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Regularity of Generalized Mean-Field G-SDEs

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Abstract

We study the regularity properties of the unique solution of a generalized mean-field G-SDE. More precisely, we consider a generalized mean-field G-SDE with a square-integrable random initial condition, establish its first- and second-order Fréchet differentiability in the stochastic initial condition, and specify the G-SDEs of the respective Fréchet derivatives. The first- and second-order Fréchet derivatives are obtained for locally Lipschitz coefficients admitting locally Lipschitz first- and second-order Fréchet derivatives respectively. Our approach heavily relies on the Grönwall inequality, which leverages the Lipschitz continuity of the coefficients.

Keywords: mean-field; McKean–Vlasov; uncertainty; sublinear expectation; SDEs; derivative; variation

MSC: 60H10

1. Introduction

Mean-field stochastic differential equations have emerged as a powerful mathematical framework for modeling the dynamics of large populations of interacting agents subject to random perturbations. Their significance lies in their ability to capture both the individual stochastic behavior of agents and the macroscopic effects of collective interactions, making them essential tools in fields such as physics, biology, economics, and quantitative finance. In particular, a mean-field SDE serves as the representation of a system whose stochastic evolution depends not only on the individual state but also on the distribution of the population. The pioneering work of Kac [1] introduced the mean-field approach in the context of kinetic theory, while McKean [2] first formalized nonlinear Markov processes whose dynamics depend on their own law. Since then, mean-field SDEs have been extensively studied and generalized, with foundational contributions by Sznitman [3] on propagation of chaos and Lasry and Lions [4,5] and Carmona and Delarue [6,7] on mean-field games and controls. These equations also underpin numerous modern applications, from systemic risk modeling in finance to synchronization phenomena in neuroscience, underscoring their broad relevance and mathematical richness.

In the 2000s, Shige Peng introduced the theory of sublinear expectations and, as a special case, the G-setting as a framework to study Knightian uncertainty; see [8–11]. A sublinear expectation can be expressed as the supremum of linear expectations over a set of probability measures; see Theorem 1.2.1 in [12]. That is, if $\hat{\mathbb{E}}$ is a sublinear expectation, then there exists a set of probability measures $\mathcal{P}$ such that $\hat{\mathbb{E}} = \sup_{P\in\mathcal{P}} E_P$, where E_P denotes the linear expectation with respect to P, and conversely, for every set of probability measures $\mathcal{P}$, the functional $\sup_{P\in\mathcal{P}} E_P$ defines a sublinear expectation. In that sense, a sublinear

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expectation can be thought of as the "worst" outcome within a class of models. The G-setting is used to quantify volatility uncertainty and consists of the so-called G-Brownian motion and the G-expectation; see Chapter 3 in [12] for more details. There have been significant advancements in the theory of sublinear expectations and the G-setting in recent years. For instance, refs. [13–16] study the construction of sublinear expectations and their properties, and refs. [17–20] study different classes of stochastic processes in a sublinear expectation framework.

Besides the probabilistic interpretation of quantifying Knightian uncertainty, there is a strong connection between sublinear expectations and fully nonlinear partial differential equations. This has been extensively studied in, e.g., [21–24] for different types of backward G-SDEs or [19,20,25,26] for forward G-SDEs. For classical mean-field SDEs, the dependence of the coefficients on the distribution of the solution results in fully nonlocal PDEs; see [27,28]. In that regard, the extension of mean-field theory to the G-expectation framework is of particular interest, as it could establish a connection between a class of fully nonlinear and nonlocal PDEs with a class of stochastic processes, which would allow the PDE to be solved numerically by simulating the associated stochastic process.

First attempts to extend mean-field theory to the G-framework can be found in [29,30]. In [29], the author considers an SDE of the form

$$\begin{aligned} dX_t &= \hat{\mathbb{E}}[b(t,x,X_t)]\Big|_{x=X_t} dt + \hat{\mathbb{E}}[h(t,x,X_t)]\Big|_{x=X_t} d\langle B\rangle_t + \hat{\mathbb{E}}[g(t,x,X_t)]\Big|_{x=X_t} dB_t, \quad 0 \le t \le T, \\ X_0 &= x, \end{aligned} \tag{1}$$

where $b, h, g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, B denotes a one-dimensional G-Brownian motion and $\hat{\mathbb{E}}$ denotes the corresponding G-expectation. More details on the G-setting are provided in Section 2 or can be found in [12]. Let $\mathrm{L}_*^{2,d}$ denote the space of all $\mathbb{R}^d$-valued random vectors ξ with finite sublinear second moment $\hat{\mathbb{E}}\left[\|\xi\|^2\right] < \infty$. For $\xi \in \mathrm{L}_*^{2,d}$, the functional F_ξ defined by

$$F_\xi : \mathrm{Lip}(\mathbb{R}^d) \to \mathbb{R}, \qquad \varphi \mapsto F_\xi(\varphi) := \hat{\mathbb{E}}[\varphi(\xi)] \tag{2}$$

can be interpreted as the "sublinear distribution" of ξ.

In [30], the approach from [29] is extended to higher dimensions and to coefficients that depend on the sublinear distribution F_{X_t} of the d-dimensional solution process X_t. That is, the authors consider an SDE of the form

$$\begin{aligned} dX_t &= b(t, X_t, F_{X_t})\, dt + h(t, X_t, F_{X_t})\, d\langle B\rangle_t + g(t, X_t, F_{X_t})\, dB_t, \qquad 0 \le t \le T, \\ X_0 &= x. \end{aligned} \tag{3}$$

In [30], the authors define a space containing all sublinear distributions and endow it with a metric allowing them to define continuity conditions on the coefficients. However, the space of sublinear distributions is not a vector space and, thus, it does not have a natural notion of differentiability, which limits the study of regularity properties of the solution; see Section 6 for a detailed discussion.

In [31], a novel formulation of a generalized mean-field G-SDE is introduced in which the coefficients depend on the solution process as random variable. More precisely, the authors consider a G-SDE of the form

$$
\begin{aligned}
dX_s^{t,\xi} &= b\Big(s,\omega,x,X_s^{t,\xi}\Big)\Big|_{x=X_s^{t,\xi}}\,ds + h\Big(s,\omega,x,X_s^{t,\xi}\Big)\Big|_{x=X_s^{t,\xi}}\,d\langle B\rangle_s \\
&\quad + g\Big(s,\omega,x,X_s^{t,\xi}\Big)\Big|_{x=X_s^{t,\xi}}\,dB_s, \qquad t\le s\le T, \\
X_t^{t,\xi} &= \xi
\end{aligned} \tag{4}
$$

with coefficients defined on $[0,T]\times\Omega\times\mathbb{R}^d\times \mathrm{L}_*^{2,d}$ and initial data $\xi\in \mathrm{L}_*^{2,d}$. This formulation generalizes the formulations introduced in [29,30] where the coefficients depend on the sublinear distribution. A significant advantage of the formulation in (4) is that $\mathrm{L}_*^{2,d}$ is a Banach space and, thus, it comes with standard notions of differentiability which are crucial for the results in our paper.

In this paper, we are interested in regularity properties of the solution of a mean-field SDE driven by G-Brownian motion. While the formulation (3) from [30] is closer to the classical formulation, as it depends on the (sublinear) distribution of the solution process, we work with the formulation (4) introduced in [31] since it allows us to consider Fréchet differentiable coefficients and study the Fréchet differentiability of the solution $X^{t,\xi}$ of (4) with respect to the random initial condition ξ. The Fréchet derivatives of $X^{t,\xi}$ capture how perturbations of the initial data propagate through the stochastic system and, thus, they are crucial for studying the sensitivity of the solution process with respect to changes in the initial data. This sensitivity analysis is a central tool for a wide range of applications. For instance, the Fréchet derivatives can be used to derive optimality conditions for stochastic control problems or establish recursive formulae for conditional expectations using the dynamic programming principle. Further, the Fréchet derivatives of $X^{t,\xi}$ can be used in numerical approximations of $X^{t,\xi}$ as well as for (sub)gradient methods for optimization problems. In particular, the Fréchet derivatives could be a useful tool for studying the properties of the value function associated to the processes $X^{t,x,\xi}$ and $X^{t,\xi}$ and establishing a Feynman–Kac-type result connecting the G-SDEs (6), (7) to a fully nonlinear and nonlocal PDE, which is the subject of ongoing research by the authors.

For simplicity and conciseness, we use the following notation.

Notation 1. *For a function f on $[0,T]\times\Omega\times\mathbb{R}^d\times \mathrm{L}_*^{2,d}$, define*

$$
f(s,\omega,\eta,\xi) := f(s,\omega,\eta(\omega),\xi) = f(s,\omega,x,\xi)\Big|_{x=\eta(\omega)} \tag{5}
$$

for any $0\le s\le T$, $\omega\in\Omega$ and $\xi,\eta\in \mathrm{L}_*^{2,d}$. Often, we suppress the explicit dependence on ω, and write $f(s,\eta,\xi)$ instead of $f(s,\omega,\eta,\xi)$.

Thus, (4) can be written as

$$
\begin{aligned}
dX_s^{t,\xi} &= b\Big(s,X_s^{t,\xi},X_s^{t,\xi}\Big)\,ds + h\Big(s,X_s^{t,\xi},X_s^{t,\xi}\Big)\,d\langle B\rangle_s + g\Big(s,X_s^{t,\xi},X_s^{t,\xi}\Big)\,dB_s, \qquad t\le s\le T,\\
X_t^{t,\xi} &= \xi.
\end{aligned} \tag{6}
$$

Under mild assumptions on the coefficients, it is shown in [31] that (6) admits a unique solution $X^{t,\xi}$; see Theorem 3.12 in [31]. For $x\in\mathbb{R}^d$, we associate to $X^{t,\xi}$ the G-SDE

$$
\begin{aligned}
dX_s^{t,x,\xi} &= b\Big(s,X_s^{t,x,\xi},X_s^{t,\xi}\Big)\,ds + h\Big(s,X_s^{t,x,\xi},X_s^{t,\xi}\Big)\,d\langle B\rangle_s + g\Big(s,X_s^{t,x,\xi},X_s^{t,\xi}\Big)\,dB_s, \quad t\le s\le T,\\
X_t^{t,x,\xi} &= x
\end{aligned} \tag{7}
$$

with deterministic initial condition $x \in \mathbb{R}^d$. The G-SDEs (6) and (7) are closely connected. More precisely, if (6) and (7) each admit a unique solution, then the process $X^{t,\xi}$ can be obtained from $X^{t,x,\xi}$ by evaluating at $x = \xi$ as formalized in Lemma 6. This allows us to infer properties of $X^{t,\xi}$ from properties of $X^{t,x,\xi}$ by using the aggregation property of the conditional sublinear expectation. More precisely, we have

$$\hat{\mathbb{E}}\Big[\Phi(X_s^{t,\xi})\Big] = \hat{\mathbb{E}}\Big[\hat{\mathbb{E}}\Big[\Phi(X_s^{t,x,\xi}) \;\Big|\; \mathcal{F}_t\Big]\Big|_{x=\xi}\Big] \tag{8}$$

and, thus, many of our auxiliary results are formulated in terms of conditional sublinear expectations of $X^{t,x,\xi}$.

Our main contribution is the derivation of first- and second-order Fréchet derivatives of the solution process as formalized in Propositions 2, 3, 4 and 5. For coefficients with Lipschitz and bounded Fréchet derivative, we establish the Fréchet differentiability of $X^{t,x,\xi}$ and $X^{t,\xi}$. Moreover, we characterize each of the Fréchet derivatives of $X^{t,x,\xi}$ and $X^{t,\xi}$ as the unique solution of a G-SDE. These results are in line with the results on classical mean-field SDEs; see [28].

This paper is structured as follows. In Section 2, we recall the G-framework before establishing preliminary results such as continuity and growth properties of the solution map $(x,\xi) \mapsto (X^{t,\xi}, X^{t,x,\xi})$ in Section 3. Section 4 is devoted to the first-order Fréchet derivatives of the solution map in x and ξ, while the second-order derivatives are studied in Section 5. Finally, in Section 6, we show how the formulation in [30] can be embedded into the formulation in [31] and develop a notion of differentiability for maps on the space of sublinear distributions.

Notation 2. *Most of our results are obtained via approximations and the Grönwall inequality. For the sake of conciseness and readability, we use the symbol $\lesssim$ to denote that the left-hand side is less than or equal a constant $C > 0$ times the right-hand side in the following sense.*

For two maps $f, g : \Theta \to \mathbb{R}$ with domain Θ, we define

$$f \lesssim g \qquad :\Longleftrightarrow \qquad \exists C > 0 : \forall \vartheta \in \Theta : f(\vartheta) \le C\, g(\vartheta). \tag{9}$$

2. Setting

In this section, we recall the generalized G-framework as introduced in Chapter 8 in [12]. Fix $n \ge 1$ and let $\Omega := C_0(\mathbb{R}_+, \mathbb{R}^n)$ denote the space of all continuous $\mathbb{R}^n$-valued paths starting at the origin equipped with the topology of uniform convergence. Let $\mathcal{F}$ denote the corresponding Borel σ-algebra. Moreover, let $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ denote the natural filtration generated by the coordinate mapping process $B : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ given by $B_t(\omega) = \omega(t)$.

Fix a convex and compact set $\Sigma \subseteq \mathbb{S}_+^n$ of symmetric non-negative definite $n \times n$-matrices and set

$$\mathcal{A}^\Sigma := \Big\{\vartheta = (\vartheta_t)_{t\ge 0} \,:\, \vartheta \text{ is } \Sigma\text{-valued and } \mathbb{F}\text{-progressively measurable}\Big\}.$$

Let P_0 denote the Wiener measure on $(\Omega, \mathcal{F})$, and define

$$\mathcal{P} := \Big\{P_0 \circ (\vartheta \bullet B)^{-1} \,:\, \vartheta \in \mathcal{A}^\Sigma\Big\},$$

where $\vartheta \bullet B := \int_0^\cdot \vartheta_s\, dB_s$ denotes the Itô integral with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P_0)$.

For $d \geq 1$ and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, let $\mathrm{B}_b^d(\mathcal{G})$ denote the space of all bounded $\mathcal{G}$-measurable maps $\xi : \Omega \to \mathbb{R}^d$. The set of probability measures $\mathcal{P}$ induces an upper expectation on $\mathrm{B}_b(\mathcal{F}) := \mathrm{B}_b^1(\mathcal{F})$, namely

$$\hat{\mathbb{E}} : \mathrm{B}_b(\mathcal{F}) \to \mathbb{R}, \qquad \xi \mapsto \hat{\mathbb{E}}[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi],$$

where E_P denotes the linear expectation with respect to P. The process B is a G-Brownian motion with respect to $\hat{\mathbb{E}}$ and $(\Omega, \mathrm{B}_b(\mathcal{F}), \hat{\mathbb{E}})$ is a sublinear expectation space. For $p \geq 1$, define the norm

$$\| \cdot \|_{\mathrm{L}_*^p} : \mathrm{B}_b^d(\mathcal{F}) \to \mathbb{R}_+, \qquad \xi \mapsto \|\xi\|_{\mathrm{L}_*^p} := \hat{\mathbb{E}}\big[\|\xi\|^p\big]^{\frac{1}{p}},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$ and let $\mathrm{L}_*^{p,d}(t)$ and $\mathrm{L}_*^{p,d}$ denote the completion of $\mathrm{B}_b^d(\mathcal{F}_t)$ and $\mathrm{B}_b^d(\mathcal{F})$ with respect to $\| \cdot \|_{\mathrm{L}_*^p}$ for $t \geq 0$. We set $\mathrm{L}_*^p(t) := \mathrm{L}_*^{p,1}(t)$ and $\mathrm{L}_*^p := \mathrm{L}_*^{p,1}$.

For $d \geq 1$ and $T > 0$, let $\mathrm{M}_b^d(0,T)$ denote the space of all maps $X : [0,T] \times \Omega \to \mathbb{R}^d$ of the form

$$X = \sum_{k=0}^{m-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}$$

with $m \in \mathbb{N}$, $0 = t_0 < \ldots < t_m = T$, and $\xi_k \in \mathrm{B}_b^d(\mathcal{F}_{t_k})$ for all $0 \leq k \leq m-1$. For $p \geq 1$, define the norms

$$\| \cdot \|_{\mathrm{M}_*^p} : \mathrm{M}_b^d(0,T) \to \mathbb{R}_+, \qquad \|X\|_{\mathrm{M}_*^p} := \left(\int_0^T \hat{\mathbb{E}}\big[\|X_s\|^p\big] \, ds \right)^{\frac{1}{p}},$$

$$\| \cdot \|_{\mathrm{H}_*^p} : \mathrm{M}_b^d(0,T) \to \mathbb{R}_+, \qquad \|X\|_{\mathrm{H}_*^p} := \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq T} \|X_s\|^p \right]^{\frac{1}{p}},$$

and let $\mathrm{M}_*^{p,d}(0,T)$ and $\mathrm{H}_*^{p,d}(0,T)$ denote the completion of $\mathrm{M}_b^d(0,T)$ with respect to $\| \cdot \|_{\mathrm{M}_*^p}$ and $\| \cdot \|_{\mathrm{H}_*^p}$, respectively. Clearly, $\mathrm{H}_*^{p,d}(0,T) \subseteq \mathrm{M}_*^{p,d}(0,T)$, and we set $\mathrm{M}_*^p(0,T) := \mathrm{M}_*^{p,1}(0,T)$, $\mathrm{H}_*^p(0,T) := \mathrm{H}_*^{p,1}(0,T)$.

Set $\mathrm{M}_b(0,T) := \mathrm{M}_b^1(0,T)$ and let B^i denote the i-th component of B for $1 \leq i \leq n$. Define the map $\mathcal{I}_i : \mathrm{M}_b(0,T) \to \mathrm{L}_*^2(T)$ by

$$\mathcal{I}_i(X) := \int_0^T X_s \, dB_s^i := \sum_{k=0}^{m-1} \xi_k \left(B_{t_{k+1}}^i - B_{t_k}^i \right)$$

for each

$$X = \sum_{k=0}^{m-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}.$$

The map $\mathcal{I}_i$ is linear and continuous with respect to $\|\cdot\|_{\mathrm{M}_*^2}$ and, thus, can be uniquely continuously extended to $\mathrm{M}_*^2(0,T)$. For $0 \leq t \leq s \leq T$ and $X \in \mathrm{M}_*^2(0,T)$, define

$$\int_t^s X_u \, dB_u^i := \mathcal{I}_i(X \mathbf{1}_{[t,s)}).$$

The quadratic variation of B is the map $\langle B \rangle : \mathbb{R}_+ \times \Omega \to \mathbb{S}_+^n$ defined componentwise by

$$\left\langle B^i, B^j \right\rangle_t := B_t^i B_t^j - \int_0^t B^i \, dB_s^j - \int_0^t B^j \, dB_s^i, \qquad t \geq 0$$

for $1 \leq i, j \leq n$. For $1 \leq i, j \leq n$, define the map $\mathcal{Q}_{ij} : \mathrm{M}_b(0,T) \to \mathrm{L}^1_*(T)$ by

$$\mathcal{Q}_{ij}(X) := \int_0^T X_s \, d\left\langle B^i, B^j \right\rangle_s := \sum_{k=0}^{m-1} \xi_k \left(\left\langle B^i, B^j \right\rangle_{t_{k+1}} - \left\langle B^i, B^j \right\rangle_{t_k} \right)$$

for each

$$X = \sum_{k=0}^{m-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}.$$

The map $\mathcal{Q}_{ij}$ is linear and continuous with respect to $\|\cdot\|_{\mathrm{M}^1_*}$ and, thus, can be uniquely continuously extended to $\mathrm{M}^1_*(0,T)$. For $0 \leq t \leq s \leq T$ and $X \in \mathrm{M}^1_*(0,T)$, define

$$\int_t^s X_u \, d\left\langle B^i, B^j \right\rangle_u := \mathcal{Q}_{ij}(X \mathbf{1}_{[t,s)}).$$

Since we consider G-SDEs with initial condition $X_t = \xi$ with $t \geq 0$, we introduce the following spaces:

$$\mathrm{H}^{p,d}_*(t,T) := \left\{ X \in \mathrm{H}^{p,d}_*(0,T) \, : \, \hat{\mathbb{E}}\left[\sup_{0 \leq s < t} \|X_s\|^p \right] = 0 \right\}$$

for $0 \leq t \leq T$, $p \geq 1$ and $d \in \mathbb{N}$. We say that the G-SDE

$$\begin{aligned} dX_s &= b(s,X)\, ds + h(s,X)\, d\langle B \rangle_s + g(s,X)\, dB_s, \qquad t \leq s \leq T, \\ X_t &= \xi_t \end{aligned} \tag{10}$$

with coefficients $b : [t,T] \times \Omega \times \mathrm{H}^{2,d}_*(t,T) \to \mathbb{R}^d$, $h : [t,T] \times \Omega \times \mathrm{H}^{2,d}_*(t,T) \to \mathbb{R}^{d \times n \times n}$ and $g : [t,T] \times \Omega \times \mathrm{H}^{2,d}_*(t,T) \to \mathbb{R}^{d \times n}$ admits a solution $X \in \mathrm{H}^{2,d}_*(t,T)$ if there exists a $X \in \mathrm{H}^{2,d}_*(t,T)$ with $X_t = \xi$ quasi-surely and the components X^k, $1 \leq k \leq d$ of X satisfy

$$X^k_s - X^k_t = \int_t^s b_k(u,X)\, du + \sum_{i,j=1}^n \int_t^s h_{kij}(u,X)\, d\left\langle B^i, B^j \right\rangle_u + \sum_{i=1}^n \int_t^s g_{ki}(u,X)\, dB^i_u$$

quasi-surely for all $t \leq s \leq T$, where b_k, h_{kij}, g_{ki} with $1 \leq k \leq d$, $1 \leq i, j \leq n$ denote the components of the coefficients b, h, g. Moreover, we say that the G-SDE (10) admits a unique solution $X \in \mathrm{H}^{2,d}_*(t,T)$ if (10) admits a solution $X \in \mathrm{H}^{2,d}_*(t,T)$ and, for any $X, Y \in \mathrm{H}^{2,d}_*(t,T)$ that solve (10), we have $\|X - Y\|_{\mathrm{H}^2_*} = 0$.

3. Preliminary Results

In [31], it is shown that the G-SDEs

$$\begin{aligned} dX^{t,\xi}_s &= b\left(s, X^{t,\xi}_s, X^{t,\xi}_s\right) ds + h\left(s, X^{t,\xi}_s, X^{t,\xi}_s\right) d\langle B \rangle_s + g\left(s, X^{t,\xi}_s, X^{t,\xi}_s\right) dB_s, \qquad t \leq s \leq T, \\ X^{t,\xi}_t &= \xi, \end{aligned} \tag{11}$$

$$\begin{aligned} dX^{t,x,\xi}_s &= b\left(s, X^{t,x,\xi}_s, X^{t,\xi}_s\right) ds + h\left(s, X^{t,x,\xi}_s, X^{t,\xi}_s\right) d\langle B \rangle_s + g\left(s, X^{t,x,\xi}_s, X^{t,\xi}_s\right) dB_s, \quad t \leq s \leq T, \\ X^{t,x,\xi}_t &= x \end{aligned} \tag{12}$$

admit unique solutions $X^{t,\xi}, X^{t,x,\xi} \in \mathrm{H}^{2,d}_*(t,T)$ under the following assumption; see Assumption 3.1 and Theorem 3.12 in [31].

Assumption 1. *The coefficients* $b : [0,T] \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \times \Omega \to \mathbb{R}^d$, $h : [0,T] \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \times \Omega \to \mathbb{R}^{d \times n \times n}$, *and* $g : [0,T] \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \times \Omega \to \mathbb{R}^{d \times n}$ *are such that the following holds for all components* $f = b_k, h_{kij}, g_{ki}$, $1 \leq i, j \leq n$, $1 \leq k \leq d$.

1. *$f(\cdot, x, \xi)\mathbf{1}_{[s,T]} \in \mathrm{M}^1_*(0,T)$ for all $x \in \mathbb{R}^d$, $\xi \in \mathrm{B}_b s^d$ and $t \le s \le T$.*
2. *There exist an integrable function $\kappa : [0,T] \to \mathbb{R}_+$, a process $K \in \mathrm{M}^1_*(0,T)$, and continuous, increasing and concave functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho_1(0) = \rho_2(0) = 0$ and*

$$\int_0^1 \frac{1}{\rho_1(r) + \rho_2(r)}\, dr = +\infty, \tag{13}$$

such that

$$|f(s,x,\xi,\omega) - f(s,y,\eta,\omega)|^2 \le \kappa(s)\,\rho_1\Big(\|x-y\|^2\Big) + K_s(\omega)\,\rho_2\Big(\|\xi-\eta\|^2_{\mathrm{L}^2_*}\Big), \tag{14}$$

$$|f(s,x,\xi,\omega)|^2 \le \kappa(s)\|x\|^2 + K_s(\omega)\Big(1 + \|\xi\|^2_{\mathrm{L}^2_*}\Big) \tag{15}$$

for all $\omega \in \Omega$, $t \le s \le T$, $x, y \in \mathbb{R}^d$, and $\xi, \eta \in \mathrm{L}^{2,d}_*(T)$.

The existence and uniqueness results in [31] are obtained using Bihari's inequality. For the sake of simplicity, in this paper, we derive existence of first- and second-order Fréchet derivatives of $X^{t,x,\xi}$ and $X^{t,\xi}$ for coefficients with locally Lipschitz first- and second-order Fréchet derivatives, respectively. Before studying the Fréchet differentiability, we establish growth and continuity properties of the solution map under the following assumptions on the coefficients.

Assumption 2. *Let $b : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \to \mathbb{R}^d$, $h : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \to \mathbb{R}^{d\times n\times n}$, and $g : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \to \mathbb{R}^{d\times n}$ be such that the following holds for all components $f = b_k, h_{kij}, g_{ki}$, $1 \le i,j \le n$, $1 \le k \le d$.*

1. *We have $f(\cdot, x, \xi)\mathbf{1}[s,T] \in \mathrm{M}^2_*(0,T)$ for all $x \in \mathbb{R}^d$, $\xi \in \mathrm{B}^d_b(\mathcal{F}_s)$ and $0 \le s \le T$.*
2. *There exists a q_0-integrable $\alpha_0 : [0,T] \to [1,\infty)$ with $q_0 \ge 2$ such that*

$$|f(s,\omega,x,\xi) - f(s,\omega,y,\eta)| \le \alpha_0(s)\Big(\|x-y\| + \|\xi-\eta\|_{\mathrm{L}^2_*}\Big)$$

for all $x, y \in \mathbb{R}^d$, $\xi, \eta \in \mathrm{L}^{2,d}_*$, $0 \le s \le T$ and $\omega \in \Omega$.

For convenience, let us define the set of coefficients

$$F := \Big\{b_k, h_{kij}, g_{ki} \,:\, 1 \le k \le d,\ 1 \le i,j \le n\Big\}.$$

Corollary 1. *If Assumption 2 is satisfied, then the following holds for all components $f = b_k, h_{kij}, g_{ki}$, $1 \le i,j \le n$, $1 \le k \le d$. There exists an integrable $\kappa : [0,T] \to [1,\infty)$ and a process $K \in \mathrm{M}^1_*(0,T)$ such that*

$$|f(s,\omega,x,\xi)|^2 \le \kappa(s)\Big(\|x\|^2 + \|\xi\|^2_{\mathrm{L}^2_*}\Big) + K_s(\omega)$$

for all $x \in \mathbb{R}^d$, $\xi \in \mathrm{L}^{2,d}_*$, $0 \le s \le T$ and $\omega \in \Omega$.

Proof. The continuity condition in Assumption 2 implies

$$\begin{aligned}|f(s,\omega,x,\xi)|^2 &\le 2|f(s,\omega,x,\xi) - f(s,\omega,0,0)|^2 + 2|f(s,\omega,0,0)|^2\\ &\le 4\alpha_0(s)^2\Big(\|x\|^2 + \|\xi\|^2_{\mathrm{L}^2_*}\Big) + 2|f(s,\omega,0,0)|^2\end{aligned}$$

and, clearly, $\kappa := 4\alpha_0^2 \ge 1$ is integrable. Finally, Assumption 2 implies that $K := |f(\cdot,0,0)|^2 \in \mathrm{M}^1_*(0,T)$, where 0 denotes the origin in $\mathbb{R}^d \subseteq \mathrm{L}^{2,d}_*(0)$. □

Thus, we conclude that Assumption 2 is stronger than Assumption 1 and, thus, Theorem 3.12 in [31] immediately yields the existence of unique solutions.

Proposition 1. *If Assumption 2 is satisfied, then the G-SDEs (11), (12) admit unique solutions* $X^{t,\xi}, X^{t,x,\xi} \in \mathrm{H}_*^{2,d}(t,T)$.

In particular, we deduce that the solution map

$$\mathbb{R}^d \times \mathrm{L}_*^{2,d}(t) \to \mathrm{H}_*^{2,d}(t,T) \times \mathrm{H}_*^{2,d}(t,T), \qquad (x,\xi) \mapsto (X^{t,x,\xi}, X^{t,\xi})$$

is well-defined. Further, Corollary 1 implies that the solution map is of linear growth. More precisely, we have the following growth properties.

Lemma 1. *If Assumption 2 is satisfied, then we have*

$$\hat{\mathbb{E}}\left[\sup_{t \le w \le T} \left\| X_w^{t,\xi} \right\|^2\right] \lesssim 1 + \|\xi\|_{\mathrm{L}_*^2}^2$$

for all $0 \le t \le T$ *and* $\xi \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4 and Corollary 1, we have for all $t \le s \le T$

$$\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{t \le w \le s} \left\| X_w^{t,\xi} \right\|^2\right] &\lesssim \|\xi\|_{\mathrm{L}_*^2}^2 + \sum_{f \in F} \int_t^s \hat{\mathbb{E}}\left[\left| f\left(u, X_u^{t,\xi}, X_u^{t,\xi}\right)\right|^2\right] du \\
&\le \|\xi\|_{\mathrm{L}_*^2}^2 + \int_t^s \hat{\mathbb{E}}\left[\kappa(u)\left(\left\| X_u^{t,\xi}\right\|^2 + \left\| X_u^{t,\xi}\right\|_{\mathrm{L}_*^2}^2\right) + K_u\right] du \\
&\lesssim 1 + \|\xi\|_{\mathrm{L}_*^2}^2 + \int_t^s \kappa(u) \hat{\mathbb{E}}\left[\left\| X_u^{t,\xi}\right\|^2\right] du,
\end{aligned}$$

and Grönwall's inequality yields the desired result. □

Lemma 2. *If Assumption 2 is satisfied, then there exists a* $K \in \mathrm{M}_*^1(0,T)$ *such that*

$$\hat{\mathbb{E}}\left[\sup_{t \le w \le s} \left\| X^{t,x,\xi} \right\|^2 \,\middle|\, \mathcal{F}_t\right] \lesssim \int_t^s \hat{\mathbb{E}}[K_u \mid \mathcal{F}_t]\, du + \|x\|^2 + \|\xi\|_{\mathrm{L}_*^2}^2$$

for all $0 \le t \le s \le T$, $x \in \mathbb{R}^d$ *and* $\xi \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4 and Corollary 1, we have

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t \le w \le s} \left\| X_w^{t,x,\xi} \right\|^2 \,\middle|\, \mathcal{F}_t\right] \\
&\lesssim \|x\|^2 + \sum_{f \in F} \int_t^s \hat{\mathbb{E}}\left[\left| f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)\right|^2 \,\middle|\, \mathcal{F}_t\right] du \\
&\le \|x\|^2 + \int_t^s \hat{\mathbb{E}}[K_u \mid \mathcal{F}_t]\, du + \int_t^s \kappa(u)\left(\hat{\mathbb{E}}\left[\left\| X_u^{t,x,\xi}\right\|^2 \,\middle|\, \mathcal{F}_t\right] + \left\| X_u^{t,\xi}\right\|_{\mathrm{L}_*^2}^2\right) du \\
&\lesssim \|x\|^2 + \int_t^s \hat{\mathbb{E}}[K_u \mid \mathcal{F}_t]\, du + \|\xi\|_{\mathrm{L}_*^2}^2 + \int_t^s \kappa(u) \hat{\mathbb{E}}\left[\left\| X_u^{t,x,\xi}\right\|^2 \,\middle|\, \mathcal{F}_t\right] du,
\end{aligned}$$

where we used Lemma 1 in the last step. Finally, Grönwall's inequality yields the desired result. □

Remark 1. *By taking the sublinear expectation, Lemma 2 immediately yields*

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|X^{t,x,\xi}\right\|^2\right] \lesssim \|x\|^2 + \|\xi\|^2_{L^2_*},$$

which is analogous to the result in Lemma 1. Many of the results for $X^{t,x,\xi}$ are stated in a conditional form so that we apply them to the concatenation $X^{t,x,\xi}|_{x=\xi}$ which, as we show in Lemma 6, is indifferent from $X^{t,\xi}$.

Lemma 3. *If Assumption 2 is satisfied, then*

$$\hat{\mathbb{E}}\left[\sup_{t\le s\le T}\left\|X^{t,\xi}_s - X^{t,\eta}_s\right\|^2\right] \lesssim \|\xi-\eta\|^2_{L^2_*}$$

for all $0 \le t \le T$ and $\xi,\eta \in L^{2,d}_*(t)$.

Proof. By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|X^{t,\xi}_w - X^{t,\eta}_w\right\|^2\right] \\
&\lesssim \|\xi-\eta\|^2_{L^2_*} + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|f\left(u, X^{t,\xi}_u, X^{t,\xi}_u\right) - f\left(u, X^{t,\eta}_u, X^{t,\eta}_u\right)\right|^2\right] du \\
&\lesssim \|\xi-\eta\|^2_{L^2_*} + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\sup_{t\le w\le u}\left\|X^{t,\xi}_w - X^{t,\eta}_w\right\|^2\right] du.
\end{aligned}$$

Finally, Grönwall's inequality yields the desired result. □

Lemma 4. *Let $1 \le p \le q_0$. If Assumption 2 is satisfied, then*

$$\hat{\mathbb{E}}\left[\sup_{t\le s\le T}\left\|X^{t,x,\xi}_s - X^{t,y,\eta}_s\right\|^p \,\middle|\, \mathcal{F}_t\right] \lesssim \|x-y\|^p + \|\xi-\eta\|^p_{L^2_*}$$

for all $0 \le t \le T$, $\xi,\eta \in L^{2,d}_*(t)$ and $x,y \in \mathbb{R}^d$.

Proof. By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|X^{t,x,\xi}_w - X^{t,y,\eta}_w\right\|^p \,\middle|\, \mathcal{F}_t\right] \\
&\lesssim \|x-y\|^p + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right) - f\left(u, X^{t,y,\eta}_u, X^{t,\eta}_u\right)\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\le \|x-y\|^p + \int_t^s \alpha_0(u)^p \left(\hat{\mathbb{E}}\left[\left\|X^{t,x,\xi}_u - X^{t,y,\eta}_u\right\|^p \,\middle|\, \mathcal{F}_t\right] + \left\|X^{t,\xi}_u - X^{t,\eta}_u\right\|^p_{L^2_*}\right) du \\
&\lesssim \|x-y\|^p + \|\xi-\eta\|^p_{L^2_*} + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\|X^{t,x,\xi}_u - X^{t,y,\eta}_u\right\|^2 \,\middle|\, \mathcal{F}_t\right] du,
\end{aligned}$$

where the last step follows from Lemma 3. Finally, Grönwall's inequality yields the desired result. □

For $\eta \in L^{1,d}_*$, we can define the concatenation

$$X^{t,\eta,\xi} : [0,T]\times\Omega \to \mathbb{R}^d, \qquad (s,\omega) \mapsto X^{t,\eta,\xi}_s(\omega) := X^{t,x,\xi}_s(\omega)\Big|_{x=\eta(\omega)}.$$

Lemma 5. *If Assumption 2 is satisfied, then* $X^{t,\eta,\xi} \in \mathrm{H}_*^{2,d}(t,T)$ *for all* $0 \le t \le T$ *and* $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$.

Proof. Lemma 4 implies $\left(X^{t,x,\xi} - X^{t,y,\xi}\right) \in \mathrm{H}_*^{2,d}(0,T) \subseteq \mathrm{M}_*^{2,d}(0,T)$ and, thus, we immediately get $X^{t,\eta,\xi} \in \mathrm{M}_*^{2,d}(t,T)$ due to Lemma A.4 in [31].

Moreover, Lemma 2 yields

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|X_w^{t,\eta,\xi}\right\|^2\right] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|X_w^{t,x,\xi}\right\|^2 \,\Big|\, \mathcal{F}_t\right]\Big|_{x=\eta}\right] \lesssim \|K\|_{\mathrm{M}_*^1} + \|\eta\|_{\mathrm{L}_*^2}^2 + \|\xi\|_{\mathrm{L}_*^2}^2 < \infty.$$

□

Lemma 6. *If Assumption 2 is satisfied, then*

$$\left\|X^{t,\xi,\xi} - X^{t,\xi}\right\|_{\mathrm{H}_*^2} = 0$$

for all $0 \le t \le T$ *and* $\xi \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|X_w^{t,x,\xi} - X_w^{t,\xi}\right\|^2 \,\Big|\, \mathcal{F}_t\right] \\
&\lesssim \|x-\xi\|^2 + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right) - f\left(u, X_u^{t,\xi}, X_u^{t,\xi}\right)\right|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\le \|x-\xi\|^2 + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|X_u^{t,x,\xi} - X_w^{t,\xi}\right\|^2 \,\Big|\, \mathcal{F}_t\right] du
\end{aligned}$$

and Grönwall's inequality yields

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|X_w^{t,x,\xi} - X_w^{t,\xi}\right\|^2 \,\Big|\, \mathcal{F}_t\right] \lesssim \|x-\xi\|^2.$$

Finally, the aggregation property implies

$$\left\|X^{t,\xi,\xi} - X^{t,\xi}\right\|_{\mathrm{H}_*^2} = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|X_w^{t,x,\xi} - X_w^{t,\xi}\right\|^2 \,\Big|\, \mathcal{F}_t\right]\Big|_{x=\xi}\right] = 0.$$

□

4. First-Order Derivatives

In this section, we show that the solution map $(x,\xi) \mapsto X^{t,x,\xi}$ is Fréchet differentiable for Fréchet differentiable coefficients with Lipschitz and bounded Fréchet derivatives. More specifically, Propositions 2, 3 and 4 formalize the Fréchet differentiability of $\xi \mapsto X^{t,\xi}$, $x \mapsto X^{t,x,\xi}$ and $\xi \mapsto X^{t,x,\xi}$ respectively. Before we turn to the differentiability results, let us agree on some definitions and recall the fundamental theorem of calculus; see, e.g., Theorem 5.1 in [32].

Definition 1. *Let* V *and* W *be normed real vector spaces with norms* $\|\cdot\|_V$ *and* $\|\cdot\|_W$*, respectively. A map* $f : V \to W$ *is called Fréchet differentiable if, for every* $v_0 \in V$*, there exists a continuous linear operator* $\mathrm{D}f(v_0) : V \to W$ *such that*

$$\lim_{\|v\|_V \to 0} \frac{\|f(v_0+v) - f(v_0) - \mathrm{D}f(v_0)v\|_W}{\|v\|_V} = 0,$$

and the map

$$\mathrm{D}f : V \to B(V,W), \qquad v \mapsto \mathrm{D}f(v)$$

is called the Fréchet derivative of f, where $B(V,W)$ denotes the space of all bounded linear operators $L : V \to W$.

A Fréchet differentiable map $f : V \to W$ is called continuously Fréchet differentiable if the Fréchet derivative $v \mapsto \mathrm{D}f(v)$ is continuous with respect to the operator norm. Let $\mathrm{C}^1(V)$ denote the space of all continuously Fréchet differentiable maps $f : V \to \mathbb{R}$.

In Section 5, we repeatedly use the following version of the fundamental theorem of calculus.

Lemma 7. *Let V and W be normed real vector spaces. If $f : V \to W$ is continuously Fréchet differentiable, then*

$$f(v_0+v) - f(v) = \int_0^1 \mathrm{D}f(v_0 + \lambda v)v \, d\lambda$$

for all $v, v_0 \in V$.

Assumption 3. *Let $b : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}_*^{2,d} \to \mathbb{R}^d$, $h : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}_*^{2,d} \to \mathbb{R}^{d\times n\times n}$, and $g : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}_*^{2,d} \to \mathbb{R}^{d\times n}$ be such that the following holds for all components $f = b_k, h_{kij}, g_{ki}$ with $1 \le i,j \le n$, $1 \le k \le d$.*

1. *We have $f(s,\omega,x,\cdot) \in \mathrm{C}^1(\mathrm{L}_*^{2,d})$ and $f(s,\omega,\cdot,\xi) \in \mathrm{C}^1(\mathbb{R}^d)$ for all $0 \le s \le T$, $\omega \in \Omega$, $x \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}$.*
2. *There exists a q_1-integrable $\alpha_1 : [0,T] \to [1,\infty)$ with $q_1 \ge 2$ such that*

$$\begin{aligned} |\,\mathrm{D}_x f(s,\omega,x,\xi)z - \mathrm{D}_x f(s,\omega,y,\eta)z| &\le \alpha_1(s)\|z\|\Big(\|x-y\| + \|\xi-\eta\|_{\mathrm{L}_*^2}\Big), \\ \big|\,\mathrm{D}_\xi f(s,\omega,x,\xi)\zeta - \mathrm{D}_\xi f(s,\omega,y,\eta)\zeta\big| &\le \alpha_1(s)\|\zeta\|_{\mathrm{L}_*^2}\Big(\|x-y\| + \|\xi-\eta\|_{\mathrm{L}_*^2}\Big), \\ \big|\,\mathrm{D}_\xi f(s,\omega,x,\xi)\eta\big| &\le \alpha_1(s)\|\eta\|_{\mathrm{L}_*^1} \end{aligned}$$

for all $x,y,z \in \mathbb{R}^d$, $\xi,\eta,\zeta \in \mathrm{L}_*^{2,d}$, $0 \le s \le T$ and $\omega \in \Omega$, where $\mathrm{D}_x f(s,\omega,x,\xi)$ and $\mathrm{D}_\xi f(s,\omega,x,\xi)$ denote the Fréchet derivatives of f with respect to x and ξ, respectively.

Remark 2. *Note that Assumption 2 yields bounds for $\mathrm{D}_x f$ and $\mathrm{D}_\xi f$ which are uniform in (ω,x,ξ) and q_0-integrable in s. To be specific, we have the following bounds for all components $f = b_k, h_{kij}, g_{ki}$, $1 \le i,j \le n$, $1 \le k \le d$,*

$$|\,\mathrm{D}_x f(s,\omega,x,\xi)y| \le \alpha_0(s)\|y\|, \qquad \big|\,\mathrm{D}_\xi f(s,\omega,x,\xi)\eta\big| \le \alpha_0(s)\|\eta\|_{\mathrm{L}_*^2} \tag{16}$$

for all $x,y \in \mathbb{R}^d$, $\xi,\eta \in \mathrm{L}_*^{2,d}$, $0 \le s \le T$ and $\omega \in \Omega$.

Moreover, Assumption 3 implies that the Fréchet derivatives of the coefficients are in $\mathrm{M}_*^2(0,T)$. More precisely, we have the following results.

Lemma 8. *If Assumptions 2 and 3 are satisfied, then the following holds for all components $f = b_k, h_{kij}, g_{ki}$ with $1 \le i,j \le n$, $1 \le k \le d$. The map*

$$\mathrm{H}_*^{2,d}(0,T) \times \mathrm{H}_*^{2,d}(0,T) \to \mathrm{M}_*^1(0,T), \qquad (X,Y) \mapsto f(\cdot,X,Y)$$

is Fréchet differentiable in each argument with Fréchet derivatives $\mathrm{D}_x f(\cdot, X, Y)$ *and* $\mathrm{D}_\xi f(\cdot, X, Y)$ *at* (X, Y)*, respectively.*

Proof. Assumption 2 implies that $f(\cdot, X, Y) \in \mathrm{M}^1_*(0,T)$ for all $X, Y \in \mathrm{H}^{2,d}_*(0,T)$; see Corollary 3.4 in [31]. Thus, the map $(X, Y) \mapsto f(\cdot, X, Y)$ is well-defined.

Let $X, Y, Z \in \mathrm{H}^{2,d}_*(0,T)$. Since $f(s, \omega, \cdot, \xi) \in \mathrm{C}^1(\mathbb{R}^d)$ for all $0 \le s \le T$, $\omega \in \Omega$ and $\xi \in \mathrm{L}^{2,d}_*$, we have

$$
\begin{aligned}
&\|f(\cdot, X+Z, Y) - f(\cdot, X, Y) - \mathrm{D}_x f(\cdot, X, Y)Z\|_{\mathrm{M}^1_*} \\
&= \int_0^T \hat{\mathbb{E}}[|f(s, X_s + Z_s, Y_s) - f(s, X_s, Y_s) - \mathrm{D}_x f(s, X_s, Y_s)Z_s|]\, ds \\
&\le \int_0^T \hat{\mathbb{E}}\left[\int_0^1 |\mathrm{D}_x f(s, X_s + \lambda Z_s, Y_s)Z_s - \mathrm{D}_x f(s, X_s, Y_s)Z_s|\, d\lambda\right] ds \\
&\le \int_0^T \alpha_1(s)\hat{\mathbb{E}}\left[\|Z_s\|^2\right] ds \\
&\le \|Z\|^2_{\mathrm{H}^2_*} \int_0^T \alpha_1(s)\, ds.
\end{aligned}
$$

Analogously, since $f(s, \omega, x, \cdot) \in \mathrm{C}^1(\mathrm{L}^{2,d}_*)$ for all $0 \le s \le T$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$, we have

$$
\begin{aligned}
&\|f(\cdot, X, Y+Z) - f(\cdot, X, Y) - \mathrm{D}_\xi f(\cdot, X, Y)Z\|_{\mathrm{M}^1_*} \\
&= \int_0^T \hat{\mathbb{E}}[|f(s, X_s, Y_s + Z_s) - f(s, X_s, Y_s) - \mathrm{D}_\xi f(s, X_s, Y_s)Z_s|]\, ds \\
&\le \int_0^T \hat{\mathbb{E}}\left[\int_0^1 |\mathrm{D}_\xi f(s, X_s, Y_s + \lambda Z_s)Z_s - \mathrm{D}_\xi f(s, X_s, Y_s)Z_s|\, d\lambda\right] ds \\
&\le \int_0^T \alpha_1(s)\|Z_s\|^2_{\mathrm{L}^2_*}\, ds \\
&\le \|Z\|^2_{\mathrm{H}^2_*} \int_0^T \alpha_1(s)\, ds.
\end{aligned}
$$

The integrability of α_1 implies

$$
\begin{aligned}
\lim_{\|Z\|_{\mathrm{H}^2_*} \to 0} \frac{\|f(\cdot, X+Z, Y) - f(\cdot, X, Y) - \mathrm{D}_x f(\cdot, X, Y)Z\|_{\mathrm{M}^1_*}}{\|Z\|_{\mathrm{H}^2_*}} &= 0, \\
\lim_{\|Z\|_{\mathrm{H}^2_*} \to 0} \frac{\|f(\cdot, X, Y+Z) - f(\cdot, X, Y) - \mathrm{D}_\xi f(\cdot, X, Y)Z\|_{\mathrm{M}^1_*}}{\|Z\|_{\mathrm{H}^2_*}} &= 0.
\end{aligned}
$$

That is, the map $(X, Y) \mapsto f(\cdot, X, Y)$ is Fréchet differentiable in each argument. □

Lemma 9. *If Assumptions 2 and 3 are satisfied, then* $\mathrm{D}_x f(\cdot, X, Y)Z, \mathrm{D}_\xi f(\cdot, X, Y)Z \in \mathrm{M}^2_*(0,T)$ *for all components* $f = b_k, h_{kij}, g_{ki}$*,* $1 \le i, j \le n$*,* $1 \le k \le d$ *and* $X, Y, Z \in \mathrm{H}^{2,d}_*(0,T)$*.*

Proof. Lemma 8 implies $\mathrm{D}_x f(\cdot, X, Y)Z, \mathrm{D}_\xi f(\cdot, X, Y)Z \in \mathrm{M}^1_*(0,T)$ for all $X, Y, Z \in \mathrm{H}^{2,d}_*(0,T)$. Moreover, the bound in (16) yields

$$
\int_0^T \hat{\mathbb{E}}\left[|\mathrm{D}_x f(s, X_s, Y_s)Z_s|^2\right] ds \le \int_0^T \alpha_0(s)^2 \hat{\mathbb{E}}\left[\|Z_s\|^2\right] ds \lesssim \|Z\|^2_{\mathrm{H}^2_*} < \infty
$$

and

$$
\int_0^T \hat{\mathbb{E}}\left[|\mathrm{D}_\xi f(s, X_s, Y_s)Z_s|^2\right] ds \le \int_0^T \alpha_0(s)^2 \|Z_s\|^2_{\mathrm{L}^2_*}\, ds \lesssim \|Z\|^2_{\mathrm{H}^2_*} < \infty
$$

since α_0 is square-integrable and $Z \in \mathrm{H}_*^{2,d}(0,T)$. Hence, $\mathrm{D}_x f(\cdot, X, Y)Z$, $\mathrm{D}_\xi f(\cdot, X, Y)Z \in \mathrm{M}_*^2(0,T)$ for all $X, Y, Z \in \mathrm{H}_*^{2,d}(0,T)$. □

Lemma 10. *If Assumptions 2 and 3 are satisfied, then the G-SDE*

$$\begin{aligned} dA_s^{t,x,\xi,y} &= \mathrm{D}_x b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) A_s^{t,x,\xi,y}\, ds + \mathrm{D}_x h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) A_s^{t,x,\xi,y}\, d\langle B\rangle_s \\ &\quad + \mathrm{D}_x g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) A_s^{t,x,\xi,y}\, dB_s, \qquad t \le s \le T, \\ A_t^{t,x,\xi,y} &= y. \end{aligned} \tag{17}$$

admits a unique solution $A^{t,x,\xi,y} \in \mathrm{H}_*^{2,d}(t,T)$ for all $0 \le t \le T$, $x, y \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$. Moreover, the map

$$\mathbb{R}^d \to \mathrm{H}_*^{2,d}(t,T), \qquad y \mapsto A^{t,x,\xi,y}$$

is linear.

Proof. By Lemma 9, the coefficients in (17) are in $\mathrm{M}_*^2(0,T)$. Moreover, they are Lipschitz continuous and, thus, (17) admits a unique solution $A^{t,x,\xi,y} \in \mathrm{H}_*^{2,d}(t,T)$ for all $0 \le t \le T$, $x, y \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$. In particular, we deduce that the map $y \mapsto A^{t,x,\xi,y}$ is well-defined.

Let $\lambda \in \mathbb{R}$. By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned} &\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\| A_w^{t,x,\xi,y+\lambda z} - A_w^{t,x,\xi,y} - \lambda A_w^{t,x,\xi,z}\right\|^2\right] \\ &\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\Big(A_u^{t,x,\xi,y+\lambda z} - A_u^{t,x,\xi,y} - \lambda A_u^{t,x,\xi,z}\Big)\right|^2\right] du \\ &\le \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\sup_{t\le w\le u}\left\| A_w^{t,x,\xi,y+\lambda z} - A_w^{t,x,\xi,y} - \lambda A_w^{t,x,\xi,z}\right\|^2\right] du. \end{aligned}$$

Finally, Grönwall's inequality yields

$$\left\| A_u^{t,x,\xi,y+\lambda z} - A_u^{t,x,\xi,y} - \lambda A_u^{t,x,\xi,z}\right\|_{\mathrm{H}_*^2} = 0.$$

Since $\lambda \in \mathbb{R}$ and $y, z \in \mathbb{R}^d$ were arbitrary, we deduce that $y \mapsto A^{t,x,\xi,y}$ is linear. □

Lemma 11. *Let $2 \le p \le q_0$. If Assumptions 2 and 3 are satisfied, then*

$$\hat{\mathbb{E}}\left[\sup_{t\le s\le T}\left\| A_s^{t,x,\xi,y}\right\|^p \,\Big|\, \mathcal{F}_t\right] \lesssim \|y\|^p$$

for all $0 \le t \le T$, $x, y \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4, we have for all $t \le s \le T$ that

$$\begin{aligned} &\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\| A_w^{t,x,\xi,y} y\right\|^p \,\Big|\, \mathcal{F}_t\right] \\ &\lesssim \|y\|^p + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big) A_u^{t,x,\xi,y}\right|^p \,\Big|\, \mathcal{F}_t\right] du \\ &\lesssim \|y\|^p + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\sup_{t\le w\le u}\left\| A_w^{t,x,\xi,y}\right\|^p \,\Big|\, \mathcal{F}_t\right] du. \end{aligned}$$

Grönwall's inequality yields the desired result. □

Proposition 2. *Let $0 \leq t \leq T$ and $\xi \in \mathrm{L}^{2,d}_*(t)$. If Assumptions 2 and 3 are satisfied, then the map*

$$\mathbb{R}^d \to \mathrm{H}^{2,d}_*(t,T), \qquad x \mapsto X^{t,x,\xi}$$

is Fréchet differentiable with Fréchet derivative

$$\mathrm{D}_x X^{t,x,\xi} : \mathbb{R}^d \to \mathrm{H}^{2,d}_*(t,T), \qquad y \mapsto \mathrm{D}_x X^{t,x,\xi} y := A^{t,x,\xi,y}$$

at $x \in \mathbb{R}^d$.

Proof. By Lemma 10, the map $\mathrm{D}_x X^{t,x,\xi} : y \mapsto A^{t,x,\xi,y}$ is linear. Set $Y := X^{t,x+y,\xi} - X^{t,x,\xi}$, then

$$\hat{\mathbb{E}}\left[\sup_{t\leq s\leq T} \|Y_s\|^4\right] = \hat{\mathbb{E}}\left[\sup_{t\leq s\leq T} \left\|X^{t,x+y,\xi}_s - X^{t,x,\xi}_s\right\|^4\right] \lesssim \|y\|^4 \tag{18}$$

due to Lemma 4. By Lemma A4, we have for all $t \leq s \leq T$ that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leq w\leq s} \left\|X^{t,x+y,\xi}_w - X^{t,x,\xi}_w - A^y_w\right\|^2\right] \\
&\lesssim \sum_{f\in F} \int_t^s \hat{\mathbb{E}}\left[\left|f\left(u, X^{t,x+y,\xi}_u, X^{t,\xi}_u\right) - f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right) - \mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right) A^{t,x,\xi,y}_u\right|^2\right] du \\
&= \sum_{f\in F} \int_t^s \hat{\mathbb{E}}\left[\left|\int_0^1 \mathrm{D}_x f\left(u, X^{t,x,\xi}_u + \lambda Y_u, X^{t,\xi}_u\right) Y_u \, d\lambda - \mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right) A^{t,x,\xi,y}_u\right|^2\right] du \\
&\lesssim \sum_{f\in F} \int_t^s \int_0^1 \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X^{t,x,\xi}_u + \lambda Y_u, X^{t,\xi}_u\right) Y_u - \mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right) Y_u\right|^2\right] d\lambda \, du \\
&\quad + \sum_{f\in F} \int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)\left(Y_u - A^{t,x,\xi,y}_u\right)\right|^2\right] du \\
&\leq \int_t^s \alpha_1(u)^2 \hat{\mathbb{E}}\left[\|Y_u\|^4\right] + \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|Y_u - A^{t,x,\xi,y}_u\right\|^2\right] du \\
&\lesssim \|y\|^4 + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\sup_{t\leq w\leq u} \left\|X^{t,x+y,\xi}_w - X^{t,x,\xi}_w - A^{t,x,\xi,y}_w\right\|^2\right] du,
\end{aligned}$$

where the last step follows from (18). Finally, Grönwall's inequality yields

$$\left\|X^{t,x+y,\xi} - X^{t,x,\xi} - A^{t,x,\xi,y}\right\|^2_{\mathrm{H}^2_*} \lesssim \|y\|^4.$$

Thus,

$$\lim_{\|y\|\to 0} \frac{\left\|X^{t,x+y,\xi} - X^{t,x,\xi} - A^{t,x,\xi,y}\right\|_{\mathrm{H}^2_*}}{\|y\|} = 0,$$

i.e., $\mathrm{D}_x X^{t,x,\xi} : y \mapsto A^{t,x,\xi,y}$ is the Fréchet derivative of $x \mapsto X^{t,x,\xi}$ at $x \in \mathbb{R}^d$. □

Next, we show that the map $x \mapsto X^{t,x,\xi}$ is continuously Fréchet differentiable.

Lemma 12. *Let $2 \leq p \leq (q_1 \wedge \frac{q_0}{2})$. If Assumptions 2 and 3 are satisfied with $q_0 \geq 4$, then*

$$\hat{\mathbb{E}}\left[\sup_{t\leq s\leq T} \left\|\mathrm{D}_x X^{t,x,\xi}_s z - \mathrm{D}_x X^{t,y,\eta}_s z\right\|^p \,\Big|\, \mathcal{F}_t\right] \lesssim \|z\|^p \left(\|x-y\|^p + \|\xi - \eta\|^p_{\mathrm{L}^2_*}\right)$$

for all $0 \leq t \leq T$, $x, y, z \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}^{2,d}_*(t)$.

Proof. By Lemma A4, we have for all $t \le s \le T$ that

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\| \mathrm{D}_x X_w^{t,x,\xi} z - \mathrm{D}_x X_w^{t,y,\eta} z\right\|^p \,\middle|\, \mathcal{F}_t\right] \\
&\quad\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right) \mathrm{D}_x X_u^{t,x,\xi} z - \mathrm{D}_x f\left(u, X_u^{t,y,\eta}, X_u^{t,\eta}\right) \mathrm{D}_x X_u^{t,y,\eta} z\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\left(u, X_u^{t,x,\xi}, X^{t,\xi}\right) \mathrm{D}_x X_u^{t,x,\xi} z - \mathrm{D}_x f\left(u, X_u^{t,y,\eta}, X^{t,\eta}\right) \mathrm{D}_x X_u^{t,x,\xi} z\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\qquad\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\left(u, X_u^{t,y,\eta}, X^{t,\eta}\right) \left(\mathrm{D}_x X_u^{t,x,\xi} z - \mathrm{D}_x X_u^{t,y,\eta} z\right)\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad\le \int_t^s \alpha_1(u)^p \hat{\mathbb{E}}\left[\left\| \mathrm{D}_x X_u^{t,x,\xi} z\right\|^p \left\| X_u^{t,x,\xi} - X_u^{t,y,\eta}\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad\le \int_t^s \alpha_1(u)^p \hat{\mathbb{E}}\left[\left\| \mathrm{D}_x X_u^{t,x,\xi} z\right\|^{2p} \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{2}} \hat{\mathbb{E}}\left[\left\| X_u^{t,x,\xi} - X_u^{t,y,\eta}\right\|^{2p} \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{2}} du \\
&\qquad + \int_t^s \alpha_1(u)^p \left\| X_u^{t,\xi} - X_u^{t,\eta}\right\|_{\mathrm{L}^2_*}^p \hat{\mathbb{E}}\left[\left\| \mathrm{D}_x X_u^{t,x,\xi} z\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\qquad + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\| \mathrm{D}_x X_u^{t,x,\xi} z - \mathrm{D}_x X_u^{t,y,\eta} z\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad\lesssim \|z\|^p \left(\|x-y\|^p + \|\xi-\eta\|_{\mathrm{L}^2_*}^p\right) + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\| \mathrm{D}_x X_u^{t,x,\xi} z - \mathrm{D}_x X_u^{t,y,\eta} z\right\|^p \,\middle|\, \mathcal{F}_t\right] du,
\end{aligned}
$$

where the last step follows from Lemmas 3, 4 and 11. Finally, Grönwall's inequality yields the desired result. □

Corollary 2. *Let $0 \le t \le T$, $\xi \in \mathrm{L}^{2,d}_*(t)$. If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then the map*

$$\mathbb{R}^d \to \mathrm{H}^{2,d}_*(t,T), \qquad x \mapsto X^{t,x,\xi}$$

is continuously Fréchet differentiable.

Proof. Lemma 12 implies that

$$\sup_{0\ne z\in\mathbb{R}^d} \frac{\left\| \mathrm{D}_x X^{t,x,\xi} z - \mathrm{D}_x X^{t,y,\xi} z\right\|_{\mathrm{H}^2_*}}{\|z\|} \lesssim \|x-y\|,$$

i.e., $x \mapsto \mathrm{D}_x X^{t,x,\xi}$ is continuous with respect to the operator norm. □

Lemma 13. *Let $0 \le t \le T$ and $\xi, \eta, \zeta \in \mathrm{L}^2_*(t)$. If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then $\mathrm{D}_x X^{t,\eta,\xi}\zeta \in \mathrm{H}^{2,d}_*(t,T)$ with*

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| \mathrm{D}_x X_w^{t,\eta,\xi}\zeta\right\|^2\right] \lesssim \|\zeta\|_{\mathrm{L}^2_*}^2,$$

where $\mathrm{D}_x X^{t,\eta,\xi}\zeta$ denotes the map

$$[0,T]\times\Omega \to \mathbb{R}^d, \qquad (s,\omega) \mapsto \mathrm{D}_x X_s^{t,\eta,\xi}\zeta(\omega) := A_s^{t,x,\xi,y}(\omega)\Big|_{x=\eta(\omega),y=\zeta(\omega)}.$$

Proof. We have $X^{t,\eta,\xi} \in \mathrm{H}_*^{2,d}(t,T)$ due to Corollary 5. Moreover, the G-SDE

$$\begin{aligned} dY_s =\ & \mathrm{D}_x b\Big(s, X_s^{t,\eta,\xi}, X_s^{t,\xi}\Big) Y_s\, ds + \mathrm{D}_x h\Big(s, X_s^{t,\eta,\xi}, X_s^{t,\xi}\Big) Y_s\, d\langle B\rangle_s \\ & + \mathrm{D}_x g\Big(s, X_s^{t,\eta,\xi}, X_s^{t,\xi}\Big) Y_s\, dB_s, \qquad t \le s \le T, \\ Y_t =\ & \zeta. \end{aligned}$$

has a unique solution $Y \in \mathrm{H}_*^{2,d}(t,T)$ since the coefficients are Lipschitz continuous and in $\mathrm{M}_*^2(0,T)$.

By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned} & \hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\| A_w^{t,x,\xi,y} - Y_w\right\|^2 \,\Big|\, \mathcal{F}_t\right] \\ & \lesssim \|y-\zeta\|^2 + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big) A_u^{t,x,\xi,y} - \mathrm{D}_x f\Big(u, X_u^{t,\eta,\xi}, X_u^{t,\xi}\Big) Y_u\right|^2 \,\Big|\, \mathcal{F}_t\right] du \\ & \lesssim \|y-\zeta\|^2 + \int_t^s \alpha_1(u)^2 \hat{\mathbb{E}}\left[\left\| X_u^{t,x,\xi} - X_u^{t,\eta,\xi}\right\|^2 \left\| A_u^{t,x,\xi,y}\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \\ & \quad + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\| A_u^{t,x,\xi,y} - Y_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \\ & \le \|y-\zeta\|^2 + \int_t^s \alpha_1(u)^2 \hat{\mathbb{E}}\left[\left\| X_u^{t,x,\xi} - X_u^{t,z,\xi}\right\|^4 \,\Big|\, \mathcal{F}_t\right]^{\frac{1}{2}}\Big|_{z=\eta} \hat{\mathbb{E}}\left[\left\| A_u^{t,x,\xi,y}\right\|^4 \,\Big|\, \mathcal{F}_t\right]^{\frac{1}{2}} du \\ & \quad + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\| A_u^{t,x,\xi,y} - Y_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \\ & \lesssim \|y-\zeta\|^2 + \|x-\eta\|^2\|y\|^2 + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\sup_{t\le w\le u}\left\| A_w^{t,x,\xi,y} - Y_w\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \end{aligned}$$

due to Lemmas 4 and 11. Grönwall's inequality implies

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| A_w^{t,x,\xi,y} - Y_w\right\|^2 \,\Big|\, \mathcal{F}_t\right] \lesssim \|y-\zeta\|^2 + \|x-\eta\|^2\|y\|^2$$

and, thus,

$$\left\| \mathrm{D}_x X^{t,\eta,\xi}\zeta - Y\right\|_{\mathrm{H}_*^2}^2 = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| A_w^{t,x,\xi,y} - Y_w\right\|^2 \,\Big|\, \mathcal{F}_t\right]\Big|_{x=\eta,\, y=\zeta}\right] = 0.$$

That is, $\mathrm{D}_x X^{t,\eta,\xi}\zeta = Y \in \mathrm{H}_*^{2,d}(t,T)$. Finally, we have

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| \mathrm{D}_x X^{t,\eta,\xi}\zeta\right\|^2\right] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| \mathrm{D}_x X^{t,x,\xi} z\right\|^2 \,\Big|\, \mathcal{F}_t\right]\Big|_{x=\eta,\, z=\zeta}\right] \lesssim \hat{\mathbb{E}}\left[\|\zeta\|^2\right]$$

due to Lemma 11. □

Corollary 3. *If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then*

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| \mathrm{D}_x X_w^{t,\eta,\xi}\zeta - \mathrm{D}_x X_w^{t,\nu,\chi}\zeta\right\|\right] \lesssim \|\zeta\|_{\mathrm{L}_*^2}\Big(\|\eta-\nu\|_{\mathrm{L}_*^2} + \|\xi-\chi\|_{\mathrm{L}_*^2}\Big)$$

for all $0 \le t \le T$ and $\xi, \eta, \zeta, \nu, \chi \in \mathrm{L}_*^{2,d}(t)$.

Proof. Lemma 13 together with the aggregation property yield

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| \mathrm{D}_x X_w^{t,\eta,\xi}\zeta - \mathrm{D}_x X_w^{t,\nu,\chi}\zeta\right\|\right] \\
&\le \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| \mathrm{D}_x X_w^{t,x,\xi}z - \mathrm{D}_x X_w^{t,y,\chi}z\right\|^2 \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{2}}\Bigg|_{x=\eta,\, y=\nu,\, z=\zeta}\right] \\
&\lesssim \hat{\mathbb{E}}\left[\|\zeta\|\left(\|\eta-\nu\| + \|\xi-\chi\|_{\mathrm{L}^2_*}\right)\right] \\
&\lesssim \|\zeta\|_{\mathrm{L}^2_*}\left(\|\eta-\nu\|_{\mathrm{L}^2_*} + \|\xi-\chi\|_{\mathrm{L}^2_*}\right).
\end{aligned}$$

□

Lemma 14. *Let $0 \le t \le T$ and $\xi \in \mathrm{L}^2_*(t)$. If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then*

$$\lim_{\|\eta\|_{\mathrm{L}^2_*}\to 0} \frac{\left\| X^{t,\xi+\eta,\xi+\eta} - X^{t,\xi,\xi+\eta} - \mathrm{D}_x X^{t,\xi,\xi}\eta\right\|_{\mathrm{H}^1_*}}{\|\eta\|_{\mathrm{L}^2_*}} = 0,$$

where the limit is taken over $\eta \in \mathrm{L}^{2,d}_*(t)$.

Proof. Due to Corollary 2, the map $x \mapsto X^{t,x,\xi+\eta}$ is continuously differentiable. In particular, we have

$$X_s^{t,x+y,\xi+\eta} - X_s^{t,x,\xi+\eta} = \int_0^1 \mathrm{D}_x X_s^{t,x+\lambda y,\xi+\eta}\eta \, d\lambda$$

q.s. for all $t \le s \le T$. Thus, Corollary 3 yields

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le s\le T}\left\| X_s^{t,\xi+\eta,\xi+\eta} - X_s^{t,\xi,\xi+\eta} - \mathrm{D}_x X_s^{t,\xi,\xi}\eta\right\|\right] \\
&\le \int_0^1 \hat{\mathbb{E}}\left[\sup_{t\le s\le T}\left\| \mathrm{D}_x X_s^{t,\xi+\lambda\eta,\xi+\eta}\eta - \mathrm{D}_x X_s^{t,\xi,\xi}\eta\right\|\right] d\lambda \\
&\lesssim \|\eta\|^2_{\mathrm{L}^2_*},
\end{aligned}$$

which implies the desired result. □

Lemma 15. *If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then the G-SDEs*

$$\begin{aligned}
dY_s^{t,\xi,\eta} &= \left[\mathrm{D}_x b\left(s, X_s^{t,\xi}, X_s^{t,\xi}\right) Y_s^{t,\xi,\eta} + \mathrm{D}_\xi b\left(s, X_s^{t,\xi}, X_s^{t,\xi}\right)\left(\mathrm{D}_x X_s^{t,\xi,\xi}\eta + Y_s^{t,\xi,\eta}\right)\right] ds \\
&\quad + \left[\mathrm{D}_x h\left(s, X_s^{t,\xi}, X_s^{t,\xi}\right) Y_s^{t,\xi,\eta} + \mathrm{D}_\xi h\left(s, X_s^{t,\xi}, X_s^{t,\xi}\right)\left(\mathrm{D}_x X_s^{t,\xi,\xi}\eta + Y_s^{t,\xi,\eta}\right)\right] d\langle B\rangle_s \\
&\quad + \left[\mathrm{D}_x g\left(s, X_s^{t,\xi}, X_s^{t,\xi}\right) Y_s^{t,\xi,\eta} + \mathrm{D}_\xi g\left(s, X_s^{t,\xi}, X_s^{t,\xi}\right)\left(\mathrm{D}_x X_s^{t,\xi,\xi}\eta + Y_s^{t,\xi,\eta}\right)\right] dB_s, \\
&\qquad t \le s \le T, \\
Y_t^{t,\xi,\eta} &= \eta, \qquad (19)
\end{aligned}$$

$$
\begin{aligned}
dY_s^{t,x,\xi,\eta} &= \Big[\mathrm{D}_x b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) Y_s^{t,x,\xi,\eta} + \mathrm{D}_\xi b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,\xi,\xi}\eta + Y_s^{t,\xi,\eta}\Big)\Big] ds \\
&\quad + \Big[\mathrm{D}_x h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) Y_s^{t,x,\xi,\eta} + \mathrm{D}_\xi h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,\xi,\xi}\eta + Y_s^{t,\xi,\eta}\Big)\Big] d\langle B\rangle_s \\
&\quad + \Big[\mathrm{D}_x g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) Y_s^{t,x,\xi,\eta} + \mathrm{D}_\xi g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,\xi,\xi}\eta + Y_s^{t,\xi,\eta}\Big)\Big] dB_s, \\
&\qquad t \le s \le T, \\
Y_t^x &= \eta
\end{aligned}
\tag{20}
$$

admit unique solutions $Y^{t,\xi,\eta}, Y^{t,x,\xi,\eta} \in \mathrm{H}_*^{2,d}(t,T)$ *for all* $0 \le t \le T$, $x \in \mathbb{R}^d$ *and* $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$. *Moreover, the map*

$$\mathrm{L}_*^{2,d}(t) \to \mathrm{H}_*^{2,d}(t,T), \qquad \eta \mapsto Y^{t,x,\xi,\eta}$$

is linear.

Proof. We have $\mathrm{D}_x X^{t,\xi,\xi}\eta \in \mathrm{H}_*^{2,d}(t,T)$ due to Lemma 13. Thus, Lemma 8 implies that the coefficients in (19) are in $\mathrm{M}_*^2(0,T)$. Since they are Lipschitz continuous, (19) admits a unique solution $Y^{t,\xi,\eta} \in \mathrm{H}_*^{2,d}(t,T)$.

Similarly, since $Y^{t,\xi,\eta} \in \mathrm{H}_*^{2,d}(t,T)$, the coefficients in (20) are in $\mathrm{M}_*^2(0,T)$ and Lipschitz continuous and, thus, (20) admits a unique solution $Y^{t,x,\xi,\eta} \in \mathrm{H}_*^{2,d}(t,T)$.

Let $\eta, \zeta \in \mathrm{L}_*^{2,d}(t)$ and $\lambda \in \mathbb{R}$. Lemma A4 yields for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\| Y_w^{t,x,\xi,\eta+\lambda\zeta} - Y_w^{t,x,\xi,\eta} - \lambda Y_w^{t,x,\xi,\zeta}\right\|^2\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\Big(Y_u^{t,x,\xi,\eta+\lambda\zeta} - Y_u^{t,x,\xi,\eta} - \lambda Y_u^{t,x,\xi,\zeta}\Big)\right|^2\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_\xi f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\Big(Y_u^{t,x,\xi,\eta+\lambda\zeta} - Y_u^{t,x,\xi,\eta} - \lambda Y_u^{t,x,\xi,\zeta}\Big)\right|^2\right] du \\
&\lesssim \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\| Y_u^{t,x,\xi,\eta+\lambda\zeta} - Y_u^{t,x,\xi,\eta} - \lambda Y_u^{t,x,\xi,\zeta}\right\|^2\right] du,
\end{aligned}
$$

and Grönwall's inequality yields $\left\| Y^{t,x,\xi,\eta+\lambda\zeta} - Y^{t,x,\xi,\eta} - \lambda Y^{t,x,\xi,\zeta}\right\|_{\mathrm{H}_*^2} = 0$. □

Lemma 16. *If Assumptions 2 and 3 are satisfied with* $q_0 \ge 4$, *then*

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\| Y_w^{t,\xi,\eta}\right\|^2\right] \lesssim \|\eta\|_{\mathrm{L}_*^2}^2$$

for all $0 \le t \le T$ *and* $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4, we have for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\| Y_w^{t,\xi,\eta}\right\|^2\right] \\
&\lesssim \|\eta\|_{\mathrm{L}_*^2}^2 + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_x f\Big(u, X_u^{t,\xi}, X_u^{t,\xi}\Big) Y_u^{t,\xi,\eta}\right|^2\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| \mathrm{D}_\xi f\Big(u, X_u^{t,\xi}, X_u^{t,\xi}\Big)\Big(\mathrm{D}_x X_u^{t,\xi,\xi}\eta + Y_u^{t,\xi,\eta}\Big)\right|^2\right] du \\
&\lesssim \|\eta\|_{\mathrm{L}_*^2}^2 + \int_t^s \alpha_0(u)^2\left(\hat{\mathbb{E}}\left[\left\| Y_u^{t,\xi,\eta}\right\|^2\right] + \left\| \mathrm{D}_x X_u^{t,\xi,\xi}\eta\right\|_{\mathrm{L}_*^2}^2\right) du \\
&\lesssim \|\eta\|_{\mathrm{L}_*^2}^2 + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\| Y_u^{t,\xi,\eta}\right\|^2\right] du
\end{aligned}
$$

due to Lemma 13. Finally, Grönwall's inequality yields the desired result. □

Lemma 17. *Let $2 \leq p \leq q_0$. If Assumptions 2 and 3 are satisfied with $q_0 \geq 4$, then*

$$\hat{\mathbb{E}}\left[\sup_{t \leq w \leq T}\left\|Y_w^{t,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] \lesssim \|\eta\|^p + \|\eta\|_{\mathrm{L}_*^2}^p,$$

$$\hat{\mathbb{E}}\left[\sup_{t \leq w \leq T}\left\|Y_w^{t,x,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] \lesssim \|\eta\|^p + \|\eta\|_{\mathrm{L}_*^2}^p$$

for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4, we have for all $t \leq s \leq T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t \leq w \leq s}\left\|Y_w^{t,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] \\
&\lesssim \|\eta\|^p + \sum_{f \in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X_u^{t,\xi}, X_u^{t,\xi}\right) Y_u^{t,\xi,\eta}\right|^p \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f \in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_\xi f\left(u, X_u^{t,\xi}, X_u^{t,\xi}\right)\left(\mathrm{D}_x X_u^{t,\xi,\xi}\eta + Y_u^{t,\xi,\eta}\right)\right|^p \,\Big|\, \mathcal{F}_t\right] du \\
&\lesssim \|\eta\|^p + \int_t^s \alpha_0(u)^p\left(\hat{\mathbb{E}}\left[\left\|Y_u^{t,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] + \left\|\mathrm{D}_x X_u^{t,\xi,\xi}\eta\right\|_{\mathrm{L}_*^2}^p + \left\|Y_u^{t,\xi,\eta}\right\|_{\mathrm{L}_*^2}^p\right) du \\
&\lesssim \|\eta\|^p + \|\eta\|_{\mathrm{L}_*^2}^p + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] du,
\end{aligned}$$

and Grönwall's inequality yields the desired result for $Y^{t,\xi,\eta}$.

Analogously, we have for $Y^{t,x,\xi,\eta}$ that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t \leq w \leq s}\left\|Y_w^{t,x,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] \\
&\lesssim \|\eta\|^p + \sum_{f \in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right) Y_u^{t,x,\xi,\eta}\right|^p \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f \in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_\xi f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)\left(\mathrm{D}_x X_u^{t,\xi,\xi}\eta + Y_u^{t,\xi,\eta}\right)\right|^p \,\Big|\, \mathcal{F}_t\right] du \\
&\lesssim \|\eta\|^p + \int_t^s \alpha_0(u)^p\left(\hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] + \left\|\mathrm{D}_x X_u^{t,\xi,\xi}\eta\right\|_{\mathrm{L}_*^2}^p + \left\|Y_u^{t,\xi,\eta}\right\|_{\mathrm{L}_*^2}^p\right) du \\
&\lesssim \|\eta\|^p + \|\eta\|_{\mathrm{L}_*^2}^p + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\eta}\right\|^p \,\Big|\, \mathcal{F}_t\right] du,
\end{aligned}$$

and Grönwall's inequality yields the desired result for $Y^{t,x,\xi,\eta}$. □

Lemma 18. *Let $0 \leq t \leq T$ and $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$. If Assumptions 2 and 3 are satisfied with $q_0 \geq 4$, then*

$$\left\|Y^{t,\xi,\eta} - Y^{t,\xi,\xi,\eta}\right\|_{\mathrm{H}_*^2} = 0,$$

where $Y^{t,\xi,\xi,\eta}$ denotes the map

$$[0,T] \times \Omega \to \mathbb{R}^d, \qquad (s,\omega) \mapsto Y_s^{t,\xi,\xi,\eta}(\omega) := Y_s^{t,x,\xi,\eta}(\omega)|_{x=\xi(\omega)}.$$

Proof. Set $Z := \mathrm{D}_x X^{t,\xi,\xi}\eta + Y^{t,\xi,\eta}$, then $\|Z\|_{\mathrm{H}^2_*} \lesssim \|\eta\|_{\mathrm{L}^2_*}$ due to Lemmas 13 and 16. By Lemma A4, we have for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y^{t,\xi,\eta}_w - Y^{t,x,\xi,\eta}_w\right\|^2 \,\Big|\, \mathcal{F}_t\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X^{t,\xi}_u, X^{t,\xi}_u\right)Y^{t,\xi,\eta}_u - \mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)Y^{t,\xi,\eta}_u\right|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)\left(Y^{t,\xi,\eta}_u - Y^{t,x,\xi,\eta}_u\right)\right|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_\xi f\left(u, X^{t,\xi}_u, X^{t,\xi}_u\right)Z_u - \mathrm{D}_\xi f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)Z_u\right|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\lesssim \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|X^{t,\xi}_u - X^{t,x,\xi}_u\right\|^2\left\|Y^{t,\xi,\eta}_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] + \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|Y^{t,\xi,\eta}_u - Y^{t,x,\xi,\eta}_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + \int_t^s \alpha_1(u)^2\|Z_u\|^2_{\mathrm{L}^2_*}\hat{\mathbb{E}}\left[\left\|X^{t,\xi}_u - X^{t,x,\xi}_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\le \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|X^{t,y,\xi}_u - X^{t,x,\xi}_u\right\|^4 \,\Big|\, \mathcal{F}_t\right]^{\frac{1}{2}}\Big|_{y=\xi}\hat{\mathbb{E}}\left[\left\|Y^{t,\xi,\eta}_u\right\|^4 \,\Big|\, \mathcal{F}_t\right]^{\frac{1}{2}} du \\
&\quad + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|Y^{t,\xi,\eta}_u - Y^{t,x,\xi,\eta}_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + \int_t^s \alpha_1(u)^2\|Z_u\|^2_{\mathrm{L}^2_*}\hat{\mathbb{E}}\left[\left\|X^{t,y,\xi}_u - X^{t,x,\xi}_u\right\|^2 \,\Big|\, \mathcal{F}_t\right]\Big|_{y=\xi} du \\
&\lesssim \|\xi - x\|^2\left(\|\zeta\|^2 + \|\zeta\|^2_{\mathrm{L}^2_*}\right) + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|Y^{t,\xi,\eta}_u - Y^{t,x,\xi,\eta}_u\right\|^2 \,\Big|\, \mathcal{F}_t\right] du.
\end{aligned}
$$

Grönwall's inequality yields

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y^{t,\xi,\eta}_w - Y^{t,x,\xi,\eta}_w\right\|^2 \,\Big|\, \mathcal{F}_t\right] \lesssim \|\xi - x\|^2\left(\|\zeta\|^2 + \|\zeta\|^2_{\mathrm{L}^2_*}\right)
$$

and, thus, the aggregation property implies

$$
\left\|Y^{t,\xi,\eta} - Y^{t,\xi,\xi,\eta}\right\|^2_{\mathrm{H}^2_*} = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|Y^{t,\xi,\eta}_w - Y^{t,x,\xi,\eta}_w\right\|^2 \,\Big|\, \mathcal{F}_t\right]\Big|_{x=\xi}\right] = 0.
$$

□

Lemma 19. *Let $2 \le p \le (q_1 \wedge \frac{q_0}{2})$. If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then*

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|Y^{t,x,\xi,\zeta}_w - Y^{t,y,\eta,\zeta}_w\right\|^p \,\Big|\, \mathcal{F}_t\right] \lesssim \|\zeta\|^p_{\mathrm{L}^2_*}\left(\|x-y\|^p + \|\xi-\eta\|^p_{\mathrm{L}^2_*}\right)
$$

for all $0 \le t \le T$, $x, y \in \mathbb{R}^d$ and $\xi, \eta, \zeta \in \mathrm{L}^{2,d}_*(t)$.

Proof. Set $Z^\xi := \mathrm{D}_x X^{t,\xi,\xi}\zeta + Y^{t,\xi,\zeta}$ and $Z^\eta := \mathrm{D}_x X^{t,\eta,\eta}\zeta + Y^{t,\eta,\zeta}$, then

$$
\left\|Z^\xi\right\|_{\mathrm{H}^2_*} + \|Z^\eta\|_{\mathrm{H}^2_*} \lesssim \|\zeta\|_{\mathrm{L}^2_*}
$$

due to Lemmas 13 and 16. Moreover,

$$\begin{aligned}\left\|Z_s^{\xi}-Z_s^{\eta}\right\|_{L^1_*} &\le \left\|D_x X_s^{t,\xi,\xi}\zeta - D_x X_s^{t,\eta,\eta}\zeta\right\|_{L^1_*} + \left\|Y_s^{t,\xi,\zeta}-Y_s^{t,\eta,\zeta}\right\|_{L^1_*} \\ &\lesssim \|\zeta\|_{L^2_*}\|\xi-\eta\|_{L^2_*} + \left\|Y_s^{t,\xi,\zeta}-Y_s^{t,\eta,\zeta}\right\|_{L^1_*}\end{aligned} \tag{21}$$

for all $t \le s \le T$ due to Corollary 3.

By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y_w^{t,x,\xi,\zeta}-Y_w^{t,y,\eta,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| D_x f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)Y_u^{t,x,\xi,\zeta} - D_x f\left(u, X_u^{t,y,\eta}, X_u^{t,\eta}\right)Y_u^{t,x,\xi,\zeta}\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| D_x f\left(u, X_u^{t,y,\eta}, X_u^{t,\eta}\right)\left(Y_u^{t,x,\xi,\zeta}-Y_u^{t,y,\eta,\zeta}\right)\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| D_\xi f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)Z_u^{\xi} - D_\xi f\left(u, X_u^{t,y,\eta}, X_u^{t,\eta}\right)Z_u^{\xi}\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left| D_\xi f\left(u, X_u^{t,y,\eta}, X_u^{t,\eta}\right)\left(Z_u^{\xi}-Z_u^{\eta}\right)\right|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\lesssim \int_t^s \alpha_1(u)^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\zeta}\right\|^p\left\|X_u^{t,x,\xi}-X_u^{t,y,\eta}\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \int_t^s \alpha_1(u)^p \left\|X_u^{t,\xi}-X_u^{t,\eta}\right\|_{L^2_*}^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\zeta}-Y_u^{t,y,\eta,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \int_t^s \alpha_1(u)^p \left\|Z_u^{\xi}\right\|_{L^2_*}^p \left(\hat{\mathbb{E}}\left[\left\|X_u^{t,x,\xi}-X_u^{t,y,\eta}\right\|^p \,\middle|\, \mathcal{F}_t\right] + \left\|X_u^{t,\xi}-X_u^{t,\eta}\right\|_{L^2_*}^p\right) du \\
&\quad + \int_t^s \alpha_1(u)^p \left\|Z_u^{\xi}-Z_u^{\eta}\right\|_{L^1_*}^p du \\
&\lesssim \int_t^s \alpha_1(u)^p \left(\|\zeta\|^p + \|\zeta\|_{L^2_*}^p\right)\left(\|x-y\|^p + \|\xi-\eta\|_{L^2_*}^p\right) du \\
&\quad + \int_t^s \alpha_1(u)^p \|\xi-\eta\|_{L^2_*}^p \left(\|\zeta\|^p + \|\zeta\|_{L^2_*}^p\right) du \\
&\quad + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\zeta}-Y_u^{t,y,\eta,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \int_t^s \alpha_1(u)^p \|\zeta\|_{L^2_*}^p \left(\|x-y\|^p + \|\xi-\eta\|_{L^2_*}^p\right) du \\
&\quad + \int_t^s \alpha_1(u)^p \|\zeta\|_{L^2_*}^p \|\xi-\eta\|_{L^2_*}^p + \alpha_1(u)^p \left\|Y_u^{t,\xi,\zeta}-Y_u^{t,\eta,\zeta}\right\|_{L^1_*}^p du \\
&\lesssim \left(\|\zeta\|^p + \|\zeta\|_{L^2_*}^p\right)\left(\|x-y\|^p + \|\xi-\eta\|_{L^2_*}^p\right) + \int_t^s \alpha_1(u)^p \left\|Y_u^{t,\xi,\zeta}-Y_u^{t,\eta,\zeta}\right\|_{L^1_*}^p du \\
&\quad + \int_t^s \alpha_0(u)^p \hat{\mathbb{E}}\left[\left\|Y_u^{t,x,\xi,\zeta}-Y_u^{t,y,\eta,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right] du
\end{aligned}$$

due to (21) and Lemmas 4 and 17. Further, Grönwall's inequality implies that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y_w^{t,x,\xi,\zeta}-Y_w^{t,y,\eta,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right] \\
&\lesssim \left(\|\zeta\|^p + \|\zeta\|_{L^2_*}^p\right)\left(\|x-y\|^p + \|\xi-\eta\|_{L^2_*}^p\right) + \int_t^s \alpha_1(u)^p \left\|Y_u^{t,\xi,\zeta}-Y_u^{t,\eta,\zeta}\right\|_{L^1_*}^p du
\end{aligned} \tag{22}$$

for all $t \le s \le T$. From Lemma 18 and (22) we obtain

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y_w^{t,\xi,\zeta}-Y_w^{t,\eta,\zeta}\right\|\right]^2 \\
&\le \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y_w^{t,x,\xi,\zeta}-Y_w^{t,y,\eta,\zeta}\right\|^2 \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{2}}\Big|_{x=\xi,\,y=\eta}\right]^2 \\
&\lesssim \hat{\mathbb{E}}\Big[\Big(\|\zeta\|+\|\zeta\|_{\mathrm{L}^2_*}\Big)\Big(\|\xi-\eta\|+\|\xi-\eta\|_{\mathrm{L}^2_*}\Big)\Big]^2+\int_t^s \alpha_1(u)^2\left\|Y_u^{t,\xi,\zeta}-Y_u^{t,\eta,\zeta}\right\|^2_{\mathrm{L}^1_*}du \\
&\lesssim \|\zeta\|^2_{\mathrm{L}^2_*}\|\xi-\eta\|^2_{\mathrm{L}^2_*}+\int_t^s \alpha_1(u)^2\left\|Y_u^{t,\xi,\zeta}-Y_u^{t,\eta,\zeta}\right\|^2_{\mathrm{L}^1_*}du,
\end{aligned}
$$

and Grönwall's inequality yields

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|Y_w^{t,\xi,\zeta}-Y_w^{t,\eta,\zeta}\right\|\right]\lesssim \|\zeta\|_{\mathrm{L}^2_*}\|\xi-\eta\|_{\mathrm{L}^2_*}.
$$

Hence, (22) becomes

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|Y_w^{t,x,\xi,\zeta}-Y_w^{t,y,\eta,\zeta}\right\|^p \,\middle|\, \mathcal{F}_t\right]\lesssim \Big(\|\zeta\|^p+\|\zeta\|^p_{\mathrm{L}^2_*}\Big)\Big(\|x-y\|^p+\|\xi-\eta\|^p_{\mathrm{L}^2_*}\Big).
$$

□

We immediately obtain the following corollary.

Corollary 4. *If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then*

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|Y_w^{t,\xi,\zeta}-Y_w^{t,\eta,\zeta}\right\|\right]\lesssim \|\zeta\|_{\mathrm{L}^2_*}\|\xi-\eta\|_{\mathrm{L}^2_*}
$$

for all $0 \le t \le T$ and $\xi,\eta,\zeta \in \mathrm{L}^{2,d}_*(t)$.

Lemma 20. *If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then*

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|X_w^{t,x,\xi+\eta}-X_w^{t,x,\xi}-Y_w^{t,x,\xi,\eta}\right\|^2 \,\middle|\, \mathcal{F}_t\right] \\
&\lesssim \|\eta\|^4_{\mathrm{L}^2_*}+\int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\Big[\Big\|X_u^{t,\xi+\eta}-X_u^{t,\xi}-\mathrm{D}_xX_u^{t,\xi,\xi}\eta-Y_u^{t,\xi,\eta}\Big\|\Big]^2du
\end{aligned}
$$

for all $0 \le t \le s \le T$, $x \in \mathbb{R}^d$ and $\xi,\eta \in \mathrm{L}^{2,d}_*(t)$.

Proof. Set

$$
\begin{aligned}
\Delta^\xi &:= X^{t,x,\xi+\eta}-X^{t,x,\xi}, &\qquad Y &:= Y^{t,x,\xi,\eta}, \\
\Delta &:= X^{t,\xi+\eta}-X^{t,\xi}, &\qquad Z &:= \mathrm{D}_xX^{t,\xi,\xi}\eta+Y^{t,\xi,\eta}.
\end{aligned}
$$

Lemmas 13 and 17 yield

$$
\|Z\|_{\mathrm{H}^2_*}+\|Y\|_{\mathrm{H}^2_*}\lesssim \|\eta\|_{\mathrm{L}^2_*}. \tag{23}
$$

Moreover, Lemma 2 implies

$$\|\Delta\|_{\mathrm{H}^2_*} \lesssim \|\eta\|_{\mathrm{L}^2_*}, \qquad \hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|\Delta^{\xi}_w\right\|^4 \,\middle|\, \mathcal{F}_t\right] \lesssim \|\eta\|^4_{\mathrm{L}^2_*}. \tag{24}$$

By Lemma A4, we have for all $t \le s \le T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|\Delta^{\xi}_w - Y^{t,x,\xi,\eta}_w\right\|^2 \,\middle|\, \mathcal{F}_t\right] \\
&\lesssim \sum_{f\in F}\int_t^s\int_0^1 \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X^{t,x,\xi}_u + \lambda\Delta^{\xi}_u, X^{t,\xi+\eta}_u\right)\Delta^{\xi}_u - \mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)\Delta^{\xi}_u\right|^2 \,\middle|\, \mathcal{F}_t\right] d\lambda\, du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)\left(\Delta^{\xi}_u - Y_u\right)\right|^2 \,\middle|\, \mathcal{F}_t\right] du \\
&\quad + \sum_{f\in F}\int_t^s\int_0^1 \hat{\mathbb{E}}\left[\left|\mathrm{D}_\xi f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u + \lambda\Delta_u\right)\Delta_u - \mathrm{D}_\xi f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)\Delta_u\right|^2 \,\middle|\, \mathcal{F}_t\right] d\lambda\, du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_\xi f\left(u, X^{t,x,\xi}_u, X^{t,\xi}_u\right)(\Delta_u - Z_u)\right|^2 \,\middle|\, \mathcal{F}_t\right] du \\
&\lesssim \int_t^s \alpha_1(u)^2\left(\hat{\mathbb{E}}\left[\left\|\Delta^{\xi}_u\right\|^4 \,\middle|\, \mathcal{F}_t\right] + \|\Delta_u\|^2_{\mathrm{L}^2_*}\hat{\mathbb{E}}\left[\left\|\Delta^{\xi}_u\right\|^2 \,\middle|\, \mathcal{F}_t\right] + \|\Delta_u\|^4_{\mathrm{L}^2_*}\right) d\lambda\, du \\
&\quad + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^{\xi}_u - Y_u\right\|^2 \,\middle|\, \mathcal{F}_t\right] + \alpha_1(u)^2\|\Delta_u - Z_u\|^2_{\mathrm{L}^1_*}\, du \\
&\lesssim \|\eta\|^4_{\mathrm{L}^2_*} + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^{\xi}_u - Y_u\right\|^2 \,\middle|\, \mathcal{F}_t\right] + \alpha_1(u)^2\|\Delta_u - Z_u\|^2_{\mathrm{L}^1_*}\, du
\end{aligned}$$

due to (24) and (23). Finally, Grönwall's inequality implies the desired result. □

Lemma 21. *Let $0 \le t \le T$ and $\xi, \eta \in \mathrm{L}^{2,d}_*(t)$. If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then*

$$\lim_{\|\eta\|_{\mathrm{L}^2_*}\to 0} \frac{\|X^{t,\xi+\eta} - X^{t,\xi} - \mathrm{D}_x X^{t,\xi,\xi}\eta - Y^{t,\xi,\eta}\|_{\mathrm{H}^1_*}}{\|\eta\|_{\mathrm{L}^2_*}} = 0,$$

where the limit is taken over $\eta \in \mathrm{L}^{2,d}_*(t)$.

Proof. By Lemmas 18 and 20, we have

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|X^{t,\xi,\xi+\eta}_w - X^{t,\xi,\xi}_w - Y^{t,\xi,\eta}_w\right\|^2\right] \\
&= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|X^{t,x,\xi+\eta}_w - X^{t,x,\xi}_w - Y^{t,x,\xi,\eta}_w\right\|^2 \,\middle|\, \mathcal{F}_t\right]\Big|_{x=\xi}\right] \\
&\lesssim \|\eta\|^4_{\mathrm{L}^2_*} + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|X^{t,\xi+\eta}_u - X^{t,\xi}_u - \mathrm{D}_x X^{t,\xi,\xi}_u\eta - Y^{t,\xi,\eta}_u\right\|\right]^2 du \\
&\lesssim \|\eta\|^4_{\mathrm{L}^2_*} + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|X^{t,\xi+\eta,\xi+\eta}_u - X^{t,\xi,\xi+\eta}_u - \mathrm{D}_x X^{t,\xi,\xi}_u\eta\right\|\right]^2 du \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|X^{t,\xi,\xi+\eta}_u - X^{t,\xi,\xi}_u - Y^{t,\xi,\eta}_u\right\|^2\right] du \\
&\lesssim \|\eta\|^4_{\mathrm{L}^2_*} + \left\|X^{t,\xi+\eta,\xi+\eta} - X^{t,\xi,\xi+\eta} - \mathrm{D}_x X^{t,\xi,\xi}\eta\right\|^2_{\mathrm{H}^1_*} \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|X^{t,\xi,\xi+\eta}_u - X^{t,\xi}_u - Y^{t,\xi,\eta}_u\right\|^2\right] du,
\end{aligned}$$

and Grönwall's inequality yields

$$\left\|X^{t,\xi,\xi+\eta} - X^{t,\xi,\xi} - Y^{t,\xi,\eta}\right\|_{\mathrm{H}^2_*} \lesssim \|\eta\|^2_{\mathrm{L}^2_*} + \left\|X^{t,\xi+\eta,\xi+\eta} - X^{t,\xi,\xi+\eta} - \mathrm{D}_x X^{t,\xi,\xi}\eta\right\|_{\mathrm{H}^1_*}. \tag{25}$$

Finally, observe that

$$
\begin{aligned}
&\left\| X^{t,\xi+\eta} - X^{t,\xi} - \mathrm{D}_x X^{t,\xi,\xi}\eta - Y^{t,\xi,\eta} \right\|_{\mathrm{H}^1_*} \\
&\leq \left\| X^{t,\xi+\eta,\xi+\eta} - X^{t,\xi,\xi+\eta} - \mathrm{D}_x X^{t,\xi,\xi}\eta \right\|_{\mathrm{H}^1_*} + \left\| X^{t,\xi,\xi+\eta} - X^{t,\xi,\xi} - Y^{t,\xi,\eta} \right\|_{\mathrm{H}^2_*} \\
&\lesssim \|\eta\|^2_{\mathrm{L}^2_*} + \left\| X^{t,\xi+\eta,\xi+\eta} - X^{t,\xi,\xi+\eta} - \mathrm{D}_x X^{t,\xi,\xi}\eta \right\|_{\mathrm{H}^1_*}
\end{aligned}
$$

due to (25) and, thus, Lemma 14 implies

$$
\lim_{\|\eta\|_{\mathrm{L}^2_*} \to 0} \frac{\| X^{t,\xi+\eta} - X^{t,\xi} - \mathrm{D}_x X^{t,\xi,\xi}\eta - Y^{t,\xi,\eta} \|_{\mathrm{H}^1_*}}{\|\eta\|_{\mathrm{L}^2_*}} = 0.
$$

□

Proposition 3. *Let* $0 \leq t \leq T$*. If Assumptions 2 and 3 are satisfied with* $q_0 \geq 4$*, then the map*

$$
\mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{1,d}_*(t,T), \qquad \xi \mapsto X^{t,\xi}
$$

is continuously Fréchet differentiable with Fréchet derivative

$$
\mathrm{D}_x X^{t,\xi} : \mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{2,d}_*(t,T), \qquad \eta \mapsto \mathrm{D}_\xi X^{t,\xi}\eta := \mathrm{D}_x X^{t,\xi,\xi}\eta + Y^{t,\xi,\eta}
$$

at $\xi \in \mathrm{L}^{2,d}_*(t)$*.*

Proof. Lemmas 10, 13, 15 and 16 imply that the map

$$
\mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{1,d}_*(t,T), \qquad \eta \mapsto \mathrm{D}_x X^{t,\xi,\xi}\eta + Y^{t,\xi,\eta}
$$

is linear and continuous.

Further, Lemma 21 implies

$$
\lim_{\|\eta\| \to 0} \frac{\| X^{t,x,\xi+\varepsilon\eta} - X^{t,x,\xi} - \mathrm{D}_x X^{t,\xi,\xi}\eta - Y^{t,\xi,\eta} \|_{\mathrm{H}^1_*}}{\|\eta\|_{\mathrm{L}^2_*}} = 0.
$$

Finally, observe that

$$
\begin{aligned}
&\left\| \mathrm{D}_x X^{t,\xi+\eta,\xi+\eta}\zeta + Y^{t,\xi+\eta,\zeta} - \mathrm{D}_x X^{t,\xi,\xi}\zeta - Y^{t,\xi,\zeta} \right\|_{\mathrm{H}^1_*} \\
&= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t \leq w \leq T} \left\| \mathrm{D}_x X^{t,x+y,\xi+\eta}_w z - \mathrm{D}_x X^{t,x,\xi}_w z + Y^{t,x+y,\xi+\eta,\zeta}_w - Y^{t,x,\xi,\zeta}_w \right\| \,\middle|\, \mathcal{F}_t \right] \Big|_{x=\xi, y=\eta, z=\zeta} \right] \\
&\lesssim \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t \leq w \leq T} \left\| \mathrm{D}_x X^{t,x+y,\xi+\eta}_w z - \mathrm{D}_x X^{t,x,\xi}_w z \right\|^2 \,\middle|\, \mathcal{F}_t \right]^{\frac{1}{2}} \Big|_{x=\xi, y=\eta, z=\zeta} \right] \\
&\quad + \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\sup_{t \leq w \leq T} \left\| Y^{t,x+y,\xi+\eta,\zeta}_w - Y^{t,x,\xi,\zeta}_w \right\|^2 \,\middle|\, \mathcal{F}_t \right]^{\frac{1}{2}} \Big|_{x=\xi, y=\eta, z=\zeta} \right] \\
&\lesssim \|\zeta\|_{\mathrm{L}^2_*} \|\eta\|_{\mathrm{L}^2_*}
\end{aligned}
$$

due to Lemmas 12, 13, 18 and 19. Thus, $\xi \mapsto \mathrm{D}_\xi X^{t,\xi}$ is continuous with respect to the operator norm. □

Proposition 4. *Let $0 \le t \le T$ and $x \in \mathbb{R}^d$. If Assumptions 2 and 3 are satisfied with $q_0 \ge 4$, then the map*

$$\mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{2,d}_*(t,T), \qquad \xi \mapsto X^{t,x,\xi}$$

is continuously Fréchet differentiable with Fréchet derivative

$$\mathrm{D}_\xi X^{t,x,\xi} : \mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{2,d}_*(t,T), \qquad \eta \mapsto \mathrm{D}_\xi X^{t,x,\xi}\eta := Y^{t,x,\xi,\eta}$$

at $\xi \in \mathrm{L}^{2,d}_*(t)$.

Proof. Lemmas 15 and 16 imply that the map

$$\mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{2,d}_*(t,T), \qquad \eta \mapsto Y^{t,x,\xi,\eta}$$

is linear and continuous. Moreover, we have

$$\left\| X^{t,x,\xi+\eta} - X^{t,x,\xi} - Y^{t,x,\xi,\eta} \right\|_{\mathrm{H}^2_*} \lesssim \|\eta\|^2_{\mathrm{L}^2_*} + \left\| X^{t,\xi+\eta} - X^{t,\xi} - \mathrm{D}_x X^{t,\xi,\xi}\eta - Y^{t,\xi,\eta} \right\|_{\mathrm{H}^1_*}$$

due to Lemma 20 and, thus, Lemma 21 yields

$$\lim_{\|\eta\|\to 0} \frac{\| X^{t,x,\xi+\eta} - X^{t,x,\xi} - Y^{t,x,\xi,\eta} \|_{\mathrm{H}^2_*}}{\|\eta\|_{\mathrm{L}^2_*}} = 0.$$

Finally, observe that

$$\left\| Y^{t,x,\xi,\zeta} - Y^{t,x,\eta,\zeta} \right\|_{\mathrm{H}^2_*} \lesssim \|\zeta\|_{\mathrm{L}^2_*} \|\xi - \eta\|_{\mathrm{L}^2_*}$$

due to Lemma 19. Thus, the map $\xi \mapsto \mathrm{D}_\xi X^{t,x,\xi}$ is continuous with respect to the operator norm. □

5. Second-Order Derivatives

In this section, we show the interchangeability in order of differentiation in Lemma 26 and establish the second-order Fréchet differentiability of $(x,\xi) \mapsto X^{t,x,\xi}$ in Propositions 5 and 7. For a normed real vector space V, let $\mathrm{C}^2(V)$ denote the space of all $f \in \mathrm{C}^1(V)$ such that $\mathrm{D}f(\cdot)v \in \mathrm{C}^1(V)$ for all $v \in V$ and, for convenience, we set $\mathrm{D}^2 f(v_0)(v_2,v_1) := \mathrm{D}\,\mathrm{D}f(v_0)v_1\,v_2$ for $v_0, v_1, v_2 \in V$.

Assumption 4. *Let $b : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \to \mathbb{R}^d$, $h : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \to \mathbb{R}^{d\times n\times n}$, and $g : [0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}^{2,d}_* \to \mathbb{R}^{d\times n}$ be such that the following holds for all components $f = b_k, h_{kij}, g_{ki}$ with $1 \le i,j \le n$, $1 \le k \le d$.*

1. *We have $f(s,\omega,\cdot,\xi) \in \mathrm{C}^2(\mathbb{R}^d)$, $\mathrm{D}_\xi f(s,\omega,\cdot,\xi)\eta \in \mathrm{C}^1(\mathbb{R}^d)$ and $\mathrm{D}_x f(s,\omega,x,\cdot)y \in \mathrm{C}^1(\mathrm{L}^{2,d}_*)$ for all $0 \le s \le T$, $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}^{2,d}_*$.*
2. *There exists a square-integrable $\alpha_2 : [0,T] \to [1,\infty)$ such that*

$$\begin{aligned}
\left| \mathrm{D}^2_x f(s,\omega,x,\xi)(y,z) - \mathrm{D}^2_x f(s,\omega,v,\xi)(y,z) \right| &\le \kappa(s)\|y\|\|z\|\|x - v\|,\\
\left| \mathrm{D}_x \mathrm{D}_\xi f(s,\omega,x,\xi)\zeta\, z - \mathrm{D}_x \mathrm{D}_\xi f(s,\omega,y,\eta)\zeta\, z \right| &\le \alpha_2(s)\|z\|\|\zeta\|_{\mathrm{L}^2_*}\left(\|x-y\| + \|\xi - \eta\|_{\mathrm{L}^2_*}\right),\\
\left| \mathrm{D}_\xi \mathrm{D}_x f(s,\omega,x,\xi) z\,\zeta - \mathrm{D}_\xi \mathrm{D}_x f(s,\omega,y,\eta) z\,\zeta \right| &\le \alpha_2(s)\|z\|\|\zeta\|_{\mathrm{L}^2_*}\left(\|x-y\| + \|\xi - \eta\|_{\mathrm{L}^2_*}\right)
\end{aligned}$$

for all $0 \le s \le T$, $\omega \in \Omega$, $v, x, y, z \in \mathbb{R}^d$ and $\xi, \eta, \zeta \in \mathrm{L}^{2,d}_*$.

Lemma 22. *Let $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$. If Assumptions 2, 3 and 4 are satisfied with $q_0 \geq 4$, then the G-SDE*

$$
\begin{aligned}
dC_s^{t,x,\xi,y,z} &= \mathrm{D}_x b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) C_s^{t,x,\xi,y,z}\, ds \\
&\quad + \mathrm{D}_x^2 b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,x,\xi} y,\, \mathrm{D}_x X_s^{t,x,\xi} z\Big)\, ds \\
&\quad + \mathrm{D}_x h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) C_s^{t,x,\xi,y,z}\, d\langle B\rangle_s \\
&\quad + \mathrm{D}_x^2 h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,x,\xi} y,\, \mathrm{D}_x X_s^{t,x,\xi} z\Big)\, d\langle B\rangle_s \\
&\quad + \mathrm{D}_x g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) C_s^{t,x,\xi,y,z}\, dB_s \\
&\quad + \mathrm{D}_{x}^2 g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,x,\xi} y,\, \mathrm{D}_x X_s^{t,x,\xi} z\Big)\, dB_s, \qquad t \leq s \leq T, \\
C_t^{t,x,\xi,y,z} &= 0
\end{aligned} \tag{26}
$$

admits a unique solution $C^{t,x,\xi,y,z} \in \mathrm{H}_*^{2,d}(t,T)$ for all $0 \leq t \leq T$, $x,y,z \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$. Moreover, the map

$$\mathbb{R}^d \times \mathbb{R}^d \to \mathrm{H}_*^{2,d}(t,T), \qquad (y,z) \mapsto C^{t,x,\xi,y,z}$$

is bilinear.

Proof. The SDE (26) has a unique solution $C^{t,x,\xi,y,z} \in \mathrm{H}_*^{2,d}(t,T)$ since the coefficients are Lipschitz and of linear growth due to Lemma 11 for any $y,z \in \mathbb{R}^d$. Thus, the map $(y,z) \mapsto C^{t,x,\xi,y,z}$ is well-defined.

Let $\lambda \in \mathbb{R}$ and $v,x,y,z \in \mathbb{R}^d$. By Lemma A4, we have for all $t \leq s \leq T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t \leq w \leq s}\left\|C_w^{t,x,\xi,y+\lambda v,z} - C_w^{t,x,\xi,y,z} - \lambda C_w^{t,x,\xi,v,z}\right\|^2\right] \\
&\lesssim \sum_{f \in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\Big(C_u^{t,x,\xi,y+\lambda v,z} - C_u^{t,x,\xi,y,z} - \lambda C_u^{t,x,\xi,v,z}\Big)\right|^2\right] du \\
&\lesssim \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|C_u^{t,x,\xi,y+\lambda v,z} - C_u^{t,x,\xi,y,z} - \lambda C_u^{t,x,\xi,v,z}\right\|^2\right] du,
\end{aligned}
$$

and Grönwall's inequality implies

$$\left\|C^{t,x,\xi,y+\lambda v,z} - C^{t,x,\xi,y,z} - \lambda C^{t,x,\xi,v,z}\right\|_{\mathrm{H}_*^2} = 0,$$

i.e., $y \mapsto C^{t,x,\xi,y,z}$ is linear. Analogously, we obtain that $z \mapsto C^{t,x,\xi,y,z}$ is linear. □

Lemma 23. *If Assumptions 2, 3 and 4 are satisfied with $q_0 \geq 4$, then*

$$\hat{\mathbb{E}}\left[\sup_{t \leq w \leq T}\left\|C_w^{t,x,\xi,y,z}\right\|^2\right] \lesssim \|y\|^2 \|z\|^2$$

for all $0 \leq t \leq T$, $x,y,z \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$.

Proof. By Lemma A4, we have for all $t \leq s \leq T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leq w\leq s}\left\|C_w^{t,x,\xi,y,z}\right\|^2\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x^2 f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)\left(\mathrm{D}_x X_u^{t,x,\xi} y,\, \mathrm{D}_x X_u^{t,x,\xi} z\right)\right|^2\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right) C_u^{t,x,\xi,y,z}\right|^2\right] du \\
&\leq \int_t^s \alpha_1(u)^2 \hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2 \left\|\mathrm{D}_x X_u^{t,x,\xi} z\right\|^2\right] + \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|C_u^{t,x,\xi,y,z}\right\|^2\right] du \\
&\lesssim \|y\|^2\|z\|^2 + \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|C_u^{t,x,\xi,y,z}\right\|^2\right] du.
\end{aligned}$$

Finally, Grönwall's inequality implies the desired result. □

Proposition 5. *Let $0 \leq t \leq T$. If Assumptions 2, 3 and 4 are satisfied with $q_0 \geq 6$ and $q_1 \geq 3$, then the map*

$$\mathbb{R}^d \to \mathrm{H}_*^{2,d}(t,T), \qquad x \mapsto X^{t,x,\xi}$$

is twice Fréchet differentiable for every $\xi \in \mathrm{L}_*^{2,d}(t)$. More precisely, for every $x \in \mathbb{R}^d$ and $\xi \in \mathrm{L}_*^{2,d}(t)$, the map

$$\mathrm{D}_x^2 X^{t,x,\xi} : \mathbb{R}^d \times \mathbb{R}^d \to \mathrm{H}_*^{2,d}(t,T), \qquad (y,z) \mapsto \mathrm{D}_x^2 X^{t,x,\xi}(y,z) := C^{t,x,\xi,y,z}$$

is bilinear and continuous and such that

$$\lim_{\|y\|\to 0} \frac{\left\|\mathrm{D}_x X^{t,x+y,\xi} z - \mathrm{D}_x X^{t,x,\xi} z - \mathrm{D}_x^2 X^{t,x,\xi}(y,z)\right\|_{\mathrm{H}_*^2}}{\|y\|} = 0$$

for all $z \in \mathbb{R}^d$.

Proof. The map $(y,z) \mapsto C^{t,x,\xi,y,z}$ is bilinear and continuous due to Lemmas 22 and 23. Set $\Delta^x := X^{t,x+y,\xi} - X^{t,x,\xi}$, then

$$\hat{\mathbb{E}}\left[\sup_{t\leq w\leq T}\|\Delta_w^x\|^6 \,\middle|\, \mathcal{F}_t\right] \lesssim \|y\|^6 \tag{27}$$

due to Lemma 4, and Lemma 12 implies

$$\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{t\leq w\leq T}\left\|\Delta_w^x - \mathrm{D}_x X_w^{t,x,\xi} y\right\|^3 \,\middle|\, \mathcal{F}_t\right] &\leq \int_0^1 \hat{\mathbb{E}}\left[\sup_{t\leq w\leq T}\left\|\mathrm{D}_x X_w^{t,x+\lambda y,\xi} y - \mathrm{D}_x X_w^{t,x,\xi} y\right\|^3 \,\middle|\, \mathcal{F}_t\right] d\lambda \\
&\lesssim \|y\|^6.
\end{aligned} \tag{28}$$

Further, set $\Delta^{x,x} := \mathrm{D}_x X^{t,x+y,\xi} z - \mathrm{D}_x X^{t,x,\xi} z$, then Lemma 12 yields

$$\hat{\mathbb{E}}\left[\sup_{t\leq w\leq T}\|\Delta_w^{x,x}\|^3\right] \lesssim \|y\|^3. \tag{29}$$

By Lemma A4, we have for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|\Delta_w^{x,x} - C_w^{t,x,\xi,y,z}\right\|^2\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\Big[\Big|\mathrm{D}_x f\Big(u, X_u^{t,x+y,\xi}, X_u^{t,\xi}\Big)\,\mathrm{D}_x X_u^{t,x+y,\xi} z - \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\,\mathrm{D}_x X_u^{t,x,\xi} z \\
&\quad - \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big) C_u^{t,x,\xi,y,z} - \mathrm{D}_x^2 f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\Big(\mathrm{D}_x X_u^{t,x,\xi} y,\, \mathrm{D}_x X_u^{t,x,\xi} z\Big)\Big|^2\Big]\,du \\
&\lesssim \int_t^s \alpha_2(u)^2\hat{\mathbb{E}}\Big[\|\Delta_u^x\|^4\Big\|\mathrm{D}_x X_u^{t,x,\xi} z\Big\|^2\Big] + \alpha_1(u)^2\hat{\mathbb{E}}\Big[\|\Delta_u^x\|^2\|\Delta_u^{x,x}\|^2\Big]\,du \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\Big[\Big\|\Delta_u^x - \mathrm{D}_x X_u^{t,x,\xi} y\Big\|^2\Big\|\mathrm{D}_x X_u^{t,x,\xi} z\Big\|^2\Big] \\
&\quad + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\Big[\Big\|\Delta_u^{x,x} - C_u^{t,x,\xi,y,z}\Big\|^2\Big] \\
&\le \int_t^s \alpha_2(u)^2\hat{\mathbb{E}}\Big[\|\Delta_u^x\|^6\Big]^{\frac{2}{3}}\hat{\mathbb{E}}\Big[\Big\|\mathrm{D}_x X_u^{t,x,\xi} z\Big\|^6\Big]^{\frac{1}{3}} + \alpha_1(u)^2\hat{\mathbb{E}}\Big[\|\Delta_u^x\|^6\Big]^{\frac{1}{3}}\hat{\mathbb{E}}\Big[\|\Delta_u^{x,x}\|^3\Big]^{\frac{2}{3}}\,du \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\Big[\Big\|\Delta_u^x - \mathrm{D}_x X_u^{t,x,\xi} y\Big\|^3\Big]^{\frac{2}{3}}\hat{\mathbb{E}}\Big[\Big\|\mathrm{D}_x X_u^{t,x,\xi} z\Big\|^6\Big]^{\frac{1}{3}} \\
&\quad + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\Big[\Big\|\Delta_u^{x,x} - C_u^{t,x,\xi,y,z}\Big\|^2\Big] \\
&\lesssim \|y\|^4\Big(1+\|z\|^2\Big) + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\Big[\Big\|\Delta_u^{x,x} - C_u^{t,x,\xi,y,z}\Big\|^2\Big]
\end{aligned}
$$

due to (27)–(29) and Lemma 11. Finally, Grönwall's inequality yields

$$\left\|\mathrm{D}_x X^{t,x+y,\xi} z - \mathrm{D}_x X^{t,x,\xi} z - \mathrm{D}_x^2 X^{t,x,\xi}(y,z)\right\|_{\mathrm{H}_*^2} \lesssim \|y\|^2(1+\|z\|)$$

which implies the desired result. □

Lemma 24. *If Assumption 2, 3 and 4 are satisfied with $q_0 \ge 6$ and $q_1 \ge 3$, then the G-SDE*

$$
\begin{aligned}
dD_s^{t,x,\xi,y,\eta} &= \mathrm{D}_x b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) D_s^{t,x,\xi,y,\eta}\,ds \\
&\quad + \mathrm{D}_x^2 b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,x,\xi} y,\, \mathrm{D}_\xi X_s^{t,x,\xi}\eta\Big)\,ds \\
&\quad + \mathrm{D}_x\,\mathrm{D}_\xi b\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\mathrm{D}_\xi X_s^{t,\xi}\eta\;\mathrm{D}_x X_s^{t,x,\xi} y\,ds \\
&\quad + \mathrm{D}_x h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) D_s^{t,x,\xi,y,\eta}\,d\langle B\rangle_s \\
&\quad + \mathrm{D}_x^2 h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,x,\xi} y,\, \mathrm{D}_\xi X_s^{t,x,\xi}\eta\Big)\,d\langle B\rangle_s \\
&\quad + \mathrm{D}_x\,\mathrm{D}_\xi h\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\mathrm{D}_\xi X_s^{t,\xi}\eta\;\mathrm{D}_x X_s^{t,x,\xi} y\,d\langle B\rangle_s \\
&\quad + \mathrm{D}_x g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big) D_s^{t,x,\xi,y,\eta}\,dB_s \\
&\quad + \mathrm{D}_x^2 g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\Big(\mathrm{D}_x X_s^{t,x,\xi} y,\, \mathrm{D}_\xi X_s^{t,x,\xi}\eta\Big)\,dB_s \\
&\quad + \mathrm{D}_x\,\mathrm{D}_\xi g\Big(s, X_s^{t,x,\xi}, X_s^{t,\xi}\Big)\mathrm{D}_\xi X_s^{t,\xi}\eta\;\mathrm{D}_x X_s^{t,x,\xi} y\,dB_s, \qquad t \le s \le T, \\
D_t^{t,x,\xi,y,\eta} &= 0
\end{aligned} \tag{30}
$$

admits a unique solution $D^{t,x,\xi,y,\eta} \in \mathrm{H}_*^{2,d}(t,T)$ for all $0 \le t \le T$, $x, y \in \mathbb{R}^d$, $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$. Moreover, the map

$$\mathbb{R}^d \times \mathrm{L}_*^{2,d}(t) \mapsto \mathrm{H}_*^{2,d}(t,T), \qquad (y,\eta) \mapsto D^{t,x,\xi,y,\eta}$$

is bilinear.

Proof. The SDE (30) has a unique solution $D^{t,x,\xi,y,\eta} \in \mathrm{H}_*^{2,d}(t,T)$ since the coefficients are Lipschitz and of linear growth due to Lemmas 11 and 17 for any $y \in \mathbb{R}^d$ and $\eta \in \mathrm{L}_*^{2,d}(t)$. Thus, the map $(y,\eta) \mapsto D^{t,x,\xi,y,\eta}$ is well defined.

Let $\lambda \in \mathbb{R}$, $y, z \in \mathbb{R}^d$ and $\eta, \zeta \in \mathrm{L}_*^{2,d}(t)$. By Lemma A4, we have for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|D_w^{t,x,\xi,y+\lambda z,\eta} - D_w^{t,x,\xi,y,\eta} - \lambda D_w^{t,x,\xi,z,\eta}\right\|^2\right] \\
&\quad\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)\left(D_u^{t,x,\xi,y+\lambda z,\eta} - D_u^{t,x,\xi,y,\eta} - \lambda D_u^{t,x,\xi,z,\eta}\right)\right|^2\right] du \\
&\quad\le \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|D_u^{t,x,\xi,y+\lambda z,\eta} - D_u^{t,x,\xi,y,\eta} - \lambda D_u^{t,x,\xi,z,\eta}\right\|^2\right] du,
\end{aligned}
$$

and Grönwall's inequality yields that

$$
\left\|D^{t,x,\xi,y+\lambda z} - D^{t,x,\xi,y} - \lambda D^{t,x,\xi,z}\right\|_{\mathrm{H}_*^2} = 0,
$$

i.e., $y \mapsto D^{t,x,\xi,y,\eta}$ is linear. Analogously, we obtain that $\eta \mapsto D^{t,x,\xi,y,\eta}$ is linear. □

Lemma 25. *If Assumption 2, 3 and 4 are satisfied with $q_0 \ge 6$ and $q_1 \ge 3$, then*

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|D_w^{t,x,\xi,y,\eta}\right\|^2\right] \lesssim \|y\|^2\|\eta\|_{\mathrm{L}_*^2}^2.
$$

Proof. By Lemma A4, we have for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|D_w^{t,x,\xi,y,\eta}\right\|^2\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right) D_u^{t,x,\xi,y,\eta}\right|^2\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x^2 f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)\left(\mathrm{D}_x X_u^{t,x,\xi} y, \mathrm{D}_\xi X_u^{t,x,\xi}\eta\right)\right|^2\right] du \\
&\quad + \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\left[\left|\mathrm{D}_x \mathrm{D}_\xi f\left(u, X_u^{t,x,\xi}, X_u^{t,\xi}\right)\mathrm{D}_\xi X_u^{t,x,\xi}\eta\ \mathrm{D}_x X_u^{t,x,\xi} y\right|^2\right] du \\
&\lesssim \int_t^s \alpha_0(u)^2 \hat{\mathbb{E}}\left[\left\|D_u^{t,x,\xi,y,\eta}\right\|^2\right] + \alpha_1(u)^2 \hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\left\|\mathrm{D}_\xi X_u^{t,x,\xi}\eta\right\|^2\right] du,
\end{aligned}
$$

and Grönwall's inequality implies

$$
\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|D_w^{t,x,\xi,y,\eta}\right\|^2\right] \lesssim \int_t^s \alpha_1(u)^2 \hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\left\|\mathrm{D}_\xi X_u^{t,x,\xi}\eta\right\|^2\right] du
$$

for all $t \le s \le T$. Finally, observe that for all $t \le s \le T$

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_s^{t,x,\xi} y\right\|^2\left\|\mathrm{D}_\xi X_s^{t,x,\xi}\eta\right\|^2\right] &= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_s^{t,x,\xi} y\right\|^2\left\|\mathrm{D}_\xi X_s^{t,x,\xi} z\right\|^2 \,\middle|\, \mathcal{F}_t\right]\Big|_{z=\zeta}\right] \\
&\le \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_s^{t,x,\xi} y\right\|^4 \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{2}} \hat{\mathbb{E}}\left[\left\|\mathrm{D}_\xi X_s^{t,x,\xi} z\right\|^4 \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{2}}\Big|_{z=\zeta}\right] \\
&\lesssim \|y\|^2\|\zeta\|_{\mathrm{L}_*^2}^2
\end{aligned}
$$

due to Lemma 11, which implies the desired result. □

Proposition 6. *Let* $0 \le t \le T$ *and* $\xi, \eta \in \mathrm{L}^{2,d}_*(t)$*. If Assumption 2, 3 and 4 are satisfied with* $q_0 \ge 6$ *and* $q_1 \ge 3$*, then the map*

$$\mathbb{R}^d \to \mathrm{H}^{2,d}_*(t,T), \qquad x \mapsto \mathrm{D}_\xi X^{t,x,\xi}\eta$$

is Fréchet differentiable with Fréchet derivative

$$\mathrm{D}_x \mathrm{D}_\xi X^{t,x,\xi}\eta : \mathbb{R}^d \to \mathrm{H}^{2,d}_*(t,T), \qquad y \mapsto \mathrm{D}_\xi X^{t,x,\xi}\eta\, y := D^{t,x,\xi,y,\eta}$$

at $x \in \mathbb{R}^d$.

Proof. By Lemmas 24 and 25, the map $y \mapsto D^{t,x,\xi,y,\eta}$ is linear and continuous.

Set $\Delta^\xi := \mathrm{D}_\xi X^{t,x+y,\xi}\eta - \mathrm{D}_\xi X^{t,x,\xi}\eta$, then Lemma 19 yields

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|\Delta^\xi_w\right\|^3 \,\middle|\, \mathcal{F}_t\right] \lesssim \|\eta\|^3_{\mathrm{L}^2_*}\|y\|^3. \tag{31}$$

As in the proof of Proposition 5, set $\Delta^x := X^{t,x+y,\xi} - X^{t,x,\xi}$, then

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\|\Delta^x_w\|^6 \,\middle|\, \mathcal{F}_t\right] \lesssim \|y\|^6, \qquad \hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left\|\Delta^x_w - \mathrm{D}_x X^{t,x,\xi}_w y\right\|^3 \,\middle|\, \mathcal{F}_t\right] \lesssim \|y\|^6. \tag{32}$$

By Lemma A4, we have for $t \le s \le T$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|\Delta^\xi_w - D^{t,x,\xi,y,\eta}_w\right\|^2\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\Big[\Big|\mathrm{D}_x f\Big(u, X^{t,x+y,\xi}_u, X^{t,\xi}_u\Big)\mathrm{D}_\xi X^{t,x+y,\xi}_u\eta + \mathrm{D}_\xi f\Big(u, X^{t,x+y,\xi}_u, X^{t,\xi}_u\Big)\mathrm{D}_\xi X^{t,\xi}_u\eta \\
&\quad - \mathrm{D}_x f\Big(u, X^{t,x,\xi}_u, X^{t,\xi}_u\Big)\mathrm{D}_\xi X^{t,x,\xi}_u\eta - \mathrm{D}_\xi f\Big(u, X^{t,x,\xi}_u, X^{t,\xi}_u\Big)\mathrm{D}_\xi X^{t,\xi}_u\eta \\
&\quad - \mathrm{D}_x f\Big(u, X^{t,x,\xi}_u, X^{t,\xi}_u\Big)D^{t,x,\xi,y,\eta}_u - \mathrm{D}^2_x f\Big(u, X^{t,x,\xi}_u, X^{t,\xi}_u\Big)\Big(\mathrm{D}_x X^{t,x,\xi}_u y, \mathrm{D}_\xi X^{t,x,\xi}_u\eta\Big) \\
&\quad - \mathrm{D}_x \mathrm{D}_\xi f\Big(u, X^{t,x,\xi}_u, X^{t,\xi}_u\Big)\mathrm{D}_\xi X^{t,\xi}_u\eta\, \mathrm{D}_x X^{t,x,\xi}_u y\Big|^2\Big] \\
&\lesssim \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^\xi_u\right\|^2\|\Delta^x_u\|^2\right] + \alpha_2(u)^2\hat{\mathbb{E}}\left[\|\Delta^x_u\|^4\left\|\mathrm{D}_\xi X^{t,x,\xi}_u\eta\right\|^2\right]du \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^x_u - \mathrm{D}_x X^{t,x,\xi}_u y\right\|^2\left\|\mathrm{D}_\xi X^{t,x,\xi}_u\eta\right\|^2\right] + \alpha_2(u)^2\hat{\mathbb{E}}\left[\|\Delta^x_u\|^4\right]\left\|\mathrm{D}_\xi X^{t,\xi}_u\eta\right\|^2_{\mathrm{L}^2_*}du \\
&\quad + \int_t^s \alpha_1(u)^2\left\|\mathrm{D}_\xi X^{t,\xi}_u\eta\right\|^2_{\mathrm{L}^2_*}\hat{\mathbb{E}}\left[\left\|\Delta^x_u - \mathrm{D}_x X^{t,x,\xi}_u\right\|^2\right] + \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^\xi_u - D^{t,x,\xi,y,\eta}_u\right\|^2\right]du \\
&\lesssim \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^\xi_u\right\|^3\right]^{\frac{2}{3}}\hat{\mathbb{E}}\left[\|\Delta^x_u\|^6\right]^{\frac{1}{3}}du \\
&\quad + \int_t^s \alpha_2(u)^2\hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\|\Delta^x_u\|^6\,\middle|\,\mathcal{F}_t\right]^{\frac{2}{3}}\hat{\mathbb{E}}\left[\left\|\mathrm{D}_\xi X^{t,x,\xi}_u\eta\right\|^6\,\middle|\,\mathcal{F}_t\right]^{\frac{1}{3}}\right]du \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\left\|\Delta^x_u - \mathrm{D}_x X^{t,x,\xi}_u y\right\|^3\,\middle|\,\mathcal{F}_t\right]^{\frac{2}{3}}\hat{\mathbb{E}}\left[\left\|\mathrm{D}_\xi X^{t,x,\xi}_u\eta\right\|^6\,\middle|\,\mathcal{F}_t\right]^{\frac{1}{3}}\right]du \\
&\quad + \int_t^s \alpha_2(u)^2\hat{\mathbb{E}}\left[\|\Delta^x_u\|^4\right]\left\|\mathrm{D}_\xi X^{t,\xi}_u\eta\right\|^2_{\mathrm{L}^2_*}du \\
&\quad + \int_t^s \alpha_1(u)^2\left\|\mathrm{D}_\xi X^{t,\xi}_u\eta\right\|^2_{\mathrm{L}^2_*}\hat{\mathbb{E}}\left[\left\|\Delta^x_u - \mathrm{D}_x X^{t,x,\xi}_u\right\|^2\right] + \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^\xi_u - D^{t,x,\xi,y,\eta}_u\right\|^2\right]du \\
&\lesssim \|y\|^4\|\eta\|^2_{\mathrm{L}^2_*} + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta^\xi_u - D^{t,x,\xi,y,\eta}_u\right\|^2\right]du
\end{aligned}$$

due to (31), (32) and Lemma 17. Finally, Grönwall's inequality yields

$$\left\|\mathrm{D}_\xi X^{t,x+y,\xi}\eta - \mathrm{D}_\xi X^{t,x,\xi}\eta - D^{t,x,\xi,y,\eta}\right\|_{\mathrm{H}^2_*} \lesssim \|y\|^2\|\eta\|_{\mathrm{L}^2_*}$$

which implies the desired result. □

Lemma 26. *If Assumption 2, 3 and 4 are satisfied, then the following holds for all components* $f = b_k, h_{kij}, g_{ki}$, $1 \le i, j \le n$, $1 \le k \le d$:

$$\mathrm{D}_x \big[\mathrm{D}_\xi f(s,x,\xi)\eta \big] y = \mathrm{D}_\xi \big[\mathrm{D}_x f(s,x,\xi) y \big] \eta$$

for all $0 \le s \le T$, $x, y \in \mathbb{R}^d$, $\xi, \eta \in \mathrm{L}^{2,d}_*$ *and* $\omega \in \Omega$.

Proof. Let $0 \le s \le T$, $x, y \in \mathbb{R}^d$, $\xi, \eta \in \mathrm{L}^{2,d}_*$ and $\omega \in \Omega$. We have

$$\begin{aligned} I &:= f(s,x+y,\xi+\eta,\omega) - f(s,x+y,\xi,\omega) - f(s,x,\xi+\eta,\omega) + f(s,x,\xi,\omega) \\ &= \int_0^1 \mathrm{D}_\xi f(s,x+y,\xi+\lambda_1\eta,\omega)\eta - \mathrm{D}_\xi f(s,x,\xi+\lambda_1\eta,\omega)\eta \, d\lambda_1 \\ &= \int_0^1 \int_0^1 \mathrm{D}_x \mathrm{D}_\xi f(s,x+\lambda_2 y,\xi+\lambda_1\eta,\omega)\eta\, y \, d\lambda_2 \, d\lambda_1 \\ &= \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y + R_1 \end{aligned}$$

with

$$\begin{aligned} R_1 &:= \int_0^1 \int_0^1 \mathrm{D}_x \mathrm{D}_\xi f(s,x+\lambda_2 y,\xi+\lambda_1\eta,\omega)\eta\, y - \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y \, d\lambda_2 \, d\lambda_1 \\ &\le \int_0^1 \int_0^1 \big| \mathrm{D}_x \mathrm{D}_\xi f(s,x+\lambda_2 y,\xi+\lambda_1\eta,\omega)\eta\, y - \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y \big| \, d\lambda_2 \, d\lambda_1 \\ &\le \int_0^1 \int_0^1 \alpha_2(s) \|y\| \|\eta\|_{\mathrm{L}^2_*} \Big(\lambda_2 \|y\| + \lambda_1 \|\eta\|_{\mathrm{L}^2_*} \Big) \, d\lambda_2 \, d\lambda_1 \\ &\le \|y\| \|\eta\|_{\mathrm{L}^2_*} \Big(\|y\| + \|\eta\|_{\mathrm{L}^2_*} \Big). \end{aligned}$$

Analogously, we have

$$\begin{aligned} I &= f(s,x+y,\xi+\eta,\omega) - f(s,x,\xi+\eta,\omega) - f(s,x+y,\xi,\omega) + f(s,x,\xi,\omega) \\ &= \int_0^1 \mathrm{D}_x f(s,x+\lambda_1 y,\xi+\eta,\omega) y - \mathrm{D}_x f(s,x+\lambda_1 y,\xi,\omega) y \, d\lambda_1 \\ &= \int_0^1 \int_0^1 \mathrm{D}_\xi \mathrm{D}_x f(s,x+\lambda_1 y,\xi+\lambda_2\eta,\omega) y\, \eta \, d\lambda_2 \, d\lambda_1 \\ &= \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y + R_2 \end{aligned}$$

with

$$\begin{aligned} R_2 &:= \int_0^1 \int_0^1 \mathrm{D}_\xi \mathrm{D}_x f(s,x+\lambda_1 y,\xi+\lambda_2\eta,\omega)\eta\, y - \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y \, d\lambda_2 \, d\lambda_1 \\ &\le \|y\| \|\eta\|_{\mathrm{L}^2_*} \Big(\|y\| + \|\eta\|_{\mathrm{L}^2_*} \Big). \end{aligned}$$

Thus, we get

$$\frac{\big| \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y - \mathrm{D}_x \mathrm{D}_\xi f(s,x,\xi,\omega)\eta\, y \big|}{\|y\| \|\eta\|_{\mathrm{L}^2_*}} \lesssim \|y\| + \|\eta\|_{\mathrm{L}^2_*}$$

for all $0 \le s \le T$, $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}^{2,d}_*(t)$. By letting $\|\eta\|_{\mathrm{L}^2_*}$ and $\|y\|$ tend to zero, we conclude the desired result. □

Proposition 7. *Let $0 \le t \le T$, $x \in \mathbb{R}^d$ and $\xi \in \mathrm{L}^{2,d}_*(t)$. If Assumptions 2, 3 and 4 are satisfied with $q_0 \ge 6$ and $q_1 \ge 3$, then the map*

$$\mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{2,d}_*(t,T), \qquad \xi \mapsto \mathrm{D}_x X^{t,x,\xi} y$$

is Fréchet differentiable with Fréchet derivative

$$\mathrm{D}_\xi \mathrm{D}_x X^{t,x,\xi} y : \mathrm{L}^{2,d}_*(t) \to \mathrm{H}^{2,d}_*(t,T), \qquad \eta \mapsto \mathrm{D}_\xi \mathrm{D}_x X^{t,x,\xi} y\, \eta := D^{t,x,\xi,y,\eta}$$

at $\xi \in \mathrm{L}^{2,d}_*(t)$.

Proof. By Lemmas 24 and 25, the map $\eta \mapsto D^{t,x,\xi,y,\eta}$ is linear and continuous.

For all components $f = b_k, h_{ijk}, g_{ik}$, $1 \le k \le d$, $1 \le i,j \le n$, we have

$$\begin{aligned} \mathrm{D}_x \mathrm{D}_\xi f(s,\omega,x,\xi)\eta\, y &= \mathrm{D}_\xi \mathrm{D}_x f(s,\omega,x,\xi) y\, \eta, \\ \mathrm{D}^2_x f(s,\omega,x,\xi)(y,z) &= \mathrm{D}^2_x f(s,\omega,x,\xi)(z,y) \end{aligned}$$

for all $0 \le s \le T$, $\omega \in \Omega$, $x,y,z \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}^{2,d}_*$ due to Lemma 26 and the symmetry of the second-order Fréchet derivative.

Fix $x, y \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}^{2,d}_*$, and set

$$\begin{aligned} \Delta &:= X^{t,\xi+\eta} - X^{t,\xi}, \\ \Delta^\xi &:= X^{t,x,\xi+\eta} - X^{t,x,\xi}, \\ \Delta^{x,\xi} &:= \mathrm{D}_x X^{t,x,\xi+\eta} y - \mathrm{D}_x X^{t,x,\xi} y. \end{aligned}$$

From Lemmas 3, 4 and 12, we obtain

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{t \le w \le T} \|\Delta_w\|^2\right] &\lesssim \|\eta\|^2_{\mathrm{L}^2_*}, \\ \hat{\mathbb{E}}\left[\sup_{t \le w \le T} \left\|\Delta^\xi_w\right\|^6\right] &\lesssim \|\eta\|^6_{\mathrm{L}^2_*}, \\ \hat{\mathbb{E}}\left[\sup_{t \le w \le T} \left\|\Delta^{x,\xi}_w\right\|^3\right] &\lesssim \|y\|^3 \|\eta\|^3_{\mathrm{L}^2_*}. \end{aligned}$$

Moreover, Lemma 19 yields

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{t \le w \le T} \left\|\Delta^\xi_w - \mathrm{D}_\xi X^{t,x,\xi}_w \eta\right\|^3\right] &\le \int_0^1 \hat{\mathbb{E}}\left[\sup_{t \le w \le T} \left\| \mathrm{D}_\xi X^{t,x,\xi+\lambda\eta}_w \eta - \mathrm{D}_\xi X^{t,x,\xi}_w \eta\right\|^3\right] d\lambda \\ &\lesssim \|\eta\|^6_{\mathrm{L}^2_*}, \end{aligned}$$

and we have

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{t \le w \le T} \left\|\Delta_w - \mathrm{D}_\xi X^{t,\xi}_w \eta\right\|\right] &\le \int_0^1 \left\| \mathrm{D}_\xi X^{t,\xi+\lambda\eta} \eta - \mathrm{D}_\xi X^{t,\xi} \eta\right\|_{\mathrm{H}^1_*} d\lambda \\ &\le \int_0^1 \left\| \mathrm{D}_x X^{t,\xi+\lambda\eta,\xi+\lambda\eta} \eta - \mathrm{D}_x X^{t,\xi,\xi} \eta\right\|_{\mathrm{H}^1_*} d\lambda \\ &\quad + \int_0^1 \left\| Y^{t,\xi+\lambda\eta,\eta} - Y^{t,\xi,\eta}\right\|_{\mathrm{H}^1_*} d\lambda \\ &\lesssim \|\eta\|^2_{\mathrm{L}^2_*} \end{aligned}$$

due to Corollaries 3 and 4.

By Lemma A4, we have for all $t \le s \le T$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\le w\le s}\left\|\Delta_w^{x,\xi} - D_w^{t,x,\xi,y,\eta}\right\|^2\right] \\
&\lesssim \sum_{f\in F}\int_t^s \hat{\mathbb{E}}\Big[\Big|\mathrm{D}_x f\Big(u, X_u^{t,x,\xi+\eta}, X_u^{t,\xi+\eta}\Big)\,\mathrm{D}_x X_u^{t,x,\xi+\eta} y - \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\,\mathrm{D}_x X_u^{t,x,\xi} y \\
&\quad - \mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big) D_u^{t,x,\xi,y,\eta} - \mathrm{D}_x^2 f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\Big(\mathrm{D}_\xi X_u^{t,x,\xi}\eta, \mathrm{D}_x X_u^{t,x,\xi} y\Big) \\
&\quad - \mathrm{D}_\xi\,\mathrm{D}_x f\Big(u, X_u^{t,x,\xi}, X_u^{t,\xi}\Big)\,\mathrm{D}_x X_u^{t,x,\xi} y\;\mathrm{D}_\xi X_u^{t,\xi}\eta\Big|^2\Big]\,du \\
&\lesssim \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|\Delta_u^\xi\right\|^4\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\right] du \\
&\quad + \int_t^s \alpha_2(u)^2\|\Delta_u\|_{\mathrm{L}_*^2}^2\left(\|\Delta_u\|_{\mathrm{L}_*^2}^2\hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\right] + \hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\left\|\Delta_u^\xi\right\|^2\right]\right) du \\
&\quad + \int_t^s \alpha_1(u)^2\hat{\mathbb{E}}\left[\left\|\Delta_u^\xi - \mathrm{D}_\xi X_u^{t,x,\xi}\eta\right\|^2\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\right] du \\
&\quad + \int_t^s \alpha_1(u)^2\left\|\Delta_u - \mathrm{D}_\xi X_u^{t,\xi}\eta\right\|_{\mathrm{L}_*^1}^2\hat{\mathbb{E}}\left[\left\|\mathrm{D}_x X_u^{t,x,\xi} y\right\|^2\right] du \\
&\quad + \int_t^s \alpha_1(u)^2\left(\hat{\mathbb{E}}\left[\left\|\Delta_u^{x,\xi}\right\|^2\left\|\Delta_u^\xi\right\|^2\right] + \|\Delta_u\|_{\mathrm{L}_*^2}^2\hat{\mathbb{E}}\left[\left\|\Delta_u^{x,\xi}\right\|^2\right]\right) du \\
&\quad + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta_u^{x,\xi} - D_u^{t,x,\xi,y,\eta}\right\|^2\right] du \\
&\lesssim \|\eta\|_{\mathrm{L}_*^2}^4\|y\|^2 + \int_t^s \alpha_0(u)^2\hat{\mathbb{E}}\left[\left\|\Delta_u^{x,\xi} - D_u^{t,x,\xi,y,\eta}\right\|^2\right] du.
\end{aligned}
$$

Finally, Grönwall's inequality yields the desired result. □

6. Application to Functions of Sublinear Distributions

In [30], the authors consider mean-field G-SDEs with coefficients that depend on the sublinear distribution of the solution process, where the sublinear distribution of a random variable ξ is defined as the mapping $\varphi \mapsto \hat{\mathbb{E}}[\varphi(\xi)]$. More precisely, they introduce the set $\mathcal{D}$ consisting of all functionals $F:\,\mathrm{Lip}(\mathbb{R}^d) \to \mathbb{R}$ which satisfy the following properties. Here, $\mathrm{Lip}(\mathbb{R}^d)$ denotes the space of all Lipschitz functions $\varphi:\,\mathbb{R}^n \to \mathbb{R}$ and $\mathrm{Lip}_1(\mathbb{R}^d) \subseteq \mathrm{Lip}(\mathbb{R}^d)$ the subspace of functions with Lipschitz constant smaller than or equal to 1.

1. Constant-Preservation: For all $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$ with $\varphi \equiv c \in \mathbb{R}$, we have $F(\varphi) = c$.
2. Monotonicity: For all $\varphi, \psi \in \mathrm{Lip}(\mathbb{R}^d)$ with $\varphi \ge \psi$ everywhere, we have $F(\varphi) \ge F(\psi)$.
3. Positive Homogeneity: For all $c \ge 0$ and $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, we have $F(c\varphi) = cF(\varphi)$.
4. Subadditivity: For all $\varphi, \psi \in \mathrm{Lip}(\mathbb{R}^d)$, we have $F(\varphi + \psi) \le F(\varphi) + F(\psi)$.
5. Boundedness: We have

$$\sup_{\varphi\in\mathrm{Lip}_1(\mathbb{R}^d)} |F(\varphi) - \varphi(0)| < \infty.$$

Further, the authors define the metric

$$d:\,\mathcal{D}\times\mathcal{D} \to \mathbb{R}, \qquad (F,G) \mapsto d(F,G) := \sup_{\varphi\in\mathrm{Lip}_1(\mathbb{R}^d)} |F(\varphi) - G(\varphi)|$$

and consider a G-SDE of the form

$$dX_t = b(t, X_t, F_{X_t})\, dt + h(t, X_t, F_{X_t})\, d\langle B\rangle_t + g(t, X_t, F_{X_t})\, dB_t, \qquad 0 \le t \le T,$$
$$X_0 = x, \tag{33}$$

where $x \in \mathbb{R}^d$ and the coefficients b, g and h are defined on $[0,T] \times \mathbb{R}^d \times \mathcal{D}$ and, for $\xi \in \mathrm{L}_*^{1,d}$, the functional $F_\xi : \mathrm{Lip}(\mathbb{R}^d) \to \mathbb{R}$ is defined by $\varphi \mapsto \hat{\mathbb{E}}[\varphi(\xi)]$. Clearly, for any X that satisfies (33), we have $X \in \mathrm{H}_*^{1,d}(t,T)$ and, in particular, $F_{X_t} \in \mathcal{D}$ for all $0 \le t \le T$; see also Remark 3.2 in [30].

The authors show that (33) admits a unique solution $X \in \mathrm{M}_*^{2,d}(0,T)$ for any initial value $x \in \mathbb{R}^d$ when the coefficients satisfy the following assumption; see Theorem 4.1 in [30].

Assumption 5. *Let $b : [0,T] \times \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^d$, $h : [0,T] \times \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^{d\times n\times n}$, and $g : [0,T] \times \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^{d\times n}$ be such that the following holds for all components $f = \tilde{b}_k, h_{kij}, \tilde{g}_{ki}$, $1 \le i,j \le n$, $1 \le k \le d$.*

1. *We have $f(\cdot, x, F) \in \mathrm{M}_*^2(0,T)$ for all $x \in \mathbb{R}^d$ and $F \in \mathcal{D}$.*
2. *There exists a constant $K > 0$ such that*

$$|f(t,x,F) - f(t,y,G)| \le K\Big(\|x-y\|^2 + d(F,G)\Big).$$

We can embed the formulation from [30] into our setting by defining coefficients $\hat{b}$, $\hat{g}$ and $\hat{h}$ on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathrm{L}_*^{2,d}$ componentwise by

$$\hat{b}_k(s,\omega,x,\xi) := b_k(s,x,F_\xi), \quad \hat{h}_{kij}(s,\omega,x,\xi) := h_{kij}(s,x,F_\xi), \quad \hat{g}_{ki}(s,\omega,x,\xi) := g_{ki}(s,x,F_\xi).$$

Note that in contrast to the general formulation in [31], the coefficients $\hat{b}$, $\hat{h}$ and $\hat{g}$ are deterministic. Moreover, for the components $\hat{f} = \hat{b}_k, \hat{h}_{kij}, \hat{g}_{ki}$, $1 \le i,j \le n$, $1 \le k \le d$, Assumption 5 yields

$$\begin{aligned}\Big|\hat{f}(t,\omega,x,\xi) - \hat{f}(t,\omega,y,\eta)\Big| &\le K(\|x-y\| + d(F_\xi, F_\eta)) \\ &\le K\Big(\|x-y\| + \|\xi-\eta\|_{\mathrm{L}_*^2}\Big)\end{aligned}$$

for all $\omega \in \Omega$, $0 \le s \le T$, $x,y \in \mathbb{R}^d$ and $\xi,\eta \in \mathrm{L}_*^{2,d}$ since

$$d(F_\xi, F_\eta)) = \sup_{\varphi \in \mathrm{Lip}_1(\mathbb{R}^d)} \Big|\hat{\mathbb{E}}[\varphi(\xi)] - \hat{\mathbb{E}}[\varphi(\eta)]\Big| \le \hat{\mathbb{E}}[\|\xi-\eta\|] = \|\xi-\eta\|_{\mathrm{L}_*^1} \le \|\xi-\eta\|_{\mathrm{L}_*^2}.$$

Further, we have $\hat{f}(\cdot,x,\xi)\mathbf{1}_{[s,T]} \in \mathrm{M}_*^2(t,T)$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathrm{B}_b^d(\mathcal{F}_s)$, $0 \le s \le T$. That is, if the coefficients b, h and g satisfy Assumption 5, then the coefficients $\hat{b}$, $\hat{h}$ and $\hat{g}$ satisfy Assumption 2. In particular, Theorem 3.12 in [31] implies Theorem 4.1 in [30].

The aim of this section is to show how our regularity results from Sections 4 and 5 can be applied to equations of type (33). Note that $\mathcal{D}$ is not a vector space and, thus, we need to consider a different notion of differentiability for functions defined on $\mathcal{D}$. In classical mean-field theory, we encounter a similar issue when considering functions defined on the space of square-integrable distributions $\mathfrak{P}_2(\mathbb{R}^d)$. By lifting a function $f : \mathfrak{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to a function $\hat{f} : L^2(\mathbb{R}^d, \Omega, \mathcal{F}, P) \to \mathbb{R}$ and considering the Fréchet derivative of the lifted function $\hat{f}$, Lions developed a useful notion of derivative which is commonly referred to as Lions derivative; see, e.g., [33] for more details. In the same manner, we might want to lift a function $f : \mathcal{D} \to \mathbb{R}$ to a function $\hat{f} : \mathrm{L}_*^{2,d} \to \mathbb{R}$ such that $\hat{f}(\xi) = f(F_\xi)$ for all $\xi \in \mathrm{L}_*^{2,d}$, but it is not immediately clear whether the space $\mathrm{L}_*^{2,d}$ is rich enough in the sense that

$$\mathcal{D} = \left\{ F_\xi : \mathrm{Lip}(\mathbb{R}^d) \to \mathbb{R},\ \varphi \mapsto \hat{\mathbb{E}}[\varphi(\xi)] \ :\ \xi \in \mathrm{L}_*^{2,d} \right\} =: \mathcal{D}_0.$$

However, it is sufficient to consider the restriction of the coefficients b, h and g in (33) to $[0,T] \times \mathbb{R}^d \times \mathcal{D}_0$ so that $\hat{b}$, $\hat{h}$ and $\hat{g}$ are the respective liftings defined on $[0,T] \times \mathbb{R}^d \times \mathrm{L}_*^{2,d}$ so that we can define a notion of differentiability for b, h and g in terms of the Gateaux or Fréchet derivatives of $\hat{b}$, $\hat{h}$ and $\hat{g}$, respectively.

In the following, we develop a notion of differentiability for a map $f : \mathcal{D}_0 \to \mathbb{R}$ in terms of the Gateaux derivative of its lifting $\hat{f}$. More specifically, for $F_\xi \in \mathcal{D}_0$, we define the map

$$\partial f(F_\xi) : \mathbb{R}^d \to \mathbb{R}, \qquad\qquad x \mapsto \partial \hat{f}(\xi; x), \tag{34}$$

where $\partial \hat{f}(\xi; x)$ denotes the Gateaux derivative of $\hat{f}$ at $\xi \in \mathrm{L}_*^{2,d}$ in the direction $x \in \mathbb{R}^d$. In particular, the construction implies that $\partial f(F_\xi)$ is well defined when the lifting $\hat{f}$ is Fréchet differentiable at $\xi \in \mathrm{L}_*^{2,d}$ since Gateaux differentiability is weaker than Fréchet differentiability. Moreover, the definition ensures that $\partial f(F_\xi)$ is such that $\partial f(F_\xi) = \partial f(F_\eta)$ for all $\xi, \eta \in \mathrm{L}_*^{2,d}$ with $F_\xi = F_\eta$.

Lemma 27. *Let $f : \mathcal{D}_0 \to \mathbb{R}$ be such that its lifting $\hat{f} : \mathrm{L}_*^{2,d} \to \mathbb{R}$ is Gateaux differentiable at $\xi \in \mathrm{L}_*^{2,d}$. If $\eta \in \mathrm{L}_*^{2,d}$ is such that $F_\xi = F_\eta$, then $\hat{f}$ is Gateaux differentiable at η and*

$$\partial \hat{f}(\xi; \zeta) = \partial \hat{f}(\eta; \zeta)$$

for all $\zeta \in \mathrm{L}_*^{2,d}$ such that ξ and η are independent of ζ, where $\partial \hat{f}(\xi; \zeta)$ denotes the Gateaux derivative of $\hat{f}$ at ξ in the direction ζ.

Proof. Since $F_\xi = F_\eta$, we have

$$\hat{\mathbb{E}}[\varphi(\xi)] = \hat{\mathbb{E}}[\varphi(\eta)]$$

for all $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$. Let $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, then $y \mapsto \varphi(y + x)$ is Lipschitz for all $x \in \mathbb{R}^d$. Since ξ and η are independent of ζ, we have

$$\hat{\mathbb{E}}[\varphi(\xi + \lambda\zeta)] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(\xi + x)] \Big|_{x = \lambda\zeta} \right] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(\eta + x)] \Big|_{x = \lambda\zeta} \right] = \hat{\mathbb{E}}[\varphi(\eta + \lambda\zeta)].$$

Since this holds for all $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, we obtain $F_{\xi + \lambda\zeta} = F_{\eta + \lambda\zeta}$ for all $\lambda > 0$. By the Gateaux differentiability of $\hat{f}$, we have

$$0 = \lim_{\lambda \to 0} \frac{\left| \hat{f}(\xi + \lambda\zeta) - \hat{f}(\xi) - \lambda \partial \hat{f}(\xi; \zeta) \right|}{\lambda} = \lim_{\lambda \to 0} \frac{\left| \hat{f}(\eta + \lambda\zeta) - \hat{f}(\eta) - \lambda \partial \hat{f}(\xi; \zeta) \right|}{\lambda}.$$

Thus, $\hat{f}$ is Gateaux differentiable at η and we conclude $\partial \hat{f}(\xi; \zeta) = \partial \hat{f}(\eta; \zeta)$ from the uniqueness of the Gateaux derivative. □

Clearly, the identity $\partial \hat{f}(\xi; \cdot) = \partial \hat{f}(\eta; \cdot)$ on $\mathrm{L}_*^{2,d}$ implies the identity on $\mathbb{R}^d$. Hence, Lemma 27, immediately yields the following corollary.

Corollary 5. *Let $f : \mathcal{D}_0 \to \mathbb{R}$ be such that its lifting $\hat{f} : \mathrm{L}_*^{2,d} \to \mathbb{R}$ is Gateaux differentiable at $\xi \in \mathrm{L}_*^{2,d}$ in the direction x for all $x \in \mathbb{R}^d$. If $\eta \in \mathrm{L}_*^{2,d}$ is such that $F_\xi = F_\eta$, then $\hat{f}$ is Gateaux differentiable at η and*

$$\partial \hat{f}(\xi; x) = \partial \hat{f}(\eta; x)$$

for all $x \in \mathbb{R}^d$.

Note that $\partial f(F_\xi) : x \mapsto \partial \hat{f}(\xi; x)$ corresponds to the restriction of the Gateaux differential to $\mathbb{R}^d$ and, thus, we can apply $F \in \mathcal{D}$ to $\partial f(F_\xi)$ if the Gateaux differential is Lipschitz.

Definition 2. *Let $f : \mathcal{D}_0 \to \mathbb{R}$. We say that f is differentiable if its lifting $\hat{f}$ is Gateaux differentiable at ξ in the direction x for all $x \in \mathbb{R}^d$ and the Gateaux differential $x \mapsto \partial \hat{f}(\xi; x)$ is Lipschitz on $\mathbb{R}^d$ for any $\xi \in \mathrm{L}_*^{2,d}$. The derivative $\partial f : \mathcal{D}_0 \times \mathcal{D}_0 \to \mathbb{R}$ is given by*

$$\partial f(F_\xi, F_\eta) := F_\eta(x \mapsto \partial \hat{f}(\xi; x)) = \hat{\mathbb{E}}\left[\partial \hat{f}(\xi; x)\Big|_{x=\eta} \right].$$

By Corollary 5, we have $\partial f(F_\xi, \cdot) = \partial f(F_\eta, \cdot)$ for all $\xi, \eta \in \mathrm{L}_*^{2,d}$ with $F_\xi = F_\eta$ if the lifting $\hat{f}$ is Gateaux differentiable. Moreover, we obtain $\partial f(F_\xi, F_\eta) = \partial f(F_\xi, F_\zeta)$ for all $\xi, \eta, \zeta \in \mathrm{L}_*^{2,d}$ with $F_\eta = F_\zeta$ if the Gateaux differential is Lipschitz. In particular, the derivative $\partial f : \mathcal{D}_0 \times \mathcal{D}_0 \to \mathbb{R}$ is well-defined if the lifting $\hat{f}$ is Fréchet differentiable. Hence, if the coefficients in (33) are sufficiently differentiable in the sense of Definition 2, we can apply the results from Sections 4 and 5 to obtain the first- and second-order variation process of the unique solution of (33).

7. Conclusions

In Sections 4 and 5, we derive G-SDEs for the first- and second-order Fréchet derivatives of the unique solution $X^{t,\xi}$, $X^{t,x,\xi}$ of the generalized mean-field G-SDEs (11), (12). These G-SDEs are analogous to the SDEs of the Fréchet derivatives for classical mean-field processes; see [28]. The main difference is that the G-SDEs are expressed in terms of the Fréchet derivatives of the coefficients, while the respective SDEs are expressed in terms of the Lions derivatives of the coefficients. However, the Lions derivative of a function f is precisely the Fréchet derivative of its lifting $\hat{f}$; see [33]. In that sense, our results are perfectly in line with the results on classical mean-field SDEs in the literature. This is expected since by choosing $\Sigma = \{\mathrm{Id}\}$, the G-Brownian motion becomes a standard Brownian motion and our setting is reduced to the classical setting with linear expectations. That is, the classical mean-field SDEs can be embedded into our setting and, thus, our results immediately yield SDEs describing the first- and second-order Fréchet derivatives of a classical mean-field process.

We note that the assumptions on the coefficients that we use throughout this paper are more general than the global Lipschitz and boundedness assumptions in [28]. Thus, by restricting our setting to standard Brownian motion, our regularity results from Sections 4 and 5 extend the results for classical mean-field SDEs to more general coefficients. For simplicity, the assumptions in our paper are chosen such that the Grönwall inequality can be applied throughout. In particular, the continuity assumptions can be weakened to consider other forms of local Lipschitz continuity, e.g., suitable for the Bihari inequality. However, we believe that the integrability assumptions on α_0, α_1 cannot be significantly relaxed when following a similar line of argument since these integrability assumptions ensure that products such as $\left\| X_u^{t,x+y,\xi} - X_u^{t,x,\xi} \right\|^4 \left\| \mathrm{D}_x X_u^{t,x,\xi} z \right\|^2$ appearing in the proof of Proposition 5 are integrable.

Future research may study a larger class of generalized mean-field G-SDEs. For instance, it could consider generalized mean-field SDEs with more general coefficients or driven by a (sub)-fractional G-Brownian motion; see [34–37] for more details on (sub)-fractional G-Brownian motion. Moreover, future research could explore the application of the obtained first- and second-order Fréchet derivatives for control and optimization problems, numerical approximation schemes or gradient methods.

Conditional Sublinear Expectation please see Appendix A.

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Appendix A. Conditional Sublinear Expectation

Lemma A1. *Let* $0 \leq t \leq T$ *and* $X \in \mathrm{M}_b(0,T)$*. Then*

$$\hat{\mathbb{E}}\left[\int_t^T X_s\, ds \,\middle|\, \mathcal{F}_t\right] \leq \int_t^T \hat{\mathbb{E}}[X_s \mid \mathcal{F}_t]\, ds.$$

Proof. Since $X \in \mathrm{M}_b(0,T)$, there exist $m \in \mathbb{N}$, $t = t_0 < \ldots < t_m = T$, and $\xi_k \in \mathrm{B}_b(\mathcal{F}_{t_k})$, $0 \leq k \leq m-1$ such that

$$X\mathbf{1}[t,T] = \sum_{k=0}^{m-1} \xi_k \mathbf{1}[t_k, t_{k+1}),$$

and

$$\int_t^T X_s\, ds = \sum_{k=0}^{m-1} \xi_k (t_{k+1} - t_k).$$

Due to the sublinearity of the conditional expectation, we obtain

$$\begin{aligned}
\hat{\mathbb{E}}\left[\int_t^T X_s\, ds \,\middle|\, \mathcal{F}_t\right] &= \hat{\mathbb{E}}\left[\sum_{k=0}^{m-1} \xi_k (t_{k+1} - t_k) \,\middle|\, \mathcal{F}_t\right] \\
&\leq \sum_{k=0}^{m-1} \hat{\mathbb{E}}[\xi_k \mid \mathcal{F}_t](t_{k+1} - t_k) \\
&= \int_t^T \hat{\mathbb{E}}[X_s \mid \mathcal{F}_t]\, ds.
\end{aligned}$$

□

Corollary A1. *Let* $p \geq 1$, $0 \leq t \leq T$ *and* $X \in \mathrm{M}_*^p(0,T)$*. Then*

$$\hat{\mathbb{E}}\left[\left|\int_t^T X_s\, ds\right|^p \,\middle|\, \mathcal{F}_t\right] \leq (T-t)^{p-1} \int_t^T \hat{\mathbb{E}}\left[|X_s|^p \mid \mathcal{F}_t\right] ds.$$

Proof. This follows immediately from the construction of $\mathrm{M}_*^p(0,T)$ and Jensen's inequality. □

Lemma A2. *Let* $a \in \mathbb{R}^n$, $p \geq 1$, $0 \leq t \leq T$ *and* $X \in \mathrm{M}_*^p(0,T)$*. Then*

$$\hat{\mathbb{E}}\left[\left|\int_t^T X_s\, d\langle B^a \rangle_s\right|^p \,\middle|\, \mathcal{F}_t\right] \leq (T-t)^{p-1} \overline{\sigma}_{aa}^{2p} \int_t^T \hat{\mathbb{E}}\left[|X_s|^p \mid \mathcal{F}_t\right] ds.$$

Proof. By Corollary 3.5.5 in [12], we have

$$\left|\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right| \le \overline{\sigma}_{aa}^2(t_{k+1}-t_k).$$

Thus, Jensen's inequality yields

$$\begin{aligned}\left|\int_t^T X_s\, d\langle B^a\rangle_s\right|^p &\le \left|\int_t^T |X_s|\overline{\sigma}_{aa}^2\, ds\right|^p \\ &\le (T-t)^{p-1}\overline{\sigma}_{aa}^{2p}\int_t^T |X_s|^p\, ds.\end{aligned}$$

Finally, Corollary A1 yields the desired result. □

Lemma A3. *Let $a \in \mathbb{R}^n$, $p \ge 2$, $0 \le t \le T$ and $X \in \mathrm{M}_*^p(0,T)$. Then*

$$\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left|\int_t^w X_s\, dB_s^a\right|^p \,\middle|\, \mathcal{F}_t\right] \le (T-t)^{\frac{p-2}{2}}\overline{\sigma}_{aa}^p\int_t^T \hat{\mathbb{E}}\left[|X_s|^p \mid \mathcal{F}_t\right] ds.$$

Proof. The Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\le w\le T}\left|\int_t^w X_s\, dB_s^a\right|^p \,\middle|\, \mathcal{F}_t\right] &\le C_p\hat{\mathbb{E}}\left[\left|\int_t^T X_s^2\, d\langle B^a\rangle_s\right|^{\frac{p}{2}} \,\middle|\, \mathcal{F}_t\right] \\ &\le C_p(T-t)^{\frac{p-2}{2}}\overline{\sigma}_{aa}^p\int_t^T \hat{\mathbb{E}}\left[|X_s|^p \mid \mathcal{F}_t\right] ds,\end{aligned}$$

where the last step follows from Lemma A2. □

Lemma A4. *Let $p \ge 2$, $0 \le t \le T$, $\xi \in \mathrm{L}_*^{p,d}(t)$ and $b_k, h_{kij}, g_{ki} \in \mathrm{M}_*^p(0,T)$ for $1 \le k \le d$, $1 \le i,j \le n$. Let X satisfy*

$$\begin{aligned}dX_s &= b(s)\, ds + h(s)\, d\langle B\rangle_s + g(s)\, dB_s, \qquad t \le s \le T \\ X_t &= \xi.\end{aligned}$$

Then

$$\begin{aligned}&\hat{\mathbb{E}}\left[\sup_{t\le s\le w}\|X_s\|^p \,\middle|\, \mathcal{F}_t\right] \\ &\lesssim \|\xi\|^p + \sum_{k=1}^d\sum_{i,j=1}^n\int_t^w \hat{\mathbb{E}}\left[|b_k(s)|^p \mid \mathcal{F}_t\right] + \hat{\mathbb{E}}\left[\left|h_{kij}(s)\right|^p \mid \mathcal{F}_t\right] + \hat{\mathbb{E}}\left[|g_{ki}(s)|^p \mid \mathcal{F}_t\right] ds.\end{aligned}$$

Proof. This follows from Corollary A1 and Lemmas A2 and A3. □

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