imprecise data  $V_i^*$ , we would still be uncertain about the distribution of the (unobserved) precise data  $V_i$ : this uncertainty is unavoidable; we call it indetermination. To formulate this mathematically, let  $P_V$  and  $P_{V^*}$  be the marginal distributions of  $V_i$  and  $V_i^*$ , respectively, corresponding to the probability measure  $P \in \mathcal{P}$ . There is statistical uncertainty about  $P_{V^*}$  in the set  $\mathcal{P}_{V^*} := \{P'_{V^*} : P' \in \mathcal{P}\}$ , but even if  $P_{V^*}$  were known, there would still be the (unavoidable) indetermination of  $P_V$  in the set

$$[P_{V^*}] := \{P'_V : P' \in \mathcal{P}, P'_{V^*} = P_{V^*}\}.$$

The sets  $[P_{V^*}]$  with  $P_{V^*} \in \mathcal{P}_{V^*}$  are the identification regions for  $P_V$  in the terminology of [24]. Each of them consists of all the distributions for the precise data  $V_i$  compatible with a particular distribution for the imprecise data  $V_i^*$ . Hence,

each set  $[P_{V^*}]$  can be interpreted as an imprecise probability distribution on  $\mathcal{V}$ . By observing the realizations of the imprecise data  $V_i^*$ , we learn something about which of the imprecise probability distributions  $[P_{V^*}]$  is the best model for the (unobserved) precise data  $V_i$ .

**Example 2.** In the situation of Example 1, the only condition on the marginal distribution of the imprecise data  $V_i^*$  is  $P(V_i^* = \emptyset) \leq \varepsilon$ . Hence,  $\mathcal{P}_{V^*}$  is the set of all probability distributions on  $2^{\{0,1\}}$  such that the probability of  $\emptyset$  is at most  $\varepsilon$ . The only condition on the joint distribution of  $V_i$  and  $V_i^*$  is given by assumption (1), which in this case can be written as

$$P(V_i = 0, V_i^* \in \{\emptyset, \{1\}\}) + P(V_i = 1, V_i^* \in \{\emptyset, \{0\}\}) \leq \varepsilon.$$

Therefore, for each  $P_{V^*} \in \mathcal{P}_{V^*}$ , the identification region  $[P_{V^*}]$  is the set of all probability distributions on  $\{0, 1\}$  such that the probability of 1 lies in the interval  $[\underline{P}_{V^*}\{1\}, \bar{P}_{V^*}\{1\}]$ , with

$$\begin{aligned} \bar{P}_{V^*}\{1\} &= \min(P_{V^*}\{\{1\}, \mathcal{V}\} + \varepsilon, 1), \\ \underline{P}_{V^*}\{1\} &= 1 - \min(P_{V^*}\{\{0\}, \mathcal{V}\} + \varepsilon, 1) = \max(P_{V^*}\{\{1\}, \emptyset\} - \varepsilon, 0). \end{aligned}$$

In particular, when  $\varepsilon = 0$ , the imprecise probability distribution  $[P_{V^*}]$  corresponds to the belief function on  $\mathcal{V}$  with basic probability assignment  $P_{V^*}$  (see for example [30]), in the sense that  $[P_{V^*}]$  is the set of all probability distributions on  $\{0, 1\}$  dominating that belief function.

## 2.2. Likelihood

The likelihood function is a central concept in statistical inference (see for example [28]). For parametric probability models, it is usually expressed as a function of the parameters: here we consider the more general formulation (as a function of the probability measures), which is applicable also to nonparametric models. The observed (imprecise) data  $V_1^* = A_1, \dots, V_n^* = A_n$  induce the (normalized) likelihood function  $lik : \mathcal{P} \rightarrow [0, 1]$  defined by

$$lik(P) = \frac{P(V_1^* = A_1, \dots, V_n^* = A_n)}{\sup_{P' \in \mathcal{P}} P'(V_1^* = A_1, \dots, V_n^* = A_n)} = \frac{\prod_{i=1}^n P_{V^*}\{A_i\}}{\sup_{P' \in \mathcal{P}} \prod_{i=1}^n P'_{V^*}\{A_i\}}$$

for all  $P \in \mathcal{P}$ . The likelihood function describes the relative ability of the probability measures  $P$  in predicting the observed (imprecise) data. Therefore, the value  $lik(P)$  depends only on the marginal distribution  $P_{V^*}$  of the imprecise data  $V_i^*$ . The likelihood function can be interpreted as the second level of a hierarchical model for imprecise probabilities, with  $\mathcal{P}$  as first level (see for example [7, 8]). In particular, for any  $\beta \in (0, 1)$ , the likelihood function can be used to reduce  $\mathcal{P}$  to the set

$$\mathcal{P}_{>\beta} := \{P \in \mathcal{P} : lik(P) > \beta\}$$

of all the probability measures that were sufficiently good in predicting the observed (imprecise) data.

Let  $g$  be a multivalued mapping from  $\mathcal{P}$  to a set  $\mathcal{G}$ , describing a particular characteristic (in which we are interested) of the models considered (mathematically,  $g : \mathcal{P} \rightarrow 2^{\mathcal{G}} \setminus \{\emptyset\}$ , but  $g$  is interpreted as an “imprecise” mapping from  $\mathcal{P}$  to  $\mathcal{G}$ ). For example,  $g$  can be the multivalued mapping from  $\mathcal{P}$  to  $\mathbb{R}$  assigning to each probability measure  $P$  the  $p$ -quantile of the distribution of  $h(V_i)$  under  $P$ , for some  $p \in (0, 1)$  and some measurable function  $h : \mathcal{V} \rightarrow \mathbb{R}$ . This is the kind of mapping  $g$  we consider in Sections 3 and 4: it is multivalued, because in general quantiles are not uniquely defined (a  $p$ -quantile of the distribution of  $h(V_i)$  is any value  $q \in \mathbb{R}$  such that  $P(h(V_i) < q) \leq p \leq P(h(V_i) \leq q)$ ). For each  $\beta \in (0, 1)$ , the set

$$\mathcal{G}_{>\beta} := \bigcup_{P \in \mathcal{P}_{>\beta}} g(P)$$

is called likelihood-based confidence region with cutoff point  $\beta$  for the values of the multivalued mapping  $g$ . This confidence region consists of all values that the characteristic described by  $g$  takes on the set  $\mathcal{P}_{>\beta}$  of all the probability measures that were sufficiently good in predicting the observed (imprecise) data. The unique function  $lik_g : \mathcal{G} \rightarrow [0, 1]$  describing these confidence regions, in the sense that

$$\mathcal{G}_{>\beta} = \{\gamma \in \mathcal{G} : lik_g(\gamma) > \beta\}$$

for all  $\beta \in (0, 1)$ , is called (normalized) profile likelihood function induced by the multivalued mapping  $g$ .

**Lemma 1.** For all  $\gamma \in \mathcal{G}$ ,

$$lik_g(\gamma) = \sup_{P \in \mathcal{P} : \gamma \in g(P)} lik(P)$$

(where the supremum is 0 when no  $P$  satisfies the condition).

*Proof.* For all  $\beta \in (0, 1)$ ,

$$lik_g(\gamma) > \beta \Leftrightarrow \gamma \in \mathcal{G}_{>\beta} \Leftrightarrow \exists P \in \mathcal{P}_{>\beta} : \gamma \in g(P) \Leftrightarrow \exists P \in \mathcal{P} : \gamma \in g(P) \wedge lik(P) > \beta \Leftrightarrow \sup_{P \in \mathcal{P} : \gamma \in g(P)} lik(P) > \beta,$$

from which the result follows, since both sides of the equation take values in  $[0, 1]$ .  $\square$

**Example 3.** In the situation of Examples 1 and 2, let  $\varepsilon = 0$ , and assume that the imprecise data  $\{0\}$ ,  $\{1\}$ , and  $\mathcal{V}$  have been observed  $n_0$ ,  $n_1$ , and  $n_{01}$  times, respectively, where  $n_0$ ,  $n_1$ , and  $n_{01}$  are positive integers. In this case the likelihood function  $lik : \mathcal{P} \rightarrow [0, 1]$  satisfies, for all  $P \in \mathcal{P}$ ,

$$lik(P) = \frac{P_{V^*} \{\{0\}\}^{n_0} P_{V^*} \{\{1\}\}^{n_1} P_{V^*} \{\mathcal{V}\}^{n_{01}}}{\sup_{P' \in \mathcal{P}} P'_{V^*} \{\{0\}\}^{n_0} P'_{V^*} \{\{1\}\}^{n_1} P'_{V^*} \{\mathcal{V}\}^{n_{01}}}$$

Consider now the mapping  $g$  from  $\mathcal{P}$  to  $[0, 1]$  assigning to each probability measure  $P$  the probability  $P_V\{1\}$  that a precise data value  $V_i$  is 1 (before observing the corresponding imprecise data value  $V_i^*$ ; as a multivalued mapping,  $g$  is defined by  $g(P) = \{P_V\{1\}\}$  for all  $P \in \mathcal{P}$ ). The induced profile likelihood function  $lik_g$  on  $[0, 1]$  is plotted in Figure 1 for the cases with  $(n_0, n_1, n_{01}) = (11, 21, 6)$  and  $(n_0, n_1, n_{01}) = (213, 651, 98)$ : solid and dashed lines, respectively (a detailed calculation of  $lik_g$  is given in Example 4).

In these two cases, the likelihood-based confidence regions with cutoff point  $\beta = 0.15$  for the probability  $P_V\{1\}$  are approximately the intervals  $[0.39, 0.84]$  and  $[0.65, 0.80]$ , respectively (the cutoff point  $\beta = 0.15$  is represented by the dotted line in Figure 1). They are (conservative) confidence intervals of approximate level 95% (see for example [22]).

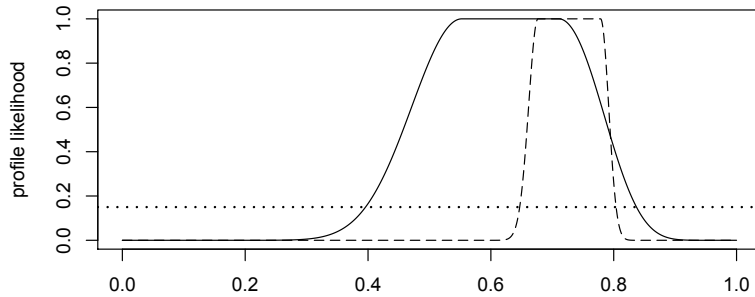


Figure 1: Profile likelihood functions from Examples 3 and 4.

### 2.3. Likelihood for Imprecise Data Models

In the situation we consider, we are actually interested in the (unobserved) precise data  $V_i$ . In this case, the characteristic of interest (described by  $g$ ) depends only on the marginal distribution  $P_V$  of the precise data  $V_i$ ; that is, we can write  $g(P) =: g'(P_V)$  for all  $P \in \mathcal{P}$ . For example, the  $p$ -quantile of the distribution of  $h(V_i)$  depends only on the distribution of  $V_i$ . By contrast, as noted at the beginning of Subsection 2.2, the value  $lik(P)$  depends only on the marginal distribution  $P_{V^*}$  of the imprecise data  $V_i^*$ . By writing  $lik(P) = lik^*(P_{V^*})$  for all  $P \in \mathcal{P}$ , we define a function  $lik^* : \mathcal{P}_{V^*} \rightarrow [0, 1]$ , which can be interpreted as the likelihood function on  $\mathcal{P}_{V^*}$ .

In order to obtain the profile likelihood function  $lik_g$ , it can be useful to consider the multivalued mapping  $g^*$  from  $\mathcal{P}_{V^*}$  to  $\mathcal{G}$  defined by

$$g^*(P_{V^*}) = \bigcup_{P_V \in [P_{V^*}]} g'(P_V)$$

for all  $P_{V^*} \in \mathcal{P}_{V^*}$ . The multivalued mapping  $g^*$  assigns to each  $P_{V^*}$  all the values that the characteristic described by  $g'$  takes on the set  $[P_{V^*}]$  of all distributions for the precise data  $V_i$  compatible with the distribution  $P_{V^*}$  for the imprecise data  $V_i^*$ . That is,  $g^*$  can be interpreted as an imprecise version of  $g'$ , assigning to each imprecise probability distribution  $[P_{V^*}]$  the corresponding imprecise value of  $g'$ .

We can now define the function  $lik_{g^*}^* : \mathcal{G} \rightarrow [0, 1]$  in analogy with the expression for the profile likelihood function  $lik_g$  given in Lemma 1:

$$lik_{g^*}^*(\gamma) = \sup_{P_{V^*} \in \mathcal{P}_{V^*} : \gamma \in g^*(P_{V^*})} lik^*(P_{V^*})$$

for all  $\gamma \in \mathcal{G}$  (where the supremum is 0 when no  $P_{V^*}$  satisfies the condition). The function  $lik_{g^*}^*$  can be interpreted as the profile likelihood function induced by the multivalued mapping  $g^*$ , when  $lik^*$  is considered as the likelihood function on  $\mathcal{P}_{V^*}$ . This profile likelihood function is particularly interesting in connection with the discussion of Subsection 2.1, because  $lik^*$  describes the statistical uncertainty about the distribution  $P_{V^*}$  of the imprecise data  $V_i^*$ , which decreases when we observe more and more (imprecise) data, while  $g^*$  describes the (unavoidable) indetermination of the values of  $g$  (in the terminology of [24], the values of  $g^*$  are the identification regions for the values of  $g$ ). Thanks to the following result, the profile likelihood function  $lik_{g^*}^*$  is not only interesting from a conceptual point of view, but also useful in order to calculate the likelihood-based confidence regions for the values of  $g$ .

**Lemma 2.**

$$lik_g = lik_{g^*}^*$$

*Proof.* From Lemma 1 and the above definitions it follows that for all  $\gamma \in \mathcal{G}$ ,

$$lik_g(\gamma) = \sup_{P \in \mathcal{P} : \gamma \in g'(P_V)} lik^*(P_{V^*}) = \sup_{P_{V^*} \in \mathcal{P}_{V^*} : \exists P' \in \mathcal{P} : P'_{V^*} = P_{V^*} \wedge \gamma \in g'(P'_V)} lik^*(P_{V^*}) = \sup_{P_{V^*} \in \mathcal{P}_{V^*} : \exists P_V \in [P_{V^*}] : \gamma \in g'(P_V)} lik^*(P_{V^*}) = lik_{g^*}^*(\gamma).$$

□

**Example 4.** The imprecise version  $g^*$  of the mapping  $g$  of Example 3 is the multivalued mapping from  $\mathcal{P}_{V^*}$  to  $[0, 1]$  assigning to each  $P_{V^*}$  the set  $\{P_V\{1\} : P_V \in [P_{V^*}]\}$ . In Example 2 we have seen that, since now  $\varepsilon = 0$ , this set is the interval

$$[P_{V^*}\{1\}, \bar{P}_{V^*}\{1\}] = [P_{V^*}\{\{1\}\}, 1 - P_{V^*}\{\{0\}\}].$$

That is,  $g^*(P_{V^*})$  is the interval probability that a precise data value  $V_i$  is 1 (before observing the corresponding imprecise data value  $V_i^*$ ) according to the imprecise probability distribution  $[P_{V^*}]$  (i.e., the belief function on  $\mathcal{V}$  with basic probability assignment  $P_{V^*}$ ).

As seen in Example 2, the only condition on the marginal distributions  $P_{V^*} \in \mathcal{P}_{V^*}$  is  $P_{V^*}\{\emptyset\} = 0$  (since now  $\varepsilon = 0$ ). That is,  $\mathcal{P}_{V^*}$  corresponds to the set of all probability distributions on the set  $\{\{0\}, \{1\}, \mathcal{V}\}$ , and can thus be parametrized by the 2-dimensional simplex

$$\mathcal{S}_2 = \{p = (p_0, p_1, p_{01}) \in [0, 1]^3 : p_0 + p_1 + p_{01} = 1\}.$$

Therefore,  $lik^* : \mathcal{P}_{V^*} \rightarrow [0, 1]$  corresponds to a (normalized) multinomial likelihood function, and we obtain

$$lik_{g^*}^*(\gamma) = \max_{p \in \mathcal{S}_2 : \gamma \in [p_1, 1 - p_0]} \frac{p_0^{n_0} p_1^{n_1} p_{01}^{n_{01}}}{\hat{p}_0^{n_0} \hat{p}_1^{n_1} \hat{p}_{01}^{n_{01}}}$$

for all  $\gamma \in [0, 1]$ , where

$$\hat{p} = \left( \frac{n_0}{n_0 + n_1 + n_{01}}, \frac{n_1}{n_0 + n_1 + n_{01}}, \frac{n_{01}}{n_0 + n_1 + n_{01}} \right)$$

is the maximum likelihood estimate of the parameter  $p \in \mathcal{S}_2$ . Hence, in particular,  $lik_{g^*}^*(\gamma) = 1$  for all  $\gamma \in [\hat{p}_1, 1 - \hat{p}_0]$ . Moreover, it can be easily proved that if  $\gamma < \hat{p}_1$ , then

$$p = \left( \hat{p}_0 \frac{1 - \gamma}{1 - \hat{p}_1}, \gamma, \hat{p}_{01} \frac{1 - \gamma}{1 - \hat{p}_1} \right)$$

maximizes  $p_0^{n_0} p_1^{n_1} p_{01}^{n_{01}}$  among all  $p \in \mathcal{S}_2$  such that  $p_1 \leq \gamma$ . Symmetrically, if  $1 - \gamma < \hat{p}_0$ , then

$$p = \left( 1 - \gamma, \hat{p}_1 \frac{\gamma}{1 - \hat{p}_0}, \hat{p}_{01} \frac{\gamma}{1 - \hat{p}_0} \right)$$

maximizes  $p_0^{n_0} p_1^{n_1} p_{01}^{n_{01}}$  among all  $p \in \mathcal{S}_2$  such that  $p_0 \leq 1 - \gamma$ . Altogether, thanks to Lemma 2, we obtain the following expression for the profile likelihood function induced by the multivalued mapping  $g$  (see also [38]):

$$\text{lik}_g(\gamma) = \text{lik}_{g^*}^*(\gamma) = \begin{cases} \left( \frac{\gamma}{\hat{p}_1} \right)^{n_1} \left( \frac{1 - \gamma}{1 - \hat{p}_1} \right)^{n_0 + n_{01}} & \text{if } 0 \leq \gamma < \hat{p}_1, \\ 1 & \text{if } \hat{p}_1 \leq \gamma \leq 1 - \hat{p}_0, \\ \left( \frac{1 - \gamma}{\hat{p}_0} \right)^{n_0} \left( \frac{\gamma}{1 - \hat{p}_0} \right)^{n_1 + n_{01}} & \text{if } 1 - \hat{p}_0 < \gamma \leq 1. \end{cases}$$

The profile likelihood function  $\text{lik}_g = \text{lik}_{g^*}^*$  on  $[0, 1]$  is plotted in Figure 1 for the two cases considered in Example 3. In the case with 38 data (solid line) there is (statistical) uncertainty also about the distribution  $P_{V^*}$  of the imprecise data  $V_i^*$ , while in the case with 962 data (dashed line) almost only the (unavoidable) indetermination described by  $g^*$  remains, in the sense that  $\text{lik}_{g^*}^*$  is almost equal to the indicator function of an identification region for  $P_V\{1\}$  (more precisely, the indicator function of the probability interval  $g^*(\hat{P}_{V^*}) = [\hat{p}_1, 1 - \hat{p}_0]$  corresponding to the maximum likelihood estimate of  $P_{V^*} \in \mathcal{P}_{V^*}$ ).

### 3. Regression

We now apply the results of Section 2 to the problem of regression with imprecisely observed variables. Hence, we assume that the (unobservable) precise data are pairs  $V_i = (X_i, Y_i)$ , where  $X_1, \dots, X_n$  are  $n$  random objects taking values in a set  $\mathcal{X}$ , and  $Y_1, \dots, Y_n$  are  $n$  random variables, with  $\mathcal{V} = \mathcal{X} \times \mathbb{R}$ . For some  $\mathcal{V}^* \subseteq 2^{\mathcal{X} \times \mathbb{R}}$  and some  $\varepsilon \in [0, 1]$ , we consider the fully nonparametric assumption  $\mathcal{P} = \mathcal{P}_\varepsilon$ . This means that we do not assume anything about the joint distribution of  $X_i$  and  $Y_i$ , while the only condition on the joint distribution of the (unobserved) precise data  $V_i$  and their imprecise observations  $V_i^*$  is given by assumption (1). In the remainder of the paper, we focus on this setting.

We want to describe the relation between  $X_i$  and  $Y_i$  by means of a function  $f \in \mathcal{F}$ , where  $\mathcal{F}$  is a particular set of (measurable) functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ . In order to assess the quality of the description by means of  $f$ , we define the (absolute) residuals

$$R_{f,i} := |Y_i - f(X_i)|.$$

The  $n$  random variables  $R_{f,1}, \dots, R_{f,n} \in [0, +\infty)$  are independent and identically distributed: the more their distribution is concentrated near 0, the better is the description by means of  $f$ .

In order to compare the quality of the descriptions by means of different functions  $f \in \mathcal{F}$ , we need to compare the concentration near 0 of the distributions of the corresponding residuals  $R_{f,i}$ . Usual choices of measures for this concentration are the second and first moments  $E(R_{f,i}^2)$  and  $E(R_{f,i})$ , respectively. However, the moments of the distribution of the residuals cannot be really estimated in the fully nonparametric setting we consider, because moments are too sensitive to small variations in the distribution (see also Subsection 4.2). In fact, if  $\varepsilon > 0$  or the set

$$\mathcal{R}_f := \{|y - f(x)| : (x, y) \in A, A \in \mathcal{V}^*\}$$

(i.e., the set of all possible values of  $R_{f,i}$  when  $V_i \in V_i^*$ ) is unbounded, then the likelihood-based confidence region for any particular moment of the distribution of the residuals is unbounded (even when only the distributions with finite moments are considered), independently of the cutoff point and of the observed (imprecise) data.

By contrast, the quantiles of the distribution of the residuals can in general be estimated even in the fully nonparametric setting we consider. Therefore, we propose to use the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  as a measure of the concentration near 0 of this distribution, for some  $p \in (0, 1)$ . The technical details of the estimation of such quantiles are given in Subsections 3.1 and 3.2.

The minimizations of the second and first moments of the distribution of the residuals can be interpreted as the theoretical counterparts of the methods of least squares and least absolute deviations, respectively. In the same sense,

the minimization of the  $p$ -quantile of the distribution of the residuals can be interpreted as the theoretical counterpart of the method of least quantile of squares (or absolute deviations), introduced in [29] as a generalization of the method of least median of squares (corresponding to the choice  $p = 0.5$ ). The method of least quantile of squares leads to robust regression estimators, with breakdown point  $\min\{p, 1 - p\}$  (that is, the highest possible breakdown point 50% is reached when  $p = 0.5$ ). By contrast, the methods of least squares and least absolute deviations lead to regression estimators with breakdown point 0, since they cannot even handle a single outlier (including leverage points; see for example [18, 26, 21]).

In the location problem (that is, when  $\mathcal{F}$  is the set of all constant functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ ), the values of the constant functions  $f$  minimizing the second and first moments of the distribution of the residuals  $R_{f,i}$  are the mean and median of the distribution of  $Y_i$ , respectively (when these exist and are unique). The value of the constant function  $f$  minimizing the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  is the  $p$ -center of the distribution of  $Y_i$  (that is, the center of the shortest interval containing  $Y_i$  with probability at least  $p$ ), when this exists and is unique. The  $p$ -center can be interpreted as a generalization of the mode of a distribution, since under some regularity conditions the mode corresponds to the limit of the  $p$ -center when  $p$  tends to 0. The  $p$ -center of a symmetric, strictly unimodal distribution corresponds to its median and mean (when this exists), independently of  $p$ . Therefore, the minimizations of the  $p$ -quantile, first moment, and second moment of the distribution of the residuals lead to the same (correct) regression function, under the usual assumptions for the error distribution: see for example [33].

**Example 5.** We consider a problem of simple linear regression: that is,  $\mathcal{X} = \mathbb{R}$  (and thus  $\mathcal{V} = \mathbb{R}^2$ ), and  $\mathcal{F} = \{f_{a,b} : a, b \in \mathbb{R}\}$  is the set of all linear functions  $f_{a,b}$  defined by  $f_{a,b}(x) = a + b x$  for all  $x \in \mathbb{R}$ . The left plot of Figure 2 shows  $n = 99$  precise data points  $V_1, \dots, V_n$ . However, we assume that the pairs of precise values  $V_i = (X_i, Y_i) \in \mathbb{R}^2$  are not known. Instead, for given partitions of the real line in intervals, we only know in which intervals  $X_i^*$  and  $Y_i^*$  lie  $X_i$  and  $Y_i$ , respectively. That is, we assume that only the imprecise data  $V_1^*, \dots, V_n^*$  are observed, where  $V_i^* = X_i^* \times Y_i^*$ , and the elements of  $\mathcal{V}^*$  (i.e., the possible imprecise observations  $V_i^*$ ) build a partition of  $\mathcal{V} = \mathbb{R}^2$  (hence, in particular,  $\mathcal{R}_f = [0, +\infty)$ ). This partition is represented in the left plot of Figure 2 (gray lines), while the right plot is the corresponding two-dimensional histogram of the data set.

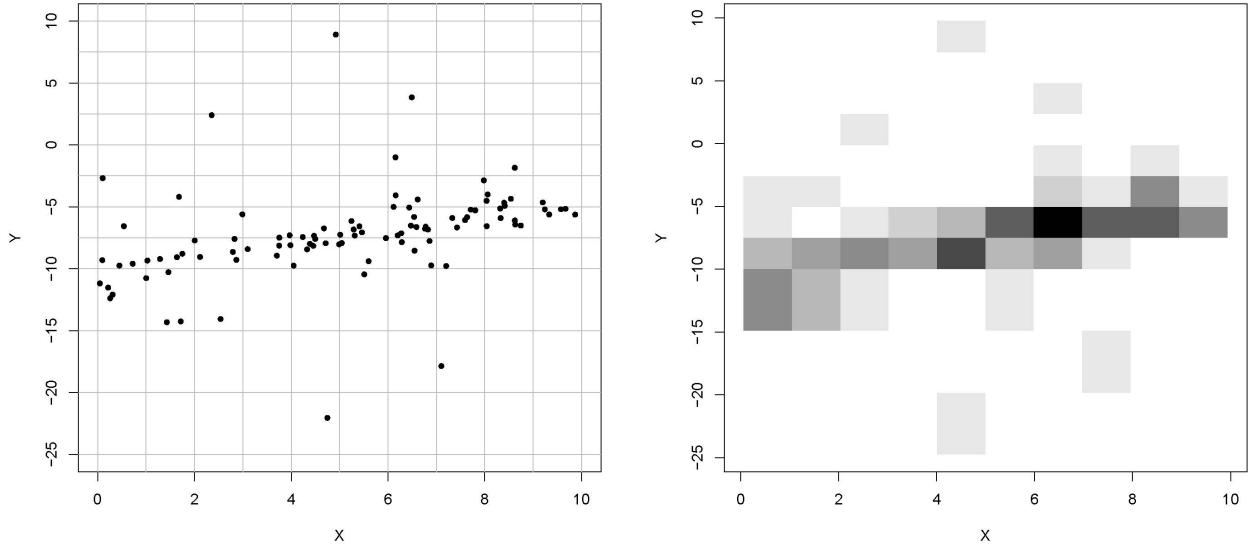


Figure 2: Data set from Examples 5, 6, 7, 8, and 9: (unobserved) precise data  $V_i = (X_i, Y_i) \in \mathbb{R}^2$  and partition of  $\mathbb{R}^2$  (left), and corresponding two-dimensional histogram representing the observed (imprecise) data  $V_i^* = X_i^* \times Y_i^* \subset \mathbb{R}^2$ .

### 3.1. Determination of Profile Likelihood Functions for Quantiles of Residuals

We want to determine the likelihood-based confidence regions for the quantiles of the distribution of the residuals: for this purpose, we calculate the profile likelihood function for such quantiles. Let  $p \in (0, 1)$ , and for each function



$f \in \mathcal{F}$ , let  $Q_f$  be the interval defined by  $Q_f = \mathcal{L}_f \cap \mathcal{U}_f$ , with

$$\mathcal{L}_f = \bigcup_{r \in \mathcal{R}_f} [r, +\infty)$$

when  $p > \varepsilon$  and  $\mathcal{L}_f = [0, +\infty)$  otherwise, while

$$\mathcal{U}_f = \bigcup_{r \in \mathcal{R}_f} [0, r]$$

when  $p < 1 - \varepsilon$  and  $\mathcal{U}_f = [0, +\infty)$  otherwise. The definition of  $Q_f$  can be interpreted as follows: if  $\varepsilon < p < 1 - \varepsilon$ , then  $Q_f$  is the smallest interval containing  $\mathcal{R}_f$ , while if  $p \leq \varepsilon$ , then this interval is extended to the left until 0 (included), and if  $p \geq 1 - \varepsilon$ , then it is extended to the right until  $+\infty$  (not included). Therefore,  $Q_f$  is the set of all possible values for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$ , since  $P(R_{f,i} \notin \mathcal{R}_f) \leq \varepsilon$  follows from assumption (1).

For each  $f \in \mathcal{F}$ , let  $Q_f$  be the multivalued mapping from  $\mathcal{P}$  to  $Q_f$  assigning to each probability measure  $P$  the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  under  $P$ . As noted in Subsection 2.2, the mapping  $Q_f$  is multivalued, because in general quantiles are not uniquely defined. We want to determine the profile likelihood function  $lik_{Q_f} : Q_f \rightarrow [0, 1]$  induced by the multivalued mapping  $Q_f$ . It is important to note that we would obtain the same results by considering only the distributions for which the  $p$ -quantile is unique (that is, the vagueness in the definition of quantiles has no influence on the resulting likelihood-based confidence regions).

Assume that the (imprecise) data  $V_1^* = A_1, \dots, V_n^* = A_n$  are observed, where  $A_1, \dots, A_n \in \mathcal{V}^* \setminus \{\emptyset\}$ . In order to obtain the profile likelihood function  $lik_{Q_f}$  for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$ , we define for each function  $f \in \mathcal{F}$  and each distance  $q \in [0, +\infty)$  the bands

$$\begin{aligned} \bar{B}_{f,q} &:= \{(x, y) \in \mathcal{V} : |y - f(x)| \leq q\}, \\ \underline{B}_{f,q} &:= \{(x, y) \in \mathcal{V} : |y - f(x)| < q\} \end{aligned}$$

and the functions  $\bar{k}_f, \underline{k}_f$  on  $[0, +\infty)$  such that

$$\begin{aligned} \bar{k}_f(q) &= \#\{i \in \{1, \dots, n\} : A_i \cap \bar{B}_{f,q} \neq \emptyset\}, \\ \underline{k}_f(q) &= \#\{i \in \{1, \dots, n\} : A_i \subseteq \underline{B}_{f,q}\} \end{aligned}$$

for all  $q \in [0, +\infty)$  (where  $\#A$  denotes the cardinality of a set  $A$ ). That is,  $\bar{k}_f(q)$  is the number of imprecise data intersecting  $\bar{B}_{f,q}$ , while  $\underline{k}_f(q)$  is the number of imprecise data completely contained in  $\underline{B}_{f,q}$ . Therefore, in particular,  $\bar{k}_f$  and  $\underline{k}_f$  are monotonically increasing functions of  $q$ , and  $\underline{k}_f(q) \leq \bar{k}_f(q)$  for all  $q \in [0, +\infty)$ . Finally, we define the function  $\lambda : [0, 1] \times (0, 1) \rightarrow (0, 1]$  as follows, for all  $s \in [0, 1]$  and all  $t \in (0, 1)$ :

$$\lambda(s, t) = \begin{cases} 1 - t & \text{if } s = 0, \\ \left(\frac{t}{s}\right)^s \left(\frac{1-t}{1-s}\right)^{1-s} & \text{if } 0 < s < 1, \\ t & \text{if } s = 1. \end{cases}$$

**Theorem 1.** For each  $f \in \mathcal{F}$ , the profile likelihood function  $lik_{Q_f}$  for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  can be expressed as follows, for all  $q \in Q_f$ :

$$lik_{Q_f}(q) = \begin{cases} \lambda\left(\frac{\bar{k}_f(q)}{n}, p - \varepsilon\right)^n & \text{if } \bar{k}_f(q) < (p - \varepsilon)n, \\ 1 & \text{if } [\underline{k}_f(q), \bar{k}_f(q)] \cap [(p - \varepsilon)n, (p + \varepsilon)n] \neq \emptyset, \\ \lambda\left(\frac{\underline{k}_f(q)}{n}, p + \varepsilon\right)^n & \text{if } \underline{k}_f(q) > (p + \varepsilon)n. \end{cases} \quad (2)$$

*Proof.* In order to prove expression (2), we use Lemma 2, which tells us that  $lik_{Q_f}(q) = lik_{Q_f^*}^*(q)$  for all  $q \in Q_f$ , where  $lik^*$  and  $Q_f^*$  are defined on the set  $\mathcal{P}_{V^*}$  of all possible distributions  $P_{V^*}$  for the imprecise data  $V_i^*$ . The function  $lik^*$  assigns to each  $P_{V^*}$  the corresponding likelihood value: in particular, it has a unique maximum in the empirical distribution (of the imprecise data)  $\hat{P}_{V^*}$ . The multivalued mapping  $Q_f^*$  assigns to each  $P_{V^*}$  all  $p$ -quantiles of the residuals  $R_{f,i}$  for all distributions of the precise data  $V_i$  compatible with  $P_{V^*}$ .

We first consider the empirical distribution (of the imprecise data)  $\hat{P}_{V^*}$ : we know that  $lik^*(\hat{P}_{V^*}) = 1$ , and we want to determine  $Q_f^*(\hat{P}_{V^*})$ . Each joint distribution of  $V_i$  and  $V_i^*$  with marginal distribution  $\hat{P}_{V^*}$  can be described by the conditional distributions of  $V_i$  given  $V_i^* = A_j$  (for each imprecise observation  $A_j$ ), since  $\hat{P}_{V^*}\{A_1, \dots, A_n\} = 1$ . In particular, for each  $q \in Q_f$  we can construct a joint distribution of  $V_i$  and  $V_i^*$  as follows: for each one of the  $\bar{k}_f(q) - \underline{k}_f(q)$  imprecise observations  $A_j$  such that  $(\bar{B}_{f,q} \setminus \underline{B}_{f,q}) \cap A_j \neq \emptyset$ , we can choose the conditional distribution of  $V_i$  given  $V_i^* = A_j$  to be concentrated on  $(\bar{B}_{f,q} \setminus \underline{B}_{f,q}) \cap A_j$ , while for all other imprecise observations  $A_j$ , we can choose the conditional distributions of  $V_i$  given  $V_i^* = A_j$  in such a way that as little probability as possible is given to  $\underline{B}_{f,q}$ , and as much as possible to  $\bar{B}_{f,q} \setminus \underline{B}_{f,q}$ , according to assumption (1). The resulting probability distribution satisfies

$$P(R_{f,i} < q) = P_V(\underline{B}_{f,q}) = \max\left(\frac{\underline{k}_f(q)}{n} - \varepsilon, 0\right) = \min_{P'_V \in [\hat{P}_{V^*}]} P'_V(\underline{B}_{f,q}),$$

$$P(R_{f,i} \leq q) = P_V(\bar{B}_{f,q}) = \min\left(\frac{\bar{k}_f(q)}{n} + \varepsilon, 1\right) = \max_{P'_V \in [\hat{P}_{V^*}]} P'_V(\bar{B}_{f,q}),$$

and therefore,

$$q \in Q_f^*(\hat{P}_{V^*}) \Leftrightarrow \frac{\underline{k}_f(q)}{n} - \varepsilon \leq p \leq \frac{\bar{k}_f(q)}{n} + \varepsilon \Leftrightarrow [\underline{k}_f(q), \bar{k}_f(q)] \cap [(p - \varepsilon)n, (p + \varepsilon)n] \neq \emptyset.$$

This proves the second case of expression (2).

We now prove the first case of expression (2), and thus assume that  $q \in Q_f$  satisfies  $\bar{k}_f(q) < (p - \varepsilon)n$ . If  $q$  is a  $p$ -quantile according to  $P \in \mathcal{P}$ , then  $P(V_i \in \bar{B}_{f,q}) = P(R_{f,i} \leq q) \geq p$ , and assumption (1) implies  $P(V_i \in V_i^* \cap \bar{B}_{f,q}) \geq p - \varepsilon$ . This is more than what the empirical distribution  $\hat{P}_{V^*}$  assigns to the  $\bar{k}_f(q)$  imprecise data intersecting  $\bar{B}_{f,q}$ , and it can be easily proved that all marginal distributions  $P_{V^*} \in \mathcal{P}_{V^*}$  maximizing  $lik^*$  among the ones satisfying  $q \in Q_f^*(P_{V^*})$  can be expressed as

$$P_{V^*} = (p - \varepsilon)P'_{V^*} + (1 - p + \varepsilon)P''_{V^*}, \quad (3)$$

where  $P'_{V^*} \in \mathcal{P}_{V^*}$  is the empirical distribution obtained when only the  $n - \bar{k}_f(q)$  imprecise data not intersecting  $\bar{B}_{f,q}$  are considered, and if  $\bar{k}_f(q) > 0$ , then  $P'_{V^*} \in \mathcal{P}_{V^*}$  is the empirical distribution obtained when only the  $\bar{k}_f(q)$  imprecise data intersecting  $\bar{B}_{f,q}$  are considered. In this case,  $P_{V^*}$  is unique, while if  $\bar{k}_f(q) = 0$ , then  $P'_{V^*} \in \mathcal{P}_{V^*}$  can be any distribution assigning the whole probability to elements of  $\mathcal{V}^*$  intersecting  $\bar{B}_{f,q}$ . Such elements of  $\mathcal{V}^*$  exist, because  $p > \varepsilon$  (since  $\bar{k}_f(q) < (p - \varepsilon)n$ ), and therefore the definition of  $Q_f$  implies that there is an  $r \in \mathcal{R}_f$  such that  $r \leq q$ . If  $\bar{k}_f(q) > 0$ , then for the unique marginal distribution  $P_{V^*}$  of the form (3) we have

$$lik^*(P_{V^*}) = \frac{\prod_{i=1}^n P_{V^*}\{A_i\}}{\prod_{i=1}^n \hat{P}_{V^*}\{A_i\}} = \frac{\left(\frac{p-\varepsilon}{\bar{k}_f(q)}\right)^{\bar{k}_f(q)} \left(\frac{1-p+\varepsilon}{n-\bar{k}_f(q)}\right)^{n-\bar{k}_f(q)}}{\left(\frac{1}{n}\right)^n} = \left(\frac{p-\varepsilon}{\bar{k}_f(q)}\right)^{\bar{k}_f(q)} \left(\frac{1-(p-\varepsilon)}{1-\frac{\bar{k}_f(q)}{n}}\right)^{n-\bar{k}_f(q)} = \lambda\left(\frac{\bar{k}_f(q)}{n}, p - \varepsilon\right)^n,$$

while if  $\bar{k}_f(q) = 0$ , then for all marginal distributions  $P_{V^*}$  of the form (3) we have

$$lik^*(P_{V^*}) = \frac{\prod_{i=1}^n P_{V^*}\{A_i\}}{\prod_{i=1}^n \hat{P}_{V^*}\{A_i\}} = \frac{\left(\frac{1-p+\varepsilon}{n}\right)^n}{\left(\frac{1}{n}\right)^n} = (1 - (p + \varepsilon))^n = \lambda(0, p - \varepsilon)^n.$$

These two expressions for  $lik^*(P_{V^*})$  are valid also when some of the imprecise observations  $A_1, \dots, A_n$  are equal, because in this case additional factors appear in the numerator as well as in the denominator of the fractions expressing the likelihood ratio of  $P_{V^*}$  and  $\hat{P}_{V^*}$  (see for instance also [27, Section 2.3]). This proves the first case of expression (2), the third one can be proved analogously.  $\square$

The expression for  $lik_{Q_f}$  given in Theorem 1 is very general, but rather involved. To obtain a simpler result about  $lik_{Q_f}$ , we define, for each function  $f \in \mathcal{F}$  and each imprecise data  $A_i$ , the infimum  $\underline{r}_{f,i}$  and the supremum  $\bar{r}_{f,i}$  of the set of all possible values of the residual  $R_{f,i}$  when  $V_i \in A_i$  (i.e., when the imprecise observation  $V_i^* = A_i$  is correct):

$$\begin{aligned}\underline{r}_{f,i} &= \inf_{(x,y) \in A_i} |y - f(x)|, \\ \bar{r}_{f,i} &= \sup_{(x,y) \in A_i} |y - f(x)|.\end{aligned}$$

As usual in statistics,  $\underline{r}_{f,(i)}$  and  $\bar{r}_{f,(i)}$  denote then the  $i$ th smallest infimum and supremum, respectively, so that with  $\underline{r}_{f,(0)} := \bar{r}_{f,(0)} := \inf Q_f$  and  $\underline{r}_{f,(n+1)} := \bar{r}_{f,(n+1)} := \sup Q_f$  we obtain  $\underline{r}_{f,(0)} \leq \dots \leq \underline{r}_{f,(n+1)}$  and  $\bar{r}_{f,(0)} \leq \dots \leq \bar{r}_{f,(n+1)}$ . Finally, we define  $\underline{i} := \max(\lceil (p - \varepsilon)n \rceil, 0)$  and  $\bar{i} := \min(\lfloor (p + \varepsilon)n \rfloor, n) + 1$ , so that  $\underline{i} \in \{0, \dots, n\}$  and  $\bar{i} \in \{1, \dots, n+1\}$ , with  $\underline{i} \leq \bar{i}$ .

**Lemma 3.** *The points of discontinuity of the restriction of  $\bar{k}_f$  to  $Q_f$ , including the endpoints of  $Q_f$ , are (in ascending order, with possible repetitions)  $\underline{r}_{f,(0)}, \dots, \underline{r}_{f,(n+1)}$ , and for all other values of  $q \in Q_f$  we have  $\bar{k}_f(q) = i$  if  $\underline{r}_{f,(i)} < q < \underline{r}_{f,(i+1)}$  with  $i \in \{0, \dots, n\}$ .*

*The points of discontinuity of the restriction of  $\underline{k}_f$  to  $Q_f$ , including the endpoints of  $Q_f$ , are (in ascending order, with possible repetitions)  $\bar{r}_{f,(0)}, \dots, \bar{r}_{f,(n+1)}$ , and for all other values of  $q \in Q_f$  we have  $\underline{k}_f(q) = i$  if  $\bar{r}_{f,(i)} < q < \bar{r}_{f,(i+1)}$  with  $i \in \{0, \dots, n\}$ .*

*Proof.* The points of discontinuity of the restrictions of  $\bar{k}_f, \underline{k}_f$  to  $Q_f$ , possibly including the endpoints of  $Q_f$ , are (for all imprecise data  $A_i$ )

$$\begin{aligned}\inf\{q \in Q_f : A_i \cap \bar{B}_{f,q} \neq \emptyset\} &= \inf\{q \in Q_f : \exists (x,y) \in A_i : |y - f(x)| \leq q\} = \inf\{|y - f(x)| : (x,y) \in A_i\} = \underline{r}_{f,i}, \\ \sup\{q \in Q_f : A_i \not\subseteq \underline{B}_{f,q}\} &= \sup\{q \in Q_f : \exists (x,y) \in A_i : |y - f(x)| \geq q\} = \sup\{|y - f(x)| : (x,y) \in A_i\} = \bar{r}_{f,i},\end{aligned}$$

respectively, because  $(x,y) \in A_i$  implies  $|y - f(x)| \in \mathcal{R}_f \subseteq Q_f$ . Hence, if  $\underline{r}_{f,(i)} < q < \underline{r}_{f,(i+1)}$  with  $i \in \{0, \dots, n\}$ , then there are exactly  $i$  imprecise data intersecting  $\bar{B}_{f,q}$  (i.e.,  $\bar{k}_f(q) = i$ ). Analogously, if  $\bar{r}_{f,(i)} < q < \bar{r}_{f,(i+1)}$  with  $i \in \{0, \dots, n\}$ , then there are exactly  $i$  imprecise data completely contained in  $\underline{B}_{f,q}$  (i.e.,  $\underline{k}_f(q) = i$ ).  $\square$

**Corollary 1.** *For each  $f \in \mathcal{F}$ , the profile likelihood function  $lik_{Q_f}$  for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  is a piecewise constant function, which can take at most  $n + 2$  different values.*

*The points of discontinuity of  $lik_{Q_f}$ , including the endpoints of  $Q_f$ , are (in ascending order, with possible repetitions)*

$$\underline{r}_{f,(0)}, \dots, \underline{r}_{f,(\underline{i})}, \bar{r}_{f,(\bar{i})}, \dots, \bar{r}_{f,(n+1)},$$

*and for all other values of  $q \in Q_f$ ,*

$$lik_{Q_f}(q) = \begin{cases} \lambda\left(\frac{i}{n}, p - \varepsilon\right)^n & \text{if } \underline{r}_{f,(i)} < q < \underline{r}_{f,(i+1)} \text{ with } i \in \{0, \dots, \underline{i} - 1\} \text{ (when } \underline{i} \geq 1), \\ 1 & \text{if } \underline{r}_{f,(\underline{i})} < q < \bar{r}_{f,(\bar{i})}, \\ \lambda\left(\frac{i}{n}, p + \varepsilon\right)^n & \text{if } \bar{r}_{f,(i)} < q < \bar{r}_{f,(i+1)} \text{ and } i \in \{\bar{i}, \dots, n\} \text{ (when } \bar{i} \leq n). \end{cases} \quad (4)$$

*Proof.* The function  $lik_{Q_f}$  can take at most  $n + 2$  different values, because in the first case of expression (2) the possible values of  $\bar{k}_f(q)$  are the integers  $k$  such that  $0 \leq k < (p - \varepsilon)n$ , while in the third case the possible values of  $\underline{k}_f(q)$  are the integers  $k$  such that  $(p + \varepsilon)n < k \leq n$  (hence, in these two cases taken together,  $lik_{Q_f}$  can take at most  $n + 1$  different values).

If  $\underline{i} \geq 1$ , then  $0 \leq \underline{i} - 1 < (p - \varepsilon)n$ , and if  $\bar{i} \leq n$ , then  $n \geq \bar{i} > (p + \varepsilon)n$ . Hence, expression (4) is well-defined, and in order to prove the second part of the corollary, it suffices to show that it holds for all  $q \in Q_f$ . This is easily done, since expression (4) is a direct consequence of Theorem 1 and Lemma 3. In the first case of expression (4), Lemma 3 implies  $\bar{k}_f(q) = i < (p - \varepsilon)n$ , in the third case it implies  $\underline{k}_f(q) = i > (p + \varepsilon)n$ , while in the second case it implies  $\bar{k}_f(q) \geq \underline{i} \geq (p - \varepsilon)n$  and  $\underline{k}_f(q) \leq \bar{i} - 1 \leq (p + \varepsilon)n$ .  $\square$

**Example 6.** In the problem of simple linear regression introduced in Example 5, let  $f \in \mathcal{F}$  be the linear function plotted in Figure 4 (left, solid line). In this situation, the sets of all possible values of the residuals  $R_{f,i}$  (when the imprecise observations  $V_i^* = X_i^* \times Y_i^*$  are correct) are intervals, and their endpoints  $\underline{r}_{f,i}, \bar{r}_{f,i}$  can be easily calculated. They can then be used in expression (4), which determines the values of the profile likelihood function  $\text{lik}_{Q_f}$  for the  $p$ -quantile of the distribution of  $R_{f,i}$  (apart in its points of discontinuity, including the endpoints of  $Q_f = \mathcal{R}_f = [0, +\infty)$ ). The function  $\text{lik}_{Q_f}$  with  $p = 0.75$  is plotted in Figure 3 for the cases with  $\varepsilon = 0$  (solid line) and  $\varepsilon = 0.1$  (dashed line).

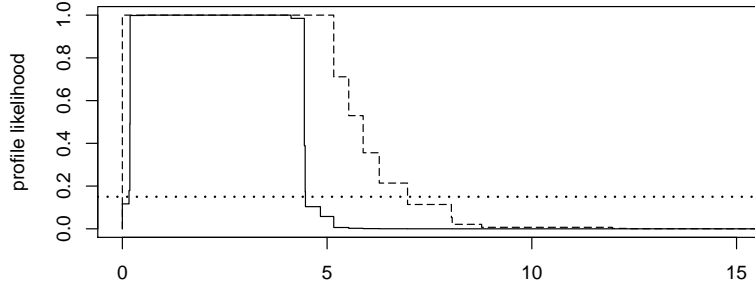


Figure 3: Profile likelihood functions from Examples 6 and 7.

### 3.2. Determination of Confidence Intervals for Quantiles of Residuals

Thanks to Theorem 1, we can now calculate, for each cutoff point  $\beta \in (0, 1)$ , the likelihood-based confidence regions for the quantiles of the distribution of the residuals  $R_{f,i}$ . We obtain in particular the following result.

**Corollary 2.** If  $\varepsilon$  is sufficiently small and  $n$  is sufficiently large so that

$$(\max\{p, 1 - p\} + \varepsilon)^n \leq \beta \quad (5)$$

holds, then

$$\begin{aligned} \underline{k} &:= \max \left\{ k \in \{0, \dots, \underline{i} - 1\} : \lambda\left(\frac{k}{n}, p - \varepsilon\right) \leq \sqrt[n]{\beta} \right\}, \\ \bar{k} &:= \min \left\{ k \in \{\bar{i}, \dots, n\} : \lambda\left(\frac{k}{n}, p + \varepsilon\right) \leq \sqrt[n]{\beta} \right\} \end{aligned}$$

are well-defined and satisfy

$$0 \leq \underline{k} < (p - \varepsilon)n \leq pn \leq (p + \varepsilon)n < \bar{k} \leq n,$$

and for each  $f \in \mathcal{F}$ , the likelihood-based confidence region with cutoff point  $\beta$  for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  is the nonempty interval

$$C_f := \left\{ q \in [0, +\infty) : [\underline{k}_f(q), \bar{k}_f(q)] \cap (\underline{k}, \bar{k}) \neq \emptyset \right\},$$

whose lower and upper endpoints are  $\underline{r}_{f,(\underline{k}+1)}$  and  $\bar{r}_{f,(\bar{k})}$ , respectively.

*Proof.* Assumption (5) implies in particular  $(p - \varepsilon)n > 0$  and  $\lambda(0, p - \varepsilon) = 1 - p + \varepsilon \leq \sqrt[n]{\beta}$ , and therefore  $\underline{k}$  is well-defined, since  $\underline{i} - 1 \geq 0$  and  $k = 0$  satisfies the condition of the maximum. Analogously,  $\bar{k}$  is well-defined, because  $\bar{i} \leq n$  and  $k = n$  satisfies the condition of the minimum, since  $(p + \varepsilon)n < n$  and  $\lambda(1, p + \varepsilon) = p + \varepsilon \leq \sqrt[n]{\beta}$  follow from assumption (5). The definitions of  $\underline{k}$  and  $\bar{k}$  imply in particular the inequalities  $0 \leq \underline{k} < (p - \varepsilon)n$  and  $(p + \varepsilon)n < \bar{k} \leq n$ .

We now prove that  $C_f$  is the likelihood-based confidence region with cutoff point  $\beta$  for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  (that is, for the values of the multivalued mapping  $Q_f$ ). From the definition of profile likelihood function given in Subsection 2.2 it follows that this confidence region is the set of all  $q \in Q_f$  such that  $\text{lik}_{Q_f}(q) > \beta$ . We can thus use Theorem 1 to determine the confidence region. It can be easily proved that for each  $t \in (0, 1)$  considered as a constant,  $\lambda$  is a continuous function of  $s \in [0, 1]$ , monotonically increasing on  $[0, t]$  and

monotonically decreasing on  $[t, 1]$ . Therefore, in the first case of expression (2) we have  $\text{lik}_{Q_f}(q) > \beta$  if and only if  $\bar{k}_f(q) > \bar{k}$ , while in the third case we have  $\text{lik}_{Q_f}(q) > \beta$  if and only if  $\underline{k}_f(q) < \bar{k}$ . Altogether, we obtain that  $\text{lik}_{Q_f}(q) > \beta$  if and only if  $[\underline{k}_f(q), \bar{k}_f(q)] \cap (\underline{k}, \bar{k}) \neq \emptyset$ , since  $[(p - \varepsilon)n, (p + \varepsilon)n] \subset (\underline{k}, \bar{k})$ . It remains to show that  $q \in C_f$  implies  $q \in Q_f$ . If  $q \in C_f$ , then  $\underline{k}_f(q) > 0$ , and so there is an  $r \in \mathcal{R}_f$  such that  $r \leq q$ . Analogously, if  $q \in C_f$ , then  $\bar{k}_f(q) < n$ , and so there is an  $r \in \mathcal{R}_f$  such that  $r \geq q$ . Hence,  $q \in C_f$  implies  $q \in Q_f$ , and therefore  $C_f$  is the desired confidence region.

The set  $C_f$  is an interval, since the functions  $\bar{k}_f, \underline{k}_f$  are monotonically increasing, and  $\underline{k}_f(q) \leq \bar{k}_f(q)$  for all  $q \in [0, +\infty)$ . Moreover, the definition of likelihood function implies that there is a probability measure  $P \in \mathcal{P}$  such that  $\text{lik}(P) > \beta$ , and therefore  $C_f$  is not empty, because  $Q_f(P) \subseteq C_f$  follows from the definition of likelihood-based confidence region. Finally, Lemma 3 implies

$$\begin{aligned} \inf C_f &= \inf \{q \in [0, +\infty) : \bar{k}_f(q) > \underline{k}\} = \underline{r}_{f,(\underline{k}+1)}, \\ \sup C_f &= \sup \{q \in [0, +\infty) : \underline{k}_f(q) < \bar{k}\} = \bar{r}_{f,(\bar{k})}, \end{aligned}$$

since  $\bar{k}_f(q) = n$  for all  $q \in [0, +\infty) \setminus \mathcal{U}_f$ , and  $\underline{k}_f(q) = 0$  for all  $q \in [0, +\infty) \setminus \mathcal{L}_f$ .  $\square$

The interval  $C_f$  defined in Corollary 2 consists of all  $q \in [0, +\infty)$  such that the band  $\bar{B}_{f,q}$  intersects at least  $\underline{k} + 1$  imprecise data, and the band  $\underline{B}_{f,q}$  contains at most  $\bar{k} - 1$  imprecise data. When  $\varepsilon = 0$ , the interval  $C_f$  is asymptotically a (conservative) confidence interval of level  $F_{\chi^2}(-2 \log \beta)$  for the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$ , where  $F_{\chi^2}$  is the cumulative distribution function of the chi-square distribution with 1 degree of freedom (see for example [27]). The finite-sample level of the (conservative) confidence interval  $C_f$  can be obtained directly from its definition, by means of simple combinatorial arguments (also when  $\varepsilon > 0$ ), but this goes beyond the scope of the present paper.

It is important to note that the confidence intervals  $C_f$  do not depend on the choice of the set  $\mathcal{V}^*$  of possible imprecise data (as far as the observed ones,  $A_1, \dots, A_n$ , are contained in it). This can be surprising, since the set  $\mathcal{P} = \mathcal{P}_\varepsilon$  of probability measures considered depends strongly on  $\mathcal{V}^*$ , as noted at the beginning of Section 2. However, the independence of the confidence intervals  $C_f$  from the choice of the set  $\mathcal{V}^*$  is not so surprising when one considers that the intervals  $C_f$  are likelihood-based confidence regions, and that likelihood inference is always conditional on the data (that is, independent of considerations about which other data could have been observed). This can be considered as a sort of robustness against misspecification of the set  $\mathcal{V}^*$  of possible imprecise data. The practical advantage is that it is not necessary to think about which other imprecise data could have been observed, besides the ones that were actually observed (that is,  $A_1, \dots, A_n$ ).

**Example 7.** In the situation of Example 6, the confidence interval  $C_f$  with  $\beta = 0.15$  is approximately  $[0.16, 4.47]$  when  $\varepsilon = 0$ , and  $[0, 6.96]$  when  $\varepsilon = 0.1$  (the cutoff point  $\beta = 0.15$  is represented by the dotted line in Figure 3).

### 3.3. Regression as a Decision Problem

The problem of minimizing the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  can be described as a statistical decision problem: the set of probability measures considered is  $\mathcal{P} = \mathcal{P}_\varepsilon$ , the set of possible decisions is  $\mathcal{F}$ , and the loss function  $L : \mathcal{P} \times \mathcal{F} \rightarrow [0, \infty)$  is defined by

$$L(P, f) = Q_f(P)$$

for all  $P \in \mathcal{P}$  and all  $f \in \mathcal{F}$ . That is, the  $p$ -quantile of the distribution of the residuals  $R_{f,i}$  is interpreted as the loss we incur when we choose the function  $f$ . In fact, the loss function  $L$  is multivalued, since in general the  $p$ -quantile is not unique:  $L(P, f)$  could be reduced to a single value by taking for example the upper  $p$ -quantile of the distribution of the residuals  $R_{f,i}$ .

The information provided by the observed (imprecise) data is described by the likelihood function  $\text{lik}$  on  $\mathcal{P}$ . A very simple way of using this information consists in reducing  $\mathcal{P}$  to the set  $\mathcal{P}_{>\beta}$  for some cutoff point  $\beta \in (0, 1)$ . The resulting set  $\mathcal{P}_{>\beta}$  can be interpreted as an imprecise probability measure, on which we can base our choice of  $f$ . For each  $f \in \mathcal{F}$ , the set of all possible values of the loss  $L(P, f)$  when  $P$  varies in  $\mathcal{P}_{>\beta}$  can be interpreted as the imprecise  $p$ -quantile of the residuals  $R_{f,i}$  under the imprecise probability measure  $\mathcal{P}_{>\beta}$ . It corresponds to the interval  $C_f$ , when condition (5) is satisfied.

Assume that condition (5) is satisfied. In order to choose a function  $f$ , we can minimize the supremum of  $C_f$ . This approach is similar to the  $\Gamma$ -minimax decision criterion with respect to the imprecise probability measure  $\mathcal{P}_{>\beta}$ , and is called LRM (Likelihood-based Region Minimax) criterion in [7]. When there is a unique  $f \in \mathcal{F}$  minimizing  $\sup C_f$  (i.e., minimizing  $\bar{r}_{f,(\bar{k})}$ ), it can be denoted by  $f_{LRM}$ , and  $\sup C_f$  can be denoted by  $\bar{q}_{LRM}$ . In this case,  $f_{LRM}$  is characterized geometrically by the fact that  $\bar{B}_{f_{LRM}, \bar{q}_{LRM}}$  is the thinnest band of the form  $B_{f,q}$  containing at least  $\bar{k}$  imprecise data, for all  $f \in \mathcal{F}$  and all  $q \in [0, +\infty)$ . Therefore, in order to find the function  $f_{LRM}$ , it suffices to adapt to the case of imprecise data the algorithms for the method of least quantile of squares (see for example [29, 36, 3]), but this goes beyond the scope of the present paper.

An interesting description of the uncertainty about the optimal choice of  $f \in \mathcal{F}$  is obtained by considering interval dominance for the imprecise  $p$ -quantiles of the residuals  $R_{f,i}$  under the imprecise probability measure  $\mathcal{P}_{>\beta}$ . When  $f_{LRM}$  exists, the undominated functions  $f \in \mathcal{F}$  are those such that  $C_f$  intersects  $C_{f_{LRM}}$ . In particular, when  $\bar{q}_{LRM} \in C_{f_{LRM}}$  (that is,  $C_{f_{LRM}}$  is right-closed), the undominated functions  $f \in \mathcal{F}$  are characterized geometrically by the fact that  $\bar{B}_{f, \bar{q}_{LRM}}$  intersects at least  $\bar{k} + 1$  imprecise data. In general, the set of undominated functions  $f$  can be interpreted as the imprecise result of the regression: it describes the complex uncertainty about the optimal choice of  $f \in \mathcal{F}$ . When we observe more and more (imprecise) data, the statistical uncertainty diminishes, but the set of undominated functions does not necessarily tend to reduce to a singleton, because of the (unavoidable) indetermination discussed in Subsection 2.1 (see also [10] for a more detailed analysis).

**Example 8.** Consider the problem of simple linear regression introduced in Example 5, with  $\varepsilon = 0$  (that is, the classification of the precise data into the elements of  $\mathcal{V}^*$  is assumed to be correct),  $p = 0.75$ , and  $\beta = 0.15$ . The thinnest band of the form  $\bar{B}_{f,q}$  (for all  $f \in \mathcal{F}$  and all  $q \in [0, +\infty)$ ) containing at least  $\bar{k} = 83$  imprecise data is represented by the dashed lines in the left plot of Figure 4. It is the band  $\bar{B}_{f_{LRM}, \bar{q}_{LRM}}$ , where  $f_{LRM}$  is also plotted in Figure 4 (left, solid line), while  $\bar{q}_{LRM} = \sup C_{f_{LRM}} = \bar{r}_{f_{LRM}, (83)} \approx 4.47$ , as we have seen in Example 7. The right plot of Figure 4 shows the undominated functions  $f \in \mathcal{F}$  (gray lines), which are characterized by the fact that the band  $\bar{B}_{f, 4.47}$  intersects at least  $\bar{k} + 1 = 67$  imprecise data.

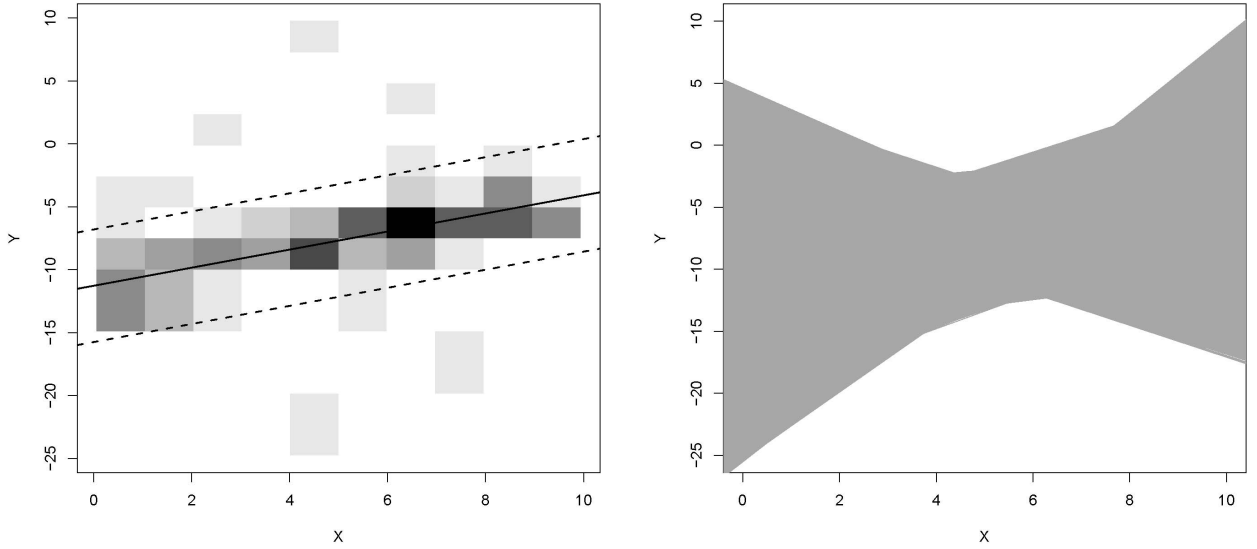


Figure 4: Function  $f_{LRM}$  (left, solid line), band  $\bar{B}_{f_{LRM}, \bar{q}_{LRM}}$  (left, dashed lines), and set of undominated functions (right, gray lines) from Example 8.

### 3.4. Prediction

Consider the case in which (instead of  $n$ ) we have  $n + 1$  pairs  $(V_i, V_i^*)$  of precise and imprecise data  $V_i = (X_i, Y_i)$  and  $V_i^*$ , respectively. We want to predict the realization of the precise data value  $V_{n+1}$  on the basis of the realization of the  $n$  imprecise data  $V_1^*, \dots, V_n^*$ . Choose  $k \in \{1, \dots, n\}$ , and assume that for each possible realization of the  $n + 1$  imprecise data  $V_1^*, \dots, V_{n+1}^*$ , there is a distance  $q' \in [0, +\infty)$  such that for some  $f' \in \mathcal{F}$  (not necessarily unique),  $\bar{B}_{f', q'}$

is a thinnest band of the form  $\bar{B}_{f,q}$  containing at least  $k$  of the  $n + 1$  imprecise data, for all  $f \in \mathcal{F}$  and all  $q \in [0, +\infty)$ . Because of symmetry, the probability that  $V_{n+1}^*$  is included in a band  $\bar{B}_{f,q'}$  containing at least  $k$  of the  $n + 1$  imprecise data (for some  $f \in \mathcal{F}$ ) is at least  $k/(n+1)$ . Hence, when  $\bar{B}_{f'',q''}$  is a thinnest band of the form  $\bar{B}_{f,q}$  containing at least  $k$  of the  $n$  imprecise data  $V_1^*, \dots, V_n^*$  (for all  $f \in \mathcal{F}$  and all  $q \in [0, +\infty)$ ), the probability that  $V_{n+1}^*$  is included in the union  $\mathcal{B}$  of all bands  $\bar{B}_{f,q''}$  containing at least  $k - 1$  of the  $n$  imprecise data  $V_1^*, \dots, V_n^*$  (for all  $f \in \mathcal{F}$ ) is at least  $k/(n+1)$ . That is,  $\mathcal{B}$  is a (conservative) prediction region of level  $k/(n+1) - \varepsilon$  for the precise data value  $V_{n+1}$ .

In particular, when condition (5) is satisfied and  $f_{LRM}$  exists, the union  $\mathcal{B}$  of all bands  $\bar{B}_{f,\bar{q}_{LRM}}$  containing at least  $\bar{k} - 1$  of the  $n$  imprecise data  $V_1^*, \dots, V_n^*$  (for all  $f \in \mathcal{F}$ ) is a (conservative) prediction region of level  $\bar{k}/(n+1) - \varepsilon$  for the precise data value  $V_{n+1}$ . Prediction regions of this form can sometimes be reduced to smaller regions thanks to the assumption that  $V_{n+1}^*$  takes values in  $\mathcal{V}^*$ . When besides the realization of the  $n$  imprecise data  $V_1^*, \dots, V_n^*$ , also the (precise or imprecise) realization of  $X_{n+1}$  has been observed, the realization of  $Y_{n+1}$  can be predicted for example by using the idea of conformal prediction (see [35]), but this goes beyond the scope of the present paper.

**Example 9.** In the situation of Example 8, the union  $\mathcal{B}$  of all bands  $\bar{B}_{f,4.47}$  containing at least 82 imprecise data (for all  $f \in \mathcal{F}$ ) corresponds to the region between the two curves in Figure 5. It is a (conservative) prediction region of level  $\bar{k}/(n+1) - \varepsilon = 0.83$  for a future precise data point.

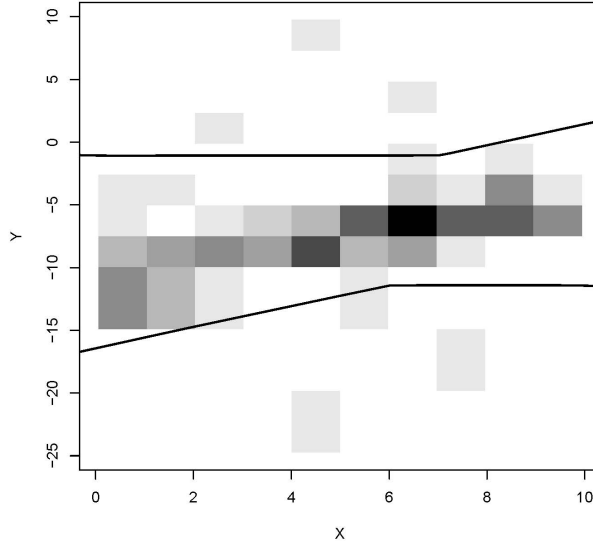


Figure 5: Prediction region from Example 9.

#### 4. Example of Application

In this section, we apply the proposed regression method to socioeconomic data from the ALLBUS (German General Social Survey). Data collection in surveys is subject to many different influences that may cause various biases in the data set (see for example [4]). Therefore, it is often reasonable to assume that the actual value lies rather in some interval around the observed value. Furthermore, data on sensitive quantities is sometimes only available in categories that form a partition of the space of possible values. A simple, ad hoc approach to analyze this kind of data is to reduce the intervals to their central values and to apply usual regression methods to the reduced, precise data. However, such an approach in general produces biased results (see [31, 2, 12]). In contrast to this, we suggest to analyze directly the interval-valued data by means of the regression method proposed in Section 3.

Here, we investigate how personal income varies with age, which is a fundamental relationship in the social sciences and a typical example in textbooks on social research methods (see for example [1]). Income is a key demographic variable in socioeconomic research questions, but asking for income in an interview is a sensitive question that some respondents refuse to answer. Therefore, survey data on personal income often include missing values. One

way to make the question less sensitive and thus to obtain better response rates is to present predefined income categories (forming a partition of the range of possible income values) to the respondent according to which the personal income shall be classified. In the ALLBUS study, income data is collected by means of a two-step design with the open question for income as first step and the presentation of a category scheme as second step. As a result, the data set contains at the same time precise values for some individuals and interval-valued observations for others. Yet, even if the respondents answer the open question, they usually give only a rough estimate of their exact income, like a rounded or a heaped income value (see [19]), where heaping refers to irregular rounding behavior (see for example [20]). Therefore, it is more reliable to regard also the precise income values as intervals, e.g. in considering as actual observations the income classes in which the precise values lie.

Data on the age of the respondents are more easily obtained, but these data are usually of limited precision. Often, the age is measured on a discrete scale, i.e.  $age \in \mathbb{N}$ . In that case, the information contained in a data value is that the actual age of the respondent lies in the interval  $[age, age + 1)$ . Furthermore, also age data are sometimes provided as a categorical variable taking values in a set of disjoint age classes forming a partition of the observation space of the continuous variable  $age$ .

#### 4.1. ALLBUS Data and Regression Model

We analyze the ALLBUS data set of 2008 containing 3 469 interviews. The considered variables are *personal income* (on average per month in euros) and *age*. For our analysis we use the categorized income variable  $v389$  with 22 possible income classes and the discrete age variable  $v154$ . (Detailed information about the data set can be found in [34].) Although the data set contains 1 063 precise income values and our regression method could also be applied to a data set with some precise and some imprecise observations (see [10]), we prefer to use the categorized income variable for the reasons mentioned above. Moreover, the age data are interpreted as intervals of length 1. Thus, for each individual  $i \in \{1, \dots, 3\,469\}$  we consider observations  $V_i^* = X_i^* \times Y_i^*$ , where  $X_i^* = [age_i, age_i + 1)$  is the interval covering the age of respondent  $i$  and  $Y_i^* = [y_i, \bar{y}_i)$  is the interval of the corresponding income category. In the given data set, there are 682 missing income values and 12 missing age values. Missing values are replaced by the entire observation space of each variable, i.e.  $X_i^* = \mathcal{X} := [18, 100)$  or  $Y_i^* = \mathcal{Y} := [0, +\infty)$ , respectively. A two-dimensional histogram of the data set is given in Figure 7 (left).

The relationship between *age* and *income* is usually modeled by a quadratic function in *age* (see for example [1]). Thus, the set of regression functions we consider here is  $\mathcal{F} = \{f_{a,b_1,b_2} : a, b_1, b_2 \in \mathbb{R}\}$ , where each function  $f_{a,b_1,b_2}$  is defined by  $f_{a,b_1,b_2}(x) = a + b_1 x + b_2 x^2$  for all  $x \in \mathcal{X}$ . We choose to minimize the median of the distribution of the absolute residuals (i.e.,  $p = 0.5$ ), which is the choice of  $p$  implying the most robust results (see the beginning of Section 3). As regards the cutoff point of the likelihood, we use a very high value:  $\beta = 0.9999$ . This choice of  $\beta$  corresponds to the special case of LIR where we consider maximum likelihood (ML) estimates to evaluate the regression functions  $f_{a,b_1,b_2} \in \mathcal{F}$  (i.e.,  $\underline{k} = \underline{i} - 1$  and  $\bar{k} = \bar{i}$ ). Note that in the present analysis the ML estimates  $C_{f_{a,b_1,b_2}}$  of the median of the absolute residuals are intervals, since the analyzed data set consists of proper sets (implying  $\underline{r}_{f,i} < \bar{r}_{f,i}$  for each imprecise observation  $A_i = X_i^* \times Y_i^*$ ). Choosing the ML intervals means to ignore the statistical uncertainty of the regression problem. A lower cutoff point  $\beta$  would imply a higher confidence level of the intervals  $C_{f_{a,b_1,b_2}}$  and lead to a more imprecise result. In the present analysis, the resulting set of undominated functions would change only a little, because there is not much statistical uncertainty given the relatively large number of observations. Finally, as we consider only the income classes, we assume that the imprecise observations are correct and set  $\varepsilon = 0$ . The effect of different choices of  $\beta$  and  $\varepsilon$  on the result of a LIR analysis has been studied thoroughly in [10].

For the present regression problem, we have implemented the LIR analysis as a grid search over the parameter space  $\mathbb{R}^3$ : First, the likelihood-based confidence regions  $C_{f_{a,b_1,b_2}}$  are computed for all regression functions corresponding to the parameter values  $(a, b_1, b_2)$  on a predefined grid. Then, we identify the parameter combination among these that minimizes the upper bound of  $C_{f_{a,b_1,b_2}}$ . The function corresponding to this parameter combination is the function  $f_{LRM}$  which is optimal according to the LRM criterion (see Subsection 3.3). Finally, the upper bound  $\bar{q}_{LRM}$  of  $C_{f_{LRM}}$  is used to determine the set of undominated regression functions.

#### 4.2. Results

We considered a grid of combinations of parameter values where  $a \in [-10\,000, 12\,000]$ ,  $b_1 \in [-200, 250]$ , and  $b_2 \in [-10, 10]$ . Corresponding to the set of undominated functions, we find the set of undominated parameter



combinations displayed in Figure 6. This set is clearly not convex. Moreover, in the case considered here, the parameters are not independent from each other, in the sense that many different combinations of parameter values  $(a, b_1, b_2)$  may lead to very similar shapes of  $f_{a,b_1,b_2}$  over  $\mathcal{X}$ . Thus, there are actually infinitely many undominated parameter combinations, but the associated curves are similar to those we find within the considered grid.

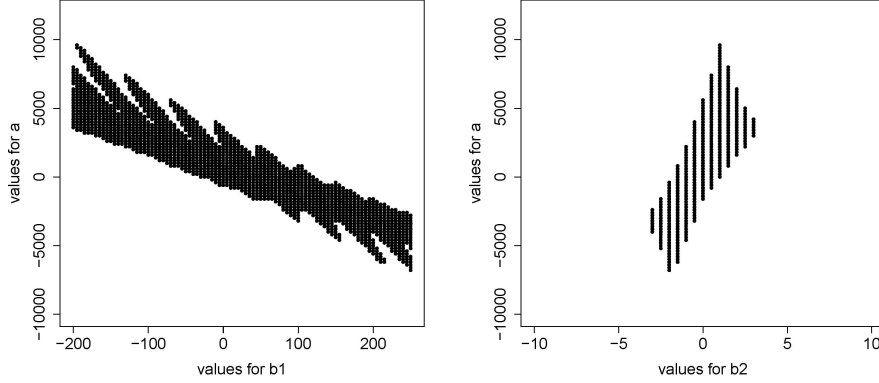


Figure 6: Two-dimensional projections of the set of undominated parameter values.

The parameter combination implying the smallest upper endpoint of the ML interval for the 0.5-quantile of the residuals is  $(600, 5, 0)$  with  $C_{f_{600,5,0}} \approx [270, 680]$ . Thus, the function  $f_{LRM}$  is a slightly increasing line. One interpretation of this function is given by the band  $\bar{B}_{f_{LRM}, \bar{q}_{LRM}}$  limited by the functions  $f_{LRM} - \bar{q}_{LRM}$  and  $f_{LRM} + \bar{q}_{LRM}$ : Among all bands (of any width) constructed around all considered functions,  $\bar{B}_{f_{LRM}, \bar{q}_{LRM}}$  is the thinnest one that contains at least  $\bar{k} = 1\,735$  imprecise observations.

The function  $f_{LRM}$  and the band  $\bar{B}_{f_{LRM}, \bar{q}_{LRM}}$  are presented in Figure 7 (right, black lines), besides the undominated functions (right, gray curves). As we considered ML estimates, no statistical uncertainty is reflected in the regression's result, thus, the extent of the set of undominated functions is only due to the imprecision of the data. It can be seen that among the undominated functions there is a large variety of shapes of the *age-income* profile, including straight lines, convex parabolic curves as well as concave ones. From a social scientist's point of view this result may be unsatisfying because it does not support only one form of the relationship between *age* and *income*. However, it is reasonable to consider all shapes consistent with the imprecise data as possible *age-income* profiles. If the observed intervals were overlapping or if they constituted a finer partition of the space of possible observations, the set of undominated functions would be smaller. The effect of different degrees of imprecision of the data on the regression's result was studied in [10], where different versions of the ALLBUS data set were analyzed and their results compared. In the present analysis, the set of undominated functions can be interpreted as the set of all plausible descriptions of the *age-income* profile that reflects at the same time the indetermination induced by the imprecise data.

The common, simple method to analyze this kind of interval data is to conduct a quadratic least squares (LS) regression based on the interval centers ignoring the indetermination induced by the imprecision of the data. In this case, an upper limit for the highest income class  $[7\,500, +\infty)$  has to be set in order to compute the interval centers. Of course, the choice of this upper limit has an impact on the estimates of the LS regression. The effect of two different choices of the upper income limit is illustrated in Figure 8 (black dashed curves). The LS curves displayed there are based on interval centers with upper income limits 10 000 and 50 000, respectively. In contrast to the LS regression based on the interval midpoints, the regression method proposed in this paper can also be applied to unbounded data. Since in the LIR method the evaluation of the regression functions is based on quantiles of the distribution of the absolute residuals, the result is not sensitive to the extremes. If there were less than  $\bar{k}$  bounded data, e.g. if there were more than 50% missing observations in the present data set, the result would be the entire set  $\mathcal{F}$  of considered regression functions.

An improvement of the approach of an LS regression based on the interval centers could be achieved by considering a robust variant of this approach, in which least median of squares (LMS) estimation is used. In this case, an upper income limit has to be fixed, but the estimated regression function is insensitive to the choice of the extreme values, since the regression is based on the median of the absolute residuals. The LMS curve estimated on the basis of

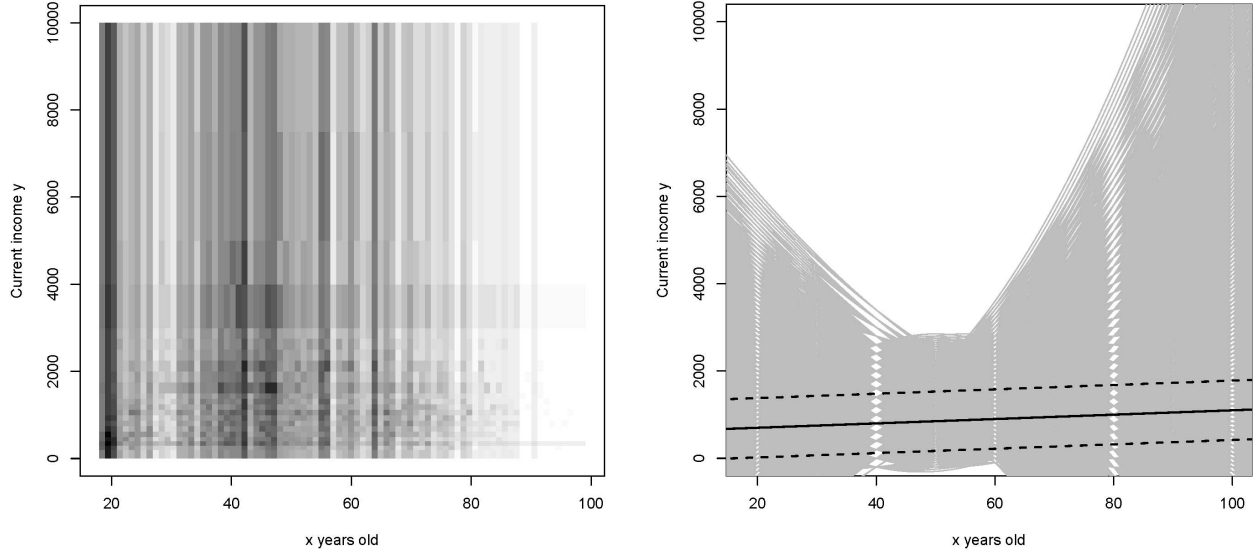


Figure 7: Two-dimensional histogram of the analyzed data set (left) and set of undominated functions (right, gray curves), minimax function  $f_{LRM}$  (right, black solid line) and band  $B_{f_{LRM}, \bar{q}_{LRM}}$  (right, black dashed lines).

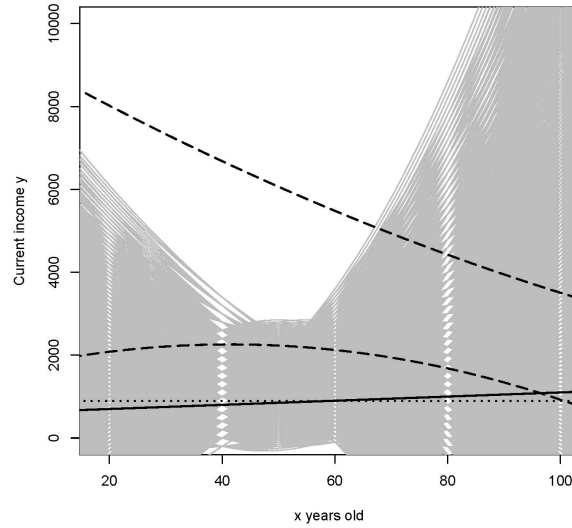


Figure 8: Set of undominated functions (gray curves) and  $f_{LRM}$  (black solid line) of the LIR analysis versus LS curves based on interval centers with upper limits 10 000 (lower black dashed curve) and 50 000 (upper black dashed curve), respectively, and LMS curve (black dotted line) based on interval centers with upper limit 50 000.

the interval centers with upper income limit 50 000 (black dotted line) and the function  $f_{LRM}$  obtained from the LIR analysis (black solid line) are also shown in Figure 8. These lines are similar to each other, which is not surprising as the proposed regression method can be seen as a generalization of the LMS regression to the case of imprecise data.

The proposed LIR method permits to identify plausible descriptions of the relationship between the socio-economic characteristics *age* and *income*. Given the imprecise data, many different shapes of the age-income profile are plausible. Further computations indicated that our findings hold for transformed income data on the logarithmic scale, too. The results are not very informative, but reflect the indetermination induced by the imprecision of the data. One idea to obtain more informative results from categorized data could be to use many different category schemes during the income data collection and thereby obtain a data set with overlapping categories.

## 5. Conclusion

In this paper, we introduced a new regression method for imprecise data, in which the error distribution is not constrained to a particular parametric family. The regression method is very robust and can be adapted to a wide range of practical settings, since it can be applied to all kinds of imprecise data, covering e.g. interval data, precise data, and missing data. In our method, the imprecise data are interpreted as the result of a coarsening process which can be informative, and even wrong with a certain probability.

The proposed method is derived from a general approach to regression with imprecise data, which we call Likelihood-based Imprecise Regression. It consists in identifying by means of likelihood inference all sufficiently plausible regression curves, which are considered as the imprecise result of the regression analysis. The extent of the imprecise result reflects both kinds of uncertainty involved in a regression problem with imprecise data: statistical uncertainty and indetermination.

In future work, we intend to improve the implementation of our regression method, and to study its statistical properties in more detail. Moreover, we plan to investigate the consequences of stronger assumptions about the error distribution and the coarsening process, and the possibility of replacing in the decision problem the quantiles of the residuals by other loss functions.

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