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ORIGINAL ARTICLE

# Frege's principle of logical parsimony, the indispensability of "$\xi = \zeta$" in *Grundgesetze*, and the nature of identity

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**Abstract**
In Section 2, I analyze Frege's principle of logical and notational parsimony in his opus magnum *Grundgesetze der Arithmetik* (vol I, 1893, vol. II, 1903). I argue *inter alia* that in order to carry out the proofs of the more important theorems of cardinal arithmetic and real analysis in *Grundgesetze* Frege's identification of the truth-values the True and the False with their unit classes in *Grundgesetze* I, §10 need not be raised to the lofty status of an axiom. Frege refrains from doing this but does not provide any reason for his restraint. In Section 3, I argue that he considered the primitive function-name "$\xi = \zeta$" indispensable in pursuit of his logicist project. I close with remarks on the nature of identity. I suggest that there is no need to interpret identity in a non-standard fashion in order to render it logically palatable and scientifically respectable.

**Keywords** Axioms · Initial stipulation · Basic Law III · Basic Law V · Logical abstraction · Value-range names · Transsortal identification · Logical object

## 1 Introductory remarks

If we canvass Frege's treatment of logical equations in his opus magnum *Grundgesetze der Arithmetik* (vol. I 1893, vol. II 1903), we are forced to conclude that he considers the function-name "$\xi = \zeta$" indispensable in pursuit of his project of laying the logical foundations of cardinal arithmetic and real analysis. Regarding the requirement of logical and notational parsimony, Frege's view is similar to Wittgenstein's in the *Tractatus-logico-philosophicus* (first published in 1918), but as we shall see, it is less radical and is apparently guided by what might be called "Frege's principle of proportionality".

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In the *Tractatus*, logical and notational parsimony is one of the main objectives that Wittgenstein attempts to achieve in his projected new concept-script. The new concept-script is intended to exceed Frege and Russell's concept-scripts in accuracy and overall correctness. The paradigmatic case in pursuit of the objective of notational parsimony is Wittgenstein's dispensation with "$=$". In the envisaged concept-script, he expresses identity of the object by identity of the sign and difference of objects by difference of the signs (see Wittgenstein, 1961, 5.53). This means that no object of the first-order domain has more than one name and that, roughly speaking, different object variables take different values. Thus, coreferential names which could be substituted one for the other in a sentence without affecting its truth-value are no longer available in a "correct" concept-script, and the free and bound variables are not treated in exactly the same way as in standard first-order logic. Despite the dispensation with "$=$", Wittgenstein adheres to identity nonetheless, but interprets it in a non-standard way, and is compelled to do so. He probably sticks to identity to ensure that the expressive power of an identity-sign free concept-script of first-order is on a par with standard first-order logic containing "$=$".[1] Note that Wittgenstein's goal of strict notational parsimony does not only concern the issue of how identity could be adequately expressed in a correct concept-script without using "$=$", but other issues of logic as well. Towards the end of Section 3, I shall briefly return to Wittgenstein's elimination of "$=$" in the concept-script that he envisages in the *Tractatus*.

My plan in this essay is as follows. In Section 2, I discuss Frege's principle of logical and notational parsimony with special emphasis on the choice of the axioms of the formal system in *Grundgesetze*. I presume that following this principle and considering aspects inherent in his logical system, Frege refrains from including three axioms that at first glance could seem to suggest themselves for additional inclusion. One of the potentially additional axioms that I have in mind would complement Basic Law VI which is: VI $\vdash a = \backslash\dot{\varepsilon}(a = \varepsilon)$. Basic Law VI governs the primitive description operator "$\backslash\xi$" only incompletely since it leaves the second clause of the elucidation of that operator out of account (cf. *Grundgesetze* I, §11). The law applies only to a concept-script name in which the description operator is applied to the name of a unit class: $\backslash\dot{\varepsilon}(a = \varepsilon)$. It does not convey anything about the case in which "$\backslash\xi$" is applied to a name that does not refer to a unit class. Frege does not mention, let alone explain the *specific* reasons for his limited choice of the axioms of *Grundgesetze*. In what follows, I shall confine myself to considering only one of three potential candidates for additional inclusion in the set of axioms. In *Grundgesetze* I, §10, Frege identifies the True and the False with their unit classes with the aim of fixing the references of value-range names (almost) completely. I argue that in order to carry out the proofs of the fundamental theorems of arithmetic the transsortal identifications in §10 need not be raised to the distinguished status of an axiom and in the light of Frege's conception of logical axioms cannot be so raised. In Section 3, I argue that in pursuing his logicist project, Frege could not

[1] On Wittgenstein's dispensation with "$=$" in the *Tractatus* see Hintikka (1956), Landini (2007), Rogers and Wehmeier (2012), Lampert and Säbel (2021) and Schirn (2024a).

have dispensed with "$\xi = \zeta$." I close the essay with some remarks on the nature of identity and suggest that there is no need to interpret identity in a non-standard way in order to ensure its logical soundness.

## 2 Frege's principle of logical and notational parsimony

When Frege sets about developing his concept-script in *Grundgesetze der Arithmetik*, logical and notational parsimony is the order of the day. Yet he does not implement his goal by hook or by crook. In his letter to Peano of 29 September 1896, Frege compares the number of primitive signs in Peano's and in his own logic and draws the following conclusion (Frege, 1976, p. 185):

> For these reasons [he had mentioned before], I do not believe that your number of primitive signs is really smaller than mine. But I also do not regard the mere counting of primitive signs as an adequate basis for judging how profound the underlying analysis is.

Thus, it seems that Frege does not submit to an iron law which inexorably prescribes the use of the smallest possible number of primitive signs in a concept-script in order to ensure its viability. The logical reducibility of the primitive horizontal function —$\xi$, under which only the True falls, to $\xi = \zeta$ (—$\xi$ and $\xi = (\xi = \xi)$ are coextensive concepts) is a case in point. Due to the reducibility of —$\xi$ to $\xi = \zeta$, —$\xi$ is not a geuine primitive function in Frege's system, although—with the exception of *Grundgesetze* I, §10—he seems to treat "—$\xi$" as a full-fledged primitive function-name.[2] The reducibility does not undermine the legitimacy of the use of "—$\xi$" in numerous places in the exposition of his concept-script and superabundantly in the proofs of the basic laws of cardinal arithmetic and real analysis. This holds quite independently of the fact that from the perspective of the operability of the two-dimensional concept-script the function-name "—$\xi$" is in fact indispensable. In particular, it cannot be replaced with "$\xi = (\xi = \xi)$". Plainly, without "—$\xi$" Frege's concept-script would truly be up in the air, that is, there would be no name of negation and of the conditional function without "—$\xi$" and almost no proof step without the conditional. Due to the horizontals attached to the vertical conditional stroke, the conditional function always maps truth-values onto truth-values. Moreover, there would be no prefixing of the judgement stroke to any formula without combining it with the horizontal and, hence, no concept-script sentence without the use of "—$\xi$," unless Frege threw the design of his concept-script into disarray.[3] All this does not mean that "—$\xi$" is indispensable from a logical point of view. Yet the fact that it was

[2] In a yet unpublished article entitled 'Frege's first-order logic in *Grundgesetze* is non-classical', Bruno Bentzen explains why Frege's reduction of —$\xi$ to $\xi = \zeta$ in *Grundgesetze* I, §10 fails in the formalism obtained from the first-order fragment of Frege's axiomatization without value-ranges.

[3] Prefixed to a truth-value name "— $\Delta$", the judgement-stroke "|" is a sign *sui generis* and semantically empty, endowed neither with a reference nor with a sense. Not so with the name "—$\xi$". It refers to a concept under which only the True falls and which could be read as "the truth-value of $\xi$'s being the True".

used in Frege's system does not offend against any notational prescription and, in particular, does not deserve to be sanctioned before Wittgenstein's tribunal of notational minimalism or "correctness."[4] What I just said may be taken into consideration if one contrasts Wittgenstein's dispensation of "=" with Frege's insistence on the indispensability of "=" in his project of laying the logical foundations of cardinal arithmetic and real analysis.

In *Grundgesetze*, Frege confines himself to laying down only those maximally general truths as axioms which he considers to be requisite for effectively carrying out his logicist project. "One must strive to reduce the number of these fundamental laws as far as possible…" (Frege, 1893, p. VI).[5] This does not rule out, however, that if a case of logical urgency had arisen in the course of carrying out his foundational project, he might have decided to install a truth that he considered to be endowed with utmost generality as an additional axiom even at a later stage of the project, if his logical system had not turned out to be inconsistent and if he had been encouraged to put his plan of a logical foundation of complex analysis in addition to the logical foundation of cardinal arithmetic and real analysis into action. There is not a trace of evidence that Frege considered his formal system to be static and non-extendable once he had laid it out by fixing the compositional semantics, introducing the axioms and inference rules, setting out the syntax, proving that the concept-script is a fully referential language, and finally by putting forward the definitions which he considered relevant for carrying out the logicist project.

A few words on the semantics of the formal language of *Grundgesetze* are in order in the context under consideration. Laying it out starts with standard elucidations of the primitive function-names. The primitive value-range operator "$\dot{\varepsilon}\varphi(\varepsilon)$" constitutes an exception. In *Grundgesetze* I, §3, it is introduced via a stipulation in terms of logical abstraction on which I comment below. The elucidations characteristically fix the references of the primitive names at one fell swoop, supplemented by the piecemeal determination of the reference of the value-range operator in *Grundgesetze* I, §3, §10–§12. The elucidations of the primitive function-names are necessarily non-definitional in character. At this early stage in the exposition of the concept-script, Frege would not be entitled, by his own lights, to frame any definition anyway. Apart from the fact that the syntactic rules are not yet available, certainty had antecedently to be attained that, assuming that the primitive names had in fact been endowed with determinate references, the formation rules preserve reference, that is, guarantee a reference for every name that is correctly formed from the primitive names. Frege prudently does not set up the first definition of his system until he gets to *Grundgesetze* I, §34. In §34, he defines the application operator

[4] Bear in mind that Frege's mindset in *Grundgesetze* is very different from Wittgenstein's in the *Tractatus*.

[5] Wittgenstein (1961, 6.1271) makes an interesting remark with respect to the number of axioms that are laid down in a logical theory like Frege's: "It is clear that the number of the 'basic laws of logic' is arbitrary, since one could derive logic from a single basic law, e.g., by simply forming the logical product of Frege's basic laws. (Frege would perhaps say that this basic law is no longer self-evident. But it is strange that a thinker as rigorous as Frege appealed to the degree of self-evidence as the criterion of a logical sentence.)"

"$\xi \cap \zeta$" by means of the primitive definite description operator "$\backslash\xi$": $\Vdash\!\!-\backslash\dot{\alpha}(\neg\forall\mathfrak{g}(u=\dot{\varepsilon}(\mathfrak{g}(\varepsilon))\rightarrow\neg\mathfrak{g}(a)=\alpha))=a\cap u$. Frege defines "$\xi\cap\zeta$" after having first set out the syntax of the formal language and subsequently carried out the proof of referentiality in §31 which is designed to demonstrate beyond doubt that the primitive names and all names that can be formed from them by iterated application of the formation rules are endowed with unique references (and unique senses).

I turn now to Frege's refusal to convert the dual stipulation that he makes in *Grundgesetze* I, §10 into an axiom. Let us first cast a quick glance at the stipulation that he makes in §3 and which is intimately related to the dual stipulation in §10. The stipulation in §3 is as follows:

> I use the words "the function $\Phi(\xi)$ has the same *value-range* as the function $\Psi(\xi)$" generally as coreferential [*gleichbedeutend*] with the words "the functions $\Phi(\xi)$ and $\Psi(\xi)$ always have the same value for the same argument."

Following Heck 2012, I call I this stipulation the "Initial Stipulation." The Initial Stipulation appears in the guise of a second-order abstraction principle.[6] Its objective is a purely semantic one: fixing *partially* the references of value-range names.[7] Frege begins §10 of *Grundgesetze* by claiming that the "Initial Stipulation" does not fix the reference of a name such as "$\dot{\varepsilon}\Phi(\varepsilon)$" completely. He then provides an argument for this claim: Suppose that $h$ is a one-to-one function of all objects of the first-order domain of Frege's logical system. In this case, the criterion of identity that applies to the object(s) referred to by value-range names, namely the coextensiveness of $\Phi(\xi)$ and $\Psi(\xi)$, also applies to the object(s) referred to by names of the form "$h(\dot{\varepsilon}\Phi(\varepsilon))$". In this case, (1) "$\dot{\varepsilon}\Phi(\varepsilon)=\dot{\alpha}\Psi(\alpha)$" and (2) "$h(\dot{\varepsilon}\Phi(\varepsilon))=h(\dot{\alpha}\Psi(\alpha))$" corefer and since by virtue of the Initial Stipulation (1) corefers with (3) "$\forall\mathfrak{a}(\Phi(\mathfrak{a})=\Psi(\mathfrak{a}))$", (2) likewise corefers with (3).

Frege concludes that the Initial Stipulation fails to fix completely the reference of a name such as "$\dot{\varepsilon}\Phi(\varepsilon)$", at least if there is a bijection $h$ such that for some value-range, say $\dot{\varepsilon}\Phi(\varepsilon)$, $\dot{\varepsilon}\Phi(\varepsilon)=h(\dot{\varepsilon}\Phi(\varepsilon))$" is false. In modern terms, Frege's argument can be presented as follows:

> (A1) Suppose that $\varphi$ is the intended assignment of objects to value-range names satisfying the Initial Stipulation. Let $h$ be a non-trivial bijection of all objects of the first-order domain of Frege's formal system, and consider the assignment $\varphi'$ of objects to value-range names which is related to $\varphi$ as follows: If $\Delta$ is assigned by $\varphi$ to "$\dot{\varepsilon}\Phi(\varepsilon)$" and $\Gamma = h(\Delta)$, then $\Gamma$ is assigned by $\varphi'$ to "$\dot{\varepsilon}\Phi(\varepsilon)$" It follows that $\varphi'$ is an assignment of objects to value-range names distinct from $\varphi$, but such that it satisfies the Initial Stipulation if $\varphi$ does.

[6] A Fregean abstraction principle is of the form: $Q(\alpha)=Q(b)=R_{eq}(\alpha, b)$. "$Q$" is here a singular term-forming operator, $\alpha$ and $b$ are free variables of the appropriate type, ranging over the members of a given domain, and "$R_{eq}$" is the sign for an equivalence relation holding between the values of $\alpha$ and $b$.

[7] In *Grundgesetze* I, §20, Frege obtains Basic Law V: $\vdash\!\!-(\dot{\varepsilon}f(\varepsilon)=\dot{\alpha}g(\alpha))=(\forall\mathrm{a}(f(\mathrm{a})=g(\mathrm{a})))$ by transforming the Initial Stipulation into the formal, assertoric mode of a concept-script sentence. In Schirn (2024c), I analyze in detail the relation between the Initial Stipulation and Basic Law V.

Frege goes on to suggest that the indeterminacy can be resolved "by determining for every function, when introducing it, which value it receives for value-ranges as arguments, just as for all other arguments." The proposed procedure eventually boils down to the question of whether the True or the False is a value-range. The question cannot possibly be decided by appeal to the Initial Stipulation. In order to find a way out of this impasse, Frege presents a variant (A2) of (A1). This variant is usually called the *permutation argument*. I formulate (A2) again in modern terms.

(A2) As in (A1), let $\varphi$ be an assignment of objects to value-range names satisfying the Initial Stipulation (or Basic Law V). Let $f(\xi)$ and $g(\xi)$ be two extensionally non-equivalent functions. Let $\varphi$ assign $a$ to "$\dot{\varepsilon}\Phi(\varepsilon)$" and $b$ to "$\dot{\alpha}\Psi(\alpha)$." Let $h$ be a function such that

(i) $h(a)$ is the True,
(ii) $h$(the True) is $a$,
(iii) $h(b)$ is the False,
(iv) $h$(the False) is $b$, and,
(v) for every argument $x$ distinct from these, $h(x) = x$.

Finally, let $\varphi'$ be an assignment of objects to value-range names related to $\varphi$ as in (A1), with respect to the special permutation $h$ just specified. Then, as in (A1), $\varphi'$ will satisfy the Initial Stipulation (or Basic Law V).

From the permutation argument Frege derives what—following Schroeder-Heister (1987)—I call the *identifiability thesis*: "Thus, without contradicting our equating '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' with '$\forall(\alpha\Phi(\alpha) = \Psi(\alpha))$', it is always possible to determine that an arbitrary value-range be the True and another arbitrary value-range be the False." By invoking the identifiability thesis, Frege identifies the True with its unit class, namely with $\dot{\varepsilon}(—\varepsilon)$, and the False likewise with its unit class, namely with $\dot{\varepsilon}(\varepsilon = \neg\forall\mathfrak{a}(\mathfrak{a} = \mathfrak{a}))$.

The purpose of the permutation argument is to ground and legitimize the transsortal identifications in §10 (let us also call them "twin stipulations") by showing that they are consistent with the Initial Stipulation, whereas the aim of the twin stipulations is to remove the referential indeterminacy of value-range names as much as possible at the stage of §10 and thus to almost complete the unfinished semantic business left by the Initial Stipulation. In my view, raising the twin stipulations to the status of an axiom (or of two axioms) was, due to the lack of utmost generality and self-evidence of such a potential axiom, probably out of the question for Frege. Maximal generality was for him the most salient feature of a *logical* axiom and thus a key requisite for laying it down in a logical theory. Regarding the requirement of self-evidence, we see that the permutation argument involves several steps (or runs through several stages) and can hardly be credited with self-evidence. Hence, the transsortal identifications that emerge from it are hardly self-evident either.

At the end of §10, Frege signalizes, in accordance with his proposed solution of the problem of the referential indeterminacy of value-range names, that in the next sections he will pursue a dual strategy: (a) further determining the value-ranges and at the same time (b) determining the primitive first-level functions that must still be introduced for the purpose of laying the logical foundations of arithmetic (and are not completely reducible to the primitive first-level functions already known) by stipulating what values those functions have for value-ranges as arguments. Clauses (a) and (b) are only two sides of the same coin.

In the long footnote to §10, Frege dismisses both a restricted and an unbounded generalization of the dual stipulation in §10 as unviable. Unfortunately, he does not explain why he considers those generalizations at all. The reason is perhaps that he takes the first-order domain of his logical system to be all-encompassing. In any event, if the stipulation "$\forall\Delta\forall\dot{\varepsilon}\Phi(\varepsilon)(\dot{\varepsilon}(\varepsilon=\Delta)=\Delta \ \wedge \ \dot{\varepsilon}(\dot{\varepsilon}\Phi(\varepsilon)=\varepsilon)=\dot{\varepsilon}\Phi(\varepsilon))$"[8] (=the unbounded generalization of the twin stipulations in *Grundgesetze* I, §10 that besides the restricted generalization "$\forall\Delta(\dot{\varepsilon}(\varepsilon=\Delta)=\Delta)$" Frege considers in the footnote to §10) were feasible, it would, in contrast to the twin stipulations, meet the constraint of utmost generality imposed on logical axioms. However, Frege rejects the unbounded generalization—it includes the identification of objects with their unit classes which are already given to us as value-ranges—on the grounds that it may be inconsistent with the criterion of identity for value-ranges embodied in the Initial Stipulation.[9]

Independently of the fact that the twin stipulations fail to meet two essential constraints that Frege imposes on logical axioms, he would have had no need to use an axiomatic version of the stipulations in the course of carrying out the proofs in *Grundgesetze*. As a matter of fact, in the informal or semi-formal proof-analyses, which always precede the purely formal proof-constructions, Frege does not refer at all to the twin stipulations. He refers to them only once again, namely in his proof of referentiality in *Grundgesetze* I, §31. We may conclude from this that he would not have relied on an axiomatized version of these stipulations in a formal proof-construction either. The fact that in *Grundgesetz*e I, §10 Frege identifies the truth-values qua primitive objects of logic with logically derivative objects (value-ranges are derived from the logically more fundamental functions) does not mean for him that in the remainder of the exposition of the concept-script (which runs from §1 to §48) standard names of a truth-value disappear from the screen altogether. Frege goes on

8 I use here "$\Delta$" as a variable ranging over singular terms distinct from every canonical value-range name.

9 The number 1 is obviously not given as a value-range by the numeral "1". According to Frege's initial argument in the second footnote to *Grundgesetze* I, §10, 1 could therefore be identified with its unit class. However, since Frege identifies cardinal numbers with equivalence classes of equinumerosity—for example, he defines 1 as the class of all classes equinumerous with $\dot{\varepsilon}(\varepsilon=0)$, that is, as $\dot{\varepsilon}(\varepsilon\backsim\dot{\varepsilon}(\varepsilon=0))$, if, for the sake of simplicity, we use here "$\backsim$" as the sign for the relation of equinumerosity between classes—he could not at the same time identify 1 with $\dot{\varepsilon}(\varepsilon=1)$ (its unit class) without contradicting Basic Law V. For an extension of this argument, see Schirn (2018). Thanks to reviewer #2 for their comments on the question of whether Frege could have converted the dual stipulation in *Grundgesetze* I, §10 into an axiom. They agree with me that Frege could not have transformed the dual stipulation into a logical axiom to govern value-ranges in the formal system in addition to Basic Law V. For a detailed discussion of the dual stipulation in the light of Frege's anti-creationism and mathematical platonism, see Schirn (2024c). I argue *inter alia* that it would not have been mere child's play for Frege to effectively defend the non-creativity of the dual stipulation against the potential charge of creativity of a creationist opponent. I further argue that in order to make sense of Frege's line of argument in *Grundgesetze* I, §10 and appreciate the impact it has, from his point of view, on the complete determination of the references of value-range names, we should not see it through the lens of his platonism.

to refer to the truth-values by using either their original metalinguistic names “the True” and “the False”, for example, when he specifies the value of a concept or a relation for fitting arguments, or by using a concept-script truth-value name.

So much on Frege’s principle of logical and notational parsimony with special emphasis on the choice of the axioms in *Grundgesetze* and his refusal to transform the dual stipulation in §10 into an axiom.[10]

## 3 The indispensability of “$\xi = \zeta$” in Frege’s foundational project in *Grundgesetze*

Setting Russell’s Paradox aside, Frege’s concept-script is a paradigm of precision, lucidity, and expressive richness, and the use of “$\xi = \zeta$” contributes to that in various ways. Without having “$\xi = \zeta$” at his disposal, the logicist project would not even have got off the ground in *Grundgesetze*.

In *Grundgesetze* I, §20, Frege introduces Basic Law III which together with the elucidation of “$\xi = \zeta$” in §7 governs the use of “=” in the formal system. In modern notation, Basic Law III has the following form: $\vdash g(a = b) \rightarrow g(\forall f(f(a) \rightarrow f(b)))$, in Frege’s words: the truth-value $\forall f(f(a) \rightarrow f(b))$ falls under every concept under which the truth-value $a = b$ falls. The reasoning which leads to the formulation of Basic Law III starts by referring back to the elucidation of “$\xi = \zeta$” in §7, and it is in line with the view that identity is a relation in which every object uniquely stands to itself (§4, §7). In *Grundgesetze* I, §50, Frege proves the principal laws of $\xi = \zeta$: IIIa – IIIi (see “Table of the basic laws and sentences immediately following from them” in *Grundgesetze* I, p. 238f.). Law IIIa corresponds to Axiom 52 of *Begriffsschrift.* In modern notation, law IIIa is: $a = b \rightarrow (f(b) \rightarrow f(a))$. Yet as expected, Frege states the content of law IIIa differently from his formulation of Axiom 52 in *Begriffsschrift*, that is, he does not state it in the metalinguistic but in the objectual mode: “If $a$ and $b$ coincide, then everything that holds of $b$ holds of $a$.” If instead of “$\xi = \zeta$” Frege had chosen “$\xi = \zeta$” as a primitive name, he could, despite first appearances, still have formulated Basic Laws III, IV, V, and VI. The formation rules of *Grundgesetze* allow the formation of “$\xi = \zeta$” from “$\xi = \zeta$” just as they allow the formation of “$\xi = \zeta$” from “$\xi = \zeta$.” However, since Frege most likely considered it essential that the axiom-thoughts of his system be expressed by truth-value names which are formed by combining via insertion only *primitive* names, the choice of “$\xi = \zeta$” as a primitive function-name was the right option for him.[11]

[10] For a critical discussion of Landini’s view of this issue in Landini (2022), see Schirn (2024c). Reviewer #2 drew my attention to Frege’s awareness in the Preface to *Grundgesetze* that the introduction of the inference rules of his logical system is for him a matter of weighing up between parsimony and simplicity on the one hand and the achievement of greater proof-theoretic flexibility by appeal to additional rules of inference on the other. Frege writes in *Grundgesetze*, p. VII: “Some might have preferred to increase the circle of permissible modes of inference and consequence, in order to achieve greater flexibility and brevity. However, one has to draw a line somewhere if one approves of my stated ideal at all; and wherever one does so, people could always say: it would have been better to allow even more modes of inference.”

[11] On identity as a logical primitive see, for example, Henkin (1975).

Frege badly needs "$\xi = \zeta$" (or $\xi = \zeta$) for the introduction of logical objects via a logical abstraction principle which yields the requisite criterion of identity for those objects.[12] The criterion of identity for value-ranges embodied in the Initial Stipulation (= the coextensiveness of two monadic first-level functions), although it is essential for the individuation and determination of value-ranges is not exhaustive and therefore its range of application must be expanded by means of additional stipulations. The stipulations are as follows. We have seen that in *Grundgesetze*, I Frege identifies the truth-values with their unit classes. He then elucidates the last two primitive first-level function-names of his formal system, in §11 the name of the description function and in §12 the name of the conditional function. In virtue of this sequence of heterogeneous but interlocking stipulations, he believes to have finally succeeded in uniquely fixing the references of value-range names. The key role of "$\xi = \zeta$" in the stipulations in §3 and §10 is obvious. Thanks to the stipulations in §3, §10–§12 and the elucidations of "$\xi = \zeta$" in §7 and of the first-order universal quantifier in §9, Frege feels entitled to state Basic Law V in §20, even before he sets out the syntax of his formal language in §26 and §30 and carries out his proof referentiality in §31 which, in accordance with the compositional character of his semantics, proceeds by induction on the complexity of concept-script names. In Schirn (2018), there are detailed arguments that Frege's proof of referentiality also essentially rests on the context principle concerning reference. The principle of compositionality regarding reference and the context principle concerning reference work hand in hand in the proof.

At this stage of my investigation, some comments on the terminology that I use are in order. Basic Law V is an equation of the form "$a = b$". The left-hand side is what I term a *canonical value-range equation* and the right-hand side is in Fregean terms a *generalized equation between function-values* or in other words: the universal closure of the coextensiveness of two monadic first-level functions. I call any term that results from the insertion of a monadic first-level function-name into the argument-place of the primitive second-level function-name "$\dot{\varepsilon}\varphi(\varepsilon)$" a *canonical value-range name*. Similarly, I call equations in which the terms flanking "=" are both canonical value-range names *canonical value-range equations*. Finally, I call equations in which one term flanking "=" is a value-range name and the other a truth-value name *mixed equations*. By a truth-value name in the formal language of *Grundgesetze*, I suggest we understand the name of a concept-value or relation-value for fitting arguments, that is, a proper name (in Frege's use of this term) which in the last step of its formation history, beginning with primitive function-names, is always obtained by inserting an admissible argument-expression or admissible argument-expressions into the argument-place(s) of a concept- or relation-expression. In short, a truth-value name is a proper name that has the syntactic structure of a declarative sentence and expresses a thought—not to be conflated with a concept-script sentence (*Begriffsschriftsatz*) "$\vdash\!\!\!-\Delta$" which consists of a truth-value name or a Latin marker of a truth-value with a judgement

[12] On Frege's method of introducing logical objects via logical abstraction see Schirn (1996), (2002a), (2002b), (2003), (2006a), (2006b), (2018), (2019a), (2023a), (2023b), (2024b) and (2024c).

stroke prefixed to the horizontal. Frege does not employ the term "truth-value name" at all—he always uses the phrase "name of a truth-value"—but the introduction of the former while retaining the latter helps to avoid ambiguity in certain contexts, notably in §31 where he assumes that names of a truth-value (what he has in mind are truth-value names) are referential before carrying out the first step in his proof of referentiality. By virtue of Frege's stipulations, the names "$\grave{\varepsilon}(—\varepsilon)$", "$\grave{\varepsilon}(\varepsilon=(\varepsilon=\varepsilon))$", "$\backslash\grave{\varepsilon}(—\varepsilon)$" and "$—\grave{\varepsilon}(\varepsilon=(\varepsilon=\varepsilon))$", for example, are coreferential. They refer to the True, but only the fourth name is a truth-value name and, hence, the only name which expresses a thought. The first three names express non-propositional senses.[13]

Basic Law V embodies an objectual identity on a larger scale, the enclosing asserted truth-value identity (partly in words): the truth-value ($\dot{\varepsilon}f(\varepsilon)=\dot{\alpha}g(\alpha)$) is the same as the truth-value $\forall\mathfrak{a}(\Phi(\mathfrak{a})=\Psi(\mathfrak{a}))$, as well as on a smaller scale, namely the enclosed value-range identity. Assertoric force is conferred on the former, not on the latter. The use of "=" is here obviously indispensable. Anachronistically speaking, Frege could never have joined forces with Wittgenstein regarding the dispensation with "=" since without his logicist key axiom (=Basic Law V) his foundational project would have collapsed like a house of cards. As I already mentioned, Basic Law VI $\vdash\!— a=\backslash\,\dot{\varepsilon}(a=\varepsilon)$ is likewise an equation of the form "$a=b$" but, in contrast to Basic Law V, it is not the case that both terms flanking "=" are truth-value names. Only "$—a$" is a (schematic) truth-value-name. Note that the definite description on the right-hand side of the axiomatic equation contains "=". "$a=\zeta$" is a first-level concept-expression and "$\dot{\varepsilon}(a=\varepsilon)$" refers to the extension of the concept $a=\zeta$ under which only one object falls.

[13] In my opinion, the logical role or function that Frege associates with the horizontal is in a sense twofold.

(a) The name "$—\xi$" transforms every concept-script proper name which is not a truth-value name into a truth-value name. If one looks at the proofs that Frege carries out in *Grundgesetze* one can see that performing this role is crucial. In the proofs, Frege is predominantly concerned with the inferential connections between thoughts, less with logical relations that may exist between non-propositional senses (and non-propositional references). It is obvious that whenever we transform an object-name "$\Delta$" which is not a truth-value name into a truth-value-name "$—\Delta$", the sense of "$—\Delta$" (=a thought) differs from that of "$\Delta$". However, if we insert a truth-value name of the form "$—\Delta$" into the argument-place of "$—\xi$", then "$—\Delta$" is converted into itself by fusing the horizontals.

(b) "$—\xi$" makes possible the transformation of a monadic first-level function-name which is not a concept name into a concept name and likewise the transformation of a dyadic first-level function-name which is not a relation name into a relation name. For example, $—\Phi(\xi)$ is a function that is composed of $—\xi$ and $\Phi(\xi)$. In 'Logik in der Mathematik' (Frege 1969, pp. 258–260), Frege would call $—\xi$ the enclosing function and $\Phi(\xi)$ the enclosed function. Plainly, one can take here only the value of $\Phi(\xi)$ for a fitting argument as an argument for $—\xi$. Similarly in the case of second-level functions. For example, the second-level function $\varphi(2)$ can be converted into the second-level concept $—\varphi(2)$, which, according to Frege (*Grundgesetze* I, p. 38), one may call *property of the number 2*. Plainly, concept names and relation names are in close proximity to truth-value names without which no proof gets off the ground. To ensure the availability of some of the thoughts which a concept-script proof is designed to involve, the argument-places of the former need only be filled with certain object names or certain function-names, as the case may be. In saying this, I ignore, for example, Frege's use of Roman object-letters and object-markers.

Frege further crucially relies on "$\xi = \zeta$" when in §18 he comes to state Basic Law (IV)—in modern notation: $\neg(—a) = (\neg b) \rightarrow (—a) = (—b)$—and formulates the *definienienta* of exactly a dozen definitions. The definitions with the use of "=" in the *definiens* are those of the application operator (Frege, 1893, §34), the single-valuedness (Frege, 1893, §37), the cardinal number *Zero* (Frege, 1893, §41), the cardinal number *One* (Frege, 1893, §42), the predecessor relation holding of each cardinal number and the next (the relation of a cardinal number to the one immediately following it, Frege, 1893, §43), the weak ancestral of a relation (the relation of an object belonging to the series of a relation starting with an object, Frege, 1893, §46), the coupling of a relation with a relation (Frege, 1893, §144), occurring in a bounded sequence (the circumstance that an object belongs to a series running from an object to an object) (Frege, 1893, §158), series of composition (Frege, 1903, §167), domain of magnitudes (Frege, 1903, §173) and positival class (Frege, 1903, §175). In every definition of *Grundgesetze*, "=" is used in the role of stipulating identity of reference and identity of sense of the *definiens* and the *definiendum.* Frege possibly thought that even for this reason, the use of "=" in his concept-script could hardly be dispensed with.

It is equally important to see that Frege needs "$\xi = \zeta$" for stating and proving a considerable number of important theorems of cardinal arithmetic and real analysis. Most prominent examples of theorems of cardinal arithmetic that essentially contain the equals sign are the right-to-left direction and the contraposed left-to-right direction of Hume's Principle (Theorems 32 and 49).[14] Further examples of sentences in which the equals sign is irreplaceable from Frege's point of view are theorems which he likewise includes in his list of the more important theorems that he has proved in laying the logical foundations of cardinal arithmetic. It is useful to mention some of them:

> "If no object falls under a concept, then Zero is the cardinal number of the objects falling under that concept" (Theorem 97). "If Zero is the cardinal number of the objects falling under a first concept, then Zero is is also the cardinal number of the objects that fall under a concept subordinated to the first" (Theorem 99). "If One is the cardinal number of objects falling under a concept and if a first object falls under this concept and likewise a second, then these objects coincide" (Theorem 117). "If Endlos is the cardinal number of a concept and if the cardinal number of another concept is finite, then Endlos is the cardinal number of the concept *falling under the first or under the second concept*" (Theorem 172). "Endlos is the cardinal number which belongs to a concept, if the objects falling under that concept can be ordered in a series that starts with a certain object and proceeds endlessly, without looping back into itself and without branching" (Theorem 263). "If one pair coincides with another, then the second member of the first coincides with the second member of the second (Theorem 219). "Any finite cardinal number is the cardinal

[14] See May and Wehmeier (2019), Heck (2012), pp. 173–178 and Schirn (2016).

number of members of the cardinal number series running from One to itself" (Theorem 314).[15]

Looking at the proofs in *Grundgesetze*, I and II show that despite its fundamental importance, "$\xi = \zeta$" is used considerably less frequently than the application operator "$\xi \cap \zeta$". The latter emerges as a kind of magic key from Frege's first definition in *Grundgesetze* I, §34 and is ubiquitous in the proofs in *Grundgesetze* I and II. Moreover, it is no accident that "$\xi \cap \zeta$" occurs in the *definiens* of almost all the twenty-six ensuing definitions. In any event, it is not least due to the availability of "$\xi = \zeta$" (or $\xi = \zeta$) and "$\xi \cap \zeta$" (or $\xi \cap \zeta$) and also to the logical interplay of these function-names (or functions) in a plethora of inference steps that Frege could live up to his proof-theoretic ambition regarding flexibility, parsimony, and transparency.

At first glance, it may come as a surprise that the use of canonical value-range equations and mixed equations plays a marginal role in the proofs that Frege carries out in *Grundgesetze*. Yet after having fixed the semantics of value-range names,[16] without which his logicist project in *Grundgesetze* would not have got off the ground, it must have been clear to him that the use of equations of the two kinds I mentioned above would take a back seat in part II of *Grundgesetze*, entitled "Proofs of the basic laws of cardinal number" and part III, 2 entitled "The theory of magnitude." By contrast, the use of "=" in other proof-theoretically relevant contexts is fairly frequent, predominantly in equations such as "$a = c$" or "$e = b$",[17] in equations in which "=" is, in a fashion affine to Hume's Principle (to its right-to-left direction), flanked by numerical terms formed from the cardinality operator (in short, CO-terms) or in equations where "=" is flanked by a CO-term and a simple numerical term introduced via a definition or by "$n$" instead of a CO-term or a numeral. And there are of course quite a few other essential and indispensable uses of "=" in the proofs such as, for example, in "$c$; $d = c$; $d$" or "$D = c$; $d$" or "$o$; $a = e$; $i$". Frege defines the two-place function-name "$o$; $a$", i.e., the name for the ordered pair, in terms of the value-range notation and the application operator; cf. Frege, 1893,

[15] For the proof of Theorem 348, see Heck (2012), pp. 180-182. Heck (2012) carefully analyzes other proofs of important theorems of cardinal arithmetic in *Grundgesetze* I as well. In Frege's foundation of real analysis, the following important theorems essentially contain "=": 485, 487, 489, 492, 495, 498, 501, 516, 532, 539, 546, 551, 556, 561, 562, 566, 568, 571, 574, 576, 578, 579, 582, 585, 588, 602, 639, 642, 644, 674, 675, and 689. In this essay, I do not try to translate all these theorems into ordinary language which would be a tall order. On Frege's theory of real numbers in *Grundgesetze* II (Frege 1903), see Boccuni and Panza (2022), Dummett (1991), Roeper (2020), Schirn (2013) and (2014), Simons (1987), Snyder and Shapiro (2019), and von Kutschera (1966). On the treatment of real numbers in the framework of neo-logicism, see Hale (2000) and Wright (2000).

[16] In his proof of referentiality, Frege intends to demonstrate beyond doubt that canonical value-range names are referential. In the first place, he intends to show that what he calls "regular value-range names" ("*rechte Wertverlaufsnamen*") are referential. In the proof, Frege calls value-range names that are formed from monadic first-level function-names, which have already been established as referential, *regular value-range names* (*rechte Wertverlaufsnamen*).

[17] As to the replacement of a Roman object-letter such as "$a$", "$b$", etc., Frege writes in §48 entitled "Summary of the rules" (at the beginning of passage 9): "When citing a sentence by its label, we may effect a simple inference by uniformly replacing a Roman object-letter within the sentence by the same proper name or the same object-marker."

§144.[18] In *Grundgesetze*, Frege even uses a few trivial equations such as "$b = b$". In his view, they nevertheless play a certain role in one or the other inference step. Otherwise, he would not have used them. Recall in this connection that in *Grundgesetze* I, §50 Frege proves IIIe: $\vdash a = a$, although in the light of his elucidation of "$\xi = \zeta$," this sentence is obvious (*selbstverständlich*). However, it is of greater generality than any of its instances.

In the light of the austerity constraint which in the *Tractatus* Wittgenstein imposes on a correct concept-script and the dispensation of "=" that he apparently considers to be well-founded, he would perhaps have outlawed the use of "$\xi = \zeta$" in the formal language of *Grundgesetze*. But he does not criticize Frege for that. In my judgement, trying to ban "=" from the logical system of *Grundgesetze* and replace the formulae in every proof, in which "=" occurs, with "equivalent" formulae without "=" along the lines of Wittgenstein's rudimentary proposal might after all turn out to be an exercise in futility. In particular, I fail to see that the elimination of "=" in *Grundgesetze*, if it could be carried out perhaps by occasional recourse to logical acrobatics, would provide any logical or notational advantage that would outweigh the potential disadvantages, such as an increase of syntactic complexity at least in some cases, and at the same time a decrease of logical transparency of several proof-structures in *Grundgesetze*.[19] In short, even if the transformation of all formulae of *Grundgesetze* in which "=" occurs essentially from Frege's point of view into equivalent formulae that do not contain "=" could in principle be implemented, it would probably be hard to spot any of the benefits that Wittgenstein may have associated with the exclusion of "=" in a correct concept-script.

What I just said with due brevity is not made out of thin air. However, mainly for reasons of space, I cannot supply exemplary and detailed evidence for my assessment. Attempting to do so, would go far beyond the scope of this essay as I designed it. I may leave this for another essay. I conclude this section with remarks on Frege's view of the logical nature of the function-name "$\xi = \zeta$" or the function $\xi = \zeta$ and of Basic Law V.

In Frege's view, both the requisite maximal generality of the basic laws of *Grundgesetze* and the requisite self-evidence of the axiom-thoughts rest crucially on how the function-names that occur in the laws are syntactically and semantically combined. The six basic laws of *Grundgesetze* are characteristically formed by combining, via the syntactic operation of insertion, only *primitive* function-names. Thanks to their pristine

[18] Heck (2012, §7.2) argues that Frege knew that it would have been possible for him to avoid using pairs in his proofs. In *Grundgesetze* II, §165–§244, Frege uses more special equations in his project of laying the logical foundations of real analysis. He regards most of them, if not all, as indispensable for effectively carrying out the proofs of a number of theorems.

[19] Sentences which contain one or more occurrences of the equals sign I call "=-sentences". In Schirn (2024a), I argue that not every first-order =-sentence that Wittgenstein lists in the *Tractatus* (see 5.531, 5.532, 5.5321, 5.534) has an equivalent identity-sign free counterpart. The law of transitivity of identity, for example (in Wittgenstein's notation: "$a = b.\ b = c.\ \supset a = c$"), defies transcription into an equivalent identity-sign free sentence. If we go beyond Wittgenstein's examples of=-sentences and their purportedly equivalent identity-sign free counterparts, we see, for example, that any attempt to appropriately "translate" polynomial, differential, linear, exponential, quadratic, parametric, etc. equations into equivalent identity-sign free sentences would quickly be stretched to its limits.

fundamentalness, these names constitute the germcell of the entire system from which everything in it sprouts. In *Grundgesetze*, Frege spares himself the trouble of arguing for the requisite purely logical nature of "$\xi = \zeta$" or $\xi = \zeta$. Nor does he argue for the logical nature of the other primitive (logically simple) function-names or primitive functions of his system. In his essay 'Über die Grundlagen der Geometrie' II (Frege, 1967), Frege emphasizes the need to supply a clear characterization of what counts as a logical inference and what is proper to logic if we want to carry out valid independence proofs. Furthermore, he points out that for this purpose, it would be necessary to formulate, in a precise manner, a basic law which one might call an emanation of the formal nature of logical laws (cf. Frege, 1967, p. 321). At the same time, Frege underscores that despite appearances logic is not purely formal. Just as concepts like *point*, *line*, *plane* and relations like *lies on*, *between*, *congruent* intrinsically belong to (Euclidean) geometry, so logic, too, has its own concepts and relations such as negation, identity, subsumption and subordination of concepts for which it allows no replacement. For Frege, this is an unmistakable mark that the relation of logic towards what is proper to it is not at all formal. He concedes, however, that here we find ourselves in unexplored territory.[20] In the light of these remarks, one might raise the question of whether in *Grundgesetze* Frege really did consider the primitive value-range function $\dot{\varepsilon}\varphi(\varepsilon)$—it is the key function in Basic Law V but the identity function also plays a crucial role in it—to belong intrinsically and irreplaceably to logic, that is, on a par with negation, identity, the conditional and the quantifier. If he did not, perhaps due to the fact that the value-range function is to a much lesser extent involved in our rational thinking and deductive reasoning than the former notions, this might provide a hint why he possibly had some doubts about the purely logical nature of Basic Law V (cf. Schirn, 2019a).

## 4 Concluding remarks

Let me conclude with a few general remarks on identity and "=". The space available at present does not suffice for further discussion of these issues. In my opinion, there is in principle nothing wrong with the use of "=" in a concept-script à la Frege or Russell, or in logic and mathematics in general. And regarding the nature of identity and the cognitive value of true statements of the form "$a = b$" in a natural language, there is, I think, nothing inexplicably arcane, unless perhaps "=" is flanked by two coreferential proper names—for example by "Claude Debussy" and "Monsier Le Croche" or "Rita Hayworth" and "Margarita Carmen Cansino"—who are supposed to express different senses. For in my view nowhere in the literature it

[20] Regarding Frege's reflections about what is proper to logic in 'Über die Grundlagen der Geometrie' II (Frege 1967, pp. 295–317), see Antonelli and May (2000), especially pp. 255ff. The authors argue that Frege's reservations about the metatheoretical method for carrying out independence proofs in the context of what they call his New Science are unjustified. They attempt to show that a characterization of the notions of logical truth and logical constant can be given by appeal to the "permutation argument" which Frege is said to use in his description of how to prove the independence of a thought from a group of thoughts (cf. Frege 1967, pp. 319–323). What the authors call Frege's *permutation argument* in this connection is not to be confused with the permutation argument in *Grundgesetze* I, §10.

is intelligibly or persuasively explained what those senses are (putting exceptions like "Hesperus" and "Phosphorus" and perhaps special examples such as "Francisco Repilado" and "Compay Segundo"[21] aside) and in what sense they should be seen to be different.[22] Frege is no exception in this respect. What I term "the coincidence view" of proper names—the view according to which the sense of a proper name and that of a coreferential definite description are supposed to be the same—is for the most part unsustainable.[23] In general (rare exceptions aside), there is no privileged definite description of an object which could be regarded as expressing *the* sense of a coreferential proper name. In the case of the name "Claude Debussy" it is neither the sense of, say, the description "the composer of *Les Nocturnes*" nor the sense of, say, "the composer of *Pelléas et Mélisande*." In an article that I recently submitted to a journal for publication, I analyze the problems to which Frege's line of argument regarding the nature of identity and the status of identity statements in the opening passage of 'Über Sinn und Bedeutung' gives rise. So I can be brief here.

In Frege's work, it remains unexplained in what specific respect the sense of a proper name (for example, of "Copernicus") is supposed to differ from the sense of a coreferential identifying description (for example, of "the founder of the heliocentric view of the planetary system"). Frege is right that the thought expressed by (a) "Copernicus = the founder of the heliocentric view of the planetary system" differs intuitively and even arguably from that expressed by (b) "Copernicus = Copernicus." Thus, he further correctly assumes that in this special case the supposed semantic difference between (a) and (b) implies a clearly specifiable difference of cognitive value of (a) and (b) across two dimensions, namely first regarding their epistemological classification and second regarding the comparative grading or ranking of their cognitive values. (1) (a) is an a posteriori truth, while (b) is an analytic or a logical truth. (2) The cognitive value of (a) is significantly greater (ranks significantly higher) than that of (b).[24] Nonetheless, Frege owes us an explanation of the specific

[21] The famous Cuban guitarist, singer and composer Francisco Repilado was also called *Compay Segundo* because he was always second voice in his musical partnerships.

[22] In his article "The Mill-Fregean Theory of Proper Names", Garcia-Carpintero (2018) develops a version of metalinguistic descriptivism for proper names. He calls it the *Mill-Frege view* of proper names. Garcia-Carpintero follows Frege in attributing not only a reference but also a sense to proper names. Probably due to the difficulties to which Frege's notion of sense for proper names gives rise, Garcia-Carpintero argues for a metalinguistic variant of sense. In particular, he argues that proper names possess metalinguistic senses "known by competent speakers on the basis of their competence, which figure in ancillary presuppositions" (p. 1107). This is considered the Fregean component of the Mill-Fregean theory, although the notion of the sense of a proper name is reinterpreted. In my view, Garcia-Carpintero fails to make sufficiently clear what the metalinguistic senses of proper names are supposed to be.

[23] Note that the sense of a proper name is something objective for Frege. Regarding his conception of objectivity see the analysis in Schirn (2019b).

[24] According to Frege, the axioms and almost all theorems of *Grundgesetze*—not only those which he lists as the more important ones in his logicist project—contain valuable knowledge. Only the theorem "$a = a$" seems to be an exception to the rule. In the third appendix to *Grundgesetze* I (pp. 242–251), Frege lists the more important theorems that he has proved for cardinal arithmetic. See also the second appendix in *Grundgesetze* II (pp. 245–252) which contains the table of the more important theorems that he has proved for real analysis (the "theory of magnitude"). For a discussion of the cognitive value of the axioms, theorems, and definitions in *Grundgesetze* and the logical hierarchy among the proved theorems on a larger and a smaller scale, see Schirn (2023b).

respect in which the sense of "Copernicus" differs from the sense of "the founder of the heliocentric view of the planetary system" and in the first place wherein the sense of "Copernicus" is supposed to consist. Since after 1892 Frege (implicitly) rejects the coincidence view of ordinary proper names, he could hardly maintain his explanation in 'Über Sinn und Bedeutung' that the sense of a proper name (he has genuine proper names and definite descriptions in mind) is that wherein the mode of presentation of the designated object is contained. These issues are passed over in conspicuous silence by him.[25] The supposed difference between the sense of a proper name and the sense of a coreferential definite description is not bound to be discussed in the context of identity statements. Statements of other logical form would basically do as well. Yet identity statements seem particularly suitable to explain the semantic and cognitive aspects regarding the use of a proper name and a coreferential definite description.

Identity, as I understand it, is neither a relation between signs for one and the same content or object as Frege thought in *Begriffsschrift* nor a substitution rule nor that which is expressed by identity of the sign (as Wittgenstein maintains in the *Tractatus*) nor something that if we wish to express it by a statement of the form "$a=b$" we move on the narrow ridge between Scylla (a tautology, if "$a=b$" is true) and Charybdis (a contradiction, if "$a=b$" is false), as Ramsey seems to assume when he sets about defining identity.[26]As unspectacular as this may sound, Frege was probably right in conceiving of identity as a binary relation in the realm of objects. It is though a very special equivalence relation since it never truly relates two objects. One might call this phenomenon *anomalous* or more emphatically *paradoxical*, but I do not think that from a logical point of view it gives rise to serious bewilderment. Although identity concerns only one object, it is not a unary relation, nor is it a hybrid of a binary and a unary relation. There is no such thing as a genuine unary equivalence relation anyway, and identity is no different. In *Grundgesetze*, "$\xi=\xi$" is obtained from the primitive name "$\xi=\zeta$" by applying the syntactic rules of the insertion of a fitting name into the argument-place of a function-name and the extraction of a function-name from a more complex name (proper name or

[25] They are discussed in an article that I recently submitted to a journal for publication.

[26] Ramsey (1931, p. 53) defines identity as follows: "So $(\phi_e).\phi_e x$ is a tautology if $x=y$, a contradiction if $x=y$. Hence it can suitably be taken as the definition of $x=y$. $x=y$ is a function in extension of two variables. Its value is [a] tautology when $x$ and $y$ have the same value, [a] contradiction when $x$, $y$ have different values." Ramsey probably designed the definition exclusively for equations of a formal language that he was envisioning under the influence of *Principia Mathematica* (Whitehead and Russell 1910) and Wittgenstein's programmatic ideas in the *Tractatus*. In my opinion, it is hard to make sense of Ramsey's definition of identity. Taken at face value, it would debar us from expressing and asserting identity in cognitively significant ways. In general, both a tautology and a contradiction lack a relevant cognitive value, which in the case of a contradiction is even more obvious. In sum, in presenting his exclusive "tautology-contradiction interpretation" of equations, Ramsey is at variance with the facts: most true equations of the form "$a=b$" undoubtedly possess a relevant cognitive value. Otherwise, mathematics would be a sterile science. Thanks to his distinction between the sense and the reference of an expression, Frege is a reliable source for advocating the cognitive significance of mathematical equations against the radical views that Ramsey and Wittgenstein (in the *Tractatus*) put forward. Note, however, that Wittgenstein strongly criticizes Ramsey's definition of identity in Wittgenstein (1964) pp. 141 ff., (1967), pp. 189 ff. and (1969), pp. 315 ff.).

function-name) by means of gap formation. In Frege's formal system, $\xi = \xi$ is a *concept* under which every object falls, whereas $\xi = \zeta$ is a dyadic *relation*. Thus, $\xi = \zeta$ and $\xi = \xi$ should never be confused with one another or lumped together in Frege's logic.

Finally, let me briefly comment on reflexivity and symmetry. Needless to say, reflexivity is not peculiar to identity. It is one of the properties that is shared by all equivalence relations. In the standard case, the relation of congruence between triangles, for example, truly relates *two* objects (triangles) *a* and *b* to one another (if the relation in fact holds between *a* and *b*). Yet from a logical point of view, this feature is in harmony with the reflexivity of the congruence relation which concerns only one object. One might perhaps say that due to the fact that identity never truly relates two objects, it is maximally compatible or congeneric with the property of being reflexive.

Consider now the case of symmetry. If we state this property for the relation of congruence between triangles or the relation of parallelism between straight lines, the terms "*a*" and "*b*" are usually intended to refer to different objects (triangles or lines).[27] By contrast, when we express that identity is symmetric by using again "*a*" and "*b*" (instead of "∀" and "*x*" and "*y*"), "*a*" and "*b*" are intended to corefer. Otherwise we would not be talking about identity. In other words, the use of different terms "*a*" and "*b*" (or different variables "*x*" and "*y*") is of course essential for expressing the property of symmetry of both congruence (or parallelism) and identity, but in general the referential role of "*a*" and "*b*" in the latter case differs from that in the former.

Here then is my résumé of the concluding remarks in a nutshell. In my opinion, identity is best conceived of in the logical tradition, namely as a dyadic relation which holds in the domain of objects. In spite of its peculiar "one-object" nature, it is not the black sheep in the family of first-level equivalence relations, nor is there any need to interpret it in a non-standard way in order to render it logically palatable and scientifically respectable.[28] As a result, I refuse to accept any approach that disparages the standard view as an outdated one.

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[27] Note that lines in a Cartesian plane can be characterized algebraically by linear equations.

[28] Wehmeier (2012) argues that the rejection of identity as a binary relation is defensible in a framework which is not committed to acknowledging an objectual identity relation, but whose expressive power is nonetheless equivalent to first-order logic with identity. According to Wehmeier, objectual identity is after all dispensable. In their article "Maximality of logic without identity" (2024), Badia, Caicedo, and Noguera comment also on the relation between identity and compactness. The authors refer in this connection to Krynicki and Lachlan (1979). The presence of identity can make a crucial difference regarding compactness. For example, monadic first-order logic with the branching Henkin quantifier is not compact and not contained in monadic first-order logic with identity since it can express the quantifier "there are at least $\aleph_0$-many elements." However, the identity-sign free fragment admits the effective elimination of the Henkin quantifier. Note that the Henkin quantifier is related to the notion of formulae in which the usual universal and existential quantifiers occur but are not linearly ordered.

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**Conflict of interest** There is no conflict of interest: I confirm that neither me nor my employer nor a sponsor have any financial, commercial, legal or professional relationship with other organizations, or with the people working with them, that could influence my research.



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