Synthese (2025) 205:19
https://doi.org/10.1007/s11229-024-04764-z

ORIGINAL RESEARCH

Frege's platonism and mathematical creation: some new perspectives

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Received: 31 October 2023 / Accepted: 28 August 2024

Abstract

In this three-part essay, I investigate Frege's platonist and anti-creationist position in Grundgesetze der Arithmetik and to some extent also in Die Grundlagen der Arithmetik. In Sect. 1.1, I analyze his arithmetical and logical platonism in Grundgesetze. I argue that the reference-fixing strategy for value-range names—and indirectly also for numerical singular terms—that Frege pursues in Grundgesetze I gives rise to a conflict with the supposed mind- and language-independent existence of numbers and logical objects in general. In Sect. 1.2 and 1.3, I discuss the non-creativity of Frege's definitions in Grundgesetze and the case of what I call weakly creative definitions. In Part II of this essay, I first deal with Stolz's and Dedekind's (intended) creation of numbers. In what follows, I focus on Grundgesetze II, §146, where Frege considers a potential creationist charge in relation to the stipulation that he makes in Grundgesetze I, §3 with the purpose of partially fixing the references of value-range names. I place equal emphasis on the related twin stipulations that he makes in Grundgesetze I, §10. In §10, Frege identifies the truth-values with their unit classes in order to fix the references of value-range names (almost) completely. He does so in a piecemeal fashion. Although in Grundgesetze II, §146 Frege refers also to Grundgesetze I, §9 and §10 in this connection, he does not explain why he thinks that the transsortal identifications in §10 and also the stipulation that he makes in §9 regarding the value-range notation may give rise to a creationist charge in addition to or in connection with the stipulation in §3, and if so, how he would have responded to it. The two main issues that I discuss in Part II are: (a) Has Frege created value-ranges in general in Grundgesetze I, §3? (b) Has he created the unit classes of the True and the False in §10? In Part III, I discuss, inter alia, the question of whether in developing the whole wealth of objects and functions that arithmetic deals with from the primitive functions of Grundgesetze by applying the formation rules Frege creates special value-ranges and special functions. This procedure is fundamentally different from the reference-fixing strategy regarding value-range names that Frege pursues in Grundgesetze I, §3, §10–12. It is just another aspect of his anti-creationism. In Grundgesetze II, §147, Frege makes a concession to an imagined creationist opponent which might suggest that he was fully convinced neither of the defensibility of his anti-creationist position regarding the syntactic development of the subject matter of arithmetic nor of his actual defence

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Published online: 23 December 2024

in §146 of the non-creativity of the introduction of value-ranges via logical abstraction in Grundgesetze I, §3 and the twin stipulations in §10. I argue that not only in Grundgesetze II, §146 but also in Grundgesetze II, §147 Frege falls short of defending his anti-creationist position. I further argue that on the face of it his creationist rival gains the upper hand in the envisioned debate in more than one respect.

Keywords Initial stipulation · Basic law V · Logical abstraction · Value-range names · Numerical singular terms · Referential indeterminacy · Twin stipulations · Strong creatinevess · Weak creativeness · Formation rules

In both *Die Grundlagen der Arithmetik* (1884) and *Grundgesetze der Arithmetik* (vol. I 1893, vol. II 1903), Frege endorses arithmetical platonism: Numbers are logical objects that exist independently of human minds. In saying this, I have a minor caveat in mind. It concerns Frege's use of the term "logical object". In Part I of this essay, I suggest that although he does not yet use this term in *Grundlagen* and does not yet introduce extensions of concepts by appeal to a basic law of logic, he *presumably* viewed numbers as logical objects nonetheless. In *Grundlagen*, Frege explicitly construes numbers as non-real or non-actual objects. He goes on to do this in *Grundgesetze der Arithmetik* I, II but in this work characterizes numbers with predilection as logical objects. Today, as already in the period of time in which Frege wrote *Grundlagen* and *Grundgesetze*, there is no unanimous agreement among philosophers of mathematics regarding the subject matter of mathematics. There is an ongoing debate about the issue of whether the subject matter of mathematics should preferably be characterized in terms of abstract objects (or in terms of logical objects roughly after the fashion of Frege) or in terms of *ante rem* structures[1] or in terms of so-called thin objects[2] or what have you. Nominalists notoriously deny that mathematics has a subject matter at all. There is ongoing disagreement among philosophers of mathematics regarding whether platonism, appropriately interpreted, represents the most suitable approach to mathematics, or if instead, constructivism, intuitionism, or nominalism, when adapted to contemporary advancements, prevail in the debate.

Turning to Frege's logicism in *Grundlagen* and *Grundgesetze*, we see that his defence of it largely goes hand in hand with his advocacy of arithmetical platonism.[3]

[1] Cf. Shapiro (1997), p. 78: "Each mathematical object is a place in a particular structure. There is thus a certain priority in the status of mathematical objects. The structure is prior to the mathematical objects it contains …".

[2] Linnebo (2018) calls an object "thin" if a comparatively weak claim—not ontologically committed to the object in question—suffices for the existence of the object. In other words, thin objects are those which do not require very much for their existence. In his exposition, Linnebo draws on Fregean abstraction principles and introduces the notion of dynamic abstraction (cf. chapter 3). First, to be an object is to be a possible referent of a singular term. Second, singular reference can be achieved by providing a suitable criterion of identity for the referent (cf. chapter 2 "Thin objects via criteria of identity" and also chapter 8 "Reference by abstraction"). The second idea facilitates a form of convenient singular reference and, thanks to the first idea, also a form of "leightweight" existence.

[3] In *Grundgesetze* II, §147, Frege does not appeal to his platonism. Yet it looms in the background of his observations. On the nature of Frege's foundational project in *Grundgesetze*, see Blanchette (2012), Heck

Some Frege scholars have doubted this.[4] But I do not think that the doubts are justified. There is ample evidence that Frege's platonism in his mature period 1891–1902 overarches his entire philosophy of arithmetic. It was meant to apply first and foremost to logical objects that he regarded as fundamental in pursuit of his logicist project, namely the True and the False and value-ranges of first-level functions which include extensions of (first-level) concepts and extensions of (first-level) relations as special cases. Despite the pristine fundamentalness of the truth-values in logic, Frege does not treat them as irreducible objects in his logical system. In *Grundgesetze* I, §10, he identifies them with their unit classes in order to remove in a first crucial step the referential indeterminacy of value-range names to which his Initial Stipulation in *Grundgesetze* I, §3 gives rise.[5] The Initial Stipulation reads as follows:

> I use the words "the function $\Phi(\xi)$ has the same value-range as the function $\Psi(\xi)$" generally as coreferential [*gleichbedeutend*] with the words "the functions $\Phi(\xi)$ and $\Psi(\xi)$ always have the same value for the same argument".

The Initial Stipulation is couched in a second-order abstraction principle and can be regarded as the informal predecessor of Basic Law V. In *Grundgesetze* I, §20, Frege converts the stipulative character of the former into the assertoric, axiomatic mode of the latter. The Initial Stipulation will play a key role in Part II of this essay when its creativeness or non-creativeness is at issue.

My plan in this essay is as follows. In Part I, I critically discuss Frege's mathematical platonism in *Grundgesetze* with side glances at *Grundlagen*. In what follows, I comment on his conception of the non-creativity of rule-governed explicit definitions with special emphasis on Frege's logicist key definition (= the definition of the cardinality operator in *Grundgesetze* I, §40) and conclude with reflections on what I call weakly creative definitions.

In Part II, I intend to provide a critical examination of Frege's position in *Grundgesetze* II, §146 combined with an attempt to figure out the extent to which his anti-creationist position could be defended against the objections that a creationist opponent may raise to the introduction of value-ranges via the Initial Stipulation and their subsequent complete determination in a piecemeal fashion (*Grundgesetze* I, §3, §10–§12)—henceforth referred to as the *first procedure*. I call Frege's syntactic development of the whole range of objects and functions that arithmetic deals with from the primitive functions of his logical system the *second procedure*. He refers to the second procedure in *Grundgesetze* II, §147, but without calling it the *second* procedure and without explaining it. However, he seems to be aware that the second procedure may also give rise to a creationist charge. This is the main topic of Part III of this essay.

Footnote 3 continued
(2011) and (2012), Panza and Sereni (2019) and Schirn (2018), (2019), (2023a) and (2023b). On Frege's conception of logic see Goldfarb (2001) and Blanchette (2012).

4 The most prominent example is probably Joan Weiner; see her "non-standard" interpretation of Frege's philosophy and logic in Weiner (1990), (2010) and (2020).

5 The phrase "Initial Stipulation" has been coined by Heck; see, for example, Heck (2012). In *Grundgesetze* I, §3, there is not yet any comment (in purely semantic terms) on the shortcoming of the Initial Stipulation. Perhaps in order to make the matter exciting for the reader, Frege does not "reveal the mystery" before he has reached §10.

I shall flesh out the scenarios of the envisioned dispute between Frege and an imagined creationist opponent in such a way that the burden of proof lies with Frege. In doing so, I shall largely side with Frege's way of presenting the creativity/non-creativity issues in *Grundgesetze* II, §146 and §147. However, I shall assume that his rival is philosophically more sophisticated than, for example, the creationist mathematicians Hankel, Stolz and Dedekind[6] and almost uncompromising. I shall further assume that the opponent is familiar with both the exposition of the concept-script in *Grundgesetze* I (§1–§48) and Frege's critique in *Grundgesetze* II of the creation of mathematical objects as well as with the arguments that he presents in §146 to invalidate a possible creationist objection to his introduction of value-ranges. Furthermore, the opponent is supposed to be familiar with Frege's observations in *Grundgesetze* II, §147. In short, in my narrative the opponent's attitude towards Frege's critique of mathematical creation and his attempted defence of the non-creativeness of the first procedure is not flattering: people who live in glasshouses should not throw stones.[7] However, putting Frege on the spot in the imagined debate—without offending against the principle of fairness—does not mean that his rival always gets the edge on him. In any event, I trust that the envisioned dispute between our two protagonists will reveal in sharp outline the pros and cons of Frege's position vis-à-vis mathematical creation in *Grundgesetze* II, §146 and §147. The two main issues that I discuss in Part II are: (a) Has Frege created value-ranges in general in *Grundgesetze* I, §3? (b) Has he created the unit classes of the True and the False in §10? In Part III, I argue that not only in *Grundgesetze* II, §146 but also in §147 Frege falls short of defending his anti-creationist position. I further argue that on the face of it his creationist rival gains the upper hand in the envisioned debate in more than one respect.

What are the new perspectives that I introduce in my essay? Let me briefly mention those which I consider the most important. On this occasion, I also say a little more about the connections between the central issues that I discuss in this essay.

First: In Part I, I discuss in detail, seemingly for the first time in the relevant literature on Frege's *Grundgesetze*, a conflict to which his reference-fixing strategy for value-range names (and also for numerical singular terms which are defined in terms of the former) gives rise in the light of his arithmetical platonism: numbers are logical objects that exist independently of human agents, their mental acts and processes and their language. Could Frege uphold this view in the formal setting of *Grundgesetze*? Probably not. The issues that I discuss in Sect. 1.1 are intimately related to my analysis of Frege's failed attempt in *Grundgesetze* II, §146 to argue convincingly for the non-creativeness of the piecemeal endowment of canonical value-range names with unique references in *Grundgesetze* I, §3 and §10 (see Part II).[8] Recall that both in *Grundgesetze* II, §146 and §147 Frege attempts to defend his anti-creationism against

[6] They are among the targets of Frege's criticisms in *Grundgesetze* II besides Cantor's and Weierstraß's views. Frege's crusade against the formalist mathematicians Heine and Thomae is occasionally long-winded. Nonetheless, I find it philosophically stimulating.

[7] In *Grundgesetze* II, §147, Frege confines himself to stressing the non-arbitrary and constrained nature of the second procedure. He does not defend its putative non-creativeness.

[8] In *Grundgesetze* I, §146, Frege mentions also *Grundgesetze* I, §9, although §9 does not play a relevant role in the reference-fixing process regarding value-range names; see sect. 2.6 of this essay. Once the second-level function-name "$\acute{\varepsilon}\varphi(\varepsilon)$" is available in the formal language of *Grundgesetze* and we are entitled to apply

the potential charges of an imagined creationist opponent. In these sections, he does not expressly invoke his logical and arithmetical platonism with the aim of backing up his anti-creationist stance, but I assume that in both sections his platonism figures in the background as a kind of backbone for his anti-creationism. Moreover, I imagine that Frege might actually have appealed to his platonism in *Grundgesetze* II, §146 and §147 had his anti-creationism been under forceful attack and had he considered himself to be in final distress to uphold his anti-creationist position. In short, in order to provide new and enlightening analyses of the topic 'Frege's platonism and mathematical creation', it is essential to discuss not only the shortcomings of Frege's defence of his anti-creationism in *Grundgesetze* II, §146 and §147 but also the serious conflict to which his platonism gives rise when in the formal environment of *Grundgesetze* he comes to fix the semantics of value-range names and numerical singular terms. More specifically, the conflict is that in the formal framework of *Grundgesetze* Frege could hardly reconcile the independence claim of his platonism with the piecemeal reference-fixing strategy concerning value-range names (= the first procedure) that he pursues in *Grundgesetze* I, §3, §10–12. Thus, there is a direct connection not only between (a) the stepwise semantic procedure regarding value-range names and Frege's advocacy of anti-creationism in *Grundgesetze* II, §146—this connection is almost omnipresent throughout Part II of my essay—but also between (b) the afore-mentioned advocacy and the viability of his platonism within his overall philosophy of arithmetic. As hinted at above, metaphorically speaking, Frege's platonism can be seen as pulling the strings behind the scenes not only in §146 but also in §147. Analyzing these key connections as well as the connections that exist between the syntactic development of the subject matter of arithmetic in *Grundgesetze* (= the second procedure) and the credibility of Frege's anti-creationism and platonism are in the focus of my investigation. It might be worth mentioning that in the present essay the connections are analyzed in detail for the first time.

Second: At the end of Part I, I discuss the notion of weak mathematical creativeness versus the notion of strong mathematical creativeness within the framework of *Grundgesetze*. As far as I can see, this is another new aspect in the literature on Frege. I consider the discussion of this aspect important since it avoids drawing a possibly biased and one-sided picture of Frege's conception of mathematical creation.

Third: As I already observed, Part II of my essay is dominated by the imagined dispute between Frege and an uncompromising creationist opponent. The dispute concerns the question of whether the piecemeal reference-fixing procedure concerning value-range names is a creation of value-ranges or not. Taking up the argument that I sketched at the beginning of this summary, it can be said that Frege's anti-creationist mission in *Grundgesetze* §146 and §147 stands or falls with the viability or non-viability of his platonism, if the latter is assessed in the light of his reference-fixing strategy regarding value-range names (and numerical singular terms) in the formal setting of *Grundgesetze*. The importance of this complex issue in Frege's overall philosophy of arithmetic is more or less obvious. Yet as I argue in my essay, Frege's anti-creationism is already in jeopardy on its own terms, since his defence of it in

Footnote 8 continued
the rule of insertion as one of the two permissible modes of forming new names from the primitive function-names, we may regard any term that results from the insertion of a monadic first-level function-name into the argument-place of "$\acute{\varepsilon}\varphi(\varepsilon)$" as a *canonical value-range name*.

Grundgesetze II, §146 and §147 does not withstand scrutiny. Returning to the innovative potential in Part II of my essay, I wish to point out that analyzing in detail Frege's anti-creationism in *Grundgesetze* II, §146 in the context of an imagined dialogue with a creationist rival who makes Frege dance to his tune with counterarguments that are difficult to refute is a novelty in the debate.

Fourth, an additional remark regarding the topic "connections" may be in order. The relevant connections between *Grundgesetze* II, §146 and *Grundgesetze* I, §3, §10–12 as well as between *Grundgesetze* II, §147 and Frege's development of the whole wealth of objects and functions that arithmetic deals with by applying the formation rules are explained in Part II and Part III of my essay and the main points are always succinctly summed up. Plainly, it would make little sense to invent connections which are not there, neither in *GGA* II, §146 and §147 nor in *GGA* I, §3, §9-§12 nor in their synopsis. Here is a case in point. If we focus on the connection that in *Grundgesetze* II, §146 Frege says to exist between *Grundgesetze* I, §3, §, 9 and §10 and a potential creationist charge, we face a slightly awkward situation regarding §9. The purported connection between the introduction of the value-range notation in §9 and the potential creationist charge—Frege thinks that an opponent may charge him with having created value-ranges in *Grundgesetze* I, §3, §9 and §10—dissolves into thin air if we take a closer look at §9 (see Sect. 2.6 of this essay).

Fifth: In Part II, I further present the first systematic and comprehensive analysis of the relation between the Initial Stipulation and Basic Law V. The analysis is crucial for appropriately assessing Frege's defence of his anti-creationism in *Grundgesetze* II, §146.

Sixth: In Part II, I add another piece of the jigsaw puzzle to the analysis of the relation between Frege's platonism and his anti-creationism. In particular, I discuss for the first time (a) the charge of creativity that Frege's antagonist may level against the transsortal identifications of the truth-values with their unit classes in *Grundgesetze* I, §10 and Frege's potential defence of his stipulations, and (b) the question of whether the potential creationist charge that in *Grundgesetze* I, §10 Frege outright offends against the canons of his mathematical platonism might be defused.

Seventh: As far as I am aware, my essay presents, in detail for the first time, the argument that Frege's development of the objects and functions that arithmetic deals with, as outlined by me in Part III, provides his creationist opponent with a real target. It is therefore not surprising that Frege fails to defend effectively his anti-creationist position in *Grundgesetze* II, §147. The purely syntactic procedure to which he most likely appeals in *Grundgesetze* II, §147 differs fundamentally from the reference-fixing procedure regarding value-range names to which he refers in *Grundgesetze* II, §146. On the face of it, the only strong connecting link between the two procedures is Frege's attempt to defend his anti-creationism from different points of view, a semantic and a syntactic. So far his development of the subject matter of arithmetic by applying the formation rules has not been analyzed in the context of *Grundgesetze* II, §147. Thus, my discussion may open up another new perspective regarding the assessment of Frege's anti-creationism and platonism in his overall philosophy of arithmetic.

By developing these new perspectives and their interrelations I intend to fill a gap in current Frege research. I trust that my essay is of value and perhaps an inspiration not only for Frege scholars but also for philosophers of mathematics who are interested in the topic *platonism and mathematical creation* in general. In this connection, it is important to note that the issues addressed in *Grundgesetze* II, §146-§147, in contrast to those dealt with in *Grundgesetze* I, §3, §10–§12, are not clearly delineated by Frege and suffer from considerable vagueness. In §147, there is not even a trace of an attempt to proceed by placing brick upon brick. The lack of clarity especially in §147 may contribute to the limited attention that these topics have received in scholarly work. Due to the intricacy of the subject I deal with, the insights that I lay out do not fall into the reader's lap. But I have tried to present them as clearly as possible, not least by giving the relevant connections their due. To be sure, providing an in-depth analysis of the most important issues involved in Frege's platonism and anti-creationism in *Grundgesetze* is not a task that could be accomplished in a short format.

1 Part I

1.1 Platonism, the reference-fixing strategy for value-range names and numerical singular terms and the existence of value-ranges and numbers in *Grundgesetze der Arithmetik*

In *Grundlagen*, §96, Frege explicitly articulates his stance against creationism in mathematics (arithmetic) and, in the same breath, seems to allude to his arithmetical platonism. He does so in the course of criticising Hankel's formal theory of arithmetic (cf. Hankel, 1867). Frege writes (§96):

> Let us create numbers that permit the diverging series to be summed! No! Even the mathematician cannot arbitrarily create something any more than the geographer; he too can only discover what is there and name it.

In this quotation, Frege tacitly presupposes the non-real or non-actual (*nicht-wirkliche*) nature of mathematical objects, in particular of numbers. In parts I–III of *Grundlagen*, he scrutinizes the views of "certain writers" on the nature of arithmetical propositions, on the concept of number and on unity and one. In several places of his critique, he makes it clear that he construes numbers as objects that are intrinsically non-real or non-actual, i.e. non-spatial, non-temporal and causally inefficacious. Numbers are neither accessible to our sensation, intuition and imagination, nor are they subjective like ideas or mental processes in general. They are objective. Note that neither in *Grundlagen* nor in *Grundgesetze* does Frege speak of abstract objects on his own account nor does he use the term "logical object" in *Grundlagen* as he does later in *Grundgesetze* II (pp. 86, 149, 153, 253, 265). Both in *Grundlagen* and *Grundgesetze*, the word "non-real" or "non-actual" is a synonym of "abstract" as present-day philosophers use this word when they deal, for example, with the objects of mathematics. In the Introduction, I suggested that already in *Grundlagen* Frege presumably regarded numbers—in particular cardinal numbers which are in the focus of his investigation—as logical objects. Perhaps he did so at least on informal grounds by appeal to their universal applicability. The question of whether he thought that in *Grundlagen*, §68 he succeeded in defining the cardinality operator in purely logical terms cannot definitely be answered. It is true that in *Grundlagen* Frege defines equinumerosity in second-order logic as the one-to-one correlation of the objects falling under F with those falling under G, although with undue delay in §72. He probably thought that this definition yields another good argument for the logical nature of cardinal numbers. Yet at the same time he presumably was aware that simply by presupposing that one knows what extensions of concepts are, he did not appropriately introduce them as the definitional target objects of cardinal numbers and overall did not provide a sufficient formal (non-intuitive) ground for their logical nature.[9]

In the quotation from *Grundlagen* above, Frege does not merely deny that the mathematician creates numbers. Rather, he specifies the denial by juxtaposing the adverb "arbitrarily". On the face of it, this seems to leave open the possibility that mathematical objects might be created if the creation is carried out in a strictly rule-governed fashion. Be that as it may, if the abstract character of mathematical objects is made explicit by appeal to related remarks in *Grundlagen*, we may extract, from the quotation above, the thesis that characterizes mathematical platonism in *Grundlagen*: Arithmetic deals with non-actual (abstract) objects that exist independently of our "mental life".[10] My interpretation, as far as it goes, reinforces the assumption that in Frege's opinion mathematical platonism goes along with anti-creationism. Note that neither in *Grundlagen* nor in *Grundgesetze* nor in any other of his writings does Frege use the term "platonism".[11] It is later philosophers who impose this term on his mathematical and logical outlook.

Not surprisingly, in *Grundgesetze* Frege goes on to endorse platonism. However, as I indicated at the outset of this essay, it now appears in a slightly modified form: numbers, in particular cardinal and real numbers, are explicitly conceived of as logical objects, as value-ranges of functions. Their identity conditions are governed by a basic law of logic. In *Grundgesetze*, Frege does not go to great lengths to explain the notion of a logical object, although it is supposed to play a key role in his logicist project.[12] The fundamental question of arithmetic (= the lynchpin of its intended logical foundation) was for him: "how do we apprehend logical objects, in particular, the numbers?" (Frege, (1903), p. 295; see also Frege, (1903), p. 153). Frege's (implicit) answer in his

9 On Frege's definitional strategy regarding cardinal numbers in *Grundlagen* see Schirn (1983), (1996), (2003) and (2024b)

10 This phrase is Frege's. Resnik's thesis (1980, pp. 164, 166) that in *Grundlagen* Frege combines, with regard to arithmetic, Kantian transcendental idealism with ontological platonism is off target and indefensible. See the critical discussion in Schirn (1987).

11 Frege does not use the term "logicism" either.

12 In *Grundlagen*, Frege probably holds that in contrast to the (second-level) relation of equinumerosity, parallelism cannot be reduced to purely logical relationships. One could perhaps account for this difference by distinguishing logically pure extensions of concepts from logically impure extensions. It is not clear whether in Frege's view the extension of the concept *parallel to line a* is less a logical object than the extension of the concept *equinumerous with the concept F*. Does he think in *Grundgesetze* that only the extension of a first-level concept or a first-level relation that in his view belongs intrinsically to logic—such as negation, identity and the conditional—is a full-fledged logical object, whereas the extension of what he considers to be a non-logical (first-level) concept or a non-logical (first-level) relation is only a thin or modest logical object? My hunch is that he does not. In any event, there is not even a hint in *Grundgesetze* or elsewhere in Frege's writings in the pre-Paradox period that he would deny straightaway that the extension of an apparently pedestrian concept like the concept *horse* is a logical object. See also Schirn (2006b).

pre-Paradox period was: We apprehend logical objects as value-ranges of functions.[13] He knew that with this answer his foundational project stands or falls.[14]

In the period of *Grundgesetze*, as in the period of *Grundlagen*, numbers are considered to exist independently of our mental acts and mental processes including our specifically mathematical and logical reasoning. Moreover, in the period of *Grundgesetze* as in the period of *Grundlagen*, Frege's platonism goes hand in hand with his advocacy of anti-creationism in arithmetic. Bearing the comparison in mind, let us now take a closer look at his arithmetical platonism in the period of *Grundgesetze*, with special emphasis on the reference-fixing strategy regarding value-range names which Frege pursues in the exposition of the concept-script before he carries out the formal proofs of the basic laws of cardinal number. For reasons that will soon become clear, I begin with observations on singular termhood in the language of arithmetic.

Frege would probably claim that the theorems of cardinal arithmetic such as "If One is the cardinal number of objects falling under a concept and if a first object falls under this concept and likewise a second, then these objects coincide" (Theorem 117) or "If Endlos is the cardinal number of a concept and if the cardinal number of another concept is finite, then Endlos is the cardinal number of the concept *falling under the first or under the second concept*" (Theorem 172) are held to be true by most if not by all academically trained mathematicians. Although from an informal point of view, Frege apparently considered the truth of these theorems indisputable, he did not regard them as self-evident, whence the need to prove them, to justify them deductively in the logical system of *Grundgesetze* once they were clad in the formal garb of concept-script sentences. According to Frege, every numerical expression in a *Grundgesetze*-theorem of, say, cardinal arithmetic—regardless of whether the relevant theorem is couched in informal (metalinguistic) or formal (concept-script) vocabulary—which satisfies his criteria of singular termhood purports to refer to a cardinal number qua object. Examples for numerical singular terms in the formal language of *Grundgesetze* are numerals introduced by definition such as "1" or "∞_1" or terms formed by applying the cardinality operator to a value-range name such as, for example, "$N\acute{\varepsilon}(\neg(\varepsilon \cap u \rightarrow \varepsilon = b))$" (cf. Frege, 1893, §93) or "$N\acute{\varepsilon}(\neg\varepsilon \cap u \rightarrow \neg(\varepsilon \cap v \rightarrow \varepsilon = c))$" (cf. Frege, 1893, §127).[15] Frege's criteria for singular termhood, which I need not spell out and discuss here in every detail, preeminently aim to spot the distinctive place that the relevant term occupies in the (logical) syntax of the theorem. Numerical singular terms are characteristically used on the two sides of the equals-sign both in an informal and a formal language. Looking at the concept-script proofs in *Grundgesetze* I we come across, for example, the cardinal number equation "$N\acute{\varepsilon}(\neg(\varepsilon \cap u \rightarrow \varepsilon = b)) = N\acute{\varepsilon}(\neg(\varepsilon \cap v \rightarrow \varepsilon = c))$" (cf. §93), whose truth-conditions

[13] The existence of value-ranges is here presupposed.

[14] In *Grundgesetze* II, §147, Frege emphasizes that without a basic law of logic that provides a means of grasping logical objects a foundation of arithmetic would be impossible. This may partly explain why in the aftermath of Russell's Paradox Frege left no stone unturned to save his logicist project from doom. It is not clear when exactly Frege came to realize that his logicist project was beyond repair, but I assume that around 1906 or even earlier he definitely knew that he had met his Waterloo.

[15] I employ Frege's symbol "∞_1" from *Grundlagen* for the number *Endlos* and dispense with the diagonal stroke crossing "0" and "1" which in *Grundgesetze* Frege uses to distinguish the cardinal numbers 0 and 1 from the numbers 0 and 1; cf. Frege (1903), §157. Likewise for the sake of convenience, I use "N" instead of Frege's special concept-script sign for the cardinality operator.

are governed by both the right-to-left direction of Hume's Principle[16] and Basic Law V: an embarassment of riches. Frege further mentions the application (in a given natural language) of the definite article to numerals and to other numerical terms as a criterion of singular termhood.[17] In short, the place of a singular numerical term in a sentence—formal or informal or semi-formal—can never be taken by a predicate or function-name, except by breaking the rule of well-formedness. But that may result in syntactic nonsense. For the time being, these remarks on singular termhood in the language of arithmetic may suffice.

It seems that in Frege's judgement the inference from the occurrence of a singular numerical term in a true arithmetical sentence to the existence of the number to which it purportedly refers is not only intuitively plausible but also legitimate. Although there is to my knowledge no direct evidence in his writings that he considers this inferential connection to be a defining characteristic of his arithmetical platonism, there is *prima facie* no compelling reason why he should refuse to do this. If I am right, then drawing the inference in question does not imply the view that numbers exist *only and exclusively in so far* as they are referred to by numerical singular terms occurring in true sentences. If it did imply this, endorsing the inference would seem to clash with Frege's view that the existence of numbers conceived of as logical objects is mind- and language-independent. In short, I fail to see why—outside the formal system of *Grundgesetze*—Frege could not consistently claim, in the same breath as it were, (a) that numbers exist independently of human minds and (b) that their existence follows from the occurrence of singular numerical terms referring to them in true sentences. Consider, by way of comparison, a simple example. Although the Eiffel Tower is a human creation—in contrast to numbers, as Frege might wish to emphasize—we would, I think, almost unanimously agree that its existence (in the year 2024) does not depend on our minds and language and, hence, not on true statements in which we refer to it. However, without contradicting this view, we could sensibly say that the existence of the Eiffel Tower follows from the fact that the statement "The Eiffel Tower is the tallest structure in Paris and its most famous landmark" is true. Some may object that the comparison I just drew between, say, the cardinal number 2 and the Eiffel Tower within the bounds of Frege's conception of mind- and language-independent existence, lacks cogency. But I do not think that it is inappropriate. In any event, in general we can say that for Frege numbers are what numerical singular terms refer to, what numerical first-level predicates and first-level function-names apply to and what first-order number-theoretic quantifiers range over.

Stepping outside the formal system of *Grundgesetze*, Frege considers cardinal numbers (and numbers in general) on informal grounds to be logical objects. In my opinion, "logicality" is an essential component of his arithmetical platonism (and not just abstractness), in addition to the assumed objecthood of the numbers which in *Grundlagen* is entirely based on syntactic considerations, whereas in *Grundgesetze* it is not specifically analyzed.[18] As I mentioned earlier, the logical objects in the two volumes

[16] On Frege's proof of Hume's Principle in *Grundgesetze* see May and Wehmeier (2019) and Schirn (2016).

[17] Wright (1983, p. 54) objects that this criterion is, at best, very weak. See his analysis of Frege's characterization of singular terms in Wright (1983), Section IX, pp. 54–64.

[18] There is no obvious reason why in Frege's foundational work the objecthood of value-ranges should be considered to be better grounded than the objecthood of numbers, say, cardinal and real numbers. Clearly,

of *Grundgesetze* are the truth-values, value-ranges and cardinal and real numbers. Like the truth-values, cardinal numbers are identified with special value-ranges. Frege also proclaimed the definitional identification of real numbers with special value-ranges, but, overshadowed by Russell's Paradox, did not proceed to define them explicitly as Relations on Relations, as he had planned before he knew of the Paradox. The attribute "logical" in the phrase "logical object" is probably meant to highlight the salient feature of logic which in Frege's view is utmost generality and universal applicability. Needless to say, logical objects in his sense are at the same time intrinsically abstract. The set of logical objects in *Grundgesetze* is a subset of the set of abstract objects. I assume that Frege considers *logical objects* to be the most relevant subject matter for his arithmetical platonism.[19] It seems that with the exception of *Grundgesetze* II, §147 (logical) functions, concepts and relations take a backseat in his platonism and anti-creationism. Let me add that according to Frege the universal applicability of cardinal numbers rests on the fact that we ascribe them to concepts under which we subsume both the spatial and temporal and the non-spatial and non-temporal. Furthermore, cardinal numbers themselves can be counted. Accordingly, the laws of cardinal arithmetic govern the domain of what is countable which for Frege is the widest domain of all since everything thinkable belongs to it.[20]

How about the third defining characteristic of Frege's arithmetical platonism as I formulated it earlier, namely the *mind-independent* existence of numbers qua logical objects?[21] If he had been urged to explain or defend his arithmetical platonism and anti-creationism independently of the strict constraints that he imposes on the formal system of *Grundgesetze*, he might have replied along these lines (although not entirely *verbatim*):

> Numbers are logical objects that we do not bring into being, neither by abstraction nor by definition. It seems to me that trying to create numbers via definition

Footnote 18 continued
if extensions of concepts (or more generally, value-ranges of functions) were not full-fledged objects for Frege, he could not define the cardinals in terms of them, for he believes to have established the objecthood of the cardinals in *Grundlagen* before he identified them with extensions of second-level concepts via the explicit definition of the cardinality operator in §68. Analogous remarks apply to Frege's definition of the cardinal numbers in *Grundgesetze*. There is not a trace of evidence in *Grundgesetze* that he changed his mind in this respect in the intervening years. Thus, it is not the objecthood of the cardinal and real numbers that has to be grounded or needs further grounding by defining them as value-ranges, but only their logical nature. For Frege, both numbers and value-ranges are objects in their own right.

19 The conception of numbers as *logical objects* is at the same time a characteristic mark of Frege's logicism.

20 In the course of developing his theory of magnitude in *Grundgesetze* II, Frege holds that the application of the real numbers is not restricted to any special types of magnitude, but rather relates to the domain of the measurable, which embraces all types of magnitude. However, the domain of the measurable is arguably not as comprehensive as the domain of the countable. Perhaps Frege thought that due to this circumstance only cardinal arithmetic is a branch of logic in the fullest sense.

21 Linnebo (2023, Sect. 4.1) discusses briefly the issue of mind- and language-independence as part of the "standard" definition of mathematical platonism which in my view also applies to Frege's position. Linnebo claims that a natural gloss on independence is the following counterfactual conditional: Had there not been any intelligent agents, or had their language, thought, or practices been suitably different, there would still have been mathematical objects. He points out, though, that the thesis of "robust independence" may seem more appropriate: Mathematical objects are metaphysically on a par with ordinary physical objects. I suggest that outside of the formal framework of *Grundgesetze* we attribute the robust version of independence to Frege.

> or some kind of abstraction, understood in a metaphysical or in an ontological sense of creation (for example, à la Stolz and Dedekind), is an exercise in futility. It is bound to fail right from the start. Strictly speaking, Stolz's definitions, for example, are not creative at all since he does not and cannot succeed in creating (bringing into existence) the numbers. He only intends to create them by means of his definitions. Calling the latter nonetheless *creative* is a *façon de parler* that I adopt for the sake of convenience. Thus, in the case of Stolz's definitions, for example, the words "creative" and "creation" apply at bottom rather to the intention underlying them than to the definitions themselves. As I say: "Creative definitions are a first-class invention" (*Grundgesetze* II, § 143). Now, what the working mathematician typically and essentially does in his or her arithmetical practice is this: he or she discovers numbers and other mathematical objects, grasps them, calculates and reasons with them. These mathematical objects I consider to exist prior to his or her discovery and grasp and independently of any (failed) attempt to conjure them into existence either by means of a definition (or what, for example, Hankel and Solz take to be a definition) or some kind of abstraction, say in the style of Dedekind's introduction of the natural numbers. In sum, numbers and logical objects in general (as I use the term "logical object") exist independently of our mental acts and mental processes and also independently of the language in which we refer to them and make statements about them. Thus, regarding the feature of mind- and language-independence, I consider numbers and logical objects in general to be on a par with physical objects.

However, it is difficult to figure out how Frege might have argued for the alleged mind- and language-independence of numbers and logical objects in general if his interlocutor or critic had insisted on a more detailed answer and some kind of justification for Frege's platonist stance. If we keep the foregoing largely informal considerations in mind and cast a glance at the reference-fixing strategy that Frege pursues in the formal framework of *Grundgesetze* regarding value-range names—recall that he defines numerical singular terms that purport to refer to cardinal numbers in terms of value-range names—a conflict seems almost inevitable, possibly overshadowing his arithmetical and logical platonism as characterized above. Might this have escaped Frege's attention or did he perhaps think that there is no conflict at all or, if there is one, that he could lightly pass over it? Note that I am arguing in controversial exegetical territory.

The problem that in my view Frege is facing in the formal context of *Grundgesetze* regarding the advocacy of arithmetical and logical platonism with the independence clause as one of its characteristic marks can roughly be described as follows. On the one hand, it seems that by taking up Frege's anti-creationist train of thought in *Grundlagen*, in the Foreword to *Grundgesetze* (cf. p. XIII–XIV, XXIV) or in *Grundgesetze* I, §66 or in *Grundgesetze* II (cf. §138-§147) Frege endorses an ontological version of arithmetical (and logical) platonism in the sense explained above, with emphasis on the mind-independent existence of mathematical and logical objects in general. On the other hand, in *Grundgesetze* the referentiality neither of value-range names nor of singular numerical terms is taken for granted—bear in mind that in Frege's view establishing the referentiality of singular terms occurring in true sentences ensures the existence of the objects referred to—nor the truth of the theorems of cardinal arithmetic and real analysis in which such terms occur, nor does Frege take as given the existence of numbers (and of value-ranges in general), nor their logical nature, nor their mind-and language-independence. As far as the arithmetical theorems in the guise of well-formed concept-script formulae are concerned, they cannot be true (or cannot be considered to be true), unless the referentiality of their constituent expressions, in particular the referentiality of canonical value-range names and numerical singular terms (simple and complex) is secured. Yet even if the referentiality of those terms is considered to be guaranteed, the theorems must still be justified within the bounds of the logicist project by carrying out gapless proofs that start from logical axioms and explicit definitions qua first premises of the proofs. From Frege's point of view, only in this way are the theorems shown to be truths of logic. In the final analysis, establishing the logical nature of cardinal numbers also depends on that. Frege thinks that Basic Law V is a powerful tool in securing the "logicality" of cardinal and real numbers, but to all appearances he does not consider it omnipotent in this respect.

In *Grundgesetze*, Frege attempts to establish and justify the supposed existence of value-ranges in general and foresightedly the existence of cardinal and real numbers qua special value-ranges by pursuing a piecemeal semantic strategy. In its core, it is an old strategy in a new guise. More specifically, it is known from the heuristic contextual definition of the cardinality operator in *Grundlagen* (if we suitably transform the case of directions to the case of cardinal numbers). The relevant difference is that the strategy in *Grundlagen* is (tentatively) definitional and informal, whereas in *Grundgesetze* it is non-definitional and semi-formal. The implementation of the strategy in *Grundgesetze* starts in §10 by fixing the values (either the True or the False) of the primitive first-level functions introduced prior to §10 "for all possible objects as arguments"—or backward-looking: in §3 with the Initial Stipulation. Carrying out the strategy specified in §10 eventually boils down to fixing the values of $\xi = \zeta$ for the truth-values and value-ranges as arguments, in contrast to the previous announcement in §10 that the values of the primitive first-level functions so far introduced have to be fixed "for all possible objects as arguments". In my view and in accordance with Frege's definition of the application operator in *Grundgesetze* I, §34 (but also in line with other remarks that he makes in *Grundgesetze* I, for example in the long footnote to §10), this phrase is to mean: fixing the values of the primitive first-level functions for objects of an all-encompassing domain.[22] Unfortunately, Frege does not explain why he confines himself to taking the truth-values and value-ranges as arguments of the primitive first-level functions when he attempts to remove the referential indeterminacy of value-range names arising from the Initial Stipulation and diagnosed at the outset of §10.

It is only in *Grundgesetze* I, §31 that Frege puts the reference-fixing procedure concerning value-range names behind himself by carrying out a proof of referentiality for his formal language. Not surprisingly, the main part of the proof deals with the referentiality of value-range names—and for that matter also with the referentiality of the primitive value-range operator "$\acute{\varepsilon}\varphi(\varepsilon)$". Thus, the overall strategy with the aim of

[22] On the size of the first-order domain of the logical system of *Grundgesetze*, see the analysis in Schirn (2018).

justifying the use of canonical value-range names and of numerical singular terms in the formal language is implemented in two steps: (1) In the first place, Frege attempts to fix the references of canonical value-range names via informal stipulations—chiefly in §3 and §10. He puts the finishing touches to that procedure in §11-§12. (2) In §31, Frege attempts to demonstrate that in §3, §10-§12 he has succeeded in conferring determinate references on canonical value-range names, first and foremost to what he calls "regular value-range names".[23] The proof in §31, which in accordance with Frege's principle of compositionality regarding reference proceeds by induction of the complexity of concept-script names, is just the "proof-theoretic" counterpart or mirror image of the (attempted) piecemeal endowment of value-range names with unique references in §3 and §10-§12. Frege knew of course that even *canonical* value-range names do not wear the requisite referentiality on their sleeves. To be on the safe side, the referentiality must be proved.

In my view, the proof in §31 essentially rests not only on the principle of compositionality regarding referentiality but, as we might say by parity of reasoning, also on the context principle from *Grundlagen*, although Frege does not make this explicit. This observation seems to be directly relevant for an adequate assessment of his semantic procedure concerning value-range names, not least in the light of Frege's platonism which is still under critical review. Far from giving rise to a head-on collision, the two semantic principles work hand in hand in the proof of referentiality. Taken at face value, the criteria of referentiality that Frege states in §29 appear as a *generalized* version of the context principle since truth-value names (sentences) are not explicitly singled out as figuring as the exclusive "semantic target names" regarding referentiality for those names whose supposed referentiality is subjected to a test and which in accordance with the criteria of referentiality are crucially constituents of the semantic target names. Nonetheless, quite in the spirit of these criteria, it can persuasively be argued that it is only in the context of a truth-value name or sentence—not to be confused with a concept-script sentence—that a regular value-range name, or more generally, a canonical value-range name, refers to something (cf. the detailed analysis in Schirn, 2018). In short, if truth-value names (sentences) are referential—and this is what Frege presupposes outright in order to get the proof off the ground – then the value-range names occurring in them likewise refer to something. In particular, picking out the true instances among the sentences in which a regular or canonical value-range name occurs, and at a later stage in the foundational project also the true sentences (formal theorems) in which concept-script terms occur that purport to refer to a cardinal or real number, entitles him, on his own understanding, to assert – within the bounds of his formal system—not only the existence of value-ranges but also that of cardinal and real numbers since cardinal numbers are defined in terms of the former and Frege intended to define the real numbers also as special value-ranges.

However, it seems that by Frege's own lights it would not be enough, in the final analysis, to highlight the logical nature of cardinal and real numbers by appeal to their definitional identification with special value-ranges and the axiom that governs them: Basic Law V. What in the formal environment of *Grundgesetze* is additionally required

[23] In *Grundgesetze* I, §31, Frege calls value-range names that are formed from referential monadic first-level function-names *regular value-range names* (*rechte Wertverlaufsnamen*).

regarding the "logicality" of, say, cardinal numbers, is invoking or arguing for the fact that the names which occur in the *definiens* of the definition of the cardinality operator are introduced either by elucidations that fix the references of primitive names most of which intrinsically belong to the fundamental part of logic or by definitions that are framed in purely logical terms. As I already pointed out, in order to establish the logical nature of cardinal numbers beyond doubt, it might also be required that the theorems of cardinal arithmetic are shown to be logical truths. Thus, it seems that a complete justification of the "logicality" of cardinal numbers (and of real numbers) in *Grundgesetze* would not have been a one-way street for Frege, had he made an attempt to argue for it, which unfortunately he did not. In my view, there is quite a bit of interdependence between several aspects to be taken into account here. In the end, the crux is that Frege merely presupposes the purely logical nature of Basic Law V but fails to argue for it; see the analysis in Schirn, (2006a), (2019). In Schirn, (2019), I explain what, from Frege's pre-Paradox point of view, might have cast doubt on the purported purely logical nature of Basic Law V.

In the preceding exposition, I gave chapter and verse for Frege's reference-fixing procedure concerning value-range names. I did so with the aim of bringing the relevant conflict into sharp focus. Let me try to pinpoint again, now in slightly different terms and based on the foregoing analysis, the problem that in my view Frege is facing with regard to his arithmetical and logical platonism in *Grundgesetze.* Could Frege reasonably claim, within the formal setting of *Grundgesetze*, that value-ranges in general and cardinal and real numbers in particular exist independently of (a) the step-by-step procedure of fixing the references of value-range names, (b) proving inductively their referentiality and (c) fixing the references of simple numerical terms via explicit definitions, and thus independently of the overall logical and linguistic framework in which the relevant semantic procedure is carried out? At present, I refrain from giving an affirmative answer. Appealing to a double standard—say, by stressing the alleged relative independence of the formal point of view of an informal point of view regarding arithmetical platonism—might be considered problematic. Throwing off the semantic shackles of Frege's concept-script seems to be of no real help here. In the formal framework of *Grundgesetze*, it seems that numbers and value-ranges in general exist *only* insofar as they are referred to by singular terms that occur in true sentences. Is there a compelling argument against this? If so, how would it go? And how about the purported non-creativeness of the reference-fixing process concerning value-range names? In Part II of this essay, we shall see that it would have been a major challenge for Frege to reject the related charges of a creationist opponent as groundless. Yet for the time being, I do not wish to rule out that thanks to a stroke of genius Frege might have been able to argue against all odds that independence and dependence regarding the existence of logical objects, appropriately interpreted, can be reconciled after all, had his platonism in *Grundgesetze* been the target of a fierce attack.

As opposed to what in an ideal scenario might have turned out to be a blessing in disguise for Frege, imagine that on second thought he realized that he had failed to endow canonical value-range names with unique references in §3, §10-§12 or alternatively that in the course of reconsidering the proof in §31 he had found a fly in the ointment regarding that part of the proof which deals with the references of value-range names. Clearly, if the reference-fixing process concerning value-range names in §3, §10-§12 miscarried in Frege's view, he could have spared himself the trouble of carrying out the proof in §31. Now, if it is correct to assume that within the formal setting of *Grundgesetze* numbers and value-ranges in general exist only insofar as they are referred to by singular terms of the formal language in true sentences Frege would have to concede that in either of the two scenarios he had failed to establish the existence of value-ranges. If the initial referential indeterminacy of value-range names that Frege states and explains at the beginning of *Grundgesetze* I, §10 had not been completely removed via his subsequent additional stipulations, then it could have affected the reference of any concept-script name that purports to refer to a cardinal or real number and, therefore, would have casted into doubt the justifiablity of asserting, within the formal system, the existence of at least some cardinal and some real numbers. In that case, Frege could not have been sure that within the formal setting of *Grundgesetze* every concept-script theorem of cardinal arithmetic and real analysis is in fact true. Furthermore, some theorems may have turned out to be neither true nor false relative to this setting if one of its constituents arguably had not been endowed with a unique reference. Such a scenario, if it arose, might cast a shadow over Frege's platonism.

1.2 The non-creativity of Frege's definitions in *Grundgesetze*

At least for those who are familiar with Frege's mature theory of definition, the non-creativity of the definitions in the two volumes of *Grundgesetze* is almost obvious. A new simple sign—a first-level function-sign or an object-sign—is introduced on the right-hand side of the definitional equation, and it is stipulated that this new sign is to have the same reference and the same sense as the *definiens*, the "old" complex function-name or complex proper name that appears on the left-hand side (cf., for example, *Grundgesetze* I, §27). If Frege had been asked "What gives you the right to consider your explicit definitions essentially non-creative?", he would probably have responded along these lines: An explicit definition that satisfies my principles of definition (see *Grundgesetze* I, §33; cf. *Grundgesetze* II, §68-§85)[24] is the arbitrary stipulation that the new simple sign on the right-hand side of the definitional equation is to have the same reference and the same sense as the complex name that appears on the left-hand side. The definition is thus nothing but a means for capturing a complex reference and a complex sense in a simple name, thereby making the name easier for us to handle. In my formal system, neither a new reference nor a new sense is ever bestowed upon a sign by means of an explicit definition, let alone brought into existence.

[24] In *Grundlagen*, Frege does not yet state principles of correct definitions. The formalism that he had developed in *Begriffsschrift* plays a relatively minor role in *Grundlagen*. It seems though that in *Grundlagen* Frege was already aware of the importance of the completeness requirement or, in other words, of the requirement of the sharp delimitation of the defined function, concept or relation; see, for example, the opening remark in *Grundlagen*, §68.

In the Foreword to *Grundgesetze*, Frege states the non-creativity of (correct explicit) definition in a way which is immediately reminiscent of his statement in *Grundlagen* which I quoted at the beginning of Part I.

> Here it is crucial to get clear about what definition is and what it can achieve. Often one seems to credit it with a creative power, although in truth nothing takes place except to make something prominent by demarcation and designate it with a name. Just as the geographer does not create a sea when he draws borderlines and says: the part of the water surface bordered by these lines I will call Yellow Sea, so too the mathematician cannot properly create anything by his definition.

Frege is of course right in denying correct explicit definitions any creative potential. The functions and objects which in *Grundgesetze* I and II he defines (there are altogether 27 definitions) form a part of the ontological repertoire *before* he embarks on defining them (more accurately: their simple names). The functions and objects that he defines must be developed in the first place. As I shall explain in more detail in Part III of this essay, the development of functions and objects proceeds by applying the formation rules of the system, starting with primitive names. Whether this development is in fact a non-creative procedure, as Frege seems to assume in *Grundgesetze* II, §147, without taking a clear stance, is open to question (see the discussion in Part III of this essay).

Frege was seemingly convinced that not only the definitions that he frames but also the standard elucidations of the primitive function-names that precede the definitions – "$\acute{\varepsilon}\varphi(\varepsilon)$" is the only primitive function-name that is not introduced via a standard elucidation – escape the potential charge of being creative. Even if he conceded to the creationist opponent that all the complex functions that he develops in *Grundgesetze* by applying the formation rules of the concept-script are the product of creative acts, this would not affect the non-creativeness of the definitions in *Grundgesetze* I and II.[25] The complex functions referred to in different ways on the two sides of the definitional equations and likewise the value-ranges whose names are integral parts of most of the *definientia* can be referred to in the formal language as soon as (a) seven of the eight primitive function-names have been introduced via standard elucidations, (b) the reference of the value-range operator has been fixed via a sequence of heterogeneous stipulations and (c) the formation rules are effectively applied by starting the formation process from primitive names via insertion. In short, regarding the non-creativeness of the definitions, it does not matter whether, for example, the complex function referred to on the two sides of the definitional equation and, roughly speaking, the value-range "embedded" in the *definiens*[26] were created in the first place or not. The simple reason is that a defined complex function could not have been created twice, first syntactically via the formation of the complex function-name (= the *definiens*) that refers to it, and second via an explicit definition that confers the sense and the reference of the *definiens* on a new simple function-name.

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25 The first definition in *Grundgesetze* is that of the application operator in volume I, §34. The *definiens* contains a value-range name.

26 We find the embedding in almost every definition that Frege frames in *Grundgesetze*.

Finally, let me clarify another point regarding the nature of Frege's definitions. The first definition in *Grundgesetze* I, §34, namely that of the application operator, is the only definition in *Grundgesetze* whose *definiens* is entirely formed from primitive names. The *definientia* of all the other definitions contain at least one simple name that has previously been defined. Thus, keeping the exception of the application operator in mind, it would be incorrect to say that the complex *names* on the left-hand side of the definitional equations are available in the concept-script as soon as (a) seven of the eight primitive names have been introduced via standard elucidations, (b) the reference of the primitive value-range operator has been fixed via a mix of non-standard and standard stipulations, and (c) the syntax of the formal language has been delivered. Yet what is referred to by Frege's *definienda* (mainly functions) and what is expressed by them (mainly "unsaturated" senses) could *in principle* always be referred to and expressed by more complex names which are exclusively formed from primitive function-names by iterated application of the formation rules and do not contain any defined name. Plainly, replacing a defined simple name with the defining name would result in an enormous increase of syntactic complexity in a large number of formulae that are used in the proofs, especially if the replacement were made in the huge definitions of key concepts in Frege's theory of magnitude: the concept of a *positival class* (*Grundgesetze* II, §175) and that of a *positive class* (*Grundgesetze* II, §197). The primary purpose of concept-script definitions is to prevent excessive notational complexity. In short, all the functions and objects (0, 1 and the cardinal number *Endlos*) that Frege defines in *Grundgesetze* can be referred to in advance in a rule-governed and evidence-based way once *Grundgesetze* I, §26 and §30-§31 are completed. Conclusive evidence is supposed to be supplied in §31, where Frege carries out a proof of referentiality for his formal language.

Thanks to having imposed rigorous constraints on correct explicit definitions in *Grundgesetze* I, §33, Frege takes the non-creativeness of his definitions in *Grundgesetze* I and II more or less for granted. In particular, he saw no need to argue specifically for the non-creativeness of what he probably considered to be the key definition in his project of laying the logical foundations of cardinal arithmetic, namely the definition of the cardinality operator in *Grundgesetze* I, §40: $\Vdash\!\!-\acute{\varepsilon}(\neg\forall\mathfrak{q}(u \cap (\varepsilon\cap\rangle\int\mathfrak{q}) \rightarrow \neg\varepsilon \cap (\mathfrak{u}\cap\rangle\mathfrak{q}))) = N(u)$.[27] He probably believed that thanks to its explicitness and its being in complete conformity with his principles of definition the non-creativeness of that definition is indisputable. In fact, the composition of both the reference and the sense of the complex *definiens* can be gaplessly traced back to the primitive function-names.[28] In sum, Frege's principles of definition in *Grundgesetze* are indeed a bulwark against any potential creationist charge if it were raised in the spirit of "tit for tat". In the case of the definition of the cardinality operator, he could in addition argue that all the previous definitions on which it rests – the definition of the application operator: $||-\backslash\ \acute{\alpha}(\neg\forall\mathfrak{g}(u = \acute{\varepsilon}(\mathfrak{g}(\varepsilon)) \rightarrow \neg\mathfrak{g}(a) = \alpha)) = a \cap u$ (Frege, 1893, §34), the definition of the mapping-into by a relation: $||-\acute{\alpha}\acute{\varepsilon}(\neg(\mathrm{I}p \rightarrow \neg\forall\mathfrak{d}(\forall\mathfrak{a}(\mathfrak{d} \cap (\mathfrak{a} \cap p) \rightarrow$

27 I use here the symbol "$\int$" for the designation of the converse of a relation instead of Frege's special sign for that function-name which he introduces (defines) in *Grundgesetze* I, §39.

28 The *definiens* is exclusively formed from what Frege calls *logically* simple names (= the primitive names) and from defined names (they are linguistically simple, but, due to the complexity of their reference and sense, are not logically simple).

$\neg \mathfrak{a} \cap \alpha) \rightarrow \neg \mathfrak{d} \cap \varepsilon))) = \rangle p$ (§38), the definition of the converse of a relation: $||\!-\acute{\alpha}\acute{\varepsilon}(\alpha \cap (\varepsilon \cap p)) = \int p$ (§39), and indirectly also on the definition of the single-valuedness of a relation: $||\!-\forall \mathfrak{e} \forall \mathfrak{d}(\mathfrak{e} \cap (\mathfrak{d} \cap p) \rightarrow \forall \mathfrak{a}(\mathfrak{e} \cap (\mathfrak{a} \cap p) \rightarrow \mathfrak{d} = \mathfrak{a})) = \mathrm{I}p$ (§37) – are non-creative and moreover purely logical in nature. To conclude Part I, I discuss the notion of *weak definitional creativeness*.

1.3 Weakly creative definitions

Frege might have said, with unerring accuracy, that any definition which does not stipulate that a new simple sign (the *definiendum*) is to have the same meaning as the *definiens* – whose meaning is completely known and not merely presupposed to be known—is potentially creative. He might also have said that any definition that does not meet his key principles of definition is prone to being creative.

There is a paucity of textual evidence that in his crusade against creative mathematical definitions in *Grundgesetze* II Frege has also definitions in mind which at his discretion are not intended to bring an object or possibly a function (concept, relation) into being but offend in a considerably weaker sense of creation against his canons of definitional non-creativeness. I call those definitions accordingly "weakly creative". Their supposed creative character should ideally be assessed on a case by case basis. Clearly, we should not read mysteries into Frege's conception of definitional creation. Frege's comments on the creation of mathematical objects by Hankel, Stolz and Dedekind are the only evidential sources for what Frege understands by mathematical (definitional and abstractionist) creation. However, I refrain from asserting that Frege categorically refuses to leave margin for a weaker, non-metaphysical sense of creation, say, regarding definitions that do not meet one or more constraints that he imposes on correct explicit definitions, disregarding those constraints which are designed to accommodate slightly idiosyncratic features of his concept-script and which probably are not relevant in other formal systems.[29] Thus, a definition that offends against Frege's definitional canons might be either strongly creative (if it is intended to bring an object or a function into existence) or weakly creative, as the case may be.[30]

Examples of weakly creative definitions might be found in the work of some of Frege's fellow mathematicians, for example, in Cantor's foundational work on irrational numbers (see the analysis of his definitions in Schirn, 2013 and 2014) or perhaps also in Peano's multiple definitions of the equals sign.[31] Frege raises objections to all

[29] Some of Frege's principles of explicit definitions such as the principle of completeness and the principle of the simplicity of the *definiendum* seem closely related to his conception of definitional non-creativeness. The principles 4, 5, 6 and 7 that he also lists in *Grundgesetze* I, §33 rest on special features of his formal system. See also the fragment 'Begründung meiner strengeren Grundsätze des Definierens' in Frege (1969), pp. 163–170.

[30] Again, Frege does not explicitly draw this distinction. The heading of §138-§147: "The creation of new objects according to R. Dedekind, H. Hankel, O. Stolz" is telling. It seems to make it clear that Frege is going to discuss, besides Dedekind's creation of numbers, some of the strongly creative definitions by Hankel (1867) and Stolz (1885). The heading further makes it clear that Frege focuses on their (intended) creation of mathematical *objects*.

[31] In Schirn 1997, I argue contra Frege (see *Grundlagen*, §96, footnote*) that Cantor does not create infinite cardinal numbers in a strong ontological or metaphysical sense. In his correspondence with Peano, Frege criticizes Peano's multiple definitions of "=" see Frege 1976, pp. 181–186). Every one of these definitions

three groups of Cantor's definitions by invoking his two key principles of definition: the principle of completeness and the principle of the simplicity of the *definiendum* (see *Grundgesetze* I, §33; II, §68-§85). In *Grundgesetze*, II, §69, Frege dismisses the definitions of the first group as flawed. The *definienda* are said to infringe his principle of the simplicity of the *definiendum*. In particular, Frege points out that the *definienda* contain the words 'greater', 'less', 'Zero' and 'equal' with which acquaintance prior to the act of framing the definitions must be assumed. In one place (*Grundgesetze* II, p. 94), he even objects that Cantor shifts these expressions back and forth between being known and being unknown. Although in his critique Frege does not complain about the (potential) creativeness of Cantor's definitions, we might classify them as weakly creative on the grounds that the *definiendum* contains an expression that has not been explained prior to setting up the definition, but whose meaning is (illicitly) assumed to be known. Compared with the assumption that underlies Frege's final definition of the cardinality operator in *Grundlagen*, §68, Cantor swaps the sides.[32] In Cantor's definitions, familiarity is presupposed with one or more constituents of the *definiendum.* Frege's charge that Cantor offends against the principle of the simplicity of the *definiendum* follows then swiftly. The non-simplicity of the *definiendum* need not per se give rise to some kind of definitional creativeness. But it may do so if, for example, it is presupposed that the meaning of a constituent of the *definiendum* is completely known, although the constituent is in need of explanation and has not been elucidated or defined in advance, as Frege's definitional methodology requires.

Frege goes on to take Cantor to task for his definitions of the elementary operations with irrational numbers. Among other things, he objects that the expressions "sum", "difference", and "product" are explained through themselves (cf. Frege, 1903, §79-§80). In this connection, Frege rebukes Cantor for having passed something off as a definition that he would have needed to prove as a theorem. Whereas according to Frege a correct explicit definition merges into an epistemically trivial truth endowed with assertoric force once the new simple sign has been defined – which can also be regarded as underpinning its non-creativeness – a mathematical theorem requiring proof characteristically possesses a relevant cognitive value or as Frege also says: it extends our knowledge. If Frege is right in claiming that Cantor "surreptitiously"[33] presents as a definition what actually is a mathematical theorem standing in need of proof, then we might consider the definition (or what is treated as a definition by Cantor) to be weakly creative. We would not say that in stating the definitions of the elementary operations Cantor intends to bring something into existence, say, a certain irrational number. He does not. We might rather say that something equally alien to a flawless explicit definition is carried into the definition or is imposed on it, namely the delivery of a piece of relevant knowledge, and that this is a creative act in

Footnote 31 continued
is said to be incomplete and even taken together they are supposed to be still incomplete. Moreover, Frege objects that Peano's definitions explain the equals sign by means of itself. For reasons of space, I cannot discuss the question of whether one or the other of Peano's definitions of "=" could reasonably be characterized as weakly creative.

32 On Frege's explicit defnition of the cardinality operator in *Grundlagen* see Schirn (1983), (1996), (2003), (2009) and (2024b).

33 Frege does not use this word, but in accordance with the trenchant jargon that he often employs in his battle against the views of some of his fellow mathematicians might have used it.

the definitional practice under consideration.[34] Frege's principles of definition strictly prohibit such a move. If a definition "wants to engender real knowledge, to save us a proof, then it degenerates into logical sleight of hand" (Frege, 1967, p. 263). In sum, in my opinion, Frege jettisons not only strongly creative definitions as inadmissible – such as Stolz's definitions – but also disapproves of weakly creative definitions as I propose to understand them: they do not bring a mathematical entity into being, nor are they intended to do this. Nonetheless, they can reasonably be described as weakly creative in the sense specified above. Clearly, if Frege was convinced that a mathematical definition violates his principles of both completeness and simplicity, he would have rejected it out of hand anyway, regardless of whether he regarded it as creative in some sense or not. Yet as I have argued, weak definitional creativeness can in some cases be seen as immediately following from an offence against Frege's strict guidelines for correct explicit definitions.

2 Part II

2.1 The creation of numbers: the positions of Stolz and Dedekind

In *Grundgesetze* II, Frege takes up the topic of the creation of new numbers via definition which he had already discussed in earlier work. In this discussion, he juxtaposes comments on Dedekind's non-definitional creation of irrational numbers. Frege's demonstration in *Grundgesetze* II, §143 that every attempt to create mathematical objects by means of a definition à la Stolz is bound to fail, I find basically convincing.[35] One of Stolz's definitions that Frege criticises is as follows:

> 6. Definition. When lim $(f{:}g)$ is a positive number or $+\infty$, a thing distinct from the moments is to exist, designated by $\mathfrak{u}(f) : \mathfrak{u}(g)$, which satisfies the equation $\mathfrak{u}(g).\{\mathfrak{u}(f) : \mathfrak{u}(g)\} = \mathfrak{u}(f)$.

Frege writes with a touch of irony (*Grundgesetze* II, §143):

> The creation, thus, takes place in distinct steps. After the first, the thing indeed is there, but it is, so to say, stark naked, lacking the most essential properties, which are attributed to it only by further acts of creation, wherepon it may be greeted as the lucky bearer of these properties.

A mathematician who wishes to introduce mathematical objects by carrying out a creative definitional act along the lines of Stolz's method must, prior to performing this act, prove that the properties he intends to assign to an object, which is initially

[34] Regarding definitions, there may be other forms of weak creativeness which, for reasons of space, I cannot adequately analyze in this article.

[35] In a footnote to *Grundgesetze* II, §145, Frege observes regarding Hankel's position vis-à-vis creation: "It is hard to say what standpoint H. Hankel adopts in his *Theorie der complexen Zahlensysteme* (Leipzig, 1867), since he makes opposing claims. Probably he has not sharply distinguished between sign and what is designated."

considered to be devoid of properties,[36] do not contradict each other. Frege correctly points out that the consistency of those properties cannot be proved save by establishing the existence of an object that possesses all those properties. Thus, the claim that the instances of the set of properties $\{F_1,...,F_\mathrm{n}\}$ do not contradict each other requires the proof that for at least one object a of the considered domain it holds: $F_1(a) \wedge \ldots \wedge F_n(a)$. Yet if one can prove this, then there is no need to create such an object by means of a definition. Thus, I basically endorse Frege's resumé in *Grundgesetze* II, §143: "Creative definitions are a first-class invention." Let us now turn to Dedekind's creation of natural and irrational numbers.

Dedekind's introduction of the natural numbers in *Was sind und was sollen die Zahlen*? (Dedekind, 1888) proceeds via a characterization of the natural number structure, that is, the form of any set of objects that has a distinguished initial object and a successor relation which satisfies the mathematical induction principle. The natural number sequence is exemplified, say, by the Zermelo numerals or the finite von Neumann ordinals. Dedekind holds that we create the natural numbers if, in considering a simply infinite system N (cf. Dedekind, 1888, §71), ordered by a mapping φ (cf. §2) we abstract completely from the specific nature of the elements, maintain only their distinguishability and focus on the relation into which they are placed to one another by φ. According to this view, the natural numbers are objects which have no properties except those they possess by virtue of having their position in the simply infinite system. Thus, Dedekind weds structure to abstraction and it is by means of the latter that he creates the natural numbers. We may refer to it as *Dedekind-abstraction* or as *structural abstraction*. However, due to the fact that Dedekind provides relatively little information about his conception of structure—we learn at least that the natural number structure is obtained by carrying out an act of abstraction of a certain kind—I hesitate to put his approach in one of the pigeonholes that the taxonomy of contemporary mathematical structuralism has in store. Let me add that Dedekind's account of the natural numbers rests on the claim that simply infinite systems exist (Dedekind, 1888, §72) and on the categoricity theorem (§132), which states that all simply infinite systems are isomorphic. Since for Dedekind performing the act of abstraction from the special properties of the elements of a given simply infinite set requires that there be a non-abstract system from which to start, he felt the need to prove the existence of an infinite system which had to be non-mathematical in character (§66). Note that Dedekind's mode of creating the natural numbers in *Was sind und was sollen die Zahlen*? differs from his earlier characterization of the creation of the positive integers in *Stetigkeit und irrationale Zahlen*. In the earlier work, it is not the specific act of abstraction by means of which the positive integers are supposed to be created but it is rather the elementary arithmetical act of counting by means of which the infinite series of the positive integers is said to be successively created (see Dedekind, 1872, §1).[37]

[36] Whatever that is supposed to mean concretely. The phrasing is reminiscent of Cantor's talk of featureless units.

[37] On Frege's relation to Dedekind, his fellow-combatant for logicism, see Reck (2013a), (2013b) and (2019) and Shapiro (2000).

Several questions arise in connection with Dedekind's creation of the natural numbers.[38] Some or even most of them probably cannot be given a definite answer. This is mainly due to the fact that Dedekind only briefly characterizes the act of creation via structural abstraction and does not provide any further clue. Here are some queries that one might raise: (1) If Dedekind held that the natural numbers exist prior to the mental act of abstraction – there is no evidence to the contrary—why then should they be created at all? (2) What does Dedekind specifically mean by the human mind who is said to carry out the act of creation? Does he tacitly appeal to "a universal human mind"? Or is the act of creation via structural abstraction essentially performed by an individual mind? I vote for the second option. (3) The fact that Dedekind considers the natural numbers to be a creation of the human mind, does not necessarily mean that he construes them as subjective mental entities. By comparison, for Frege judging is a mental act—the abstractive transformations from right to left in Hume's Principle and in Basic Law V are likewise mental acts – but what is acknowledged as being true, namely the thought expressed by a true declarative sentence, is something objective for him as is its truth. Unlike Frege, Dedekind does not comment on the notion of objectivity. But it would be too hasty to infer from this that in his work on the foundations of mathematics Dedekind did not care two figs about the notion of objectivity. I tend to assume that he understood the simply infinite system—the subject matter from which we are supposed to start the process of creating the natural numbers—as something objective and presumably the natural numbers too, although it may be difficult to figure out which argument he might have adduced in favour of the objectivity of the natural numbers. (4) Regarding the creation of numbers, Dedekind does not follow an "asymmetrical" path. He takes both the natural and the real numbers to be creations of the human mind, but the creations are performed in different ways.[39] In my judgement, Dedekind most likely understands the creation of the natural numbers by means of structural abstraction in a strong, ontological sense of creation. To all appearances, this applies also to his creation of irrational numbers to which I now turn.

In Dedekind, 1872, irrational numbers are not created by way of structural abstraction in the sense explained above but by what I call "cut generation". Structure is here important too, but in a different respect. Dedekind describes the creation of an irrational number as follows (Dedekind, 1872, p. 13):

[38] In *Grundgesetze* II, §138 and §139, Frege leaves Dedekind's creation of the natural numbers in Dedekind (1888) out of consideration. The omission is presumably due to the fact that in this volume Frege is primarily concerned with the foundations of real analysis. Strangely enough, regarding Weierstraß, we come across the opposite situation. In *Grundgesetze* II, §148-§155, Frege criticises, spiced with plenty of irony, Weierstraß's treatment of the natural numbers but disregards his important contribution to the foundations of real analysis. Frege only observes that Weierstraß simply creates the negative, rational and irrational numbers – without providing any evidence for his observation—and that, due to the unsafe basis of Weierstraß's approach to real analysis, a more thoroughgoing treatment of it is unnecessary. Yet giving Weierstraß such a dressing-down is not only unfair but also unproductive.

[39] In *Principles of Mathematics* (Russell 1903, chapter XXX, pp. 245–251), Russell criticises Dedekind's theory of number. One point he makes is that Dedekind's definition of cardinals is unnecessarily complicated and that the dependence of cardinals on order is only apparent.

> Whenever, then, we are presented with a cut (A_1, A_2) that is not generated by a rational number, we create for ourselves a new, irrational number α, which we regard as completely defined by this cut.

I assume that in Dedekind's view the irrational number α is uniquely individuated by the cut (A_1, A_2), that is, that by virtue of the cut (A_1, A_2) α is uniquely identifiable, re-identifiable and distinguishable from any number β in the real number structure, provided that β in turn is uniquely defined by a cut (B_1, B_2), which is essentially different from (A_1, A_2). As Dedekind says, two numbers are considered to be unequal only if they correspond to essentially different cuts.

Under what conditions are two cuts completely identical? The answer is: If every number a_1 contained in class A_1 of (A_1, A_2) is also contained in class B_1 of (B_1, B_2), and if every number b_1 that is contained in B_1 is also contained in A_1, then A_1 and B_1 are completely identical. Yet in this case A_2 is also completely identical with B_2 and, as a result, (A_1, A_2) is completely identical with (B_1, B_2), what Dedekind indicates by $\alpha = \beta$ or $\beta = \alpha$.[40] If class A_1 contains at least two distinct numbers $a_1 = b'_2$ and $a''_1 = b''_2$, which are not contained in class B_1, then there are infinitely many such numbers since all the infinitely many numbers that lie between a'_1 and a''_2 are contained in A_1 but not in B_1. In this case, Dedekind calls the numbers α and β, which correspond to the two essentially different cuts (A_1, A_2) and (B_1, B_2), likewise distinct from one another (cf. Dedekind, 1872, p. 14f.).

It is not clear whether in this context Dedekind uses the word "*hervorbringen*" exclusively in the sense of "produce" or "generate" and not in the possibly somewhat stronger sense of "create" ("*erschaffen*", "*schöpfen*"). If the second option applied, then we would be facing the following situation: If we are given a cut (A_1, A_2) that is not created by a rational number, we create an irrational number α which in turn creates the very cut to which it corresponds. If the first option regarding the choice of terminology applied, it would be useful to know what the difference between the use or meaning of "*schöpfen*" or "*erschaffen*" on the one hand and that of "*hervorbringen*" on the other is supposed to be in the relevant context.[41]

Neither Dedekind's creation of the natural numbers in Dedekind, (1888) by means of structural abstraction nor his creation of irrational numbers in Dedekind, (1872) by means of cut generation should be lumped together with or assimilated to the property-listing creation of mathematical objects à la Stolz, although both seem to understand the creation they describe in an ontological sense. Thus, I disagree with Frege when in *Grundgesetze* II, §145 he writes: "Dedekind's conception of creation agrees with that of Stolz."[42] With a mild proviso, one might even say that Dedekind's creation of the natural numbers is diametrically opposed to the procedure applied by Stolz.

40 Plainly, (a) if to every cut corresponds exactly one rational or irrational number and (b) if two cuts (A_1, A_2) and (B_1, B_2) are completely identical, then there is only one rational or irrational number that corresponds to both (A_1, A_2) and (B_1, B_2).

41 It could seem that Dedekind employs the words "*Schöpfung*" and "*erschaffen*" only in order to designate with them free acts of the human mind. If so, he could hardly mean that an irrational number creates the corresponding cut.

42 Frege adds: "for him [Stolz] too the numbers are not signs but the references of the number-signs." Yet this remark does not justify Frege's previous assimilation of Dedekind's conception of creation to that of Stolz.

More specifically, abstracting from the specific nature of the elements of a simply infinite system N ordered by a mapping φ is in a sense just the opposite of listing or enumerating (the) properties of an object with the aim of bringing it into being in this way.[43] Let us now turn to *Grundgesetze* II, §146.

2.2 *Grundgesetze* II, §146 and *Grundgesetze* I, §3, §9 and §10: Does Frege create value-ranges via non-definitional stipulations? An imaginary dispute

The heading of *Grundgesetze* II, §146 reads as follows: "Our introduction of value-ranges is different from the creation of numbers by the mathematicians". In §146, Frege shows his awareness of a related problem to which especially the Initial Stipulation in *Grundgesetze* I, §3 may give rise. He observes that the latter may provoke the objection that it is a definitional creation of value-ranges or that it is at least akin to a creative definition and, hence, more or less on a par with the (strongly) creative definitions of other mathematicians which in *Grundgesetze* II he intends to tear to pieces. Frege responds to the possible creationist charge by first pointing out what he did not do in *Grundgesetze* I, §3, §9 and §10. He then tells us what he actually did. He writes:

> We did not list properties and then say: we create a thing that has these properties. Rather we said: if one function (of first level with one argument) and a second function are so constituted that both always have the same value for the same argument, then one may say instead: the value-range of the first function is the same as the value-range of the second ... That we have the right so to acknowledge what is common to both functions, and that, accordingly, we can convert the generality of an equality into an equality (identity) must be regarded as a basic law of logic.

Pointing out what he (Frege) has not done in *Grundgesetze* I, §3 is hardly an argument for the non-creativity of the first procedure nor is the description of what he actually did in §3 per se an efficient argument for non-creativity. I shall say more about this from the point of view of Frege's opponent in due course. In the quotation, Frege refers to the conversion embodied in the Initial Stipulation. Almost in the same breath, he appeals to Basic Law V without mentioning it by name. The conversion enshrined in Basic Law V is of course endowed with an axiomatic status. It therefore has all the features by virtue of which (logical) axioms differ from non-axiomatic truths. In *Grundgesetze* II, §146, Frege thus argues for the legitimacy of the act of recognizing something in common to two (monadic first-level) functions, and consequently for the rightfulness of the Initial Stipulation, by invoking a basic law of logic (which he has not yet presented in the guise of a concept-script sentence). However, so far a forceful argument for the non-creativeness of the Initial Stipulation has not been forthcoming. Note that in making the Initial Stipulation in *Grundgesetze* I, §3 Frege does not yet appeal to a (basic) law of logic. It is only in *Grundgesetze* I, §9 where he contends that the possibility of transforming the generality of an equality into a value-range equality

43 In *Grundgesetze* II, §139, Frege correctly observes that Dedekind's creation of irrational numbers is quite different from the introduction of figures in formal arithmetic.

and vice versa must be regarded as a logic law.[44] For reasons that I shall consider later, Frege states the concept-script version of Basic Law V not before he has reached §20 in the exposition of the concept-script.

As I have observed in Sect. 1.1, in *Grundgesetze* II, §146 Frege relates the potential charge of having created value-ranges not only to the Initial Stipulation but also to the stipulations that he makes in *Grundgesetze* I, §9 and §10. But he confines himself to arguing for the non-creativeness of the Initial Stipulation. In particular, he does not mention any objection which his creationist opponent might raise to the stipulations in §9 and §10. In Sect. 2.6, I shall argue that the stipulation concerning "$\grave{\varepsilon}\Phi(\varepsilon)$" in *Grundgesetze* I, §9 does not provide a coign of vantage for Frege's opponent if he intended to convict Frege of falling prey to the creation of value-ranges in general. Thus, Frege had no reason to worry about the possibly creative status of his stipulation in §9. However, an inveterate foe may charge him of having randomly created two special value-ranges in *Grundgesetze* I, §10. I shall discuss this charge in Sect. 2.7 of this essay.

2.3 The non-definitional nature of the Initial Stipulation and the weakness of Frege's argument of non-creativity

As I said in Part I, a definition which is clad in concept-script garb and satisfies Frege's principles of definition (see again Frege, 1893, §33, Frege, 1903 II, §56–§67) bestows the reference and the sense of a complex function-name or complex proper name on a new simple function-name or a new simple proper name.[45] We have seen that Frege assigns a reference conferring task also to the Initial Stipulation. Its purpose is arguably to fix partially the reference of the metalinguistic analogue of "$\grave{\varepsilon}\varphi(\varepsilon)$", or as he says in §10, the reference of a name such as "$\grave{\varepsilon}\Phi(\varepsilon)$".[46] If Frege thought that he did not achieve this, he could have buried his foundational project before it was gaining momentum. Note that irrespective of Frege's introduction of the value-range

44 In *Grundgesetze* II, §146 and §147, Frege refers only to the transformation from right to left incorporated in both the Initial Stipulation and Basic Law V and not to the mutual transformation for which they stand, due to the use of "=" between the equation on the left and the generalized sentence on the right in the case of Basic Law V or due to the stipulation of coreferentiality in the case of the Initial Stipulation. If with the formulation "instead of saying … one may say …" Frege intends to be faithful to the Initial Stipulation, then the same-saying relation must be understood here as coreferentiality and not as thought-identity. (The same-saying relation is usually considered to be in close vicinity to synonymy.) I know only of two places in *Grundgesetze*, namely I, §9 and I, §20, where Frege not only mentions the possibility of converting the generality of an equality into a value-range equality but also the converse.

45 In *Grundgesetze* I, §34, Frege sets up the first definition of his system, namely that of the name of the application function "$\xi \cap \zeta$". He can do this at this stage of the exposition of the concept-script only because (a) he has introduced all primitive function-names by standard elucidations (with the exception of the metalinguistic analogue of the value-range operator), (b) stated the formation rules, followed by an explanation of how they interlock in the formal language, and (c) laid down the principles governing concept-script definitions (cf. §33). Moreover, a successful proof of referentiality (cf. *Grundgesetze* I, §31) was presumably considered a further condition for the acceptability of every concept-script definition.

46 The Initial Stipulation is structurally very much akin to the tentative contextual definition of the cardinality operator in *Grundlagen* (if we transfer Frege's remarks on directions to the case of cardinal numbers). The difference is only that in *Grundgesetze* I, §3 we have a *non-definitional* stipulation. By their very nature, it holds for stipulations, regardless of whether they are definitional or non-definitional and informal in character, that they do not assert anything, according to Frege.

notation in §9, particular concept-script value-range names are, strictly speaking, not yet available at the stage of §10, let alone at the stage of §3. In Sect. 2.7, I shall argue that this circumstance plays into the hands of Frege's creationist rival.

Besides the reference-fixing task, the Initial Stipulation and a concept-script definition share another characteristic. Neither one contains a judgement (= the acknowledgement of the truth of a thought) and an assertion (= the manifestation of the judgement). Thus, I assume that in putting forward the Initial Stipulation Frege does not regard it as a sentence which is uttered with assertoric force. But what is the status of the Initial Stipulation immediately after it has been made? Regarding the concept-script sentence that results from replacing the double stroke of definition with the judgement-stroke—the replacement is characteristically carried out before a definition is drawn upon in a proof—it goes without saying that it contains a judgement and an assertion and, hence, can be used like an axiom as a (first) premise in a deductive proof.[47] (Note that the concept-script definition marked as such by the double stroke of definition is not a concept-script sentence in Frege's use of this term.) However, due to Frege's silence about the status of the Initial Stipulation immediately after its reference conferring role has come into effect, I refrain from jumping to conclusions by appealing to the paradigm case of transforming a sentence without assertoric force into one with assertoric force, which in Frege's system is the transformation of a definition into a concept-script sentence. One thing seems clear. If Frege thought that, similar or in analogy to the transformation of a definition into a concept-script sentence, the Initial Stipulation, immediately after having bestowed a (yet incomplete) reference on the term "the value-range of the function φ", is turned into a declarative sentence carrying assertoric force,[48] then in §3 he would almost have arrived at Basic Law V. Only the necessary notational changes would still have to be made to state Basic Law V in the form in which Frege presents in *Grundgesetze* I, §20. But as I argue below, it is not likely that he assessed the situation in this way.[49]

Towards the end of *Grundgesetze* II, §146, Frege argues that it would be a mistake to construe the Initial Stipulation as a definition. He points out that the sentence "The value-range of the first function is the same as that of the second function" "is composite and contains the expression 'the same' which has to be regarded as completely known".[50] Construing the Initial Stipulation as a definition would therefore

47 In a deductive proof, the concept-script sentence resulting from the definition may only formally adopt the role of an axiom. In contrast to any axiom of the system of *Grundgesetze*, it can be proved, but just as an axiom does not require proof (= deductive justification), since it is self-evident and even epistemically trivial. By contrast, the self-evident axiom is supposed to contain real knowledge. Thus, while in the case of a concept-script sentence that emerges from a definition the self-evidence comes, as it were, free of charge the requisite self-evidence of an axiom cannot always be taken for granted. By Frege's own lights, Basic Law V is a well-known example for that.

48 Due to the informal character of the Initial Stipulation, the conversion would not be made typographically visible by replacing a sign that indicates the stipulative character of the sentence with a sign that carries assertoric force.

49 If, after having introduced the value-range notation (cf. §9), Frege presented the Initial Stipulation qua stipulation in a formal outfit, he could not use, to avoid ambiguity, the double- stroke of definition since its use is exclusively designed for explicit concept-script definitions.

50 Note that Frege elucidates "$\xi = \zeta$" only in *Grundgesetze* I, §7.

be an offense against Frege's principle of the simplicity of the *definiendum*. Yet this argument fails to show that the Initial Stipulation is non-creative.

There are further arguments for the non-definitional nature of the Initial Stipulation which Frege does not mention.

(a) What Frege puts under taboo regarding a concept-script definition – namely fixing the reference of th*e definiendum* in a piecemeal manner – he licenses in the case of the Initial Stipulation. In §10, the Initial Stipulation is said to fix the reference of a name such as "$\grave{\varepsilon}\Phi(\varepsilon)$" only incompletely. It is supplemented by further stipulations to achieve referential uniqueness of "$\grave{\varepsilon}\Phi(\varepsilon)$".[51] Thus, if Frege had considered the Initial Stipulation a definition, he would also have offended against his principle of completeness. From this principle, he infers the inadequacy and inadmissibility of piecemeal definitions.

(b) Unlike a concept-script definition, the Initial Stipulation is framed in metalinguistic vocabulary.[52] *Grundgesetze* contains only concept-script definitions. They reveal themselves as such by the use of the double stroke of definition which is attached to the horizontal, followed by a concept-script equation, where "=" is flanked either by a simple and a complex function-name or by a simple and a complex proper name. (The second variant applies only in three cases.) There is not a single definitional stipulation in the exposition of the concept-script until Frege has reached *Grundgesetze* I, §34.

(c) On the face of it, the Initial Stipulation appears less arbitrary than an explicit definition if we take Frege's remarks in *Grundgesetze* I, §9 on the Initial Stipulation into consideration. Whether Frege possibly thought that the Initial Stipulation is not arbitrary at all despite its stipulative nature, and if so, why, we simply do not know. We can only speculate in this respect. Frege might wish to explain what he – if I put it cautiously – possibly views as a non-arbitrary feature of the Initial Stipulation by pointing out that the possibility of converting the generality of an identity into a value-range identity must be regarded as a logical law (cf. *Grundgesetze* I, §9). In any event, from the assumption that the Initial Stipulation is perhaps to a lesser degree arbitrary than an explicit definition (again, Frege does not provide any clue in this respect), we cannot infer that in the former truth essentially comes into play for Frege. In my view, it does not.

It could seem that at the beginning of *Grundgesetze* II, §147, Frege intends to present a second argument in favour of the alleged non-creativeness of the Initial Stipulation. "We are thus not really doing anything new by means of this conversion; but we do it in full awareness and by appealing to a basic law of logic." However, Frege's claim that the conversion embodied in the Initial Stipulation is nothing new in logic (around the turn of the twentieth century) is debatable. It is true that already

[51] It could seem that in *Grundgesetze* II, §58, Frege mitigates a little his critique of the praxis of piecemeal definitions in mathematics by granting that the "development of the science which occurred in the conquest of ever wider domains of numbers, almost inevitably demands such a practice; and this demand could be used as an apology."

[52] In *Grundgesetze*, there is not a single non-definitional stipulation that is completely formalized. Couched in concept-script notation, the dual stipulation in *Grundgesetze* I, §10 might be written as follows: $|||{-}\acute{\varepsilon}(-\varepsilon) = \forall \mathfrak{a}(\mathfrak{a} = \mathfrak{a}); |||{-}\acute{\varepsilon}(\varepsilon = \neg\forall \mathfrak{a}(\mathfrak{a} = \mathfrak{a})) = \neg\forall \mathfrak{a}(\mathfrak{a} = \mathfrak{a})$. I use here three vertical strokes in order to distinguish the two formalized stipulations from concept-script definitions.

in the logical work of Leibniz we find a formulation which, on the face of it, has a certain affinity to Frege's conversion of coextensional functions into a value-range identity. To see this, let us take a brief look at what Leibniz says. I use the letters A and B to represent concepts. According to Leibniz (cf. Leibniz 1875–1890, vol. 7, pp. 238–240), we have: If $A = B$, then A is in B and B is in A. In his view, this amounts to saying that one of two coinciding concepts is in the other. Leibniz also states the converse: If A is in B and B is in A, then $A = B$. Concepts which stand in the relation of mutual inclusion to one another (Frege calls it mutual subordination) are said to coincide. Under an extensional interpretation of Leibniz's logic of concepts, we obtain: $E(A)$ and $E(B)$ (that is, the extensions of A and B respectively) coincide if and only if $E(A) \subseteq E(B)$, and conversely $E(B) \subseteq E(A)$. (In this connection, see Frege, 1969, pp. 16f. where he appeals to Boole; see also Frege, 1893, §9.) However, I doubt that Leibniz conceived of extensions of concepts exactly in the way Frege did. He most likely did not construe them as forming a subset of a more comprehensive set of logical objects, as did Frege when he introduced value-ranges of functions. Analogous remarks apply to Boole.[53] Frege construes extensions of first-level concepts as value-ranges of monadic first-level functions. In his view, every sharply delimited first-level concept has an extension, regardless of whether two or more objects fall under it, or exactly one object falls under it or no object falls under it as is the case with $x \neq x$, for example. This view is entirely independent of the question of whether a *logical* object falls under the given concept or not. In short, the conversion incorporated in the Initial Stipulation is at least to some extent a novelty in logic around the turn of the twentieth century. It deals not only with extensions of concepts but more generally with value-ranges of monadic first-level functions.[54] By pointing out that he is doing nothing new with the conversion Frege would hardly have convinced his creationist opponent that for this reason it does not involve an act of creation.

The upshot so far is that neither the well-established non-definitional nature of the Initial Stipulation nor the claim that the conversion embodied in it is nothing new in logic around the turn of the twentieth century significantly support the alleged non-creativeness of the Initial Stipulation, let alone establish it.

53 In *Grundgesetze* I, §9, Frege remarks that the entire calculating logic of Leibniz and Boole rests on the logical law that governs the conversion of the generality of an equality into a value-range equality.

54 In *Grundgesetze* I, §36, Frege introduces value-ranges of dyadic first-level functions which include extensions of relations (= Relations). He does so after having defined the application operator "$\xi \cap \zeta$" (in §34). This function-name is almost omnipresent, not only in the definitions of *Grundgesetze* I and II – the three exceptions are: the definitions of Zero (*Grundgesetze* I, §41), of One (*Grundgesetze* I, §42) and of series of composition (*Grundgesetze* II, §167) – but also in the proofs which Frege carries out in these two volumes. The introduction of double value-ranges, unlike the introduction of simple value-ranges in §3, does not proceed via an informal stipulation which states an identity criterion for them in its own right. The most likely reason for Frege's eschewal is that he saw no need to lay down, in addition to Basic Law V, a basic law that governs double-value-ranges: $(\acute{\alpha}\acute{\varepsilon}(f(\varepsilon, \alpha)) = \acute{\alpha}\acute{\varepsilon}(g(\varepsilon, \alpha))) = (\forall x \forall y(f(x, y) = g(x, y)))$. See Heck (2019) and Schirn (2023b) on Frege's principle of logical parsimony with respect to his choice of the axioms of *Grundgesetze*. Note that the names for double value-ranges can be formed by means of the notation which is available for simple value-ranges. On double value-ranges in Frege's formal system, see Heck (2012) and (2019), Simons (2019) and Schirn (2018), (2023a) and (2023b).

2.4 Further charges of the creationist opponent

Let us suppose that Frege's antagonist acknowledges without further ado that the Initial Stipulation is not a definition, at least not according to the theory of definition that Frege presents in *Grundgesetze*. The antagonist might nevertheless object that the Initial Stipulation has creative potential. By putting it forward, so his possible complaint, Frege fails to introduce logical objects in line with his platonism. In particular, Frege is said to bring value-ranges partially into being by way of logical abstraction couched in a stipulation. I used the word "partially" since by assumption the opponent knows that from Frege's point of view the determination of value-ranges in *Grundgesetze* I, §3 is yet incomplete. Note that in raising his objection the opponent feels free to transform Frege's wording of the incomplete determination of the references of value-range names into creationist terms.

Frege's rival may further argue that the view that value-ranges exist prior to the transformation incorporated in the Initial Stipulation (and in Basic Law V) is an unfounded ontological or metaphysical assumption which defies verifiability. As I hinted at in connection with Leibniz, value-ranges of monadic and dyadic first-level functions, as Frege characteristically construes them, do not appear in any mathematical or logical theory prior to their introduction in the logical theory of *Grundgesetze*. Thus, when shortly after having published *Grundlagen* Frege was in search of a comprehensive set of fundamental and irreducible logical objects with which he intended to identify all numbers to give logicism its due and which, in contrast to *Grundlagen*, contained extensions of concepts and of relations as proper subsets, he could not have *discovered* such objects in the work of any fellow mathematician. The creationist rival may jump on the bandwagon by pointing out that, true to the motto "Necessity is the mother of invention", Frege was bound to *invent* value-ranges of functions in pursuit of his logicist project, contrary to the view to which he avowed himself in *Grundlagen*, pp. 107f. (see also *Grundgesetze* I, p. XIII).[55]

The words which I have just put into the mouth of the creationist opponent may be granted some weight in the debate. They may even override Frege's arguments in *Grundgesetze* II, §146 for the non-creativity of the Initial Stipulation, weak as they are. Yet in the present scenario, the opponent is far from checkmating Frege. Unfortunately, it is difficult to figure out how Frege might have struck back had he considered the creationist's objections a true challenge for his anti-creationist and platonist position. Thus, for the time being I leave the distribution of "winning points" to our protagonists undecided.

However, Frege's opponent may play another trump in the dispute as far as it goes. The assumption that only a definitional stipulation in mathematics or logic might be prone to being creative is far from compelling. In particular, any stipulation in mathematical discourse—regardless of whether it appears in the guise of a definition or not—which could be "subject to the suspicion" of being creative, need not be a listing of properties followed by the declaration that one creates or has created a thing that possesses all those properties. Frege was most likely aware of the diversity

[55] I think that in general Wittgenstein is right when in *Bemerkungen über die Grundlagen der Mathematik* he observes, not least with Frege in mind, that the mathematician is not a discoverer but an inventor (Wittgenstein 1974, p. 111).

of the modes of mathematical creation among some of his fellow mathematicians. Nonetheless, the definitional property-listing creation of mathematical objects is the main target of his critique. In *Grundgesetze* II, §143, Frege seems to assume that there is a danger lurking in a creation à la Stolz and Hankel which he himself had not to fear when he put forward the Initial Stipulation: a mathematician who pursues the creative property-listing method may get entangled in a contradiction before he knows it. Yet this does not invalidate the opponent's objection. And we know that the formal counterpart of the Initial Stipulation, namely Basic Law V, caused a contradiction in the logical system of *Grundgesetze*.[56] In his letter to Russell of 22 June 1902, Frege remarks, in slightly different wording, that his proof of referentiality miscarried. Prior to Russell's discovery, Frege presumably thought that in *Grundgesetze* I, §31 he had succeeded in demonstrating the consistency of his logical system by establishing that all well-formed names of the formal language (uniquely) refer to something. There is some evidence for that.

Frege, in his response to the creationist charge, may call to mind that in the first place one should be clear about what a creation of mathematical objects is and what it is not. He may declare that laying down identity conditions for new logical objects does not amount to creating them in any reasonable sense of creation. He may further draw attention to his objective in *Grundgesetze* I, §3, namely of fixing at least partially the references of value-range names via the delivery of a criterion of identity for value-ranges. And he may point out that the dual stipulation in *Grundgesetze* I, §10, together with the Initial Stipulation (and the stipulations in *Grundgesetze* I, §11-§12), is designed to ensure that every canonical value-range name of his concept-script has a unique reference and that achieving referential uniqueness via a series of non-definitional (but heterogeneous) stipulations is essentially a non-creative activity.[57] However, the unyielding opponent may again shrug off these arguments as inconclusive. He may insist that the Initial Stipulation marks only the first crucial step in a process of piecemeal creation. Unless Frege advanced a more substantial argument for the non-creativity of the reference-fixing procedure in *Grundgesetze* I, §3 and §10 regarding value-range names, he would hardly be scoring against his rival. Thus, if Frege wished to have the final say in the debate, as far as it goes, he would not have an easy task.

To recap: In response to the charge of having created value-ranges, Frege claims that in §3, §9 and §10 he did not introduce them via the definitional property-listing method à la Stolz but rather by converting the coextensiveness of two monadic first-level functions into a value-range identity, based on a basic law of logic. This claim does not yet establish non-creativity, unless Frege argued convincingly that there is only one way of creating mathematical objects: the definitional property-listing procedure which he rules out for himself. Yet regardless of his inappropriate conformity statement—"Dedekind's conception of creation agrees with that of Stolz"—Frege

[56] In another place (see Schirn 2019), I claim that regarding the viability or failure of Frege's logicist project, the die is cast once the Initial Stipulation has been made in §3, irrespective of its informal character, that is, regardless of its non-axiomatic and non-definitional status. (In *Grundgesetze*, both axioms and definitions are invariably formulated in the formal language.) Thus, it is already in *Grundgesetze* I, §3 that mischief takes its course in Frege's logicist project.

[57] Note that the term "introduction" has an ontologically noncommittal ring to it.

must have known that he could not provide such an argument. Dedekind's creations of the natural and irrational numbers are just two prominent counter-examples to the creation of mathematical objects along the lines of Stolz's procedure.

2.5 The relation between the Initial Stipulation and Basic Law V under scrutiny

Bearing in mind the previous comparison between definitions and the Initial Stipulation, I shall now comment on the relation between the latter and Basic Law V.[58] Frege says relatively little about this relation. We only learn that the Initial Stipulation is grounded on a basic law of logic.

(1) Basic Law V is supposed to have all the characteristics that according to Frege belong to logical axioms: truth, self-evidence,[59] utmost generality, formal unprovability (in the system of *Grundgesetze*) and relevant cognitive value.[60] Imposing the constraint of formal unprovability on the informal, metalinguistic stipulation in *Grundgesetze* I, §3 would hardly make sense, even though in *Grundgesetze* I, §9 Frege appeals to a logical law that is said to ground the conversion contained in the Initial Stipulation. Note that in §9 Frege does not yet speak of a *basic* law of logic but only of a logical law. A logical law that is not selected as an axiom of a theory T may of course be provable in T. In any event, saying that the possibility of converting the generality of an equality into a value-range equality and vice versa "must be regarded as a logical law" is not tantamount to saying that the Initial Stipulation *is* a logical law or a basic law of logic. To reemphasize, it stipulates coreferentiality but does not assert it. Plainly, the Initial Stipulation is not only non-definitional but also non-axiomatic in character and it is methodologically prior to Basic Law V. Basic Law V requires that the value-range operator has been endowed with a unique reference, whereas the Initial Stipulation almost trivially

58 Heck (2012, p. 116) likewise emphasizes the difference between the Initial Stipulation and Basic Law V. For a detailed discussion of Heck's interpretation of *Grundgesetze* II, §146–§147 see Ebert and Rossberg (2019b), pp. 330–333.

59 Frege's statement in the Afterword to *Grundgesetze* that he had already pointed out in the Foreword to the first volume that Basic Law V is not as evident as must be required of a logical law, should be taken with a pinch of salt. Strictly speaking and curiously enough, his statement in the Afterword is false. In the Foreword, the notion of self-evidence is not even mentioned in connection with Basic Law V. It is not mentioned at all in the Foreword. Frege is only envisioning a *potential* dispute about Basic Law V without explaining why he thinks that a dispute *might* be roused by someone (perhaps about the assumed purely logical nature of Basic Law V or perhaps about its purported self-evidence or …). This does not imply that he himself had a concrete doubt about Basic Law V. Nor does Frege's remark in the Foreword rule out that when writing it he was aware that Basic Law V lacked the requisite degree of self-evidence. Yet *if* he was (silently) aware of this, this would outright clash with his statement at the end of the Foreword (p. XXVI) that he considers his logical system to be irrefutable. It is also possible that in the Foreword Frege threw caution to the wind or downplayed the concern that he *might* have had in regard to the requisite self-evidence of Basic Law V. Incidentally, in the Afterword, Frege should have said: Basic Law V is not as evident as must properly be required of a *primitive* logical law. A logical law that is proved in a theory T need not be self-evident.

60 See the discussion of axioms in general and of Axiom V in particular with special emphasis on the requirement of self-evidence in Schirn (2006a), (2019) and (2023b). On logical axioms in Frege's work see also Blanchette (2012) and (2019) and Pedriali (2019). Parsons's claim ((1965), p. 190) that Basic Law V is a partial contextual definition of the value-range operator is patently false. Even the Initial Stipulation is not a (contextual) definition of that operator. Kneebone ((1965), p. 179) commits the same error as Parsons.

does not require that. So the right order for assessing the relationship between the two is to start with the latter and then move to the former, and this is exactly what Frege does. Now if we were to cast a backward glance from Basic Law V in §20 to the Initial Stipulation in §3 (which Frege does of course not prohibit), shall we say anything different of what so far we have said about the Initial Stipulation? In particular, shall we say that seen in this reverse direction the Initial Stipulation inherits the axiomatic status from Basic Law V? My answer is a clear "no". The difference of status between the former and the latter is inviolable, no matter how and in which order we look at them.

(2) On the face of it, Frege's characterization, in *Grundgesetze* II, §146, of the relation between the Initial Stipulation and the basic law of logic to which he appeals without mentioning it by name seems to have the air of circularity. In saying this, I assume (a) that the law Frege has in mind is Basic Law V and (b) that the dependence of the latter on the Initial Stipulation and the twin stipulations emerges clearly in *Grundgesetze* I, §10 if we also look ahead to §20 where Basic Law V makes its first appearance in *Grundgesetze*. Bear in mind that in *Grundgesetze* II, §146 the rightfulness of the abstractive conversion embedded in the Initial Stipulation is said to rest on a basic law of logic. Yet there was still a long way to go in the exposition of the concept-script to introduce this previously unnamed law as Basic Law V. More specifically, the right to couch the transformation embodied in the Initial Stipulation in an axiomatic concept-script sentence depended essentially on the reference-fixing accomplishment of the Initial Stipulation regarding "$\acute{\varepsilon}\varphi(\varepsilon)$" (or its metalinguistic analogue) in combination with the twin stipulations. If we now adopt the viewpoint that Frege takes in *Grundgesetze* II, §146, we seem to face a mutual dependence between the Initial Stipulation and the basic law of logic to which he appeals in §146. On the one hand, there is no way to introduce this law without relying on the prior stipulations with regard to the value-range operator. On the other hand, Frege thinks that he can lay claim to the legitimacy of the transformation embodied in the Initial Stipulation (see also *Grundgesetze* I, §9), and hence to the legitimacy of the Initial Stipulation itself, only if he invokes a basic law of logic whose content is precisely this transformation. However, as was said, the formal version of this law is not yet available at this stage in the exposition of the concept-script.

(3) Once Basic Law V has been enthroned to play the key role among the axioms of the formal system in pursuit of the logicist project (cf. *Grundgesetze* I, §9), it is independent of both the Initial Stipulation and the twin stipulations in *Grundgesetze* I, §10. It is, as it were, self-sufficient. By contrast, as far as the objective of the Initial Stipulation is concerned, it does not stand on its own. As we have seen, it is in need of being supplemented by further stipulations in order to accomplish its reference-fixing task. The Initial Stipulation gives rise to referential indeterminacy, whereas Basic Law V does not.

(4) Although the Initial Stipulation is in a sense the precursor of Basic Law V, it cannot, by itself, ensure the truth of the latter. In the ideal case in which Frege succeeds in uniquely fixing the references of canonical value-range names, the Initial Stipulation may be seen to ensure the truth of Basic Law V only jointly with the twin stipulations, and, following his explicit strategy in *Grundgesetze*

I, §10, in combination with the stipulations in §11-§12 as well, which in the relevant literature is almost entirely neglected. The stipulations in §11-§12 are standard elucidations of the last two primitive function-names of first level that Frege introduces in his system: the definite description operator (§11) and the name of the conditional function (§12). Thus, without the contribution that in his view these standard elucidations make to uniquely fixing the reference of the value-range operator—in addition to the reference-fixing contributions of the other stipulations concerning "$\acute{\varepsilon}\varphi(\varepsilon)$" or its metalinguistic counterpart—Frege could not have laid down Basic Law V. Note that by means of the elucidations of the definite description operator and the name of the conditional function Frege is able to kill two birds with one stone: "a determination of the value-ranges as well as of [those] function[s]" (*Grundgesetze* I, §10). Nor does Frege's statement at the beginning of *Grundgesetze* I, §9 that "the possibility of converting the generality of an equality into a value-range identity must be regarded as a logical law" empower him to install the conversion at issue as an axiom in his formal system. But the statement could be seen as a belated justification of the legitimacy of the Initial Stipulation.

Clearly, if at the outset of the exposition of the concept-script Frege had considered the (metalinguistic) term "the value-range of the function $\Phi(\xi)$" to possess a (unique) reference all along, he could have done without the Initial Stipulation and, hence, without the twin stipulations, but not without the elucidations of the definite description operator and the name of the conditional function. In that hypothetical case, the role of Basic Law V would not undergo any change which is governing value-ranges in the formal system, guaranteeing to some extent their purely logical nature and providing the appropriate cognitive access to them. Note that the names which besides "$\acute{\varepsilon}\varphi(\varepsilon)$" occur in Basic Law V (they are all primitive) must of course also refer to something which I omitted to mention above. (The judgement-stroke is not a name.) I suppose that at the stage of *Grundgesetze* I, §20 Frege was confident that he had met the conditions of referentiality for Basic Law V, but with an eye to his proof of referentiality in §31, which follows on the heels of the exposition of the syntax of the formal language, he may have thought: prevention is better than a cure. Thanks to the (alleged) success of that proof, the referentiality of Basic Law V was established for him beyond doubt.

A few more words about the issue under discussion may be in order. The stipulation of coreferentiality of the sentence "The function $\Phi(\xi)$ has the same value-range as the function $\Psi(\xi)$" and the sentence "The functions $\Phi(\xi)$ and $\Psi(\xi)$ always have the same value for the same argument" could basically assure the truth of Basic Law V if both sentences could justifiably be considered to be fully referential. Regarding the second sentence, Frege seems to take this for granted. With regard to the first he does not since this would amount to illicitly presupposing that the Initial Stipulation uniquely fixes the reference of the term "the value-range of the function $\Phi(\xi)$". One might be inclined to think that stipulating the coreferentiality of any two sentences only makes sense if it can be assumed that both have a determinate reference in the first place. Yet this is not how Frege proceeds in *Grundgesetze* I, §3. He feels entitled to stipulate the coreferentiality of two specific metalinguistic sentences as a means of fixing at least partially the reference of the value-range operator, as his line of argument in §10 makes clear. If the Initial Stipulation were fully successful, Frege would get at one fell swoop what he particularly needs for the introduction of Basic Law V: the referentiality of a name such as "$\acute{\varepsilon}\Phi(\varepsilon)$" plus the coreferentiality of the coextensiveness statement and the corresponding value-range equation. In saying this, I momentarily disregard the condition of referentiality concerning the other semantically relevant constituents of Basic Law V – which Frege considered to be satisfied, thanks to the standard elucidations that he had provided for "$-\xi$", "$\xi = \zeta$" and "$\forall\mathfrak{a}\varphi(\mathfrak{a})$" – and the requirement of self-evidence which I briefly consider now.

(5) The Initial Stipulation taken by itself not only falls short of ensuring the truth of Basic Law V but it also fails to guarantee the requisite self-evidence of that law. This is due to the fact that the Initial Stipulation stipulates only the coreferentiality of the sentence expressing the generality of an equality and the sentence expressing the corresponding value-range identity. To all appearances, it is sense-identity of the two sentences that from Frege's point of view would be required to establish the requisite self-evidence of Basic Baw V beyond doubt. Yet the Initial Stipulation is designed to guarantee only the coreferentiality of the two truth-value names flanking the central occurrence of "=" in Basic Law V.[61] It is astonishing that Frege does not say one word about the dilemma he is facing with regard to the reconcilability of self-evidence and relevant cognitive value in the case of Basic Law V.

In sum: The logical status of the Initial Stipulation and that of Basic Law V are essentially distinct. In Frege's view, this difference involves that the Initial Stipulation and Basic Law V play essentially different roles and fulfill essentially different tasks in the exposition of the concept-script (mind you: in the *exposition* of the system). Clearly, in the formal proofs of the basic laws of arithmetic the Initial Stipulation plays no role at all, whereas Basic Law V plays a fundamental role at least in the proof of Hume's Principle. (Strictly speaking, Frege proves the right-to-left direction and the contraposed right-to-left direction of Hume's Principle.) The Initial Stipulation is appealed to a last time in the proof of referentiality in *Grundgesetze* I, § 31. This is a clear sign that Basic Law V did not swallow it along the way. Unlike the Initial Stipulation, Basic Law V itself plays no role in the proof of referentiality. Due to its axiomatic status, it exerts no influence on the construction of the semantics of the concept-script. By contrast, the Initial Stipulation is crucial for fixing the semantics of value-range names. Regardless of the difference of status and role of the Initial Stipulation and Basic Law V, they have something fundamental in common. Both are couched in the same second-order abstraction principle and, hence, express the same thought. If that were not so, Frege could hardly have converted the Initial Stipulation into Basic Law V. In Schirn, (2023b), I argue that it would be incoherent if Frege assumed that the two sides of Basic Law V express the same thought. From a semantic point of view, the Initial Stipulation legitimizes and guarantees for Basic Law V only what it stipulates, namely the coreferentiality and not the synonymy of two sentences.

[61] For more on this problem see Schirn (2006a) and Schirn (2023b).

2.6 Frege's stipulation in *Grundgesetze* I, §9 regarding "$\acute{\varepsilon}\Phi(\varepsilon)$" closer examined

It is time to take also a closer look at the relationship between the Initial Stipulation and the stipulation in *Grundgesetze* I, §9 regarding "$\acute{\varepsilon}\Phi(\varepsilon)$". Recall that in *Grundgesetze* II, §146 Frege mentions not only the Initial Stipulation in §3 but also §9 and §10 of *Grundgesetze* I as a possible target for a creationist charge. On the face of it, the stipulation in §9 bears a certain similarity to the standard elucidations that Frege provides for the primitive function-names (exempting "$\acute{\varepsilon}\varphi(\varepsilon)$"). It reads as follows: Generally speaking, "$\acute{\varepsilon}\Phi(\varepsilon)$" shall refer to the value-range of the function $\Phi(\xi)$. This statement marks a crucial step in the introduction of the value-range notation followed by a supplementary explanation of which function is to be regarded as the corresponding function $\Phi(\xi)$ in each case. Yet in my view the statement is not meant as a standard elucidation of "$\acute{\varepsilon}\varphi(\varepsilon)$" or of a name such as "$\acute{\varepsilon}\Phi(\varepsilon)$". In the reference-fixing process regarding "$\acute{\varepsilon}\varphi(\varepsilon)$", the stipulation in §9 therefore does not override the Initial Stipulation. Yet for the sake of argument suppose (counterfactually) that Frege intended to put the stipulation in §9 on a par with the standard elucidations of the other primitive function-names. In that case, the stipulation in §9 should have bestowed a determinate reference on "$\acute{\varepsilon}\varphi(\varepsilon)$" in a single step—this is how standard elucidations are supposed to work—instead of proceeding in a piecemeal manner. Consequently, the stipulations in §3 and §10 could have been considered dispensable with respect to fixing the reference of "$\acute{\varepsilon}\varphi(\varepsilon)$" or "$\acute{\varepsilon}\Phi(\varepsilon)$". Moreover, the otherwise indispensable elucidations of the last two primitive function-names of first level in §11 and §12 would not have performed one of their two intended functions, namely filling the final gap in the reference-fixing process concerning "$\acute{\varepsilon}\varphi(\varepsilon)$". Yet from Frege's point of view the stipulations in §3, §10–§12 are crucial for endowing canonical value-range names with unique references.

At the outset of §10, Frege does not mention the stipulation concerning "$\acute{\varepsilon}\Phi(\varepsilon)$" in §9. He revealingly refers only to the Initial Stipulation. I conclude from this that the stipulation in §9 is not intended to contribute anything essential to the complete determination of the reference of a name such as "$\acute{\varepsilon}\Phi(\varepsilon)$". It is exclusively the stipulations in §10-§12 that are intended to complete the unfinished reference-fixing business left in §3. Thus, in the context of the envisioned scenario, the stipulation in §9 does not provide the creationist opponent with another target for attack. Note that a standard elucidation of "$\acute{\varepsilon}\varphi(\varepsilon)$", say, "The value of "$\acute{\varepsilon}\varphi(\varepsilon)$" for every monadic first-level function $\Phi(\xi)$ as argument shall be the value-range of $\Phi(\xi)$", was for Frege out of the question. It would have rested on the illicit assumption that prior to the introduction of value-ranges in §3 the reader of *Grundgesetze* was familiar with them, just as he is supposed to be familiar with the True and the False all along. Thus, I suppose that in *Grundgesetze* Frege assiduously avoided the methodological blunder that he had committed in *Grundlagen* §68 when immediately after having stated the explicit definition of the cardinality operator he proclaimed: "I assume that one knows what the extension of a concept is." In *Grundgesetze*, with the wisdom of hindsight, Frege probably thought: once bitten, twice shy. In his view, the methodologically correct introduction of the logical "target objects" in the definitions of the cardinal numbers and the envisaged definitions of the real and complex numbers meant an immense progress in *Grundgesetze* over *Grundlagen.*

Let me end this section with a brief preview of what I plan to do in the next section. I shall argue that the creationist rival may feel encouraged to beat Frege at his own game by claiming that the first procedure involves successive creations of different kinds in §3 and §10 or, more specifically, that the dual stipulation in §10 qua second step in the piecemeal creation of value-ranges in general is at the same-time a creation of two special value-ranges, namely of the unit classes "$\acute{\varepsilon}(-\varepsilon)$" and "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$". By creating them almost entirely out of the blue at this early stage in the exposition, Frege – so the reproach runs – fails to meet the constraints of rigour and lawfulness which, according to his view in *Grundgesetze* II, must be imposed on any mathematical creation that he might not wish to pooh-pooh outright. Thus, in the face of this scenario Frege may have had to defend the non-creativeness of the first procedure at two fronts and certainly with better arguments than those that he presents in *Grundgesetze* II, §146.

2.7 The twin stipulations in *Grundgesetze* I, §10 seen from the point of view of the creationist opponent

At the stage of *Grundgesetze* I, §10, the syntax of the formal language is at best *in statu nascendi*. Frege is yet unable to form, for example, the name "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" which he uses in §10 in the stipulation that $\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$ be the False (or that "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" is to refer to the False, cf. for this alternative formulation *Grundgesetze* I, §31). Plainly, only if we knew that "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" is a well-formed name according to the formation rules that govern the syntax of the concept-script might we be confident that it has been endowed with a unique reference by virtue of the stipulations that Frege makes in §3 and §10-§12. It is true that in §9 he introduces the value-range notation, but the three rules for the extraction of function-names from more complex names by means of what I call gap formation are only stated in §26. And the rule which governs the insertion of a suitable argument expression into the argument-place of a first-, second- or third-level function-name is stated and explained with considerable delay in §30. Even at the initial stage in the exposition of the concept-script, Frege seems to take the application of the rule of insertion for granted but he is not entitled to do so. The name "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$", for example, can only be obtained by applying the rule of insertion first to "$\xi = \zeta$" and then to "$\xi = \varDelta$"—the resulting name is "$\varDelta = \varDelta$", where "$\varDelta$" is an auxiliary name—followed by an application of the first gap formation rule to "$\varDelta = \varDelta$" (we obtain then "$\xi = \xi$", cf. *Grundgesetze* I, §26 and §30) and by subsequently applying four times the rule of insertion involving besides "$\xi = \xi$" the primitive names "$\forall\mathfrak{a}\varphi(\mathfrak{a})$", "$\neg\xi$", "$\xi = \zeta$" and "$\acute{\varepsilon}\varphi(\varepsilon)$" in exactly this order. It follows that due to the unavailability of any gap formation rule in §10 and, hence, of "$\xi = \xi$", Frege's choice of "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" as a name that in §10 is declared to refer to the False is inadmissible, even if prior to §10 the device of insertion had already been explained and declared to be one of the formation rules of the concept-script. If the explanation had been made and "$\acute{\varepsilon}\varphi(\varepsilon)$" had been introduced before §10 – recall that in §3 Frege introduces only the metalinguistic analogue of "$\grave{\varepsilon}\varphi(\varepsilon)$" – he could at least have formed the name that he chooses in §10 to refer to the unit class of the True, namely "$\grave{\varepsilon}(\text{—}\varepsilon)$". This name is formed in one step by inserting "—ξ" into the argument-place of "$\grave{\varepsilon}\varphi(\varepsilon)$".

Assuming, for the sake of argument, that Frege's antagonist is aware of all these details, he might argue against Frege as follows: Contrary to what you hold, you do (partially) create value-ranges in general in §3 of your exposition of the concept-script and in doing so you offend against your anti-creationist credo and at the same time against your platonism. However, your creation in *Grundgesetze* I, §3 and §10 characteristically proceeds in a piecemeal manner. Let me mention in passing that in *Grundgesetze* II, §146 you do not expressly rule out that the Initial Stipulation has creative potential. The piecemeal character of your creation strikes me as vulnerable if we consider it in the light of your methodology. With the sole exception of the name of the value-range function, you elucidate the primitive function-names of your concept-script in one step. Moreover, by appeal to your principle of definitional completeness you reject piecemeal definitions in mathematics out of hand, and in your system you define every function-name and proper name on which you rely in pursuit of your logicist project at one fell swoop. I concede that what I consider your first act of creation in *Grundgesetze* I, §3 could be seen as having been carried out in a constrained and lawful manner if you had invoked the supposed lawful character of the conversion of a coextensiveness statement into an identity statement and vice versa in due course, that is, as early as in §3. However, when you set about removing the referential indeterminacy of value-range names in §10, you do not shy away from pulling two special value-ranges out of your hat. In doing so, you create two objects ad hoc and fail to meet your strict methodological standards. Furthermore, in §10 you seem to ignore the fact that by your own lights you did not yet completely fix the references of value-range names in general. Thus, what gives you the right to identify the truth-values with yet incompletely determined value-ranges and why should we accept this as a solution to your indeterminacy problem? I assume that you recognize that the truth-values *are* value-ranges once you identified them with their unit classes.[62]

How could Frege have saved his neck if he had been confronted with the opponent's objections? With an eye to *Grundgesetze* II, §146, it is difficult to say, since Frege does not explain why he thinks that the dual stipulation in *Grundgesetze* I, §10, in addition to the Initial Stipulation, may give rise to a creationist charge. It seems to me that it would have required some effort on Frege's part if he had intended to add a substantial argument to the advocacy of his anti-creationist standpoint that I tentatively suggested in Sect. 2.4. Recall that in the scenario I was envisioning his opponent would probably not have been convinced by the defence. I any event, simply sweeping away the objections of the opponent as ungrounded would have been a weak strategy.

Let me close this section with a simple thought experiment. Suppose that Frege had juxtaposed his presentation of the permutation argument and the ensuing dual stipulation between his explanation of the interplay of the formation rules in §30 and the proof of referentiality in §31 – perhaps in a slightly modified form – and thus at a stage in his exposition where the syntax of the concept-script was already in the bag. In

62 The opponent might even argue that in *Grundgesetze* I, §2 Frege creates the True and the False by means of his informal explanations.

my view, Frege might have derived much benefit from this for his development of the semantics of value-range names. If the case I described above applied, the creationist opponent would have to acknowledge that what he considers to be a creation of two special value-ranges is rule-governed and thus meets the constraints of rigour and non-arbitrariness.[63] Frege would probably have insisted that no matter where the dual stipulation is placed in the exposition of the concept-script, it does not constitute a creation at all.

2.8 *Grundgesetze* I, §10 again: a potential conflict with Frege's platonism

In what follows, I slightly change the perspective in the imagined debate of our two protagonists. Note that in what follows the word "creation" is not used at all. Even if the creationist opponent did not charge Frege with a creation of value-ranges, he could feel encouraged to raise an objection concerning Frege's platonism. It is this.

The opponent might object that the twin stipulations in *Grundgesetze* I, §10 not only involve a costly conversion of logically primitive objects into logically derivative objects, but also clash outright with Frege's platonism. In particular, the opponent may argue that for Frege it should be an objective fact whether, say, the True, is a value-range, and if it is one, which one it is and which function it may belong to. From the point of view of his platonism, this is a matter of metaphysics that must be settled in the mind-independent universe of value-ranges. Hence, this can never be a matter of a stipulation, even if the formal legitimacy of the stipulation is backed up by a logically valid argument.

So much for the potential creationist charge. However, with a moderate proviso I want to take sides with Frege in this respect. I do not think that we should accuse him of betraying, in §10, the platonist position that he defends elsewhere in *Grundgesetze*. Admittedly, there is a tension between Frege's platonism and the dual stipulation in §10. Assuming that he was aware of the tension, he may have thought that he could lightly pass over it without jeopardizing the authenticity of his platonism. For he may have wished to argue that in the exposition of the concept-script he is entitled to make certain stipulations – even if it at first glance they seem to contravene his platonism – with the sole aim of solving an urgent semantic problem. Frege may have pointed out that his solution of the indeterminacy concerning value-range names rests on the demonstration that the transsortal identifications in §10 are consistent with the Initial Stipulation, that this guarantees their formal legitimacy and that no more and no less is required. He may have added that his commitment to platonism and the principles that underlie his logic must strictly be kept apart. Thus, it stands to reason that it is chiefly for pragmatic or utilitarian considerations that in §10 Frege sets his platonism temporarily at naught. By contrast, in the long footnote to §10 platonist concerns apparently do play a certain role. Yet in the footnote Frege does not make any semantically relevant stipulation. Rather he dismisses both a restricted and an unbounded generalization of the dual stipulation in §10 as unviable. Unfortunately, he

[63] In Part III of this essay, I shall show that the creationist opponent may argue that Frege's development of the whole spectrum of objects and functions that arithmetic deals with by applying the formation rules of the system is a creation. Frege seems to have been aware of this.

does not explain why he considers those generalizations at all. The reason is perhaps that he takes the first-order domain of his logical system to be all-embracing.

To sum up: On the face of it, the strategy that Frege pursues in §10 is in conflict with the platonism which he advocates elsewhere in *Grundgesetze*. In particular, it seems that in §10 he does not care about the question of whether prior to the twin stipulations there is a fact of the matter about whether the True or the False is a value-range and if so, which one it is. This may also explain why prior to the stipulations he does not expressly rule out that the True and the False are classes containing more than one object or no object at all.[64] So, I suggest that in order to make sense of Frege's line of argument in §10 and appreciate the impact it has, from his point of view, on the complete determination of the references of value-range names, we should not see it through the lens of his platonism.

A final point. If Frege had argued persuasively that the Initial Stipulation is not a creation of value-ranges (which he did not), then it would have been hard for the creationist rival to make a convincing case for the claim that Basic Law V involves a creation of value-ranges. Appealing to the axiomatic status of this law would have been ineffective anyway in this regard. Now suppose, for the sake of argument, that the opposite assumption applies, namely that the opponent succeeded in convicting Frege of having created, via the Initial Stipulation, value-ranges in general. In that case, Basic Law V could hardly have been the target of an additional creationist charge. The reason is that value-ranges in general could not have been created twice by logical abstraction, first via the transformation embodied in the Initial Stipulation and a second time via that very transformation incorporated in Basic Law V. Thus, in my view the situation is as follows. Regardless of whether or not Frege could have adduced a persuasive argument for the non-creativeness of the Initial Stipulation, he could at least have defended the non-creativeness of Basic Law V if he had been challenged to do so. Regarding a potential creationist charge against Basic Law V, the Initial Stipulation is a kind of risk insurance for Basic Law V.

3 Summary of Part II

In the preceding sections, I have presented a number of arguments in connection with Frege's platonist and anti-creationist position in *Grundgesetze der Arithmetik* II, §146. I began by commenting on Stolz's and Dedekind's views of mathematical creation. Following roughly Frege's line of argument in §146, I placed special emphasis on the potentially creative character of his introduction and determination of value-ranges

[64] See the discussion of Frege's line of argument in *Grundgesetze* I, §10 in Heck (1999) and Heck (2012); see also the analysis of §10 in Schröder-Heister 1987 and Wehmeier and Schröder-Heister 2005. Bentzen (2019) discusses the question of whether or not Frege's intended solution, in §10 of *Grundgesetze* I, of the problem of the referential indeterminacy of value-range names (arising from the Initial Stipulation in §3) offends against his principle that a (first-level) predicate must be defined for all objects whatsoever. In the light of Frege's argumentation in §10, the relevant first-level predicate in this connection is "$\xi = \zeta$". (Note that Frege does not introduce the predicate "*a* is a value-range" ("*VR*(*a*)"). When he thought that he had succeeded in fixing completely the reference of "$\acute{\varepsilon}\varphi(\varepsilon)$", he could have defined "*VR*(*a*)" as follows: $VR(a) := \exists\varphi(\acute{\varepsilon}\varphi(\varepsilon) = a)$.) In my view, much depends in this context on the size of the first-order domain of Frege's logical system. I have analyzed this issue in several places, more recently in Schirn (2018).

in *Grundgesetze* I, §3 and §10. I described the scenarios of the envisioned debate between Frege and a creationist opponent in such a way that the burden of proof lies with Frege. In what follows, I shall summarize the main points that I have made.

(1) Frege's defence of arithmetical platonism and anti-creationism goes largely hand in hand with his advocacy of logicism. In particular, Frege did not regard the transformation embodied in the Initial Stipulation as a creation of value-ranges in general.
(2) The fact that in *Was sind und was sollen die Zahlen (*1888), Dedekind regards the natural numbers as a creation of the human mind does not necessarily mean that he construes them as subjective mental entities.
(3) In Dedekind's earlier work *Stetigkeit und irrationale Zahlen* (1872), it is not the specific act of (structural) abstraction by means of which the natural numbers are supposed to be created but it is rather the elementary arithmetical act of counting by means of which the infinite series of the positive integers is said to be successively created.
(4) I disagree with Frege when in *Grundgesetze* II, §145 he writes: "Dedekind's conception of creation agrees with that of Stolz." Neither Dedekind's creation of the natural numbers in Dedekind, (1888) nor his creation of irrational numbers in Dedekind, (1872) should be lumped together with or assimilated to the definitional property-listing creation of mathematical objects à la Stolz.
(5) Frege concedes that a creationist opponent may charge him with having created value-ranges via the Initial Stipulation in *Grundgesetze* I, §3 and the twin stipulations in *Grundgesetze* I, §10. Yet in *Grundgesetze* II, §146 Frege fails to adduce a convincing argument for the non-creativeness of the Initial Stipulation. In particular, stressing the non-definitional nature of the Initial Stipulation does not support its alleged non-creativeness, let alone establish it. In *Grundgesetze* II, §146, Frege points out that that his introduction of value-ranges by converting the coextensiveness of two monadic first-level functions into a value-range identity in *Grundgesetze* I, §3 was based on a logical law. This claim likewise does not establish non-creativity, unless Frege argued persuasively that there is only one way of creating mathematical or logical objects: the procedure à la Stolz. Yet irrespective of Frege's inappropriate statement – "Dedekind's conception of creation agrees with that of Stolz" – he must have known that he could not provide such an argument. Dedekind's creations of the natural and the real numbers are just two counter-examples to the creation of mathematical objects à la Stolz.
(6) Frege's claim that the conversion incorporated in the Initial Stipulation is nothing new (around the turn of the twentieth century) is disputable. If the claim is meant to support the non-creativity of the Initial Stipulation, it fails to do so.
(7) Frege's rival may argue that the view that value-ranges exist prior to the transformation enshrined in the Initial Stipulation (and in Basic Law V) is an ungrounded ontological or metaphysical assumption that defies verifiability. It is difficult to figure out how Frege might have reacted to this charge. I only tentatively suggested what his response might have been.

(8) Frege obtains Basic Law V from the Initial Stipulation by (a) transforming the stipulative mode of the latter into the assertoric mode of the former, (b) converting the stipulated coreferentiality of two metalinguistic sentences into an objectual identity (= truth-value identity) and (c) by fitting out the objectual identity with a formal guise. The Initial Stipulation and Basic Law V play essentially different roles and fulfill essentially different tasks in the exposition of the concept-script. Regardless of the difference of status and role of the Initial Stipulation and Basic Law V, they have something fundamental in common. Both are couched in the same second-order abstraction principle and, hence, express the same thought. From a semantic point of view, the Initial Stipulation legitimizes and guarantees for Basic Law V only what it stipulates, namely the coreferentiality and not the synonymy of the two sentences. Hence, the former cannot ensure the requisite self-evidence of the latter: self-evidence requires synonymy. Furthermore, the Initial Stipulation, taken by itself, cannot guarantee the truth of Basic Law V. If we adopt the viewpoint that Frege takes in *Grundgesetze* II, §146, we seem to face a mutual dependence between the Initial Stipulation and the basic law of logic to which he appeals and which I take to be Basic Law V. This law cannot be introduced without relying on the Initial Stipulation and the twin stipulations. However, Frege apparently can lay claim to the legitimacy of the transformation embodied in the Initial Stipulation only if he invokes a basic law of logic whose content is precisely this transformation. And this is what he actually does in *Grundgesetze* I, §9 and again, although in retrospect, in *Grundgesetze* II, §146 and §147.

(9) In *Grundgesetze* II, §146, Frege relates the potential charge of having created value-ranges not only to the Initial Stipulation but also to stipulations that he makes in §9 and §10. But he confines himself to defending the non-creativeness of the Initial Stipulation. I argue that the stipulation concerning "$\acute{\varepsilon}\Phi(\varepsilon)$" in *Grundgesetze* I, §9 does not provide a coign of vantage for Frege's opponent. By contrast, in the scenario that I am envisioning it could seem that the dual stipulation in *Grundgesetze* I, §10 was an easy target for the creationist opponent. He objects that Frege creates the unit classes of the True and the False almost entirely out of the blue and consequently fails to meet the constraints of non-arbitrariness or non-haphazardness, rigour and lawfulness that he imposes on any creation that he might tolerate, though would not adopt for himself. Unfortunately, Frege does not explain why he thinks that the dual stipulation in *Grundgesetze* I, §10, in addition to the Initial Stipulation, may give rise to a creationist charge and if so, how he would have responded to it. In any case, simply dismissing the charges of the opponent as unsubstantial in the scenario that I am envisioning would hardly have been convincing.

(10) On the face of it, Frege's identification of the truth-values with their unit classes in *Grundgesetze* I, §10 is in conflict with his platonism. However, if we wish to make sense of these transsortal identifications and appreciate the impact they have on Frege's attempt to completely fix the references of value-range names, we should not evaluate them from the viewpoint of his platonism. I suggested that in this respect we may take up the cudgels for Frege. His illicit use of "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" in §10 is another, though related matter. Even the use of

"$\acute{\varepsilon}(-\varepsilon)$" in §10 may have turned out to be unlicensed. The issue was analyzed to some extent in Sect. 2.7 of this essay.

(11) Frege could have defended the non-creativity of Basic Law V with more or less ease if if he had been challenged to do so.

4 Part III

Part II of this essay was devoted to Frege's partial defence, in *Grundgesetze* II, §146, of the alleged non-creativeness of the stipulations in *Grundgesetz*e I, §3 und §10 which are designed to fix the references of value-range names in a piecemeal fashion. I called this process the *first procedure*. In what follows, I critically examine Frege's line of argument in *Grundgesetze* II, §147 and argue that as in §146 he falls short of advocating his anti-creationist standpoint. The question of whether by applying the formation rules of the concept-script Frege could be said to create the whole range of objects and functions that arithmetic deals with (henceforth called the *second procedure*) probably cannot be given a definitive answer. At present, as I put the finishing touches to this essay, l tend to assume that an affirmative answer appears no less likely than a negative, unless someone succeeded in adducing a powerful argument in favour of the putative non-creative nature of Frege's second procedure. I am sure that at the time when Frege began writing the sections on mathematical creation in *Grundgesetze* II (including §146 and §147), he would have answered the question outright in the negative, if he thought that he was in possession of a conclusive argument. Yet curiously enough, in §147 he confines himself to raising the different question of whether his procedure – he refers at least to the second procedure and possibly to the first as well – can be *called* a creation. Even in response to this question, which plays only a subordinate role in the envisioned dispute, Frege remains rather vague. Whatever reason he may have had for his restraint and for making a concession to an imagined creationist rival, an argument that the second procedure is across the board non-creative is not to be forthcoming in *Grundgesetze* II, §147. I presume that Frege knew this. I further presume that he knew that it would have been a tall order to defend the non-creativeness of the second procedure against a massive charge of an uncompromising sceptic, if he had felt constrained to do so. As in Part II of this essay, I describe several scenarios in which an imaginary creationist mathematician is Frege's opponent.

It is astonishing that neither in *Grundgesetze* II, §146 nor in §147 does Frege appeal to what he apparently considers the driving force behind his arithmetical platonism in order to defend his anti-creationist position. Instead, he repeatedly stresses the importance of the requirements of non-arbitrariness, boundedness and lawfulness (or compliance) that he thinks he has met with respect to both the first and the second procedure and which he may have put in the balance with even greater emphasis if his anti-creationist position had been challenged in substance, not just verbally. Despite the indecision that Frege displays in §147 when he raises, almost in passing, the creativity/non-creativity issue, and despite the verbal concession that he makes to an imagined opponent, his commitment to arithmetical platonism seems to loom large in the background. Why in §146 and §147 he does not bring it to the fore as a source for a possibly more efficient advocacy of the putative non-creativity of both the first and the second procedure is an enigma to me.

4.1 What is at issue in *Grundgesetze* II, §147?

In *Grundgesetze* II, §147, Frege writes:

> If there are logical objects at all — and the objects of arithmetic are such — then there must also be a means to grasp them [*sie zu fassen*], to recognize them [*zu erkennen*]. The basic law of logic that permits the transformation of the generality of an equality into an equality serves for this purpose. Without such a means, a scientific foundation of arithmetic would be impossible. For us it serves the purposes that other mathematicians intend to achieve by the creation of new numbers. Our hope is thus that from the eight functions whose names are listed in I, §31, we can develop, as from one seed, the whole wealth of objects and functions that mathematics deals with. Can our procedure be called a creation? The discussion of this question can easily degenerate into a quarrel over words. In any case, our creation, if one wishes so to call it, is not unrestricted and arbitrary, but rather the way of proceeding, and its admissibility, are established once and for all. And with this, all the difficulties and concerns that otherwise call into question the logical possibility of creation vanish; and by means of our value-ranges we may hope to achieve everything that these other approaches fall short of.

At the beginning of this passage, Frege does not justify his inference from the presupposed existence of logical objects to the necessary existence of a means of grasping them. In any event, he seems to identify a scientific foundation of arithmetic with a logical foundation, and the prospect of success for the latter depends, in his view, essentially on a methodologically sound introduction of fundamental logical objects in terms of which all numbers could be defined and, thanks to those definitions, uniformly governed by a basic law of logic. However, when Frege goes on to assert that this law serves towards the ends that other mathematicians intend to attain by creating new numbers he does not argue for that. Clearly, the aim in *Grundgesetze* I and II is laying the logical foundation of cardinal arithmetic and real analysis.[65] Yet, pursuing and achieving this aim did not by any means coincide with the aim that, for example, Hankel and Stolz sought to achieve with their creative definitions. Among the creationist mathematicians contemporary to Frege, it is only Dedekind who had set himself a goal which apparently was *grosso modo* akin to the goal that Frege had taken up the cause in pursuit of his foundational project: providing unfailing cognitive access to the numbers – conceived of as logical objects by Frege, though not necessarily by Dedekind. According to Frege, gaining such access had to proceed uniformly for all numbers via logical abstraction (via Basic Law V),[66] whereas for Dedekind our cognitive access

[65] On Frege's theory of real numbers see von Kutschera (1966), Simons (1987), Dummett (1991), Schirn (2013), (2014), Snyder and Shapiro (2019) and Boccuni and Panza (2022).

[66] Heck (2011), p. 15 interprets Frege's remark "And to this end serves us that basic law of logic that permits the transformation of the generality of an equality into an equality" as follows: "And it is important

to the natural numbers (see Dedekind, 1888) and to the real numbers (see Dedekind, 1872) proceed in different ways, namely by what I called "structural abstraction" in the first case and "cut generation" in the second. I do not think that the use of the phrase "cognitive access" is out of place when we consider Dedekind's approach to number theory and real analysis, although, unlike Frege, he does not directly comment on epistemological topics in a narrower sense. Dedekind's Preface to the first edition of *Was sind und was sollen die Zahlen*?, for example, may give us some idea of the philosophical attitude that underlies his formal construction of arithmetic.

The second part of Frege's exposition in §147 is overshadowed by vagueness and a conspicuous lack of explicitness. To begin with, I find his abrupt transition from his comments on Basic Law V to the development of the rich spectrum of objects and functions that arithmetic deals with (= the second procedure) irritating. Frege's use of the words "our hope is *thus* [my emphasis]" may suggest that the development he has in mind, but omits to characterize, is closely related to the benefits that he derives from Basic Law V in pursuit of his logicist project. Furthermore, in *Grundgesetze* II, §147 it is up to guessing what the specific nature of the relation of the second procedure to the first is supposed to be, especially with regard to potential creationist charges. In any event, by drawing the reader's attention, towards the end of §147, to the development of the arithmetically relevant objects and functions followed by the question "Can our procedure be called a creation?", Frege unexpectedly turns over a new leaf in his remarks on mathematical creation in *Grundgesetze*.[67] Without

Footnote 66 continued
to appreciate that the 'fundamental law of logic' for whose acceptance Frege is arguing here is not Law V itself. It is, rather, something that is a law of logic in a quite different sense and that serves to justify Law V. This law is what justifies our 'recognizing something common', so that 'accordingly we may transform an equality holding generally into an equation' (Gg, v. II, §146)." I disagree. There simply is no basic law of logic in Frege's logical system which justifies another basic law of logic. In my opinion, it is beyond doubt that in the quotation Frege appeals to Basic Law V for lack of any alternative. It seems pointless to me to speculate about what possibly motivated him to mention this basic law neither in §146 nor in §147 by its well-known name. I have not even an inkling what the law of logic that according to Heck's interpretation Frege considers to justify Basic Law V is supposed to be.

[67] In the heading of *Grundgesetze* II, §147 "Our procedure is not really new, is performed in full awareness of its logical admissibility. Without it, a scientific justification of arithmetic would be impossible.", Frege can reasonably refer only to the first procedure. The reason is straightforward. It is the transformation in the Initial Stipulation which is said to be nothing new. By contrast, the second procedure is largely new in logic at the end of the nineteenth century. While it is clear that in Frege's view in his pre-Paradox period a logical foundation of arithmetic would be impossible without relying on the first procedure—first and foremost on the Initial Stipulation and its formal counterpart Basic Law V—it is less clear whether he thought that such a foundation would be impossible without relying on the second procedure in every respect. Without doubt, from his point of view the syntax of the formal language used in a logical foundation of arithmetic had to be devised in such a way that the special objects and special functions that arithmetic deals with could be provided. Yet Frege *might* have thought that the syntax of *Grundgesetze* with its specific formation rules and their symbiotic interaction in the formation of concept-script-names is perhaps not the only possible syntax to achieve this. In the fourth and final passage of §147, after having highlighted the indispensability of a special basic law of logic for a scientific foundation of arithmetic, there is a shift of reference in Frege's use of the phrase "our procedure". He now mentions the second procedure but does not provide any related information. The Initial Stipulation and Basic Law V, along with the justifying role of the latter for the transformation enshrined in the former dominate almost the entire line of argument not only in §146 but also in §147. Thus, it seems plausible to assume that in raising the question "Can our procedure be called a creation?" towards the end of §147 Frege has not only the second but also the first procedure in mind, in particular the Initial Stipulation and possibly the dual stipulation in *Grundgesetze* I, §10 as well, although he does not comment on the latter.

going into detail at this point, the way in which the objects and functions that he considers to belong essentially to arithmetic sprout from the seed of the primitive functions of his system differs essentially (a) from the step of logical abstraction (= the transformation from right to left incorporated in both the Initial Stipulation and its formal counterpart, Basic Law V) and (b) from the twin stipulations in *Grundgesetze* I, §10 (= the identification of the truth-values with their unit classes).[68] Note that this is my own analysis, not Frege's. When in *Grundgesetze* II, §146 Frege formulates a possible creationist charge, he deals exclusively with the Initial Stipulation, although he seems to be aware that the twin stipulations may also give rise to a creationist charge. In any case, with the stipulations concerning value-range names in *Grundgesetze* I, §3, §9 and §10 in his hands, together with a handful of standard elucidations of primitive function-names prior to §10 and after §10 (in §11 and §12) and the extension of the notation for generality in *Grundgesetze* I, §19-§20 (which actually goes until §25) – Frege can, from his viewpoint, present the concept-script version of Basic Law V (in §20). Moreover, by virtue of these stipulations and equipped with the criteria of referentiality for proper names, first-level, second-level and third-level function-names in §29 and the development of the syntax of the concept-script in §26 and §30, Frege apparently thinks that he can successfully carry out the proof of referentiality on which all depends in his project. In particular, it seems that only after he had provided this basis did he feel entitled to initiate the development of particular value-ranges and particular functions that "contain" a value-range as a "constituent". In other words, only after having proved in §31 that value-range names – in the first place regular value-range names – are in fact referential did Frege set the wheels of the syntactic machinery laid out in §26 and §30 in motion to form individual value-range names which demonstrably refer to particular value-ranges as well as special function-names by means of which he defines (in *Grundgesetze* I) the simple names of all those complex functions that he considers indispensable for laying the logical foundations of cardinal arithmetic. As far as the formation and the use of special value-range names is concerned, there is at least one exception to the strategy that I just characterized. In *Grundgesetze* I, §10, Frege appears to disregard the methodological imperative which, I assume, he had intended to follow (almost) throughout the exposition of the concept-script: deliver the syntax of the formal language before you form and use particular value-range names. In my view, it is chiefly for pragmatic reasons that in *Grundgesetze* I, §10 Frege temporarily offends against this methodological guideline, although in his view without grave consequences for his platonism which otherwise remains intact.

There is another aspect worth mentioning in this connection. Frege does not set up any definition prior to carrying out the proof of referentiality.[69] By contrast, he states Basic Laws V and VI before he carries out this proof and even prior to setting out the

[68] In *Grundgesetze* I, §10, Frege formulates the twin stipulations in the objectual mode. In §31, he couches them in semantic terms: "Owing to our stipulations that '$\acute{\varepsilon}\Psi(\varepsilon) = \acute{\varepsilon}\Phi(\varepsilon)$' is always to be coreferential with '$\forall\mathfrak{a}(\Psi(\mathfrak{a}) = \Phi(\mathfrak{a}))$', that '$\acute{\varepsilon}(-\varepsilon)$' is to refer to the True and that '$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$' is to refer to the False, every proper name of the form '$\Gamma = \Delta$' is guaranteed a reference if 'Γ' and 'Δ' are regular value-range names or names of truth-values."

[69] On Frege's proof of referentiality see Thiel (1975), Resnik (1986), Linnebo (2004), Heck (1997) and (2012) and Schirn (2018).

syntax of the formal language. In accordance with his requirement of utmost generality for logical axioms, Frege formulates Basic Laws V and VI by using schematic value-range names: they do not refer to a particular value-range. Yet when he comes to formulate these laws, he apparently presupposes that in *Grundgeseze* I, §3 and §10-§12 he has succeeded in conferring a unique reference on the value-range operator "$\acute{\varepsilon}\varphi(\varepsilon)$". The main part of the proof of referentiality is designed to establish this beyond doubt.

I think that at this stage of my investigation there is not much more to say about the relation of the second procedure to the first. It is true that both procedures deal with value-ranges and functions but, as I already indicated and in a moment shall explain in more detail, they do this in fundamentally different ways. However, one remark that I consider important still has to be made in this section. Regarding the issue of creation, I hold that even if by virtue of a persuasive argument Frege had succeeded in establishing the non-creativeness of the first procedure, he could not have inferred from this that the second procedure is likewise non-creative throughout. As a matter of fact, in *Grundgesetze* II, §147 he refrains from drawing this inference and I presume that he does so deliberately. I further presume that Frege knew that in §146 he had failed to advance a cogent argument for the non-creativeness of the first procedure. Nor does he argue in §147 for the non-creativeness of the second procedure independently of the first. The truth is that in §147 he does not argue at all in either direction.

4.2 The second procedure under scrutiny

It is unfortunate that in *Grundgesetze* II, §147 Frege does not briefly characterize the development of the whole spectrum of objects and functions that arithmetic deals with. For the reader's awareness of what is specifically at issue not least with respect to the creativity/non-creativity issue, a few explanatory remarks would have been very useful. Frege should also have informed the reader why he thinks that the second procedure is not a creation instead of making short work of it. Nevertheless, we can plausibly assume that he has the application of the formation rules of his concept-script in mind when he mentions the development of the whole wealth of objects and functions that arithmetic deals with.

For the time being, here are some of the queries to which Frege's exposition towards the end of §147 may give rise.

(i) To begin with, the creationist opponent may complain that Frege not only spares himself the trouble of briefly characterizing the second procedure but also fails to specify and delimit the range of the objects and functions that arithmetic deals with and which are said to derive from the primitive functions of his formal system. In particular, it is not clear whether in Frege's view arithmetic, if it is given a logical foundation, comprises not only special functions that are obtained from the primitive functions by iterated application of the formation rules but the primitive functions as well. It could seem that regarding the functions which Frege considers to belong essentially to arithmetic, he does not draw a sharp dividing line (at least not in *Grundgesetze* II, §147) between (a) functions that in his view belong intrinsically to the core of logic and for which logic therefore allows no replacement (cf. Frege, 1967, p. 32),[70] such as negation, identity, the conditional and the first-order and second-order universal quantifiers (they are all primitive in *Grundgesetze*), and (b) specifically arithmetical functions such as the cardinality function (*Grundgesetze* I, §40), the concept of cardinal number (§42) and the relation in which one member of the cardinal number series stands to that immediately following it (§43) (= the predecessor relation). While these functions essentially belong to cardinal arithmetic, the strong ancestral of a relation (§45) and the weak ancestral of a relation (§46), for example, play a crucial role not only in cardinal arithmetic but also in real analysis, as conceived of by Frege. The functions (or their names) mentioned under (b) as well as the strong and the weak ancestral of a relation are introduced via explicit definitions in the exposition of the concept-script. Clearly, if Frege had succeeded in establishing cardinal arithmetic and real analysis as (highly developed) branches of logic, then, from his point of view, the functions that he defines in *Grundgesetze* I and II would belong to logic too, although not initially.

(ii) At the stage of *Grundgesetze* II, §147, Frege has not yet defined any function-name which in addition to some of the "old" function-names that he defines in *Grundgesetze* I he needs specifically for the development of his theory of magnitude. This theory is laid out in the semi-formal proof-analyses and the ensuing formal proof-constructions in *Grundgesetze* II, §165–§245, but ends abruptly in §245, overshadowed by Russell's Paradox. Thus, in *Grundgesetze* II, §147, when Frege mentions the rich spectrum of objects and functions that arithmetic deals with, he is primarily looking back on the functions and objects that he has already introduced via definitions in the course of laying the logical foundations of cardinal arithmetic (and perhaps on some of the functions that he has developed as well by applying the formation rules but does not define). His wording – "Our *hope* is thus that from the eight functions … we *can* develop …. the whole wealth of objects and functions that arithmetic deals with" [my emphasis] – suggests though that at the same time he is looking ahead to the development of the objects and functions that play an essential role in the projected logical foundation of real analysis, in particular to the construction of new central functions that he will define. As I indicated above, regarding the repertoire of the functions and objects that play an important role in the theory of magnitude, not everything is new. In the proofs that Frege carries out, he utilizes, besides the primitive functions introduced in *Grundgesetze* I and a number of newly defined functions, such as *domain of magnitudes*, *limit*, *positival class*, *positive class* and *Archimedian condition*,[71] several of the "old" functions which he had defined in

[70] Besides negation and identity, Frege mentions in 'Über die Grundlagen der Geometrie' 1906, III subsumption and subordination of concepts as functions that belong intrinsically to logic (see Frege 1967, p. 322). I think that regarding identity Wittgenstein would basically have agreed with Frege in the *Tractatus*. Despite his dispensation with "=" in a correct concept-script, Wittgenstein adheres to identity nonetheless in the *Tractatus* but construes it neither as a relation in which every object uniquely stands to itself nor as coreferentiality or mutual substitutivity. A correct concept-script à la Wittgenstein does not contain coreferential names. He expresses identity of the object (which is not conceived of as a relation) by identity of the sign. For details see Schirn (2024a).

[71] Frege does not introduce a specific term for this dyadic function. For the sake of brevity, he defines the corresponding simple name in *Grundgesetze* II, §199. Frege needs this function-name in order to prove

Grundgesetze I: the relation of an object falling within the extension of a concept (the application function, *Grundgesetze* I, §34); the single-valuedness of a relation (*Grundgesetze* I, §37); the converse of a relation (*Grundgesetze* I §39); the relation that is composed from two relations (*Grundgesetze* I, §54); and the two functions that I already mentioned in this connection: The following of an object after an object in the series of a relation (the strong ancestral of a relation) and the relation of an object belonging to a series of a relation starting with an object (the weak ancestral of a relation).

(iii) In *Grundgesetze* II, §147, Frege does not inform the reader whether he appeals only to the development (from the primitive functions) of those arithmetically relevant functions that he defines or also to those complex functions which he uses in the course of carrying out the proofs of the basic laws of arithmetic but does not highlight by definitions. Be that as it may, the primitive function-names (or functions) together with the defined function-names (or functions) take centre stage in Frege's logicist project. Moreover, the names of functions of these two kinds play a crucial role in the construction of function-names which are not defined but are nevertheless requisite in conducting the proofs and which Frege probably includes in the range of objects and functions that arithmetic deals with.[72]

(iv) Regarding the objects that together with functions form the subject matter of arithmetic, Frege has in *Grundgesetze* II, §147 most likely special value-ranges in mind, such as equivalence classes of equinumerosity, ordered pairs (cf. *Grundgesetze* I, §144), classes of finite cardinal numbers (cf. *Grundgesetze* II, §164), and with a foresighted eye on the imminent development of the theory of magnitude: magnitudes (Relations), domains of magnitudes, classes of Relations, Relations belonging to a positival class, Relations belonging to a positive class and last but not least Relations on Relations (ratios of magnitudes).[73] This aspect is likewise passed over in silence in §147.

In what follows, I close another gap in Frege's exposition in §147 by taking a look behind the scenes. I have the formation rules in *Grundgesetze* in mind, the way they work and interact and what, from the point of view of Frege's opponent, the creative

Footnote 71 continued
what he calls the *Archimedian Axiom* (see *Grundgesetze* II, p. 191). "If two Relations belong to the same positive class, then there is a multiple of the one that is not less than the other." Regarding Frege's definition of the Archimedian condition, see Cook (2013), pp. A–41 f.). On Frege's definitions in *Grundgesetze* in their entirety see Cook (2013); see also the comments of Cook and Ebert (2016) and of Kremer (2019) on some selected *Grundgesetze*-definitions.

[72] Here is just one simple example the reader may look at in this connection. I select the proof of the proposition that the cardinal *Endlos* is not a finite cardinal number. The relatively short proof proceeds by showing that the cardinal number *Endlos*, in contrast to every finite cardinal number, follows after itself in the cardinal number series. The proof involves the use of several primitive and defined function-names as well as the use of complex function-names which are formed from the former but are not defined (see the proof-construction in *Grundgesetze* I, §123 that follows the proof-analysis in §122). If in one of the much longer proofs in *Grundgesetze* one picks out a concept-script sentence of great syntactic complexity with a fair amount of embedded conditionals and reconstructs its complete formation pedigree, one may come across an even richer variety of function-names.

[73] A Relation is the value-range of a dyadic first-level function whose value for every fitting pair of arguments is either the True or the False. In short, it is the extension of a first-level relation.

potential of the formation rules might be regarding the rich spectrum of objects and functions that arithmetic deals with.

4.3 A look behind the scenes. The syntax of the concept-script: the backbone of the development of the subject matter of arithmetic

Those readers of *Grundgesetze* II, §147 who have skipped *Grundgesetze* I are hardly able to judge whether the second procedure is creative or not. Also for this reason, raising the question "Can our procedure be called a creation?" is almost pointless in §147, at least regarding the second procedure. I shall say more about this in due course.

In *Grundgesetze* I, §26, Frege states three rules of forming function-names. These rules govern the formation of monadic first-level function-names, dyadic first-level function-names and monadic second-level function-names with an argument-place of the second kind (it is suitable to take monadic first-level function-names) or of the third kind (it is suitable to take dyadic first-level function-names). In all three cases, the formation proceeds by extracting new function-names from more complex names via what I call *gap formation*. For the sake of brevity, I call these rules accordingly *gap formation rules*; for more details see Schirn, (2018). Frege applies and must apply a second syntactic device of forming concept-script-names which is designed to interlock with the first. I call it the *rule of insertion*. This rule permits the insertion of fitting argument-expressions into the argument-place(s) of well-formed concept-script function-names. (Frege most likely considers the primitive names to be well-formed on assumption.) Hence, in contrast to the gap formation rules, the rule of insertion governs the formation of more complex names from simpler names. In that respect, the latter is – roughly speaking – the converse or mirror image of the former. Only by initially applying the rule of insertion to primitive function-names can the entire formation process get off the ground. It enables Frege not only to form the proper names (function-value names) on which he crucially relies in the logical construction of arithmetic but also complex monadic function-names that are likewise required in that construction. In §30, on the brink of carrying out his proof of referentiality, he explains – regarding the twin stipulations in §10, with undue delay – the interplay between insertion and gap formation and relates the formation of names to the criteria of referentiality for (a) monadic first-level function-names, (b) proper names, (c) dyadic first-level function-names, (d) second-level function-names with an argument-place of the second kind and (e) the only third-level function-name (the second-order universal quantifier) that he actually uses in his system and which is primitive (cf. §29). Note that in these criteria Frege mentions only the operation of inserting names of a certain kind into the argument-place(s) of function-names of a certain kind and, hence, not any construction of a function-name by means of gap formation. Clearly, any name of type (a) – (d) to be shown to be referential, and thus any name that results from combining, according to the corresponding criterion of referentiality, a name of type (a) – (d) with a name of the appropriate syntactic category via insertion, may include gap formation in its constructional history. Due to the primitiveness of the second-order universal quantifier, this obviously does not apply to it.

I conclude this brief foray into the syntax of Frege's concept-script with one more comment. When we trace back the formation sequence of a concept-script sentence to its first member, we may come across the formation by insertion either from two primitive names or from a primitive name and a defined name or from two defined names. Since all function-names that Frege defines in the two volumes of *Grundgesetze* are of first level,[74] the initial formation step in the genesis of a concept-script sentence can never proceed by combining two defined function-names. The third variant mentioned above is, in principle, not excluded precisely because Frege also defines some proper names, namely those of the cardinal numbers 0 (§41), 1 (§42) and *Endlos* (§122).

4.4 Facing the potential charge of a creation of special value-ranges and special functions

In what follows, I make some remarks on the potential charge of the creationist opponent that by applying the formation rules Frege creates special value-ranges and special functions.

First, if the potential charge of the syntactic creation of special value-ranges and special functions that arithmetic deals with preyed on Frege's mind, then he should have got this off his chest straightaway. Instead of asking whether the second procedure can be called a creation, he should then rather have raised the question "*Is* our procedure a creation?" How the procedure can be called is not that relevant, although it is not completely irrelevant either. By contrast, what the second procedure intrinsically is or is taken to be by Frege is of paramount importance for his philosophy of arithmetic and for asssessing it appropriately. Even a discussion of the question "Can our procedure be called a creation?", which seems to have a rethorical ring to it, need not degenerate into a quarrel over words. Frege does not claim that it inevitably does. He only says that it *can* easily degenerate into a quarrel over words.[75] I assume that he thought he was able to draw a more or less clear-cut dividing line between what is a creation of mathematical or logical objects and what is not, at least in relevant cases. Otherwise, he could hardly have discredited creationist tendencies in the work of some fellow mathematicians with such unwavering conviction.

Second, if Frege had raised the question "Is our procedure a creation?" instead of "Can our procedure be called a creation?" and if he had answered it with a clear "no", it would have been incumbent upon him to argue outright for the non-creativeness (at least) of the second procedure.

[74] By defining the application operator "$\xi \cap \zeta$" Frege creates the possibility of representing both monadic and dyadic functions of first level by their value-ranges – "although of course not in such a way that they simply concede their places to them, for that is impossible" (*Grundgesetze* I, §34). In this way, he provides an efficient device of using first-level functions instead of second-level functions in the formal system (see *Grundgesetze* I, §34-§36). Stepping down from function-level 2 to function-level 1 is intended to bring about flexibility, conceptual parsimony as well as proof-theoretic conciseness. There is a revealing discussion of the status and the role of the application operator in Frege's logical system in Cook and Ebert (2016). Cf. also Cook (2023) and Schirn (2018), (2023b).

[75] Clearly, the question of whether in making the Initial Stipulation and the twin stipulations Frege offends against his anti-creationist credo might also degenerate into a quarrel over words.

Third, suppose that Frege thought he could convincingly argue that the second procedure is not a creation of arithmetically relevant objects and functions. In that case, he should not have allowed that his creationist rival may call the second procedure a creation nonetheless. What arguably is not a creation cannot reasonably be called a creation. At any rate, in the scenario that I am envisioning Frege's antagonist is not concerned with a sheer verbal dispute but "accuses" Frege of actually having created special value-ranges and special functions via the syntactic procedure.

Fourth, although in *Grundgesetz*e II, §147 (and earlier in §146) Frege offers an open flank to his creationist rival and despite the verbal concession he makes to him, pointless as it is in the relevant context, he most likely continued to believe in the non-creativity not only of the first but also of the second procedure and therefore in the sustainability of his platonism. There is no evidence that his long-standing platonism reaching back to *Grundlagen* underwent any significant change in *Grundgesetze* II, §146-§147, let alone that he was pondering over its posssible abandonment. However, when he wrote these sections Frege apparently had no conclusive argument at hand to effectively defend his anti-creationism.

4.5 Ebert and Rossberg on *Grundgesetze* II, §146-§147

In their interesting essay 'Mathematical Creation in Frege's *Grundgesetze*', Ebert and Rossberg (2019b) offer a different interpretation of what Frege probably means by "our procedure" when he raises the question "Can our procedure be called a creation?" They write (p. 334f.):

> Crucially, then, the "procedure" Frege has in mind is neither Basic Law V nor, as Heck urges, the Initial Stipulation. Rather, it is his procedure of introducing "mathematical" objects and functions as special kinds of logical objects (i.e., value-ranges) into his formal system. As a result, we are here concerned with a different kind of creation. While in §146 Frege rejects the idea that value-ranges *per se* are created by means of his transformation, in the last paragraph of §147 he ponders whether the introduction of "the whole wealth of mathematical objects and functions" by means of his value-ranges can be called a creation. And, naturally, Frege shows little interest in this kind of dispute. After all, that issue is insubstantial: provided that value-ranges are not created but objective and non-actual, calling the introduction of "new" mathematical objects as a special group of these value-ranges a creation does not challenge his platonism—it really is a mere quarrel about words.

(1) Ebert and Rossberg contend that in *Grundgesetze* II, §146 Frege rejects "the idea that value-ranges per se are created by means of his transformation". Well, not quite. Frege only reports (a) what he has not done in *Grundgesetze* I, §3, §9 and §10 – he did not list properties of a thing and then said we create a thing that has these properties – and goes on to declare (b) what he actually has done (in §3): he stipulated that the coextensiveness of two monadic first-level functions can be transformed into a value-range identity. Literally, he stipulated the coreferentiality of the two corresponding sentences. Yet neither (a) nor (b), nor the conjunction of

(a) and (b) imply that in §146 Frege rejects the idea that value-ranges are created by means of this transformation. As I pointed out in Part II, the property-listing procedure à la Stolz is not the only method of creating mathematical objects that some of Frege's fellow mathematicians pursued, and I assume that Frege knew this. Both Dedekind's creation of the natural numbers via structural abstraction and his creation of irrational numbers via cut-generation are just two examples of mathematical creation that differ significantly from the procedure à la Stolz. In sum, in *Grundgesetze* I, §146 Frege does not explicitly repudiate the idea that value-ranges are created by means of the transformation, nor does he explicitly deny in §147 that at this stage of his logicist project—he has not yet made the first step in laying the logical foundations of real analysis—he has created, from the primitive functions of his system, special objects and special functions that cardinal arithmetic deals with (by means of the syntactic procedure that he does not mention, let alone briefly characterize in this connection). Admittedly, it is nevertheless highly plausible if we assume that in Frege's view the transformation embedded in the Initial Stipulation is not a creation of value-ranges. But this does not include that he has a forceful argument to defend the non-creativeness of the Initial Stipulation. In Part II of this essay, I argued that the creationist opponent may effectively challenge the putative non-creativeness of the first procedure.

(2) Suppose that Ebert and Rossberg wished to argue as follows, which, in the light of what they say at the end of the quotation above, is perhaps not too far-fetched: Provided that with his response to the potential creationist charge in *Grundgesetze* II, §146 Frege succeeds in convincing us that in *Grundgesetze* I, §3 and §10 he has not created value-ranges in general, he might infer from this that by applying the formation rules of the concept-script neither has he created particular value-ranges nor special functions that arithmetic deals with. That is what Ebert and Rossberg seem to suggest as a possible defence by Frege of his platonism and anti-creationism at least regarding *special* value-ranges. As to the development of the functions that arithmetic deals with, restricting it to those which are defined and whose *definiens* "contains" a value-range as a "constituent", which allegedly has not been created and thus is supposed to be immune to the charge of having been created, cannot be maintained in full generality. First, not every *definiens* of a defined function-name in *Grundgesetze*, which Frege considers to play a crucial role in arithmetic and its logical foundation, contains a value-range name. There are three exceptions among the 27 definitions that Frege sets up in the two volumes of *Grundgesetze*: neither the *definiens* of the definition of the name of single-valuedness of a relation (*Grundgesetze* I, §37), nor the *definiens* of the definition of the name of the cardinal number *Endlos* (*Grundgesetze* I, §122), nor the *definiens* of the definition of the function *positive class* (*Grundgesetze* II, §197) contains a value-range name.[76] Admittedly, these are only a very few exceptions but they should not be ignored in the context under discussion. Second, as I mentioned earlier, in carrying out the proofs in *Grundgesetze* Frege develops complex functions which do not "include" a value-range as a "constituent" and

[76] Note that the use of names of double value-ranges in the definitions in *Grundgesetze* I outweighs the use of names of single value-ranges.

are not defined either. I further pointed out that he presumably considers those functions nonetheless to be relevant for laying the logical foundation of arithmetic,[77] although to a lesser extent than the complex functions that he defines. Due to the lofty status that the latter possess, Frege develops them in the first place and then proceeds to define the simple names that he introduces for them.

(3) In what follows, let us prescind from complex functions that do not "include" a value-range. If Ebert and Rossberg intend to offer a convincing argument that in §147 Frege assumes the correctness of the inference from the supposed non-creativity of the first procedure to the non-creativity of the second, they fail to offer one. Recall that there is not a scrap of textual evidence for that, neither regarding the construction of special value-ranges nor regarding the construction of special functions that "contain" a value-range as a "constituent". Why should Frege believe that he could justifiably draw this conclusion? If he thought that in *Grundgesetze* II, §146 he has succeeded in advancing a cogent argument for the non-creativity of the first procedure, then his self-confidence would seem to border on overconfidence. Yet regardless of what *he* thought, I fail to see a compelling argument for the validity of the inference in question even if the first procedure could be established as non-creative.

(4) If Frege thought that discussing a possible creationist charge against the second procedure would inevitably lead to a fruitless quarrel about words, why then does he raise the issue at all?

(5) What motivates Frege to emphasize that what in §147 he calls "our procedure" is neither boundless nor arbitrary, or more specifically: that it meets the constraint of proceeding according to a logical law (in the case of the first procedure) and according to rigid, lawlike rules (in the case of the second)? In *Grundgesetze* II, §147, Frege imposes precisely this constraint on what he allows his opponent to call a creation. Regrettably, he spares himself the trouble of illustrating the constraint. It therefore remains unclear what it is supposed to mean that the way of performing "our creation, if one wishes so to call it … and its admissibility, are established once and for all."

(6) In sum, in *Grundgesetze* II, §147 Frege seems to show at least some concern regarding the defensibility of the non-creativeness of the second procedure (and possibly of the first as well). Believing in the non-creativeness of the two procedures, or more generally: in arithmetical platonism, is one thing, advocating it with a persuasive argument against possible objections is another.

(7) Due to all that is left unsaid in *Grundgesetze* II, §147, some of the issues that Ebert and Rossberg and I interpret differently cannot be settled definitively, either for their benefit or mine. However, I agree with the authors that Frege's observations in §146 and §147 are essential for assessing his overall view about mathematical creation on the one hand and mathematical platonism on the other. We further agree that in §146 and §147 two different forms of potential creation are at issue which is almost entirely ignored in the literature: in §146, the potential creation

[77] If he does, this would mean that he includes those functions in the development of "the whole wealth of objects and functions that arithmetic deals with".

of value-ranges in general via logical abstraction clad in the garb of an informal stipulation (supplemented by further stipulations) and in §147, the potential creation of special value-ranges and special functions dealt with in arithmetic via the application of the formation rules of the concept-script. I reemphasize that despite Frege's vacillating attitude in §147 he does not intend to call into question his platonism, neither in §147 nor in any other place of *Grundgesetze*. Why should he do this without urgent need or completely voluntarily? It would appear self-defeating if he did. Also in this respect I seem to be in agreement with Ebert and Rossberg. Keep in mind that in *Grundgesetze* II, §146 and §147 Frege is not facing any *real* creationist charge. However, considering a fictitious debate between him and a creationist opponent seems entirely appropriate to me.[78]

4.6 Frege's appeal to the constrained, non-arbitrary and rule-governed nature of the second procedure: some critical comments

Suppose that in §147 Frege actually argues that the second procedure is not a creation of arithmetically relevant objects and functions. Suppose further that regarding this procedure he is nonetheless facing a substantial creationist charge of an imaginary opponent, not just a charge which may only lead to a quarrel over words. Even in that case, Frege would not have to "seek his salvation" by invoking the constrained, non-arbitrary and strictly rule-governed character of the second procedure which, as I criticized before, he omits to characterize. The methodological merits and achievements which Frege claims for the second procedure – and for the first procedure as well, plus the logical lawfulness of the conversion embodied in the Initial Stipulation – have no direct impact on the answer to the question of whether it is a creation or not, unless perhaps he had ruled out that a creation which avoids possibly fatal unboundedness and otherwise keeps the reins tight, is possible. But his remarks in *Grundgesetze* II, §147 do not suggest anything along these lines. Nor do they suggest that in Frege's view a creation of mathematical objects is bound to be rampant or arbitrary. I am of course aware that at the end of *Grundgesetze* II, §145 he doubts whether an haphazard creation of mathematical objects is possible at all. His doubt seems to concern both unregulated and regulated creation, although presumably in different ways. I further appreciate that Frege begins *Grundgesetze* II, §146 by saying that in the foregoing sections it became plausible "that creating proper is not available to the mathematician, or at least, that it is tied to conditions that make it worthless". However, Frege spares himself the trouble of spelling out (a) why he doubts in general – and not only regarding the definitional property-listing method à la Stolz and Dedekind's creation of irrational numbers – that creating proper is unavailable to the mathematician, (b)

[78] Linnebo (2019, p. 111) seems to assume that towards *Grundgesetze* II, §147 Frege is still exclusively concerned with the question of whether the first procedure is a creation of value-ranges. The second procedure which I think Frege definitely has in mind (even though not exclusively) when he raises the question of whether his procedure can be called a creation Linnebo lets go by the board in his account. The question of whether the conversion of "an instance of the left-hand side of Basic Law V to the corresponding instance of the right-hand side should be seen as a form of creation" is probably considered irrelevant by Frege since abstraction in Basic Law V (and in any other Fregean abstraction principle) intrinsically proceeds from right to left.

what he means more exactly by "creating proper" and (c) why he thinks that the conditions he associates with creating proper, in particular the condition of performing an act of creation only if it is sanctioned by certain laws, render it worthless.[79]

Despite the lack of unequivocal explicitness concerning (a) – (c), I note that Frege's remarks do not seamlessly conform to what he writes towards the end of *Grundgesetze* II, §147. There he does not doubt the logical possibility of creating proper nor does he depreciate a creation that meets the conditions of non-arbitrariness, boundedness and lawfulness as worthless. (Recall that his earlier statement at the beginning of *Grundgesetze* II, §146 seems to suggest such depreciation.) Otherwise it would have been pointless to highlight the fulfilment of these conditions by the second procedure which, to reemphasize, is not expressly denied creative potential. Yet if Frege thought that he was bound to give mathematical platonism its due, then he should have defended the non-creativeness of the second procedure by advancing a sound argument instead of confining himself to praising its non-arbitrary and law-abiding character and instead of making a futile verbal concession to an imagined opponent. Recall that by pointing out the fundamental procedural difference between the Initial Stipulation and the definitional property-listing method à la Stolz as well as by underscoring the non-definitional nature of the Initial Stipulation and the groundedness of the transformation of coextensiveness into identity in a basic law of logic Frege fails to present a powerful argument for the non-creativeness of the first procedure. At least in these two respects, his opponent seems to gain the upper hand in the debate. I would therefore refrain from saying that in the scenario I am envisioning we are facing a stalemate between the two adversaries.[80] In sum, especially in *Grundgesetze* II, §147 Frege misses the chance of arguing forcefully for his anti-creationist and platonist position.[81]

79 In *Grundgesetze* II, §143 – the heading is: 'O. Stolz's creative definitions. Highly consequential restriction of the power of creation' – Frege relates the worthlessness of "the creative power that many mathematicians award themselves" only to the definitional property-listing method à la Stolz. In this paragraph, Frege does not refer to creation proper, nor to the laws that must be obeyed while creating, nor to their requisite justification before carrying out an act of creation.

80 Hallett (2019, pp. 318–320) quotes extensively from *Grundgesetze* II, §147 but, contrary to what he promises to do, does not "bring out the main points of what is being asserted by Frege here". In particular, I fail to see that any of Hallett's five points sheds new light on what Frege writes in §147. The interpretative problems to which the second half of §147 gives rise are not even touched upon by Hallett.

81 Frege's reflections in *Grundgesetze* I, §66 are devoted to the analysis of the ensuing formal proof of the proposition that the relation of a cardinal number to that immediately following it in the cardinal number series is single-valued; see the proof in §67-§87 which at a crucial point involves the use of the right-to-left direction of Hume's Principle (Theorem 32). In the preceding analysis of the proof (§66), Frege writes: "What is it what we are doing when we correlate objects for the purpose of a proof? Seemingly something similar to drawing an auxiliary line in geometry. Euclid, whose method can still often serve as a model of rigour, has his postulates for this purpose, stating that certain lines may be drawn. However, the drawing of a line should no more be regarded as a creation than the specification of a point of intersection. Rather, in both cases we merely bring to attention, apprehend what is already there." This is a clear statement endorsing mathematical platonism and in a sense is reminiscent of what Frege writes about thirteen years later in 'Über Schoenflies: Die logischen Paradoxien der Mengenlehre' (Frege 1969, p. 197): "By means of our logical faculties we lay hold upon [*bemächtigen uns*] the extension of a concept, by starting out from the concept." In Frege's view, the phrase "lay hold upon" is here probably replaceable with "grasp" or "apprehend". He may have added: but we do not create the extension of a concept. I assume that Frege has here the transformation of functional coextensiveness into a value-range identity governed by Basic Law V in mind, which in his own express opinion is not brought down by Russell's Paradox. In *Grundgesetze* I, §66, Frege goes on to write: "Correlations also have to be possible of infinitely many objects, but only a

Let us take stock for a moment. A fair assessment of the pros and cons of Frege's platonist and anti-creationist position in *Grundgesetze* II, §146-§147 is a difficult task for the interpreter. In any event, it turned out, as far as it goes in this essay, that Frege would have faced an uphill battle if his anti-creationist position had been challenged by an uncompromising creationist opponent. In the case of works of art, for example, we seem to have an almost unfailing judgement regarding their creativeness. Virtually nobody who is thoroughly familiar with, say, Bach's *Goldberg Variations* or Proust's *À la recherche du temps perdu* or Picasso's *Guernica* would seriously doubt that these are creations in the fullest sense of creation. By comparison, in mathematics, we may be going down a slippery slope if we intend to draw a clear-cut dividing line between what a creation is and what it is not. The procedures that are at issue in *Grundgesetze* II, §146 and §147 are just two examples in this respect. As I mentioned earlier, Frege probably thought that regarding the evaluation of the positions of some of his fellow mathematicians he could draw a clear dividing line between what a creation is and what it is not. Yet it seems that regarding a convincing defence of his anti-creationist position in *Grundgesetze* II, §146 and §147 he was, in his own opinion, in a less comfortable position. The weakness of the arguments which he advances in §146 and his failure in §147 to deliver an argument for his anti-creationist standpoint are probably not due to sheer laziness or oversight. Naturally, this does not mean that when writing these sections Frege was beset by nagging doubts about the platonistically requisite non-creativeness of the two procedures.

4.7 Concluding remarks

Suppose (i) that the alleged non-creativeness of the second procedure had been challenged by the creationist opponent. Suppose further (ii) that Frege had agreed that the opponent's charge should not be "watered down" to a sheer verbal dispute but ought to be taken seriously. Suppose finally (following more or less the line of thought in §147) (iii) that Frege had responded to the charge by appealing in the first place to the constrained, rule-governed and non-arbitrary nature of the second procedure. However, in that case his response would have been a weak defence strategy, indeed a blunt sword. A (potential) creation that is subject to lawlike rules is still a creation and as such runs afoul of Frege's platonism and anti-creationism. This is not to say that in §147 he endorses a lawful creation of mathematical objects for himself. He does not. Yet the way he argues and formulates may suggest that he is not miles away from tolerating it. Thus, if Hankel and Stolz, for example, had created mathematical objects in a non-arbitrary and lawful manner – recall, however, that it is not entirely clear whether Frege considered this to be possible – he may have criticised their creations to a much lesser extent. But I presume that they would not entirely have escaped his critique. To make a long story short. Once Frege had committed himself to mathematical platonism without any ifs or buts – and he had done so in *Grundlagen* and with equal emphasis in some places of *Grundgesetze* I and II – he should, for the sake of

Footnote 81 continued
few of these infinitely many correlations could actually be carried out if correlating were a creative activity of the mind." It seems *prima facie* plausible to conclude *from* the premise that it is impossible to carry out infinitely many correlations *to* the non-creativity of correlations.

coherence, not have tolerated any form of mathematical creation, be it random and boundless or rule-governed and constrained.

Throughout Part II and Part III of this essay, I have argued that in an imaginary debate a creationist opponent might challenge the putative non-creativeness of both Frege's first and second procedure by turning the tables. In particular, in Part II I argued that Frege's arguments for the non-creativity of the first procedure in *Grundgesetze* I, §146 have not much force and that he was probably aware of this. In Part III, I have argued, *inter alia*, (a) that in *Grundgesetze*, §147 Frege probably refers to both the first and the second procedure when he raises the question "Can our procedure be called a creation?"; (b) that there is not a trace of textual evidence that he appeals to the (alleged) non-creativity of the first procedure with the aim of justifying the (purported) non-creativity of the second; (c) that the inference from the (supposed) non-creativity of the first procedure to the non-creativity of the second would be far from compelling anyway. The first procedure is purely semantic and deals with value-ranges or value-range names in general. In *Grundgesetze* I, §10, Frege uses two special value-range names in order to fix the references of value-range names in general. By contrast, the second procedure is purely syntactical and exclusively concerned with the development of particular value-ranges and particular functions that in Frege's view play an important role in arithmetic and its logical foundation. By appeal to the Initial Stipulation, he can never get hold or avail himself of any particular value-range. He can achieve this only by applying the formation rules of the concept-script, there being no other way. In this respect, the dual stipulation in *Grundgesetze* I, §10 must be considered an exception to the rule. The formation rules which license the formation of "$\acute{\varepsilon}(-\varepsilon)$" and "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" are not yet stated. Frege formulates them in *Grundgesetze* I, §26 and §30. In §30, he also explains how the formation rules interact.

Thus, Frege's use in §10 of the names "$\acute{\varepsilon}(-\varepsilon)$" and "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" to refer to the unit classes of the True and the False is, at this stage, premature and, strictly speaking, illicit. Even if Frege had laid out the formation rules of his concept-script prior to §10, he would not dispose of any canonical value-range name whose reference has been completely fixed before making the transsortal identifications – and, according to his own assessment, not before he has introduced the description operator in §11 and the name of the conditional function in §12 via standard elucidations. In my view, the crux of the twin stipulations is that Frege is bound to presuppose what by his own lights he is not entitled to presuppose, namely that the intended references of the names "$\acute{\varepsilon}(-\varepsilon)$" and "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$" are already completely fixed when he uses them in the twin stipulations in §10. In other words, it is hard to see that Frege might achieve his aim of fixing (almost) completely the references of canonical value-range names in general, including "$\acute{\varepsilon}(-\varepsilon)$" and "$\acute{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$", as if by magic, namely by appeal to value-range names which at this stage of the piecemeal reference-fixing process share the fate of referential indeterminacy with all other value-range names.[82]

[82] Value-ranges do not inherit the properties of primitiveness and logical simplicity from the value-range function $\acute{\varepsilon}\varphi(\varepsilon)$. Unlike the latter, they are by their very nature logically complex. This applies also to the value-range of every primitive first-level function. According to Frege, logical complexity is a necessary condition for definability. Yet he does not define value-ranges in general, nor does he define any particular value-range. To reemphasize, the twin stipulations in *Grundgesetze* I, 10 are non-definitional stipulations.

I further argued (d) that Frege falls short of presenting any argument for the alleged non-creative nature of the second procedure and that putting forward a convincing defence of it would not have been mere child's play for him. Finally, I argued (e) that if we impartially assess the two opposite positions in the imaginary dispute, the conclusion that the creationist opponent may gain the upper hand in more than one respect does not appear unwarranted.

Frege would certainly have acknowledged that the development of the whole wealth of objects and functions that arithmetic deals with proceeds by (syntactic) *constructions* but he did not regard those constructions as *creations*. Charles Dickens, whom I always admired as a writer, explains the difference between *constructio*n and *creation* as follows: "The whole difference between construction and creation is exactly this: that a thing constructed can only be loved after it is constructed; but a thing created is loved before it exists." Might Frege have acquired a taste for that when he was writing §147 of *Grundgesetze* II?

Acknowledgements I am very grateful to Daniel Mook for carefully reading both the original and the penultimate versions of my essay and for offering valuable suggestions to improve it. Special thanks are also due to reviewer #2 for their extensive and detailed comments, both appreciative and critical, which inspired me to write and add Part I when I revised the paper. I thank reviewer #1, Otávio Bueno, Patricia Blanchette, Riki Heck, Bruno Bentzen, Philip A. Ebert and Marcus Rossberg for the inspiration they provided while I was working on my essay. Finally, my thanks go to Esther Rentmeester, Team Manager Journal Production, for kindly helping me with the proofreading.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest There is no conflict of interest: I confirm that neither me nor my employer nor a sponsor have any financial, commercial, legal or professional relationship with other organizations, or with the people working with them, that could influence my research.

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