Not quite Intuitionism

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Abstract

This is an investigation of M. Dummett's claim that a theory of meaning based on verification conditions should lead to the abandonment of classical logic in favor of intuitionistic logic. I especially concentrate on his suggestion in Dummett [1] that, in order to give the meaning of negations and conditionals, we should also take on board *falsification* conditions. Taken seriously, however, this route takes us not to intuitionistic logic, but rather to one of the Nelson logics.

Michael Dummett has argued that a theory of meaning expressed in terms of truth conditions that might be beyond our recognitional grasp cannot make sense of how we acquire and use language[1]. For Dummett, the prime example of a more adequte semantic theory is the account of mathematical statements in Brouwer's intuitionism: Such a statement is true only if it is constructively provable, which brings the intuitionists to reject certain classical inferences and laws, chief among them the Law of Excluded Middle ($\vDash (A \vee -A)$): We can't assume that for every mathematical statement, we will eventually come across a proof of it or its negation, therefore this classical tautology can not be assumed.

The so called Brouwer-Heyting-Kolmogorov interpretation (BHK for short) of intuitionistic logic spells out the meanings of the logical constants directly in terms of proofs:

- c is a proof of $A \wedge B$ iff c is a pair $(c1, c2)$ such that $c1$ is a proof of A and $c2$ is a proof of B
- c is a proof of $A \vee B$ iff c is a pair $(i, c1)$ such that $i = 0$ and $c1$ is a proof of A or $i = 1$ and $c1$ is a proof of B

[1] Parts of this argument can be found in most of his papers and books. Here, the essay "What is a Theory of Meaning (II)", reprinted in 1, is taken as the primary source.

- c is a proof of $A \supset B$ iff c is a construction that converts each proof d of A into a proof $c(d)$ of B
- nothing is a proof of $\bot$
- c is a proof of $-A$ iff c is a construction which transforms each proof of A into a proof of $\bot$.

$\bot$ is a mathematical absurdity, such as $1 = 0$.

A formal semantics that corresponds to this interpretation was given by S. Kripke:

A Kripke-model for intuitionistic logic is a structure, $[W, \leq, v]$, where W is a set of worlds or information states, $\leq$ is a partial order on these worlds.

The valuation function v assigns a truth value, 1 or 0, to each atomic statement p at each world[2], satisfying a *heredity constraint*:

For each p: if $w \leq w$' and $v_w(p) = 1$ then $v_{w'}(p) = 1$.

Now, for the truth conditions of the logical operators:

For all $w \epsilon W$:

$w \Vdash_1 A \wedge B$ iff $w \Vdash_1 A$ and $w \Vdash_1 B$

$w \Vdash_1 A \vee B$ iff $w \Vdash_1 A$ or $w \Vdash_1 B$

$w \Vdash_1 A \supset B$ iff for all $x \geq w$, $x \Vdash_0 A$ or $x \Vdash_1 B$

$w \Vdash_1 -A$ iff for all $x \geq w$, $x \Vdash_0 A$

Consequence is defined as follows:

$\Gamma \vDash A$ iff in every model and every $w \in W$, if $w \vDash_1 B$ for every $B \in \Gamma$, then $w \vDash_1 A$.

Given these semantics, it is not hard to show the characteristic features of intuitionistic logic: Excluded Middle ($\vDash (A \vee -A)$ of course fails. Furthermore, even though double negation introduction holds ($\vDash A \supset --A)$, double negation elimination fails ($\nvDash --A \supset A$), as well as one of the de Morgan's Laws ($-(A \wedge B) \nvDash (-A \vee -B)$).

It is Dummett's aim to bring the ideas of intuitionistic mathematics to other, more empirical subject matters. Transferred to those empirical subject matters, the intuitionistic mathematician's quest for proofs is replaced by a quest for *verifications*. The content of a statement, so the verificationistic theory of meaning, is given by what would verify it.

[2]For $v_w(p) = 1$ I will write $w \Vdash_1 p$, and for $v_w(p) = 0$ I will write $w \Vdash_0 p$.

An important part of this idea is that some BHK-style interpretation of the logical vocabulary in terms of verifications can be given. The most straightforward adaptation would be simply to write “verification” instead of “proof” in the BHK-clauses. $\bot$ would be taken to be some empirical absurdity, such as the claim that every sentient being is actually a cactus.

Dummett, however, is not quite happy with the resulting interpretation:

[A] proof of the negation of any arbitrary statement then consists of an effective method for transforming any proof of that statement into a proof of some false numerical equation. Such an explanation relies on the underlying presumption that, given a proof of a false numerical equation, we can construct a proof of any statement whatsoever. It is not obvious that, when we extend these conceptions to empirical statements, there exists any class of decidable atomic statements for which a similar presumption holds good; and it is therefore not obvious that we have, for the general case, any similar uniform way of explaining negation for arbitrary statements.

It would therefore remain well within the spirit of a theory of meaning of this type that we should regard the meaning of each statement as being given by the simultaneous provision of a means for recognizing a verification of it and a means for recognizing a falsification of it, where the only general requirement is that these should be specified in such a way as to make it impossible for any statement to be both verified and falsified. (Dummett [1]p.71)

The new concept to enter the picture, then, is that of *falsification*. There is no indication that he thought that this move would have any effect on the logic, i.e. that anything else than intuitionistic logic should supply the correct rules of inference in this case. However, if we proceed in the most natural way here, we are not at all lead to intuitionistic logic.

To see what the logic will turn out to be, we have to ask not just for the verification, but also for the falsification conditions of compound statements.

It is very plausible to say that a conjunction is falsified by a falsification of *either* conjunct and a disjunction by a falsification of the first *and* a falsification of the second disjunct.

The most natural interpretation of what Dummett says about negation is this: A negation is verified (falsified) if the negated statement is falsified (verified).

The conditional, as Dummett also notes in the essay, has a very clear falsification condition as well: The antecedent has to be verified, the succedent falsified. For the verification condition, the question is not all that clear, but the understanding of intuitionistic logic seemed to give a good approximation.

We can once again collect and express the preceding thoughts in a list of BHK-style clauses:

- c is a verification of $A \wedge B$ iff c is a pair $(c1, c2)$ such that $c1$ is a verification of A and $c2$ is a verification of B

- c is a falsification of $A \wedge B$ iff c is a pair $(i, c1)$ such that $i = 0$ and $c1$ is a falsification of A or $i = 1$ and $c1$ is a falsification of B

- c is a verification of $A \vee B$ iff c is a pair $(i, c1)$ such that $i = 0$ and $c1$ is a verification of A or $i = 1$ and $c1$ is a verification of B
- c is a falsification of $A \vee B$ iff c is a pair $(c1, c2)$ such that $c1$ is a falsification of A and $c2$ is a falsification of B

- c is a verification of $A \supset B$ iff c is a procedure that converts each verification d of A into a verification $c(d)$ of B
- c is a falsification of $A \supset B$ iff c is a pair $(c1, c2)$ such that $c1$ is a verification of A and $c2$ is a falsification of B

- c is a verification of $-A$ iff c is a falsification of A
- c is a falsification of $-A$ iff c is a verification of A

These clauses (together with the demand that nothing should be both verified and falsified) correspond to the Nelson logic N_3, developed independently and in different guises by David Nelson (Nelson [4]) and Franz von Kutschera (2) (cf. López-Escobar [3] and Wansing [6]). We will go straight to the Kripke semantics:

A model for N_3 is once again a structure $[W, \leq, v]$, W again being a set of partially ordered ($\leq$) worlds and$v$ a valuation function from formulas to 1 and 0. This time, though, we allow v to be a *partial* function[3].

We will have hereditary constraints for both 1 and 0:

For all p, and all worlds w and w', if $w \leq w'$ and $w \Vdash_1 p$, then $w' \Vdash_1 p$, and

for all p, and all worlds w and w', if $w \leq w'$ and $w \Vdash_0 p$, then $w' \Vdash_0 p$.

As $w \Vdash_0 A$ is not equivalent to $w \nVdash_1 A$, we have to give separate clauses for $\Vdash_1$ and $\Vdash_0$ when defining the connectives:

$w \Vdash_1 A \wedge B$ iff $w \Vdash_1 A$ and $w \Vdash_1 B$

$w \Vdash_0 A \wedge B$ iff $w \Vdash_0 A$ or $w \Vdash_0 B$

$w \Vdash_1 A \vee B$ iff $w \Vdash_1 A$ or $w \Vdash_1 B$

$w \Vdash_0 A \vee B$ iff $w \Vdash_0 A$ and $w \Vdash_0 B$

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[3]If we want to give up the requirement that there are no statements that are both verified and falsified, we are lead to the system N_4, which utilizes a valuation relation instead of a function, i.e. allows for the assignment of both 1 and 0 to statements.

$w \Vdash_1 A \supset B$ iff for all $x \geq w$, $x \nVdash_1 A$ or $x \Vdash_1 B$

$w \Vdash_0 A \supset B$ iff $w \Vdash_1 A$ and $w \Vdash_0 B$

$w \Vdash_1 -A$ iff $w \Vdash_0 A$

$w \Vdash_0 -A$ iff $w \Vdash_1 A$

As for consequence:

$\Gamma \vDash A$ iff in every model and every $w \in W$, if $w \Vdash_1 B$ for every $B \in \Gamma$, then $w \Vdash_1 A$.

This logic has been studied extensively in Wansing [6, 7], Odintsov [5] . Some characteristics of it are: Excluded Middle fails, just like in intuitionistic logic, but unlike intuitionistic logic, both double negation laws and the de Morgan laws are all valid.

An unusual feature of N_3 is that $-(A \wedge -A)$ is not valid. This corresponds to the failure of Excluded Middle, and it is instructive to see how close the connection is. $-(A \wedge -A)$ will be true if there is a falsification of $(A \wedge -A)$. Such a falsification, under the present proposal, will consist in either a falsification of A or a falsification of $-A$. It would be just as preposterous to claim to be able to be able to supply either a falsification for every statement or its negation as it was preposterous to claim to be able to verify either of them. Therefore, in a theory based on verifications and falsifications, both $\vDash (A \vee -A)$ and $\vDash -(A \wedge -A)$ will have to go.

How about $\vDash --A \supset A$ and$-(A \wedge B) \vDash (-A \vee -B)$? The thought that there is something non-constructive about these (especially double negation elimination) is deeply ingrained. However, given the new explanation in terms of verifications and falsifications, there is no reason to object to either of them from a constructivist point of view.

Moreover, given that Dummett wants to revise everyday logic, the closer the constructive logic comes to embracing intuitive valid reasoning, the easier it will become to swallow this pill. Double negation elimination and the intuitionistically invalid de Morgan law constitute inferences that are commonly felt to be valid, so this seems to speak rather favourably for the adoption of Nelson logic.

References

[1] M. Dummett. *The Seas of Language*. Clarendon Press, Oxford, 1993.

[2] F. Kutschera. Ein verallgemeinerter Widerlegungsbegriff für Gentzenkalküle. *Archive for Mathematical Logic*, 12(3):104–118, 1969.

[3] E. López-Escobar. Constructions and negationless logic. *Studia Logica*, 30 (1):7–19, 1972.

[4] D. Nelson. Constructible falsity. *Journal of Symbolic logic*, 14/1:16–26, 1949.

[5] S. Odintsov. *Constructive negations and paraconsistency*. Springer Verlag, 2008.

[6] H. Wansing. *The Logic of Information Structures*. Lecture Notes in AI 681. Springer, Berlin, 1993.

[7] H. Wansing. *Displaying Modal Logic*. Kluwer Academic Publishers, 1998.