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Cwikel–Lieb–Rozenblum type inequalities for Hardy–Schrödinger operator

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A B S T R A C T

We prove a Cwikel–Lieb–Rozenblum type inequality for the number of negative eigenvalues of the Hardy–Schrödinger operator $-\Delta - (d-2)^2/(4|x|^2) - W(x)$ on $L^2(\mathbb{R}^d)$. The bound is given in terms of a weighted $L^{d/2}$-norm of W which is sharp in both large and small coupling regimes. We also obtain a similar bound for the fractional Laplacian.

R É S U M É

Nous prouvons une inégalité de type Cwikel–Lieb–Rozenblum pour le nombre de valeurs propres négatives de l'opérateur de Hardy–Schrödinger $-\Delta - (d-2)^2/(4|x|^2) - W(x)$ sur $L^2(\mathbb{R}^d)$. La borne est donnée en termes d'une norme pondérée $L^{d/2}$ de W, qui est optimale à la fois dans les régimes de couplage fort et faible. Nous obtenons également une borne similaire pour le Laplacien fractionnaire.

1. Introduction and main results

The celebrated Cwikel–Lieb–Rozenblum (CLR) inequality [3,18,26] states that for all dimensions $d \geq 3$, the number of negative eigenvalues of the Schrödinger operator $-\Delta - V$ on $L^2(\mathbb{R}^d)$, with a real-valued potential $V \in L^{d/2}(\mathbb{R}^d)$, satisfies

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$$N(0,-\Delta-V) \lesssim_d \int_{\mathbb{R}^d} V(x)_+^{d/2}\mathrm{d}x \tag{1}$$

where $V(x)_+ = \max(V(x),0)$. Here the notation $\lesssim_d$ means that the implicit constant on the right hand side depends only on the dimension d. In particular, since $N(0,-\Delta-V)$ is always an integer, (1) implies that $N(0,-\Delta-V)=0$ if $\|V_+\|_{L^{d/2}(\mathbb{R}^d)}$ is small enough, which can be deduced from Sobolev's inequality

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\mathrm{d}x \geq S_d \left(\int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2}}\mathrm{d}x\right)^{\frac{d-2}{d}} \tag{2}$$

via the duality argument

$$\inf_{\|V_+\|_{L^{d/2}(\mathbb{R}^d)}\leq S_d} \langle u,(-\Delta-V)u\rangle = \|\nabla u\|^2_{L^2(\mathbb{R}^d)} - S_d\|u\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \geq 0. \tag{3}$$

However, the CLR inequality (1) is much deeper than Sobolev's inequality since it captures correctly the semiclassical behavior which is usually described by Weyl's law in the large coupling regime

$$\begin{aligned} N(0,-\Delta-\lambda V) &= \frac{1}{(2\pi)^d}|\{(p,x)\in\mathbb{R}^d\times\mathbb{R}^d : |p|^2-\lambda V(x)<0\}| + o(\lambda^{d/2})_{\lambda\to\infty} \\ &= \frac{|B|}{(2\pi)^d}\int_{\mathbb{R}^d}(\lambda V(x))_+^{d/2}\mathrm{d}x + o(\lambda^{d/2})_{\lambda\to\infty} \end{aligned} \tag{4}$$

where $|B|$ is the volume of the unit ball $B=\{x\in\mathbb{R}^d : |x|<1\}$. We refer to [8] for a textbook introduction to (1), (4) and related estimates.

In the present paper, we are interested in potentials of the form

$$V(x) = \frac{(d-2)^2}{4|x|^2} + W(x), \quad W \in L^{\frac{d}{2}}(\mathbb{R}^d)$$

where the singular part comes from Hardy's inequality

$$\mathcal{L} = -\Delta - \frac{(d-2)^2}{4|x|^2} \geq 0 \qquad \text{on } L^2(\mathbb{R}^d). \tag{5}$$

It was proved in [4] that if the Hardy–Schrödinger operator $\mathcal{L}-W(x)$ has negative eigenvalues $\{E_n\}_{n\geq1}$, then

$$\sum_{n\geq1}|E_n|^\gamma \lesssim_{\gamma,d} \int_{\mathbb{R}^d} W(x)_+^{\gamma+d/2}\mathrm{d}x \tag{6}$$

for all $d\geq3$ and $\gamma>0$. The Hardy–Lieb–Thirring inequality (6) is an improvement over standard Lieb-Thirring inequalities [20,21] concerning similar estimates for the Schrödinger operator $-\Delta-W$.

On the other hand, it is well-known that (6) does not hold for $\gamma=0$ [4]. In fact, even the corresponding Sobolev inequality does not hold with $\|u\|^2_{\dot{H}^1}$ replaced by $\langle u,\mathcal{L}u\rangle$. However, there is a remarkable replacement for the Sobolev inequality in the restricted case where $\mathbb{R}^d$ is replaced by the unit ball B. To be precise, it was proved by Filippas–Tertikas in [12] (see also Musina's remarks in [23]) that

$$\langle u,\mathcal{L}u\rangle \gtrsim_d \left(\int_B \frac{|u(x)|^{\frac{2d}{d-2}}}{(1+|\ln|x||)^{1+\frac{d}{d-2}}}\mathrm{d}x\right)^{\frac{d-2}{d}}, \quad u\in C_c^\infty(B) \tag{7}$$

where the power of the logarithmic weight is optimal. By a duality argument similar to (3), the Hardy–Sobolev inequality (7) is equivalent to the fact that $\mathcal{L} - W \geq 0$ on $L^2(B)$, as quadratic forms with Dirichlet boundary conditions, if

$$\int_B W(x)_+^{\frac{d}{2}} (1 + |\ln|x||)^{d-1} \mathrm{d}x$$

is sufficiently small.

Our first new result is an extension of the above Hardy–Sobolev inequality concerning the number of negative eigenvalues of $\mathcal{L} - W$ on $L^2(\mathbb{R}^d)$.

Theorem 1 *(CLR type bound for Hardy–Schrödinger operator). For every dimension $d \geq 3$, there exists a constant $C_d > 0$ independent of the real-valued potential W such that*

$$N(0, \mathcal{L} - W) \leq 1 + C_d \int_{\mathbb{R}^d} W(x)_+^{\frac{d}{2}} (1 + |\ln|x||)^{d-1} \mathrm{d}x \,. \tag{8}$$

Here when the right-hand side of (8) is finite, $\mathcal{L} - W$ is bounded from below with the core domain $C_c^\infty(\mathbb{R}^d \backslash \{0\})$ and extended to be a self-adjoint operator by Friedrichs' method.

Remark 2. The number 1 on the right-hand side of (8) cannot be removed. This follows from the fact that the operator $\mathcal{L} - \lambda W$ has a negative eigenvalue for all $\lambda > 0$, whenever $W \geq 0$ and $W \not\equiv 0$; see, e.g., [4, Remark 1.4 and Proposition 3.2]. In this situation, our bound (8) implies that $\mathcal{L} - \lambda W$ has exactly one negative eigenvalue for $\lambda > 0$ small, and hence it is optimal in the small coupling regime. Our bound also captures the optimal $\lambda^{d/2}$-behavior of $N(0, \mathcal{L} - \lambda W)$ for $\lambda > 0$ large.

Remark 3. The number 1 on the right-hand side can be removed if we restrict the consideration to any ball $B_R = \{x \in \mathbb{R}^d : |x| < R\} \subset \mathbb{R}^d$ with Dirichlet boundary conditions. More precisely, it follows from our proof that if we restrict the quadratic form of $\mathcal{L} - W$ on the ball B_R, then

$$N(0, \mathcal{L}_{B_R} - W) \leq C_d \int_{B_R} W(x)_+^{\frac{d}{2}} (1 + |\ln|x/R||)^{d-1} \mathrm{d}x \,. \tag{9}$$

We note that the logarithmic weight in (9) is a consequence of the presence of the critical Hardy potential $-(d-2)^2/(4|x|^2)$, which is singular at the origin. If instead we use the critical Hardy potential $-1/(4(R - |x|)^2)$, which is singular at the boundary, the inequality holds without the logarithmic weight, as shown in [11].

In the proof of Theorem 1, we will use an improvement of Hardy's inequality on the orthogonal complement of radial functions, where the singular potential $(d-2)^2/(4|x|^2)$ is not critical, and then restrict the consideration to radial functions. This strategy has been used extensively in the literature; two examples (not necessarily the earliest) are the paper by Solomyak [24] and by Birman–Laptev [1]. These ingredients also appear in the proof of Hardy–Lieb–Thirring inequalities (6) in [4] as well as the Hardy–Sobolev inequality (7) in [12,23]. The original ingredient in our paper is the treatment on the subspace of radial functions, for which we do not know a precedent. In particular, we will prove the following Strauss type inequality, which is of independent interest.

Lemma 4 *(Strauss type estimate for radial functions). Consider the operator $\mathcal{L}_B$ defined by the quadratic form in (5) restricted to $L^2(B)$ with Dirichlet boundary conditions. Then for all radial functions $\{u_n\}_{n \geq 1}$ in the quadratic form domain of $\mathcal{L}_B$ satisfying*

$$\sum_{n\geq 1} |\sqrt{\mathcal{L}_B}u_n\rangle\langle\sqrt{\mathcal{L}_B}u_n| \leq 1 \qquad \textit{in } L^2(B), \tag{10}$$

we have the pointwise estimate

$$\rho(x) := \sum_{n\geq 1} |u_n(x)|^2 \leq \frac{C_d}{|x|^{d-2}}(1+|\ln|x||) \qquad \textit{for a.e. } x \in B. \tag{11}$$

Remark 5. The bound (11) is reminiscent of Strauss' pointwise decay $|u(x)|^2 \lesssim_d |x|^{-(d-1)}$ of a single radial function in $H^1(\mathbb{R}^d)$ [25, Lemma 1]. Our proof strategy of Lemma 4 also allows to show that if radial functions $\{v_n\}_{n\geq 1} \subset H^1_0(B)$ satisfy the orthogonality (10) with $\mathcal{L}_B$ replaced by the usual Dirichlet Laplacian $-\Delta_B$, then $\sum_{n\geq 1} |v_n(x)|^2 \lesssim_d |x|^{-(d-2)}$ for $d \geq 3$ (see Remark 9). In contrast, (11) is slightly worse than the latter bound since it contains a logarithmic weight, which is however optimal due to the effect of the Hardy potential.

Our proof of Lemma 4 uses an analogue of Rumin's method [27], plus the precise spectral property of $\mathcal{L}_B$ which has been understood by Vázquez–ZuaZua [30]. Although the result there holds only on the unit ball, its application to the whole $\mathbb{R}^d$ is made possible due to the relation

$$N(0, P(\mathcal{L}_{\mathbb{R}^d} - W)P) \leq 1 + N(0, P(\mathcal{L}_B - \mathbb{1}_B W)P) + N(0, P(\mathcal{L}_{B^c} - \mathbb{1}_{B^c} W)P) \tag{12}$$

where P is the projection onto radial functions. The bound (12) follows from the fact that imposing the Dirichlet boundary condition at $|x| = 1$ in one-dimension is a rank-one perturbation. The same idea was used by Kovařík–Vugalter–Weidl [15] to derive CLR type estimates for Schrödinger operators in 2D. The conclusion of Theorem 1 then follows from an inversion in the unit sphere which allows to control $N(0, P(\mathcal{L}_{B^c} - \mathbb{1}_{B^c} W)P)$ via $N(0, P(\mathcal{L}_B - \mathbb{1}_B W)P)$.

Note that the Hardy–Lieb–Thirring inequality (6) has been extended to the fractional Laplacian $(-\Delta)^s$ with singular potential $|x|^{-2s}$ [9,6], which in the case $s = 1/2$ is relevant to applications in the stability of relativistic matter. Again, the corresponding bound has been known only for eigenvalue moments $\gamma > 0$. The case $\gamma = 0$ is the content of our next result.

Let $d \geq 1$ and $0 < s < \min(1, d/2)$. Let $(-\Delta)^s$ be the fractional Laplacian on $L^2(\mathbb{R}^d)$ defined via the quadratic form

$$h_s[u] = \langle u, (-\Delta)^s u\rangle = a_{s,d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} \mathrm{d}x\,\mathrm{d}y, \quad a_{s,d} = 2^{2s-1}\frac{\Gamma(\frac{d+2s}{2})}{\pi^{\frac{d}{2}}|\Gamma(-s)|}. \tag{13}$$

Consider the fractional Hardy–Schrödinger operator

$$\mathcal{L}_s = (-\Delta)^s - \frac{\mathcal{C}_{s,d}}{|x|^{2s}} \geq 0 \quad \text{ on } L^2(\mathbb{R}^d), \quad \mathcal{C}_{s,d} = 2^{2s}\frac{\Gamma^2\left(\frac{d+2s}{4}\right)}{\Gamma^2\left(\frac{d-2s}{4}\right)}, \tag{14}$$

where $\mathcal{C}_{s,d}$ is the optimal constant in the fractional Hardy inequality [13]. We have the following extension of Theorem 1 to the fractional case.

Theorem 6 *(Fractional CLR type bound). For every dimension $d \geq 1$ and $0 < s < \min(1, d/2)$, there exists a constant $C_{s,d} > 0$ independent of the real-valued potential W such that*

$$N(0, \mathcal{L}_s - W) \leq C_{s,d}\left(1 + \int_{\mathbb{R}^d} W(x)_+^{\frac{d}{2s}}(1+|\ln|x||)^{\frac{d-s}{s}}\mathrm{d}x\right).$$

We remark that the same inequality holds in the presence of a magnetic field.

The proof of Theorem 6 deviates substantially from that of Theorem 1. On the one hand, we will split again $\mathbb{R}^d$ into B and B^c, and use crucially the fractional Hardy–Sobolev inequality by Tzirakis [29] on B. On the other hand, the generalization from the one-body inequality to the many-body inequality on each domain (B or B^c) is done via an abstract equivalence of Sobolev and CLR inequalities, a strategy proposed in [10]. The conclusion also requires a careful implementation of the localization method in the fractional case.

In fact, the proof of Theorem 6 is also valid when $s = 1$ and $d \geq 3$ and simplifies at some points. Thus, we obtain an alternative proof of Theorem 1. In the local case, however, we feel that the first proof is more direct, which motivated us to present it first.

We will prove Theorem 1 and Theorem 6 in Section 2 and Section 3, respectively.

2. Local case

2.1. Improved Hardy inequalities

We denote by P the projection onto radially symmetric functions in $L^2(\mathbb{R}^d)$ and set $P^\perp = 1 - P$. On the non-radial part, the following improved Hardy inequality is well-known (see e.g. [24,1,4,12,23]).

Lemma 7. *We have the operator inequality on $L^2(\mathbb{R}^d)$*

$$P^\perp(-\Delta)P^\perp \geq \left(\frac{(d-2)^2}{4} + (d-1)\right) P^\perp |x|^{-2} P^\perp. \tag{15}$$

The above estimate comes from the fact that the lowest nontrivial eigenvalue of the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^{d-1}$ is equal to $d - 1$.

On the radial part, we recall the following result from Musina [23].

Lemma 8 *([23, Proposition 1.1]). For every radial function $u \in H_0^1(B)$, we have*

$$\langle u, \mathcal{L}u \rangle \geq \frac{1}{4} \int\limits_B \frac{|u(x)|^2}{|x|^2 |\ln |x||^2} \mathrm{d}x. \tag{16}$$

2.2. Strauss type estimate

In this subsection we restrict to radial functions and prove the pointwise estimate in Lemma 4.

Proof of Lemma 4. Let $z_{0,k}$ be the k-th zero of the Bessel function J_0. From the spectral property of $\mathcal{L}_B$ studied in [30], we have

$$P\mathcal{L}_B = \sum_{k \geq 1} \lambda_{0,k} |\varphi_{0,k}\rangle\langle\varphi_{0,k}|$$

with the (L^2-normalized) eigenfunctions

$$\varphi_{0,k}(x) = \frac{1}{|x|^{\frac{d-2}{2}} \sqrt{|\mathbb{S}^{d-1}| \int_0^1 r J_0^2(z_{0,k} r) \mathrm{d}r}} J_0\left(z_{0,k}|x|\right), \quad x \in B \setminus \{0\},$$

and the corresponding eigenvalues $\lambda_{0,k} = z_{0,k}^2$. Here $|\mathbb{S}^{d-1}|$ is the surface area of the unit sphere in $\mathbb{R}^d$. Consequently, for a.e. $x \in B$,

$$\rho(x) = \sum_{n\geq 1} |u_n(x)|^2 = \sum_{n\geq 1} \left| \sum_{k\geq 1} \langle \varphi_{0,k}, u_n \rangle \varphi_{0,k}(x) \right|^2$$
$$= \sum_{n\geq 1} \left| \int_B \sum_{k\geq 1} \frac{1}{\sqrt{\lambda_{0,k}}} \overline{\varphi_{0,k}(y)} \varphi_{0,k}(x) \sqrt{\mathcal{L}_B} u_n(y) \mathrm{d}y \right|^2$$
$$\leq \int_B \left| \sum_{k\geq 1} \frac{1}{\sqrt{\lambda_{0,k}}} \overline{\varphi_{0,k}(y)} \varphi_{0,k}(x) \right|^2 \mathrm{d}y \leq \sum_{k\geq 1} \frac{|\varphi_{0,k}(x)|^2}{\lambda_{0,k}}.$$

Here we used Bessel's inequality via the (sub)orthogonality of $\{\sqrt{\mathcal{L}_B} u_n\}_n$, or more precisely the condition (10). Using the asymptotic properties of Bessel function (see [31, p. 199])

$$J_0(0) = 1, \quad J_0(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi}{4}\right) + O(r^{-1})_{r\to\infty},$$

we have

$$J_0^2(r) \lesssim \min(1, r^{-1}), \quad z_{0,k} \sim k, \quad \lambda_{0,k} \int_0^1 r J_0^2(z_{0,k} r) \mathrm{d}r = \int_0^{z_{0,k}} r J_0^2(r) \mathrm{d}r \sim k.$$

Hence,

$$\rho(x) \leq \sum_{k\geq 1} \frac{|\varphi_{0,k}(x)|^2}{\lambda_{0,k}} \lesssim_d \frac{1}{|x|^{d-2}} \sum_{k\geq 1} \frac{1}{k} \min\left(1, \frac{1}{k|x|}\right) \lesssim_d \frac{1}{|x|^{d-2}} (1 + |\ln|x||),$$

which is the desired pointwise estimate (11). □

Remark 9 *(Laplacian case).* If radial functions $\{v_n\}_{n\geq 1} \subset H_0^1(B)$ satisfy the orthogonality (10) with $\mathcal{L}_B$ replaced by the Dirichlet Laplacian $-\Delta_B$ in dimensions $d \geq 3$, then following the above proof of Lemma 4 and using the well-known spectral properties of $-\Delta_B$ (see, e.g., [28, Sec 6.4.4]) we find that

$$\sum_{n\geq 1} |v_n(x)|^2 \leq \sum_{k\geq 1} \frac{|\varphi_k(x)|^2}{\lambda_k}$$

where

$$\varphi_k(x) = \frac{1}{|x|^{\frac{d-2}{2}} \sqrt{|\mathbb{S}^{d-1}| \int_0^1 r J_{\frac{d-2}{2}}^2 (z_k r) \mathrm{d}r}} J_{\frac{d-2}{2}}(z_k |x|)$$

are (L^2-normalized) radial eigenfunctions of $-\Delta_B$ with eigenvalues $\lambda_k = z_k^2$, and z_k is the k-th zero of $J_{\frac{d-2}{2}}$. From the asymptotic properties of Bessel function (see [31, p. 40, 199])

$$J_{\frac{d-2}{2}}(r) \sim_{r\to 0} \frac{1}{\Gamma(d/2)} \left(\frac{r}{2}\right)^{\frac{d-2}{2}}, \quad J_{\frac{d-2}{2}}(r) \sim_{r\to\infty} \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{(d-1)\pi}{4}\right) + O(r^{-1}),$$

we obtain

$$J^2_{\frac{d-2}{2}}(r) \lesssim \min(r^{d-2}, r^{-1}), \quad z_k \sim k, \quad \lambda_k \int_0^1 r J^2_{\frac{d-2}{2}}(\sqrt{\lambda_k} r)\mathrm{d}r = \int_0^{\sqrt{\lambda_k}} r J^2_{\frac{d-2}{2}}(r)\mathrm{d}r \sim \sqrt{\lambda_k},$$

and hence

$$\sum_{n\geq 1} |v_n(x)|^2 \lesssim_d \frac{1}{|x|^{d-2}} \sum_{k\geq 1} \frac{J^2_{\frac{d-2}{2}}(\sqrt{\lambda_k}|x|)}{\sqrt{\lambda_k}} \lesssim_d \frac{1}{|x|^{d-2}} \sum_{k\geq 1} \frac{1}{k} \min\left((k|x|)^{d-2}, \frac{1}{k|x|}\right) \lesssim_d \frac{1}{|x|^{d-2}}. \tag{17}$$

The bound (17) is slightly better than (11) as it does not contain a logarithmic weight.

2.3. Conclusion of Theorem 1

Since $\mathcal{L} - W \geq \mathcal{L} - W_+$, by the min-max principle it suffices to assume that $W \geq 0$. Recall that P is the projection onto radially symmetric functions in $L^2(\mathbb{R}^d)$ and $P^\perp = 1 - P$. Then, using $W \geq 0$, we have the Cauchy–Schwarz inequality

$$W \leq 2PWP + 2P^\perp W P^\perp,$$

and hence

$$N(0, \mathcal{L} - W) \leq N(0, P(\mathcal{L} - 2W)P) + N(0, P^\perp(\mathcal{L} - 2W)P^\perp). \tag{18}$$

On the non-radial part, using the improved Hardy inequality in Lemma 7 we have

$$P^\perp \mathcal{L} P^\perp \gtrsim_d P^\perp(-\Delta)P^\perp.$$

Therefore, by the standard CLR inequality,

$$N(0, P^\perp(\mathcal{L} - 2W)P^\perp) \leq N(0, P^\perp(C_d^{-1}(-\Delta) - 2W)P^\perp) \lesssim_d \int_{\mathbb{R}^d} W(x)^{\frac{d}{2}} \mathrm{d}x. \tag{19}$$

On the radial part, using the Hoffmann–Ostenhof inequality [14], the improved Hardy inequality in Lemma 8, and the pointwise estimate in Lemma 4, we get

$$\begin{aligned} \sum_{n\geq 1} \|\sqrt{\mathcal{L}}_B u_n\|^2_{L^2(B)} &\geq \|\sqrt{\mathcal{L}}_B \sqrt{\rho}\|^2_{L^2(B)} \\ &\geq \frac{1}{4} \int_B \frac{\rho(x)}{|x|^2 |\ln|x||^2} \mathrm{d}x \gtrsim_d \int_B \frac{\rho(x)^{\frac{d}{d-2}}}{(1 + |\ln|x||)^{1+\frac{d}{d-2}}} \mathrm{d}x \end{aligned} \tag{20}$$

for all radial functions $\{u_n\}_{n\geq 1}$ satisfying (10). By a standard duality argument (see e.g. [7]), the kinetic inequality (20) implies that

$$N(0, P(\mathcal{L}_B - 2\mathbb{1}_B W)P) \lesssim_d \int_B W(x)_+^{\frac{d}{2}} (1 + |\ln|x||)^{d-1} \mathrm{d}x. \tag{21}$$

Next, we use (12), namely

$$N(0, P(\mathcal{L}_{\mathbb{R}^d} - 2W)P) \leq 1 + N(0, P(\mathcal{L}_B - 2\mathbb{1}_B W)P) + N(0, P(\mathcal{L}_{B^c} - 2\mathbb{1}_{B^c} W)P). \tag{22}$$

To control $N(0, P(\mathcal{L}_{B^c} - 2\mathbb{1}_{B^c} W)P)$, we use an inversion in the unit sphere. Let us introduce some notation. Let $\mathcal{Q}$ be the form domain of the operator

$$-r^{1-d}\partial_r\left(r^{d-1}\partial_r\right) - \frac{(d-2)^2}{4r^2} \quad \text{in } L^2((1,\infty), r^{d-1}dr)$$

with a Dirichlet boundary condition at $r = 1$. Similarly, let $\tilde{\mathcal{Q}}$ be the form domain of the operator

$$-s^{1-d}\partial_s\left(s^{d-1}\partial_s\right) - \frac{(d-2)^2}{4s^2} \quad \text{in } L^2((0,1), s^{d-1}ds)$$

with a Dirichlet boundary condition at $s = 1$ (and a 'Dirichlet boundary' condition at $s = 0$).

Lemma 10. *Assume that $u \in \mathcal{Q}$ and $\tilde{u} \in \tilde{\mathcal{Q}}$ are related by*

$$u(r) = r^{2-d}\tilde{u}(1/r) \qquad \textit{for all } r \in (1,\infty)\,.$$

Then

$$\int_1^\infty \left(u'(r)^2 - \frac{(d-2)^2}{4r^2}u(r)^2\right)r^{d-1}\,\mathrm{d}r = \int_0^1 \left(\tilde{u}'(s)^2 - \frac{(d-2)^2}{4s^2}\tilde{u}(s)^2\right)s^{d-1}\,\mathrm{d}s\,.$$

Proof. By an approximation argument, we may assume the $u \in C_c^2(1,\infty)$ and $\tilde{u} \in C_c^2(0,1)$. Then the assertion follows by a straightforward computation, which we omit. $\square$

Corollary 11. *Assume that W defined on $\overline{B}^c$ and $\tilde{W}$ defined on B are related by*

$$W(x) = |x|^{-4}\tilde{W}(x/|x|^2) \qquad \textit{for all } x \in \overline{B}^c\,.$$

Then

$$N(0, P(\mathcal{L}_{B^c} - \mathbb{1}_{B^c}W)P) = N(0, P(\mathcal{L}_B - \mathbb{1}_B\tilde{W})P)\,.$$

Proof. Clearly, the assertion only depends on the spherical means of W and $\tilde{W}$, which we denote by w and $\tilde{w}$. By Glazman's lemma (see e.g. [8, Theorem 1.25]), we have

$$\begin{aligned}&N(0, P(\mathcal{L}_{B^c} - \mathbb{1}_{B^c}W)P)\\ &= \sup\Big\{\dim M : \int_1^\infty \left(u'(r)^2 - \frac{(d-2)^2}{4r^2}u(r)^2\right)r^{d-1}\,\mathrm{d}r < \int_1^\infty w(r)u(r)^2r^{d-1}\,\mathrm{d}r\ \forall 0 \neq u \in M\Big\}\end{aligned}$$

and

$$\begin{aligned}&N(0, P(\mathcal{L}_B - \mathbb{1}_B\tilde{W})P)\\ &= \sup\Big\{\dim \tilde{M} : \int_0^1 \left(\tilde{u}'(s)^2 - \frac{(d-2)^2}{4s^2}\tilde{u}(s)^2\right)s^{d-1}\,\mathrm{d}s < \int_0^1 \tilde{w}(s)\tilde{u}(s)^2s^{d-1}\,\mathrm{d}s\ \forall 0 \neq \tilde{u} \in \tilde{M}\Big\}\end{aligned}$$

where M and $\tilde{M}$ run through subspaces in $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$, respectively. The claimed equality therefore follows from the identity in the lemma, the identity

$$\int_1^\infty w(r)u(r)^2 r^{d-1}\,\mathrm{d}r = \int_0^1 \tilde{w}(s)\tilde{u}(s)^2 s^{d-1}\,\mathrm{d}s,$$

together with the fact that the correspondence $u \mapsto \tilde{u}$ is bijective on form cores consisting of functions vanishing near the origin and near infinity, respectively. □

It is now easy to finish the proof of Theorem 1. Clearly, we have, under the conditions of Corollary 11,

$$\int_{B^c} W(x)^{\frac{d}{2}}(1+|\ln|x||)^{d-1}\,\mathrm{d}x = \int_B \tilde{W}(x)^{\frac{d}{2}}(1+|\ln|x||)^{d-1}\,\mathrm{d}x\,.$$

This identity, together with Corollary 11 and inequality (21), yields

$$N(0, P(\mathcal{L}_{B^c} - 2\mathbb{1}_{B^c}W)P) \lesssim_d \int_{B^c} W(x)^{\frac{d}{2}}(1+|\ln|x||)^{d-1}\,\mathrm{d}x\,. \tag{23}$$

Therefore, inserting (21) and (23) into (22), we obtain

$$N(0, P(\mathcal{L}_{\mathbb{R}^d} - 2W)P) \le 1 + C_d \int_{\mathbb{R}^d} W(x)_+^{\frac{d}{2}}(1+|\ln|x||)^{d-1}\mathrm{d}x\,.$$

This, together with (18) and (19), completes the proof of Theorem 1. □

3. Nonlocal case

3.1. Fractional Hardy–Sobolev inequalities

Let $0 < s < \min(1, d/2)$ and let $\mathcal{L}_s$ be defined in (14). Recall the following results of Tzirakis [29].

Lemma 12 *([29, Theorem 3 and Theorem 5]). For all $v \in C_c^\infty(B)$, we have*

$$\langle v, \mathcal{L}_s v\rangle \gtrsim_{s,d} \left(\int_B |v(x)|^{\frac{2d}{d-2s}}(1+|\ln|x||)^{-\frac{2(d-s)}{d-2s}}\,\mathrm{d}x\right)^{\frac{d-2s}{d}},$$

and

$$\langle v, \mathcal{L}_s v\rangle \gtrsim_{s,d} \int_B \frac{|v(x)|^2}{(1+|\ln|x||^2)|x|^{2s}}\,\mathrm{d}x.$$

3.2. Equivalence of Sobolev and CLR inequalities

In this part, we recall the equivalence of Sobolev and CLR inequalities from [10]. Let X be a separable measure space. We consider the measure on X as fixed and denote integration with respect to this measure by $\mathrm{d}x$. By $L^p(X)$ for $1 \le p \le \infty$ we denote the usual L^p space with respect to this measure.

Let t be a non-negative quadratic form with domain dom t that is closed in the Hilbert space $L^2(X)$ and let T be the corresponding self-adjoint operator.

Throughout this paper we work under the following assumption, which depends on a parameter $1 < \kappa < \infty$.

Assumption 13 *(Generalized Beurling–Deny conditions).*

(a) if $u, v \in \operatorname{dom} t$ are real-valued, then $t[u+iv] = t[u] + t[v]$,
(b) if $u \in \operatorname{dom} t$ is real-valued, then $|u| \in \operatorname{dom} t$ and $t[|u|] \le t[u]$,
(c) there is a measurable, a.e. positive function μ such that, if $u \in \operatorname{dom} t$ is non-negative then $\min(u, \mu) \in \operatorname{dom} t$ and $t[\min(u,\mu)] \le t[u]$. Moreover, there is a form core $\mathcal{Q}$ of t such that $\mu^{-1}\mathcal{Q}$ is dense in $L^2(X, \mu^{2\kappa/(\kappa-1)}\mathrm{d}x)$.

Theorem 14 *(Equivalence of Sobolev and CLR inequalities in the presence of weights). Let Assumption 13 be satisfied for some $\kappa > 1$ and let w be a nonnegative, measurable function on X that is finite a.e. Then the following are equivalent:*

(i) *T satisfies a weighted Sobolev inequality with exponent $q = 2\kappa/(\kappa-1)$, that is, there is a constant $S > 0$ such that for all $u \in \operatorname{dom} t$,*

$$t[u] \ge S \left(\int_X |u|^q w^{-(q-2)/2} \, \mathrm{d}x \right)^{2/q} . \tag{24}$$

(ii) *T satisfies a weighted CLR inequality with exponent κ, that is, there is a constant $L > 0$ such that for all $0 \le V \in L^\kappa(X, w\,\mathrm{d}x)$,*

$$N(0, T-V) \le L \int_X V^\kappa w \, \mathrm{d}x \, . \tag{25}$$

The respective constants are related according to

$$S^{-\kappa} \le L \le e^{\kappa-1} S^{-\kappa} \, . \tag{26}$$

Proof. This theorem for $w = 1$ appears in the paper [10]. It is based on a method due to Li and Yau [17] with an improvement in [2] and generalizes a theorem of Levin and Solomyak [16].

We now prove the result for general w as in the statement of the theorem. The implication (ii $\Longrightarrow$ i) follows by a standard application of Hölder's inequality. Thus, we only need to prove (i $\Longrightarrow$ ii). For the proof, we note that we may assume that w is bounded away from zero. Indeed, once the implication is proved under this extra assumption, we can apply it to $w_\varepsilon := w + \varepsilon$ in place of w. This w_ε still satisfies (24) and it satisfies the extra condition $w_\varepsilon \ge \varepsilon$. Thus, we obtain (25) with w_ε in place of w. Since the constant is independent of ε, we can let $\varepsilon \to 0$ and obtain, by monotone convergence, the claimed inequality (25) with w.

Thus, assume that w is bounded away from zero. To better explain the strategy of the proof, let us first assume, in addition, that $w \in L^\infty(X)$. Then the Hilbert space $\mathfrak{H} := L^2(X, w^{-(q-2)/2}\mathrm{d}x)$ coincides, with equivalent norm, with the Hilbert space $L^2(X)$. We consider the quadratic form t in the Hilbert space $\mathfrak{H}$. It is clearly nonnegative and, by our assumptions on w, it is closed. (It is in the proof of closedness that we use the assumption $w \in L^\infty(X)$.) Thus it generates a nonnegative operator A in $\mathfrak{H}$. Moreover, it satisfies Assumption 13, since we are assuming that the corresponding assumption is satisfied for the original form t in the Hilbert space $L^2(X)$. Applying [10, Theorem 2.1] to the operator A, we obtain that for any $0 \le U \in L^\kappa(X, w^{-(q-2)/2}\mathrm{d}x)$ one has

$$N(0, A-U) \le e^{\kappa-1} S^{-\kappa} \int_X U^\kappa \, w^{-\frac{q-2}{2}} \, \mathrm{d}x \, .$$

At the point we notice that the quadratic form of the operator $A - U$ is

$$t[v] - \int_X U|v|^2 w^{-\frac{q-2}{2}}\,\mathrm{d}x\,.$$

Thus, by Glazman's lemma (see e.g. [8, Theorem 1.25]),

$$N(0, A-U) = \sup\Big\{\dim M:\ t[v] - \int_X U|v|^2 w^{-\frac{q-2}{2}}\,\mathrm{d}x < 0,\ \forall 0 \neq v \in M\Big\} = N(0, T - Uw^{-\frac{q-2}{2}})\,,$$

where we emphasize that the operator $A - U$ acts in $\mathfrak{H}$ and the operator $T - Uw^{-\frac{q-2}{2}}$ acts in $L^2(X, \mathrm{d}x)$. (We emphasize that the crucial point in this 'trick' is that Glazman's lemma only sees the quadratic form, but not the norm in the Hilbert space.) Writing $V = Uw^{-\frac{q-2}{2}}$ we have shown that

$$N(0, T-V) \leq e^{\kappa-1} S^{-\kappa} \int_X V^\kappa w\,\mathrm{d}x\,.$$

This is the assertion (ii) that we wanted to prove under the assumption $w \in L^\infty(X)$.

Now we consider the general case, where w is finite a.e. As we mentioned before, we may also assume that w is bound away from zero. The Hilbert space $\mathfrak{H}$ is defined as before, but now we only know that $L^2(X) \subset \mathfrak{H}$. Given $\delta > 0$ we consider the quadratic form $v \mapsto t_\delta[v] := t[v] + \delta \int_X |v|^2\,\mathrm{d}x$ in the Hilbert space $\mathfrak{H}$ with form domain $\operatorname{dom} t$. Let us show that t_δ is closed in $\mathfrak{H}$. Thus, let $(v_j) \subset \operatorname{dom} t$ and $v \in \mathfrak{H}$ such that $v_j \to v$ in $\mathfrak{H}$ and such that (v_j) is Cauchy with respect to t_δ. Since t is nonnegative, we infer that (v_j) is Cauchy with respect to the norm in $L^2(X)$ and hence convergent in $L^2(X)$. Passing to a.e.-convergent subsequences and recalling that w is finite a.e., it is easy to see that $v \in L^2(X)$ and that $v_j \to v$ in $L^2(X)$. Therefore the assumed closedness of t implies that $v \in \operatorname{dom} t$ and that $t[v_j - v] \to 0$. Thus, we have shown that t_δ is closed in $\mathfrak{H}$.

Let us denote by A_δ the nonnegative operator in $\mathfrak{H}$ generated by t_δ. The form t_δ satisfies Assumption 13 and we deduce, as before, that

$$N(0, A_\delta - U) \leq e^{\kappa-1} S^{-\kappa} \int_X U^\kappa\, w^{-\frac{q-2}{2}}\,\mathrm{d}x\,.$$

Meanwhile, again by Glazman's lemma,

$$N(0, A_\delta - U) = N(0, T + \delta - Uw^{-\frac{q-2}{2}}) = N(-\delta, T - Uw^{-\frac{q-2}{2}})\,.$$

Thus, we conclude that for $V \geq 0$ satisfying $\int_X V^\kappa w\,\mathrm{d}x < \infty$ the spectrum of $T - V$ in the interval $(-\infty, -\delta)$ is finite and

$$N(-\delta, T-V) \leq e^{\kappa-1} S^{-\kappa} \int_X V^\kappa w\,\mathrm{d}x\,.$$

Letting $\delta \searrow 0$, we obtain the claimed inequality. □

We return now to our operator $\mathcal{L}_s$ and prove the corresponding CLR inequality in the ball B.

Lemma 15. *Let $W \geq 0$. Consider the operator $\mathcal{L}_s - W$ on $L^2(B)$ defined by the quadratic form restricted to functions vanishing outside of B. We have*

$$N(0, \mathcal{L}_s - W) \lesssim_{s,d} \int_B W(x)^{\frac{d}{2s}} (1 + |\ln|x||)^{\frac{d-s}{s}} \, \mathrm{d}x \, .$$

Proof. We apply Theorem 14 in the measure space $X = B$ with Lebesgue measure and with $t[u] = \|\mathcal{L}_s^{1/2} u\|^2$. Items (a) and (b) in Assumption 13 are clearly satisfied. The first part of item (c) is satisfied with $\mu(x) = |x|^{-(d-2s)/2}$, as follows from the ground state substitution formula in the [10] on Hardy–Lieb–Thirring. The second part is satisfied, for any $1 < \kappa < \infty$, since $C_c^\infty(B \setminus \{0\})$ is a form core of $\mathcal{L}_s$. Thanks to Lemma 12, the Sobolev inequality (24) is satisfied with $q = \frac{2d}{d-2s}$ and $w = (1 + |\ln|x||)^{\frac{d-s}{s}}$. The assertion follows now from the CLR inequality (25). □

Similarly as in the previous section, one can use the inversion in the unit sphere to derive a corresponding inequality in the complement of the unit ball.

Lemma 16. *Let $W \geq 0$. For the operator $\mathcal{L}_s - W$ defined on $L^2(\overline{B}^c)$ with Dirichlet boundary conditions, we have*

$$N(0, \mathcal{L}_s - W) \lesssim_{s,d} \int_{B^c} W(x)^{\frac{d}{2s}} (1 + |\ln|x||)^{\frac{d-s}{s}} \, \mathrm{d}x \, .$$

Proof. We keep the notation $\mathcal{L}_s$ for the operator in $L^2(\overline{B}^c)$ that appears in the lemma and denote the one in $L^2(B)$ from Lemma 15 temporarily by $\tilde{\mathcal{L}}_s$. Assume that functions u and $\tilde{u}$ in the form domains of $\mathcal{L}_s$ and $\tilde{\mathcal{L}}_s$ are related by

$$u(x) = |x|^{2s-d} \tilde{u}(x/|x|^2) \qquad \text{for all } x \in \mathbb{R}^d \setminus \{0\}.$$

We claim that

$$\langle u, \mathcal{L}_s u \rangle = \langle \tilde{u}, \tilde{\mathcal{L}}_s \tilde{u} \rangle \, . \tag{27}$$

Since $C_c^2(B \setminus \{0\})$ and $C_c^2(\overline{B}^c)$ are form cores for the relative operators, it suffices to assume that u and $\tilde{u}$ belong to these sets. For such functions it is well known that

$$\langle u, (-\Delta)^s u \rangle = \langle \tilde{u}, (-\Delta)^s \tilde{u} \rangle \, .$$

This appears, in a dual form, for instance in [19]; see also [5, Lemma 2.2]. Additionally, by changing variable $y = \frac{x}{|x|^2}$, we have

$$\int_{B^c} \frac{|u(x)|^2}{|x|^{2s}} \mathrm{d}x = \int_{B^c} \frac{|x|^{2(2s-d)} |\tilde{u}\left(\frac{x}{|x|^2}\right)|^2}{|x|^{2s}} \mathrm{d}x = \int_B \frac{|\tilde{u}(y)|^2}{|y|^{2s}} \mathrm{d}y \, ,$$

thus proving the claimed identity (27).

Identity (27) is the analogue of Lemma 10. As in the proof of Corollary 11, we deduce that

$$N(0, \mathcal{L}_s - W) = N(0, \tilde{\mathcal{L}}_s - \tilde{W}) \, ,$$

where

$$\tilde{W}(y) := |y|^{-4s} W(y/|y|^2) \qquad \text{for all } y \in B \backslash \{0\} \, .$$

Since

$$\int_B \tilde{W}(y)^{\frac{d}{2s}}(1+|\ln|y||)^{\frac{d-s}{s}}\,\mathrm{d}y = \int_{B^c} W(x)^{\frac{d}{2s}}(1+|\ln|x||)^{\frac{d-s}{s}}\,\mathrm{d}x\,,$$

we deduce the inequality in the lemma from that in Lemma 15. □

Remark 17. We record another use of the inversion method employed in the previous proof, which will be useful later. Namely, we have the bound

$$\langle u,(-\Delta)^s u\rangle \geq \mathcal{C}_{s,d}\int_{B^c}\frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x + C_{s,d}\int_{B^c}\frac{|u(x)|^2}{(|\ln|x||^2+1)|x|^{2s}}\,\mathrm{d}x.$$

Indeed, this follows from the second bound in Lemma 12 applied to the function $v=\tilde{u}$ from the previous proof.

3.3. Localization

Consider smooth partition functions $\chi,\eta:\mathbb{R}^d\to[0,1]$ satisfying

$$\chi^2(x)+\eta^2(x)\equiv 1,\quad \operatorname{supp}\chi\subset\{|x|\leq 2\},\quad \operatorname{supp}\eta\subset\{|x|\geq 1\}. \tag{28}$$

The following localization estimate for the fractional Laplacian is of independent interest.

Lemma 18. *Let $0<s<\min(1,d/2)$ and $\delta\in(0,1)$. Then for every $\delta>0$, there exists $C=C(s,d,\delta)>0$ such that*

$$\mathcal{L}_s \geq \chi(\mathcal{L}_s - C)\chi + (1-\delta)\eta(\mathcal{L}_s - C\mathbb{1}_{B_3})\eta,$$

where $\mathbb{1}_{B_3}$ is the indicator function of $B_3=\{|x|<3\}\subset\mathbb{R}^d$.

Proof. For every $u\in C_c^\infty(\mathbb{R}^d)$, we have the IMS formula

$$\langle u,\mathcal{L}_s u\rangle = \langle u,(\chi\mathcal{L}_s\chi)u\rangle + \langle u,(\eta\mathcal{L}_s\eta)u\rangle - \langle u,\mathcal{H}u\rangle \tag{29}$$

where $\mathcal{H}$ is the bounded operator on $L^2(\mathbb{R}^d)$ with integral kernel

$$H(x,y)=a_{s,d}\frac{(\chi(x)-\chi(y))^2+(\eta(x)-\eta(y))^2}{|x-y|^{d+2s}}.$$

This formula is due to Michael Loss and appeared in [22].

By the triangle inequality we have the pointwise estimate

$$0\leq H(x,y)\lesssim_{s,d}\frac{\mathbb{1}_{B_3}(x)\mathbb{1}_{B_3}(y)}{|x-y|^{d+2s-2}}+\frac{\mathbb{1}_{B_3}(x)\mathbb{1}_{B_3^c}(y)}{(1+|y|)^{d+2s}}+\frac{\mathbb{1}_{B_3}(y)\mathbb{1}_{B_3^c}(x)}{(1+|x|)^{d+2s}}\quad\text{for } x\neq y.$$

When $d\geq 2$, combining with the Hardy–Littlewood–Sobolev (HLS) inequality and the Hölder inequality, we get

$$\begin{aligned}\langle u, \mathcal{H}u\rangle &\lesssim_{s,d} \int\limits_{B_3}\int\limits_{B_3} \frac{|u(x)||u(y)|}{|x-y|^{d+2s-2}}\mathrm{d}x\,\mathrm{d}y + \int\limits_{B_3}\int\limits_{B_3^c} \frac{|u(x)||u(y)|}{(1+|y|)^{d+2s}}\mathrm{d}x\,\mathrm{d}y \\ &\lesssim_{s,d} \|\mathbb{1}_{B_3}u\|^2_{L^{\frac{2d}{d-2s+2}}(\mathbb{R}^d)} + \|\mathbb{1}_{B_3}u\|_{L^1(\mathbb{R}^d)} \left\|\frac{\mathbb{1}_{B_3^c}(y)u(y)}{|y|^s(1+|\ln|y||)}\right\|_{L^2(\mathbb{R}^d)} \left\|\frac{|y|^s(1+|\ln|y||)}{(1+|y|)^{d+2s}}\right\|_{L^2(\mathbb{R}^d)} \\ &\lesssim_{s,d} \delta \left\|\frac{\mathbb{1}_{B_3^c}(y)u(y)}{|y|^s(1+|\ln|y||)}\right\|^2_{L^2(\mathbb{R}^d)} + (1+\delta^{-1})\|\mathbb{1}_{B_3}u\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

for all $\delta > 0$. In dimension $d = 1$, the exponent $d + 2s - 2$ is negative for $s < \frac{1}{2}$, and hence instead of the HLS inequality we can use $|x-y|^{-(d+2s-2)} \lesssim 1$ for $x, y \in B_3$, leading to the same final estimate.

By the Hardy inequality with remainder term in Remark 17,

$$\left\|\frac{\mathbb{1}_{B_3^c}(y)u(y)}{|y|^s(1+|\ln|y||)}\right\|^2_{L^2(\mathbb{R}^d)} \le \left\|\frac{\eta(y)u(y)}{|y|^s(1+|\ln|y||)}\right\|^2_{L^2(\mathbb{R}^d)} \lesssim_{s,d} \langle u, (\eta\mathcal{L}_s\eta)u\rangle.$$

Thus in summary, for every $\delta \in (0,1)$ we have

$$\langle u, \mathcal{H}u\rangle \le \delta\langle(\eta\mathcal{L}_s\eta)u, u\rangle + C_{s,d,\delta}\|\mathbb{1}_{B_3}u\|^2_{L^2(\mathbb{R}^d)}.$$

The conclusion follows by inserting the latter bound in (29). □

3.4. Conclusion of Theorem 6

It suffices to assume that $W \ge 0$. Let $\chi^2 + \eta^2 = 1$ as in (28). By Lemma 18, we have the following quadratic form estimate on $L^2(\mathbb{R}^d)$

$$\mathcal{L}_s - W \ge \chi\Big(\mathcal{L}_s - W - C\Big)\chi + \frac{1}{2}\eta\Big(\mathcal{L}_s - 2W - C\mathbb{1}_{B_3}\Big)\eta.$$

Therefore,

$$N(0, \mathcal{L}_s - W) \le N(0, \chi(\mathcal{L}_s - W - C)\chi) + N(0, \eta(\mathcal{L}_s - 2W - C\mathbb{1}_{B_3})\eta).$$

Using Lemma 15 (with B replaced by B_3, the result remains true with a possible change of the implicit constant), we have

$$N(0, \chi(\mathcal{L}_s - W - C)\chi) \lesssim_{s,d} \int\limits_{\mathbb{R}^d} \Big[\chi^2(x)(W(x)+C)\Big]^{\frac{d}{2s}} (1+|\ln|x||)^{\frac{d-s}{s}}\,\mathrm{d}x\,.$$

Similarly, by Lemma 16,

$$N(0, \eta(\mathcal{L}_s - 2W - C\mathbb{1}_{B_3})\eta) \lesssim_{s,d} \int\limits_{\mathbb{R}^d} \Big[\eta^2(x)(W(x)+C\mathbb{1}_{B_3}(x))\Big]^{\frac{d}{2s}} (1+|\ln|x||)^{\frac{d-s}{s}}\,\mathrm{d}x\,.$$

Thus we conclude that

$$N(0, \mathcal{L}_s - W) \lesssim_{s,d} \int\limits_{\mathbb{R}^d} \Big[\chi^2(x)(W(x)+C)\Big]^{\frac{d}{2s}} (1+|\ln|x||)^{\frac{d-s}{s}}\,\mathrm{d}x$$

$$+\int_{\mathbb{R}^d}\Big[\eta^2(x)(W(x)+C\mathbb{1}_{B_3}(x))\Big]^{\frac{d}{2s}}(1+|\ln|x||)^{\frac{d-s}{s}}\,\mathrm{d}x$$
$$\lesssim_{s,d} 1+\int_{\mathbb{R}^d}W(x)^{\frac{d}{2s}}(1+|\ln|x||)^{\frac{d-s}{s}}\,\mathrm{d}x.$$

The proof of Theorem 6 is complete. □

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