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Born Geometry via Künneth Structures and Recursion Operators

M. J. D. Hamilton[1], **D. Kotschick**[1], **P. N. Pilatus**[2]

[1] Mathematisches Institut, LMU München, Theresienstr. 39, 80333 Munich, Germany
E-mail: mark.hamilton@math.lmu.de; dieter@math.lmu.de
[2] Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany.
E-mail: paula.pilatus@uni-hamburg.de

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Abstract: We propose a simple definition of a Born geometry in the framework of Künneth geometry. While superficially different, this new definition is equivalent to the known definitions in terms of para-quaternionic or generalized geometries. We discuss integrability of Born structures and their associated connections. In particular we find that for integrable Born geometries the Born connection is obtained by a simple averaging under a conjugation from the Künneth connection. We also give examples of integrable Born geometries on nilmanifolds.

1. Introduction

Born geometry was introduced by Freidel, Leigh and Minic in [10] in the context of T-duality in string theory. These authors proposed to describe T-duality by replacing the target space of a theory by a manifold of twice the dimension, carrying a geometric structure which they call a Born structure or a Born geometry. The name was chosen because Max Born [4] had suggested a long time ago that, in order to unify quantum mechanics and general relativity, the momentum space should be allowed to have curvature too. Born geometry and its application in high-energy physics have been developed further in many papers, see e.g. [11–14,25,26].

The mathematical definition of a Born geometry that appears for example in the paper of Freidel et al. [14] was in terms of para-quaternionic and para-hermitian structures, and was influenced by the setup of generalized geometry in the sense of Hitchin. An equivalent description of Born structures which is purely in the language of generalized geometry is discussed in [21].

In this paper we propose a streamlined definition of a Born geometry as a diagram

$$\begin{array}{ccc} & g & \\ {}^{A}\swarrow & & \searrow^{B} \\ \omega & \xrightarrow{\;-J\;} & h \end{array},$$

where g and h are pseudo-Riemannian metrics and ω is a non-degenerate 2-form, such that the intertwining operators or tensor fields satisfy $A^2 = B^2 = -J^2 = \mathrm{Id}$. (Recall that given two non-degenerate bilinear forms on the tangent bundle of a smooth manifold, there is a unique field of invertible endomorphisms intertwining them, see Sect. 2.2 below.) We do not assume that the operators define any kind of (para-)quaternionic structure, since this turns out to be automatically true. Further, we make no assumption about the signatures of the pseudo-Riemannian metrics g and h. It turns out that g is automatically of neutral signature, but the signature of h is arbitrary as long as it is of the form $(2p, 2q)$.

This definition of a Born structure is motivated by the definition of a hypersymplectic structure in [3]. This structure was originally defined by Hitchin [20] using a neutral metric and certain symplectic forms that are parallel for its Levi–Civita connection. It was shown in [3], see also [17, Chapter 8], that Hitchin's definition is equivalent to giving only a diagram of symplectic forms

$$\begin{array}{ccc} & \omega & \\ {}^{A}\swarrow & & \searrow^{B} \\ \alpha & \xrightarrow{\;J\;} & \beta \end{array}$$

such that the intertwining operators satisfy $A^2 = B^2 = -J^2 = \mathrm{Id}$. All other relations between the operators and the existence of a neutral metric defined in terms of these symplectic forms are a consequence of the diagram.

In our definition of a Born geometry it turns out that the (± 1)-eigenbundles of the involution A are Lagrangian for the form ω, and so constitute an almost bi-Lagrangian [9] or almost para-Kähler [7] or almost Künneth structure [17]. As usual, the use of the almost qualification means that no integrability has been assumed. Conversely, we will see that every Born geometry is just an almost Künneth structure together with a choice of an isomorphism between the two Lagrangian subbundles.

The existence of such almost structures is purely a matter of algebraic topology and bundle theory. However, once one imposes integrability, it turns out that these structures are geometrically interesting, and their existence can be quite subtle. We will discuss the (partial) integrability of Born structures, and exhibit some consequences. For example, every almost Künneth structure on a manifold gives rise to a preferred affine connection, the Künneth connection, with respect to which the whole structure is parallel. The Künneth connection is torsion-free if and only if the structure is integrable. Now Freidel et al. [14] proved that every Born structure gives rise to a unique connection for which the structure is parallel, and which satisfies a variation of torsion-freeness suggested by generalized geometry. The expressions given for the Born connection in [14] are very complicated, and are difficult to work with. We will prove that in the integrable case the Born connection arises in a very simple way from the Künneth connection of the underlying Künneth structure, although in general it does not equal the Künneth connection.

Structure of the paper. In Sect. 2 we recall some known facts, mostly about Künneth geometry, which we need for our discussion of Born geometry. In Sect. 3 we discuss our definition of Born geometry, and we develop simple properties. We also discuss integrability and the relationship to hypersymplectic geometry. In Sect. 4 we compare the different connections that arise from a Born geometry, and we prove that in the integrable case the Born connection can be recovered directly from the Künneth connection. Finally in Sect. 5 we give examples of integrable Born geometries on some nilmanifolds.

2. Preliminaries

In this section we recall some definitions we need for our formulation of Born geometry. All the missing details are contained in [17].

2.1. Künneth structures. The following is our basic structure.

Definition 1. An *almost Künneth structure* on a smooth manifold M consists of a non-degenerate 2-form ω together with two complementary ω-isotropic subbundles F and G. An almost Künneth structure is *Künneth* if it is integrable, meaning that ω is closed and therefore symplectic, and F and G are integrable to Lagrangian foliations $\mathcal{F}$ and $\mathcal{G}$.

These structures are also known under several other names, the most common of which is that of a para-Kähler structure, cf. [7]. It is proved in [17, Section 6.3] that an almost Künneth structure is equivalent to an almost para-Hermitian structure. Moreover, in the integrable case, a Künneth structure is equivalent to a para-Kähler structure.

One can think of Definition 1 as a metric-free definition of para-Kähler structures. The metric is recovered by the following construction. The splitting of the tangent bundle $TM = F \oplus G$ defines an almost product structure

$$\begin{aligned} I\colon TM &\longrightarrow TM \\ X_F + X_G &\longmapsto X_F - X_G \ , \end{aligned}$$

which, together with ω, yields:

Proposition 2. [17, Proposition 5.9] *For an almost Künneth structure* (ω, F, G) *the map*

$$\begin{aligned} g\colon TM \times TM &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto \omega(IX, Y) \end{aligned}$$

defines a pseudo-Riemannian metric of neutral signature.

Note that, while in the definition of an almost Künneth structure the two subbundles are treated symmetrically, after defining the almost product structure, this symmetry is broken, and the subbundles are distinguished by the property of being either the (+1)- or the (-1)-eigenbundles of I. Therefore, the almost product structure and the neutral metric associated with an almost Künneth structure are well-defined only up to an overall sign.

There is a canonical connection associated with an almost Künneth structure, the *Künneth connection*, whose definition goes back to Hess [18,19].

Theorem 3. [17, Theorem 6.6] *Let* (ω, F, G) *be an almost Künneth structure on a smooth manifold* M*. There exists a unique affine connection* ∇^K *on* M*, the* Künneth connection*, that preserves both* F *and* G*, is compatible with* ω*, and whose mixed torsion vanishes identically, i.e.* $T^K(X, Y) = 0$ *for* $X \in F$ *and* $Y \in G$ *or vice versa, where* T^K *is the torsion tensor of* ∇^K*.*

The Künneth connection can be expressed as follows. Let

$$D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

be the map defined by $i_{D(X,Y)}\omega = L_X i_Y \omega$. Then for $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(F)$ the formula

$$\nabla^K_X Y = D(X_F, Y)_F + [X_G, Y]_F$$

defines a connection on F. Similarly, for $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(G)$

$$\nabla^K_X Y = D(X_G, Y)_G + [X_F, Y]_G$$

defines a connection on G. Here, for $X \in \mathfrak{X}(M)$, X_F and X_G denote the projections to F and G, respectively. The Künneth connection is then given by

$$\nabla^K_X Y = \nabla^K_X Y_F + \nabla^K_X Y_G$$

for $X, Y \in \mathfrak{X}(M)$. In other words, it is the direct sum of the connections defined separately on F and G.

The torsion of the Künneth connection is related to the integrability of the almost Künneth structure:

Theorem 4. [17, Theorem 6.8] *An almost Künneth structure is Künneth if and only if its Künneth connection is torsion-free.*

Since the Künneth connection is compatible with ω and commutes with the almost product structure I defined by the subbundles F and G, it is also compatible with the corresponding pseudo-Riemannian metric g. Therefore, we obtain the following:

Theorem 5. [17, Theorem 6.10] *If (ω, F, G) is a Künneth structure, then the Levi–Civita connection of the associated neutral metric g is the Künneth connection.*

2.2. Recursion operators. The almost product structure I is a particular instance of what we call a *recursion operator*. These will play a central role in our definition of Born structures. The terminology of recursion operators originates from the theory of bi-Hamiltonian systems, and was developed in a way that is relevant to this paper in [3,17].

Given two non-degenerate bilinear forms a and b on the tangent bundle of a smooth manifold M, there is a unique field of invertible endomorphisms A, such that $a(A\cdot, \cdot) = b(\cdot, \cdot)$. Then, A is called the *recursion operator* from a to b and we will depict the situation as $a \xrightarrow{A} b$. In this case, A^{-1} is the recursion operator from b to a, i.e. $b \xrightarrow{A^{-1}} a$. Given a third bilinear form c, such that $b \xrightarrow{B} c$, it follows that $a \xrightarrow{AB} c$.

In the special case where a recursion operator satisfies $A^2 = \mathrm{Id}$ and $A \neq \pm \mathrm{Id}$, the tangent bundle TM splits as the direct sum of the (± 1)-eigenbundles of A, see the discussion in Subsection 3.1 of [3].

3. Born Geometry

We can now formulate our definition of Born geometry.

Definition 6. Let M be a smooth manifold. A *Born structure* on M is a triple (g, h, ω), where g and h are pseudo-Riemannian metrics and ω is a non-degenerate 2-form, such that the recursion operators specified in the diagram

$$\begin{array}{ccc} & g & \\ {\scriptstyle A}\swarrow & & \searrow{\scriptstyle B} \\ \omega & \xrightarrow{\;-J\;} & h \end{array}$$

satisfy $A^2 = B^2 = -J^2 = \mathrm{Id}$.

Note that the existence of a non-degenerate 2-form implies that M must be even-dimensional.

A diagram of recursion operators as in this definition always commutes, because of the uniqueness of recursion operators. For example, $A \circ (-J)$ is a recursion operator from g to h, and so is B. Therefore, $-AJ = B$.

3.1. Algebraic identities. We will now discuss basic properties of Born structures that can be deduced immediately from the definition. In the following, let M be a smooth manifold of dimension $2n$ with a Born structure (g, h, ω) with recursion operators A, B and J as in Definition 6.

Lemma 7. *The recursion operators A, B and J pairwise anti-commute and $ABJ = \mathrm{Id}$.*

Proof. The second assertion follows immediately from $AB = -J$ and $J^2 = -\mathrm{Id}$. Moreover,

$$AB = -J = J^{-1} = -B^{-1}A^{-1} = -BA\ .$$

This implies

$$AJ = -A^2 B = ABA = -JA$$

and

$$JB = -AB^2 = BAB = -BJ\ .$$

□

An immediate consequence of Lemma 7 is the following:

Lemma 8. *The almost complex structure J interchanges the (± 1)-eigenbundles of A. Similarly, it interchanges the (± 1)-eigenbundles of B. Furthermore, A interchanges the eigenbundles of B and vice versa.*

Proof. Let $X \in \mathfrak{X}(M)$ such that $AX = \pm X$. Then

$$A(JX) = -JAX = \mp JX\ .$$

Hence, JX is in the (∓ 1)-eigen-subbundle of A. The same argument applies to the eigen-subbundles of B and to the second statement. □

The (± 1)-eigenbundles of A play a special role in the Born structure, which will become apparent in Proposition 12. From now on, we will denote them by $L_\pm$. By Lemma 8, the endomorphism field J defines isomorphisms

$$J : L_\pm \longrightarrow L_\mp\ .$$

In particular, L_+ and L_- both have rank equal to $n = \frac{1}{2}\dim(M)$.

We will now study the transformation properties of ω, h and g under the three recursion operators.

Lemma 9. *The 2-form* ω *transforms under the fields of endomorphisms* A, B *and* J *as*

$$\omega(JX, JY) = \omega(X, Y) = -\omega(AX, AY) = -\omega(BX, BY) .$$

Proof. For the first equality, we use the symmetry of h and the anti-symmetry of ω to obtain

$$\omega(JX, JY) = -h(X, JY) = -h(JY, X) = -\omega(Y, X) = \omega(X, Y) .$$

Similarly, using the anti-symmetry of ω and the symmetry of g we find

$$\omega(AX, AY) = g(X, AY) = g(AY, X) = \omega(A^2Y, X) = \omega(Y, X) = -\omega(X, Y) .$$

Finally, $B = -AJ$, and the first two equalities yield

$$\omega(BX, BY) = \omega(AJX, AJY) = -\omega(JX, JY) = -\omega(X, Y) .$$

This completes the proof. □

Lemma 10. *The pseudo-Riemannian metric* g *satisfies*

$$g(BX, BY) = g(X, Y) = -g(AX, AY) = -g(JX, JY) .$$

Proof. We apply Lemma 9 several times and employ the fact that the recursion operators anti-commute. For the first equality, we have

$$g(BX, BY) = \omega(ABX, BY) = -\omega(BAX, BY) = \omega(AX, Y) = g(X, Y) .$$

Similarly, we proceed for A, obtaining

$$g(AX, AY) = \omega(X, AY) = -\omega(AX, Y) = -g(X, Y) .$$

For J, it follows that

$$g(JX, JY) = \omega(AJX, JY) = -\omega(JAX, JY) = -\omega(AX, Y) = -g(X, Y) .$$

□

Proceeding analogously for h, we find that it is invariant under all three recursion operators:

Lemma 11. *The pseudo-Riemannian metric* h *is invariant under the three recursion operators, i.e.*

$$h(AX, AY) = h(X, Y) = h(BX, BY) = h(JX, JY).$$

In particular this shows that the pair (h, J) forms an almost pseudo-Hermitian structure with fundamental 2-form ω. This explains the choice of sign for J in Definition 6.

The transformation properties of the three bilinear forms are summarized in Table 1.

We will now prove further properties of the eigenbundles of A and B and the bilinear forms.

Proposition 12. *The* (± 1)*-eigenbundles of* A *are Lagrangian for* ω*. In particular,* (ω, L_+, L_-) *is an almost Künneth structure. The pseudo-Riemannian metric* g *of the Born structure is the neutral metric associated with this almost Künneth structure.*

Table 1. Transformation properties of g, h and ω under A, B and J.

$g(AX, AY) = -g(X, Y)$	$h(AX, AY) = h(X, Y)$	$\omega(AX, AY) = -\omega(X, Y)$
$g(AX, Y) = -g(X, AY)$	$h(AX, Y) = h(X, AY)$	$\omega(AX, Y) = -\omega(X, AY)$
$g(BX, BY) = g(X, Y)$	$h(BX, BY) = h(X, Y)$	$\omega(BX, BY) = -\omega(X, Y)$
$g(BX, Y) = g(X, BY)$	$h(BX, Y) = h(X, BY)$	$\omega(BX, Y) = -\omega(X, BY)$
$g(JX, JY) = -g(X, Y)$	$h(JX, JY) = h(X, Y)$	$\omega(JX, JY) = \omega(X, Y)$
$g(JX, Y) = g(X, JY)$	$h(JX, Y) = -h(X, JY)$	$\omega(JX, Y) = -\omega(X, JY)$

Proof. Let $X, Y \in \mathfrak{X}(M)$ such that X and Y are both in L_+ or both in L_-. Then by Lemma 9, we have

$$\omega(X, Y) = \omega(AX, AY) = -\omega(X, Y).$$

Hence, the (± 1)-eigenbundles of A are isotropic for ω. Since they are complementary, they are also Lagrangian.

Note that the almost product structure A agrees (up to sign) with the almost product structure I associated to the almost Künneth structure that we defined in Sect. 2.1. Since the neutral metric associated with the almost Künneth structure was defined by plugging I into ω, it follows that it is the same as the pseudo-Riemannian metric g of the Born structure. □

The relation between almost Künneth structures and Born structures will be explored further in Sect. 3.3.

Corollary 13. *The signature of g is neutral and the subbundles $L_\pm$ are null for g.*

Proof. This is a reformulation of Proposition 2. The fact that the $L_\pm$ are null for g can either be deduced from the transformation property of g under A or by observing that $g|_{L_\pm} = \pm\omega|_{L_\pm}$. Note that this holds for any neutral metric associated with an almost Künneth structure. □

Proposition 14. *The (± 1)-eigenbundles of B are g-orthogonal.*

Proof. Let $X, Y \in \mathfrak{X}(M)$ such that $BX = X$, $BY = -Y$. Then

$$g(X, Y) = g(BX, BY) = -g(X, Y),$$

and hence $g(X, Y) = 0$. □

Similarly, using Lemma 11, we find

Proposition 15. *The (± 1)-eigenbundles of A and B are h-orthogonal.*

With the help of Proposition 15, we can also say something about the signature of h:

Proposition 16. *The signature of h is of the form $(2p, 2q)$, where $p + q = n$.*

Proof. As the subbundles $L_\pm$ are h-orthogonal, the signature of h is the sum of the signatures of the restrictions $h|_{L_+}$ and $h|_{L_-}$. Since $J\colon L_+ \to L_-$ is an isometry for h, both restricted metrics have the same signature (p, q) and the assertion follows. □

We have now established the most basic properties of Born structures and are ready to construct first examples.

Example 17. Consider $\mathbb{C}^n$ with the standard Hermitian metric h and the standard Kähler form ω. The recursion operator J from h to ω is the standard complex structure on $\mathbb{C}^n$. Furthermore, thinking of $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, let $L_\pm$ be the Lagrangian subbundles defined by the foliations given by the copies $\mathbb{R}^n \times \{y\}$ and $\{x\} \times \mathbb{R}^n$ of the two factors in the product decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Then we define A to be the almost product structure determined by $L_\pm$. We denote by g the neutral metric defined by ω and A. Then (g, h, ω) is a Born structure on $\mathbb{C}^n$.

Example 18. Since the Born structure described in Example 17 is invariant under translations, it descends to tori $\mathbb{C}^n/\Lambda$, where $\Lambda \subset \mathbb{C}^n$ is a lattice.

In view of Proposition 16 we can slightly modify Example 18 to show that on tori T^{2n}, the signature of h can take any form $(2p, 2q)$ with $p + q = n$:

Example 19. Think of T^{2n} as the product $T^{2p} \times T^{2q}$ and define ω and g as in Example 18. However, for the almost complex structure, we choose now $J = J_p \oplus -J_q$, where J_p and J_q are the complex structures on the factors T^{2p} and T^{2q} coming from the standard complex structure on $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively. The pseudo-Hermitian metric defined by ω and J is now of signature $(2p, 2q)$.

3.2. Integrability of Born structures. For integrability of a Born structure, there are several conditions that can be imposed. First of all, there is the condition of closedness of the 2-form. Second of all, we could ask for integrability of the recursion operators. Since these all square to $\pm\,\mathrm{Id}$, a necessary and sufficient condition for their integrability is the vanishing of the *Nijenhuis tensor*.

Definition 20. The *Nijenhuis tensor* of an endomorphism field T is the $(1, 2)$-tensor field defined by

$$N_T(X, Y) = [TX, TY] + T^2[X, Y] - T[TX, Y] - T[X, TY]\,.$$

That the vanishing of the Nijenhuis tensor is equivalent to the integrability of an almost complex structure is just the Newlander–Nirenberg theorem. For an almost product structure T, the statement follows by observing that for $X, Y \in TM$, we have

$$N_T(X, Y) = 2 \cdot ([X_+, Y_+]_- + [X_-, Y_-]_+)\,,$$

where the subscripts $\pm$ denote the projections to the (± 1)-eigenbundles of T. Therefore, the vanishing of the Nijenhuis tensor is equivalent to the Frobenius integrability of the eigen-subbundles of T and hence, to the integrability of T.

The recursion operators of a Born structure have the special feature that the integrability of any two of them implies the integrability of the third one.

Proposition 21. *If two out of the three recursion operators in a Born structure are integrable, then so is the third one.*

Proof. For the Nijenhuis tensors of A, B and J a straightforward calculation leads to

$$\begin{aligned} &N_J(X, Y) + N_J(AX, AY) \\ &\quad = N_A(BX, BY) - N_A(X, Y) - A\left(N_B(AX, Y) + N_B(X, AY)\right)\,. \end{aligned}$$

If we assume $N_A \equiv N_B \equiv 0$, then this formula proves the vanishing of $N_J(X, Y)$ if X and Y are in the same eigenbundle of A. For $X \in L_+$, $Y \in L_-$, we find

$$\begin{aligned} N_J(X, Y) =&[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] \\ =& - [BX, BY] - [X, Y] - BA[BX, Y] + BA[X, BY] \\ =& - N_B(X, Y) = 0 \,. \end{aligned}$$

Here, we used for the third equality that since B interchanges the subbundles $L_\pm$ and A is integrable, we have $[BX, Y] \in L_-$ and $[X, BY] \in L_+$. In the same way, one shows that $N_J \equiv N_A \equiv 0$ implies that N_B vanishes identically.

If we assume $N_J \equiv N_B \equiv 0$, then we obtain $N_A(X, Y) = N_A(BX, BY)$. It follows that $N_A(X, Y)$ vanishes if X and Y are in different eigenbundles for B. If X and Y are both in the (+1)-eigenbundle of B, we compute

$$\begin{aligned} N_A(X, Y) &= [AX, AY] + [X, Y] - A[AX, Y] - A[X, AY] \\ &= -B[JX, JY] + B[X, Y] + BJ[JX, Y] + BJ[X, JY] \\ &= -BN_J(X, Y) = 0 \,, \end{aligned}$$

where we used $A = -JB = BJ$, and that the eigenbundles of B are closed under commutators. In the same way one proves that if X and Y are both in the (-1)-eigenbundle of B, then

$$N_A(X, Y) = BN_J(X, Y) = 0 \,.$$

This completes the proof. □

A statement equivalent to Proposition 21 appeared in [29] with a different proof.

Remark 22. In the situation of Proposition 21 the pair (J, A) forms a complex product structure in the sense of Andrada and Salamon [2], cf. Lemma 7. This is sometimes useful in order to exclude existence of integrable Born structures in the sense of the following definition, see Example 41 below.

Definition 23. A Born structure is *integrable*, if the two-form ω is closed and at least two, and therefore all, recursion operators are integrable.

We have already encountered first examples of integrable Born structures. The Born structures constructed on $\mathbb{C}^n$ in Example 17 and on tori in Examples 18 and 19 are clearly integrable.

Integrability of a Born structure implies that (h, J) is a pseudo-Kähler structure with Kähler form ω and (ω, L_+, L_-) is a Künneth structure.

3.3. Born structures as enhanced almost Künneth structures. We have seen in Proposition 12 that every Born structure induces an almost Künneth structure. It turns out that the converse is also true and every almost Künneth structure can be realized as part of a Born structure. Indeed, given an almost Künneth structure (ω, F, G), we can choose an isomorphism

$$\tilde{J} \colon F \to G \,,$$

satisfying

$$\omega(\tilde{J}X, Y) = -\omega(X, \tilde{J}Y), \quad X, Y \in F \,.$$

Locally, this can be realized as follows. Choose a local frame $\{f_i, g_j\}_{i,j=1}^n$ such that the $\{f_i\}$ and $\{g_i\}$ are a local frame for F and G, respectively, and $\omega(f_i, g_j) = \delta_{ij}$. Then set $\tilde{J}(f_i) = g_i$.

Using $\tilde{J}$, we can define an endomorphism field J on TM by setting

$$J\,|_F := \tilde{J}, \quad J\,|_G := -\tilde{J}^{-1}.$$

By construction, J is an almost complex structure which interchanges the Lagrangian subbundles of the almost Künneth structure and such that $h(X, Y) = \omega(X, JY)$ defines a pseudo-Riemannian metric h. Then we obtain a Born structure (g, h, ω), where g is the neutral metric associated with the almost Künneth structure. This makes sense even though g is only well-defined up to sign. Choosing the neutral metric with a different sign still yields a Born structure $(-g, h, \omega)$, where the recursion operators A and B are replaced by $-A$ and $-B$.

It follows that a Born structure is an almost Künneth structure, together with a choice of an almost complex structure J, which is compatible with the almost Künneth structure in the sense that it interchanges the Lagrangian subbundles and it satisfies $\omega(J\cdot, \cdot) = -\omega(\cdot, J\cdot)$. In particular, an integrable Born structure is a Künneth structure together with a compatible complex structure.

Remark 24. Note that if J is defined by an isomorphism $\tilde{J}$, which is locally of the form described above, the resulting h will always be positive definite. The reason is that the condition $\omega(f_i, g_j) = \delta_{ij}$ implies that $\{f_i\}$ is a local orthonormal frame of L_+ with respect to the h so constructed. To obtain a Born structure with an indefinite h in an analogous construction, we need a local frame $\{f_i, g_i\}_{i=1}^n$ such that $\omega(f_i, g_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$. However, in general this will not lead to a globally defined pseudo-Riemannian metric.

3.4. Relation to hypersymplectic geometry. According to [3], hypersymplectic structures in the sense of Hitchin [20] can be defined by a diagram of symplectic forms

$$\begin{array}{ccc} & \omega & \\ {\scriptstyle A}\swarrow & & \searrow{\scriptstyle B} \\ \alpha & \xrightarrow{\;J\;} & \beta \end{array}$$

such that the recursion operators satisfy $-J^2 = A^2 = B^2 = \mathrm{Id}$. This looks very similar to our definition of a Born structure, the only difference being that the symplectic forms α and β are replaced by pseudo-Riemannian metrics.

It was proved in [17, Section 8.3] that every hypersymplectic structure induces an S^1-family of Künneth structures. The hypersymplectic metric from Hitchin's definition is in fact the neutral metric g associated with any one of these Künneth structures. By Sect. 3.3 above, every Künneth structure in this family gives rise to a Born structure. In some situations, this even yields an S^1-family of integrable Born structures:

Theorem 25. *Let M be a smooth manifold admitting a hypersymplectic structure (ω, α, β) with hypersymplectic metric g. Assume that there is an almost complex structure $\tilde{J}$ on M, which anti-commutes with A and B and such that $g(\tilde{J}X, \tilde{J}Y) = -g(X, Y)$ for all X, Y in TM. Then*

$$\begin{array}{ccc} & g & \\ {\scriptstyle I_\theta}\swarrow & & \searrow{\scriptstyle \tilde{B}_\theta} \\ \beta_\theta & \xrightarrow{\;-\tilde{J}\;} & h_\theta \end{array}$$

defines an S^1-family of Born structures, where

$$\beta_\theta = -\sin(\theta)\alpha + \cos(\theta)\beta, \quad I_\theta = \cos(\theta)A + \sin(\theta)B,$$

and $\tilde{B}_\theta = \tilde{J} I_\theta$. In particular, if $\tilde{J}$ is integrable, then every Born structure in this family is integrable.

Proof. Recall from [17, Section 8.3] the S^1-family of Künneth structures $(\alpha_{\theta+\pi/2}, \mathcal{F}_\theta, \mathcal{G}_\theta)$ that arises from the hypersymplectic structure, where

$$\alpha_\theta = \cos(\theta)\alpha + \sin(\theta)\beta,$$

and $\mathcal{F}_\theta, \mathcal{G}_\theta$ are the eigen-foliations of the product structure

$$I_\theta = \cos(\theta)A + \sin(\theta)B.$$

Since $\tilde{J}$ anti-commutes with A and B, it anti-commutes with I_θ for each θ and since every hypersymplectic structure in the family has the same associated neutral metric g and $g(\tilde{J}X, Y) = g(X, \tilde{J}Y)$ by assumption, it follows that $\alpha_\theta(\tilde{J}X, Y) = -\alpha_\theta(X, \tilde{J}Y)$ for each θ. Therefore, every Künneth structure in the S^1-family is compatible with $\tilde{J}$ in the sense of Sect. 3.3 and we obtain an S^1-family of Born structures

$$\begin{array}{ccc} & g & \\ {\scriptstyle I_\theta}\swarrow & & \searrow{\scriptstyle \tilde{B}_\theta} \\ \beta_\theta & \xrightarrow{\;-\tilde{J}\;} & h_\theta \end{array} ,$$

where $\beta_\theta := \alpha_{\theta+\pi/2}$ and $\tilde{B}_\theta = \tilde{J} I_\theta$.

If $\tilde{J}$ is integrable, then, since β_θ is closed and I_θ is integrable for each θ, every Born structure in this family is completely integrable because of Proposition 21. □

We will discuss an example of this construction on $Nil^3 \times \mathbb{R}$ in Sect. 5 below.

3.5. Comparison to the previous literature. In [14] a Born structure was defined as an almost para-Hermitian structure (g, I) with corresponding 2-form ω, together with a pseudo-Riemannian metric h such that the recursion operator from g to h squares to Id and the one from ω to h squares to $-$Id. From Proposition 12 it follows that this is equivalent to our Definition 6.

However, our notation differs from the notation that has been used in the physics literature. In [14,26,27], the metric h is denoted by $\mathcal{H}$ and the metric g is called η. Moreover, the almost product structures A and B are denoted by K and J, while the almost complex structure J is called I.

The structure formed by A, B and J is called an *almost para-quaternionic structure* and was first introduced in [24] and later studied in [22,28,29], and in many other papers.

4. Connections Associated with a Born Structure

A Born structure on a smooth manifold M determines the following affine connections:

- ∇^g, the *Levi–Civita connection* of the pseudo-Riemannian metric g,
- ∇^h, the *Levi–Civita connection* of the pseudo-Riemannian metric h,
- ∇^K, the *Künneth connection* of the almost Künneth structure (ω, L_+, L_-),
- ∇^c, the *canonical connection* of the almost Künneth structure (ω, L_+, L_-),
- ∇^B, the *Born connection*, which is compatible with ω, g and h and has vanishing *generalized torsion*.

We already discussed the Künneth connection in Sect. 2.1. We will now define the canonical connection and the Born connection and discuss their properties. Furthermore, we will explain the relations between the different connections.

Since the so-called Bismut connection is more relevant to the approach via generalized geometry [21], which is different from our approach via Künneth geometry, we do not discuss it here.

4.1. The canonical connection. Given an almost Künneth structure (ω, L_+, L_-) with almost product structure A, whose eigenbundles are $L_\pm$, we have the associated neutral Riemannian metric g defined in Proposition 2. Its Levi–Civita connection ∇^g does not necessarily commute with A, meaning that $\nabla^g_X(AY)$ and $A(\nabla^g_X Y)$ do not always agree. This failure motivates the definition of the canonical connection.

For $X, Y \in \mathfrak{X}(M)$ the *canonical connection* ∇^c is defined by averaging ∇^g under conjugation with A:

$$\nabla^c_X Y = \frac{1}{2}\left(\nabla^g_X Y + A\nabla^g_X AY\right) . \tag{1}$$

This does indeed define a connection which commutes with A. Furthermore, this new connection is still compatible with g, but may have non-trivial torsion. Since the canonical connection is compatible with g and commutes with A, it is also compatible with ω. This means that the whole almost Künneth structure is parallel with respect to ∇^c.

One can rewrite the definition of the canonical connection as

$$\nabla^c_X Y = \left(\nabla^g_X Y_+\right)_+ + \left(\nabla^g_X Y_-\right)_- ,$$

where the subscripts $\pm$ denote the projections to the subbundles $L_\pm$.

It follows immediately from this and the fact that the subbundles $L_\pm$ are null for g that

$$g\left(\nabla^c_X Y_+, Z_-\right) + g\left(Y_+, \nabla^c_X Z_-\right) = g\left(\nabla^g_X Y_+, Z_-\right) + g\left(Y_+, \nabla^g_X Z_-\right) = L_X\left(g(Y_+, Z_-)\right).$$

Moreover, using again that the $L_\pm$ are null for g, we find

$$g\left(\nabla^c_X Y_\pm, Z_\pm\right) + g\left(Y_\pm, \nabla^c_X Z_\pm\right) = 0$$

and

$$L_X\left(g\left(Y_\pm, Z_\pm\right)\right) = 0 .$$

We have seen in Sect. 2.1, that for the Künneth connection, just like the for canonical connection, the whole almost Künneth structure is parallel. Therefore, the question arises whether there are conditions under which these connections coincide with each other, and how they are related in general. By the uniqueness of the Künneth connection, it is clear that a necessary and sufficient condition for both connections to agree with each other is the vanishing of the mixed torsion of the canonical connection. For the purpose of stating the relation between the connections more precisely, we will need the following lemma:

Lemma 26. *The following equalities hold for $X, Y, Z \in \mathfrak{X}(M)$:*

i) $\nabla^g_X \omega\,(Y_\pm, Z_\mp) = 0$,
ii) $\nabla^g_X \omega\,(Y_\pm, Z_\pm) = -2\omega\left(\nabla^g_X Y_\pm, Z_\pm\right) = -2\omega\left(Y_\pm, \nabla^g_X Z_\pm\right)$.

Proof. The first statement follows from

$$\begin{aligned}\nabla^g_Z \omega(X_+, Y_-) &= L_Z\,(\omega(X_+, Y_-)) - \omega(\nabla^g_Z X_+, Y_-) - \omega(X_+, \nabla^g_Z Y_-)\\ &= L_Z\,(\omega(X_+, Y_-)) - \omega(\nabla^c_Z X_+, Y_-) - \omega(X_+, \nabla^c_Z Y_-)\\ &= 0,\end{aligned}$$

where in the second line we used the definition of ∇^c and in the last step that ω is parallel for ∇^c.

For the second statement, we observe

$$\omega\left(Y_\pm, \nabla^g_X Z_\pm\right) = \pm g\left(Y_\pm, \nabla^g_X Z_\pm\right) = \mp g\left(\nabla^g_X Y_\pm, Z_\pm\right) = \omega\left(\nabla^g_X Y_\pm, Z_\pm\right),$$

where we used for the second equality that the $L_\pm$ are null for g. This yields

$$\nabla^g_X \omega\,(Y_\pm, Z_\pm) = -\omega\left(\nabla^g_X Y_\pm, Z_\pm\right) - \omega\left(Y_\pm, \nabla^g_X Z_\pm\right) = -2\omega\left(\nabla^g_X Y_\pm, Z_\pm\right).$$

□

With the help of Lemma 26, we obtain the following result:

Proposition 27. *The Künneth connection and the canonical connection are related by*

$$\omega\left(\nabla^K_X Y, Z\right) = \omega\left(\nabla^c_X Y, Z\right) - \Omega^K(X, Y, AZ),$$

where

$$\Omega^K(X, Y, AZ) = \frac{1}{2}\left\{\mathrm{d}\omega(X, Y_+, Z_-) + \mathrm{d}\omega(X, Y_-, Z_+)\right\}.$$

Proof. Using the definition of the Künneth connection, we write

$$\begin{aligned}\omega\left(\nabla^K_X Y_+, Z_-\right) &= L_{X_+}\omega\,(Y_+, Z_-) - \omega\,(Y_+, [X_+, Z_-]) + \omega\,([X_-, Y_+], Z_-)\\ &= \omega\left(\nabla^c_{X_+} Y_+, Z_-\right) + \omega\left(Y_+, \nabla^c_{X_+} Z_-\right) - \omega\,(Y_+, [X_+, Z_-]) + \omega\,([X_-, Y_+], Z_-),\end{aligned}$$

where we used for the second equality that the canonical connection is compatible with ω. Using the definition of ∇^c and that ∇^g is torsion-free, we observe that

$$\omega\left(Y_+, \nabla^c_{X_+} Z_- - [X_+, Z_-]\right) = \omega\left(Y_+, \nabla^g_{X_+} Z_- - [X_+, Z_-]\right) = \omega\left(Y_+, \nabla^g_{Z_-} X_+\right),$$

and similarly,

$$\omega\,([X_-, Y_+], Z_-) = \omega\left(\nabla^g_{X_-} Y_+ - \nabla^g_{Y_+} X_-, Z_-\right) = \omega\left(\nabla^c_{X_-} Y_+, Z_-\right) - \omega\left(\nabla^g_{Y_+} X_-, Z_-\right).$$

It follows that

$$\omega\left(\nabla^K_X Y_+, Z\right) = \omega\left(\nabla^c_X Y_+, Z\right) + \omega\left(Y_+, \nabla^g_{Z_-} X_+\right) - \omega\left(\nabla^g_{Y_+} X_-, Z_-\right).$$

By analogous arguments we obtain

$$\omega\left(\nabla^K_X Y_-, Z\right) = \omega\left(\nabla^c_X Y_-, Z\right) + \omega\left(Y_-, \nabla^g_{Z_+} X_-\right) - \omega\left(\nabla^g_{Y_-} X_+, Z_+\right).$$

Therefore, we have

$$\omega\left(\nabla^K_X Y, Z\right) = \omega\left(\nabla^c_X Y, Z\right) - \Omega^K(X, Y, AZ),$$

where

$$\begin{aligned}\Omega^K(X, Y, AZ) = &- \omega\left(Y_+, \nabla^g_{Z_-} X_+\right) + \omega\left(\nabla^g_{Y_+} X_-, Z_-\right)\\ &- \omega\left(Y_-, \nabla^g_{Z_+} X_-\right) + \omega\left(\nabla^g_{Y_-} X_+, Z_+\right).\end{aligned}$$

Using Lemma 26 twice, we find

$$\begin{aligned}\omega\left(Y_+, \nabla^g_{Z_-} X_+\right) &= \frac{1}{2}\nabla^g_{Z_-}\omega(X_+, Y_+)\\ &= \frac{1}{2}\left\{\nabla^g_{X_+}\omega(Y_+, Z_-) - \nabla^g_{Y_+}\omega(Z_-, X_+) + \nabla^g_{Z_-}\omega(X_+, Y_+)\right\}\\ &= \frac{1}{2}\,\mathrm{d}\omega(X_+, Y_+, Z_-).\end{aligned}$$

Proceeding similarly with the other terms, it follows that

$$\begin{aligned}\Omega^K(X, Y, AZ) =& \frac{1}{2}\,\mathrm{d}\omega(X_+, Y_+, Z_-) + \frac{1}{2}\,\mathrm{d}\omega(X_-, Y_+, Z_-)\\ &+ \frac{1}{2}\,\mathrm{d}\omega(X_-, Y_-, Z_+) + \frac{1}{2}\,\mathrm{d}\omega(X_+, Y_-, Z_+)\\ =& \frac{1}{2}\,\mathrm{d}\omega(X, Y_+, Z_-) + \frac{1}{2}\,\mathrm{d}\omega(X, Y_-, Z_+).\end{aligned}$$

□

The subbundles $L_\pm$ induce a bigrading on the differential forms of M. In particular, if the subbundles are integrable, then the exterior derivative splits as $d = d_+ + d_-$, where d_+ and d_- are of bidegree $(1, 0)$ and $(0, 1)$, respectively. By Proposition 12, the 2-form ω is of type $(1, 1)$. Using this bigrading, we obtain the following necessary and sufficient conditions for the vanishing of the difference of the Künneth connection and the canonical connection:

Corollary 28. *The Künneth connection and the canonical connection agree with each other if and only if*

$$\mathrm{d}\omega^{(1,2)} = 0 = \mathrm{d}\omega^{(2,1)}.$$

In particular, if ω is closed, then $\nabla^c = \nabla^K$. If A is integrable, then $\nabla^c = \nabla^K$ if and only if ω is closed.

Using Theorem 5, see also [9, Theorem 3], this implies:

Corollary 29. *If (ω, L_+, L_-) is Künneth, then $\nabla^c = \nabla^K = \nabla^g$.*

4.2. The Born connection. We will now discuss connections compatible with the full Born structure.

Definition 30. A connection ∇ is *compatible with the Born structure* (g, h, ω) if

$$\nabla h = \nabla \omega = \nabla g = 0.$$

First we show that compatible connections exist, using again the averaging mechanism used in the definition of the canonical connection.

Proposition 31. *Let* (g, h, ω) *be a Born structure, and* ∇^K *the Künneth connection of the underlying almost Künneth structure. Then*

$$\nabla_X Y = \frac{1}{2}\left(\nabla^K_X Y + B\nabla^K_X BY\right) = \frac{1}{2}\left(\nabla^K_X Y - J\nabla^K_X JY\right)$$

defines a connection that is compatible with the Born structure.

Proof. We first check that the two expressions involving B and J respectively agree. For this we use that $B = JA$, that J and A anti-commute, and the fact that ∇^K commutes with A. Together, these identities imply

$$B\nabla^K_X BY = JA\nabla^K_X JAY = JA\nabla^K_X(-AJY) = -JA^2\nabla^K_X JY = -J\nabla^K_X JY\ .$$

Thus the two expressions do indeed agree. They define an affine connection ∇, for which we want to show that it is compatible with the Born structure.

Recall that the Künneth connection ∇^K is compatible with g and ω. It commutes with A, but not necessarily with B. However, ∇ does commute with B:

$$\begin{aligned}
\nabla_X BY &= \frac{1}{2}\left(\nabla^K_X BY + B\nabla^K_X B^2 Y\right) \\
&= \frac{1}{2}\left(\nabla^K_X BY + B\nabla^K_X Y\right) \\
&= B\left(\frac{1}{2}\left(B\nabla^K_X BY + \nabla^K_X Y\right)\right) \\
&= B\nabla_X Y\ .
\end{aligned}$$

Moreover, ∇ is still compatible with g and ω, and commutes with A. Since it also commutes with B it commutes with J as well, and is also compatible with h. □

In general there are many connections compatible with a Born structure, but from our point of view, the connection constructed above is the most natural, and would deserve to be called the Born connection. However, this name is already in use for a connection that does not always agree with this ∇.

Freidel et al. [14] used a condition analogous to torsion-freeness in the case of the Levi–Civita connection of a pseudo-Riemannian metric in order to single out a preferred connection in the space of all connections compatible with a Born structure. Their condition is the following:

Definition 32. Let ∇^c be the canonical connection for an almost Künneth structure with associated neutral metric g on a smooth manifold M. Then an affine connection ∇ on TM has *vanishing generalized torsion* if

$$g(\nabla_X Y - \nabla_Y X, Z) + g(\nabla_Z X, Y) = g(\nabla^c_X Y - \nabla^c_Y X, Z) + g(\nabla^c_Z X, Y) \qquad (2)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

This definition does not arise naturally in our setup, and we refer to [14] for its motivation.

The following was proved in [14]:

Theorem 33. [14, Theorem 1] *Given a Born structure* (g, h, ω)*, there is a unique connection, the* Born connection ∇^B*, that is compatible with the Born structure and has vanishing generalized torsion.*

In the case of an integrable Born structure the Born connection does agree with our candidate ∇ constructed by averaging the Künneth connection under the conjugation with B or J, although this is far from obvious.

Theorem 34. *Let* (g, h, ω) *be an integrable Born structure. Then the connection* ∇ *defined in Proposition 31 agrees with the Born connection* ∇^B.

Proof. By Proposition 31, ∇ is compatible with the Born structure. Therefore, by Theorem 33, it suffices to show that ∇ has vanishing generalized torsion, and for this we use integrability.

From Corollary 29 we know that in the integrable case the canonical and Künneth connections coincide. Therefore, on the right-hand side of (2) we can use the Künneth connection. Thus, using the definition of ∇ in terms of ∇^K, the condition (2) is equivalent to

$$g(\nabla^K_X Y - \nabla^K_Y X, Z) + g(\nabla^K_Z X, Y) = g(B\nabla^K_X BY - B\nabla^K_Y BX, Z) + g(B\nabla^K_Z BX, Y). \tag{3}$$

Since in the integrable case ∇^K is torsion-free in the usual sense, the first summand on the left-hand side of (3) simplifies to $g([X, Y], Z)$. Now the integrability of B implies

$$[X, Y] = B[BX, Y] + B[X, BY] - [BX, BY] ,$$

where on the right-hand side we can rewrite commutators in terms of ∇^K since this is torsion-free:

$$[X, Y] = B\nabla^K_{BX} Y - B\nabla^K_Y BX + B\nabla^K_X BY - B\nabla^K_{BY} X - [BX, BY] .$$

Substituting this into the left-hand side of (3) we see that certain summands on the two sides agree and therefore drop out. We are left with the condition

$$g(B\nabla^K_{BX} Y - B\nabla^K_{BY} X - [BX, BY], Z) + g(\nabla^K_Z X, Y) = g(B\nabla^K_Z BX, Y) . \tag{4}$$

Lemma 35. *The identity* (4) *holds if* X *and* Y *are in the same eigenbundle for* B.

Proof. Suppose $BX = X$ and $BY = Y$. Then (4) becomes

$$g(B\nabla^K_X Y - B\nabla^K_Y X - [X, Y], Z) + g(\nabla^K_Z X, Y) = g(B\nabla^K_Z X, Y) . \tag{5}$$

Since the eigenbundles of B are integrable, we have $[X, Y] = B[X, Y]$, and so the first summand on the left-hand side vanishes by the torsion-freeness of ∇^K. For the second summand we have:

$$g(\nabla^K_Z X, Y) = g(B\nabla^K_Z X, BY) = g(B\nabla^K_Z X, Y) .$$

This show that (4) holds if $BX = X$ and $BY = Y$. The proof also works if $BX = -X$ and $BY = -Y$. □

Lemma 36. *The identity* (4) *holds if X and Y are in different eigenbundles for B.*

Proof. Suppose $BX = X$ and $BY = -Y$. Then (4) becomes

$$g(B\nabla^K_X Y + B\nabla^K_Y X + [X, Y], Z) + g(\nabla^K_Z X, Y) = g(B\nabla^K_Z X, Y) \,. \tag{6}$$

The right-hand side is

$$g(B\nabla^K_Z X, Y) = g(\nabla^K_Z X, BY) = -g(\nabla^K_Z X, Y) \,,$$

which is the negative of the second summand on the left-hand side. The first summand there can be rewritten using once more the torsion-freeness of ∇^K:

$$\begin{aligned} g(B\nabla^K_X Y + B\nabla^K_Y X + [X, Y], Z) &= g(B\nabla^K_X Y + B\nabla^K_Y X + \nabla^K_X Y - \nabla^K_Y X, Z) \\ &= g((B\nabla^K_X Y + \nabla^K_X Y) + (B\nabla^K_Y X - \nabla^K_Y X), Z) \,. \end{aligned} \tag{7}$$

Note that $B\nabla^K_X Y + \nabla^K_X Y$ is in the (+1)-eigenbundle of B, and $B\nabla^K_Y X - \nabla^K_Y X$ is in the (−1)-eigenbundle. These eigenbundles are g-orthogonal by Proposition 14, and therefore we now consider separately the two cases where Z is in one of them.

If $BZ = Z$, then using (7) the first summand on the left-hand side of (6) is

$$\begin{aligned} 2g(\nabla^K_X Y, Z) &= 2L_X g(Y, Z) - 2g(Y, \nabla^K_X Z) \\ &= -2g(Y, \nabla^K_Z X + [X, Z]) \\ &= -2g(Y, \nabla^K_Z X) \,, \end{aligned}$$

which shows that (6) holds in this case. In this calculation we used that ∇^K is compatible with g, that $[X, Z]$ is in the (+1)-eigenbundle of B by the integrability of this subbundle, and that this is g-orthogonal to the (−1)-eigenbundle containing Y.

If $BZ = -Z$, then using (7) the first summand on the left-hand side of (6) is

$$\begin{aligned} -2g(\nabla^K_Y X, Z) &= -2L_Y g(X, Z) + 2g(X, \nabla^K_Y Z) \\ &= 2g(X, \nabla^K_Z Y + [Y, Z]) \\ &= 2g(X, \nabla^K_Z Y) \\ &= 2L_Z g(X, Y) - 2g(\nabla^K_Z X, Y) \\ &= -2g(\nabla^K_Z X, Y) \,, \end{aligned}$$

where we use the same arguments as before. This shows that (6) holds in the case $BZ = -Z$. By linearity it holds for all Z.

In exactly the same way one proves the case when $BX = -X$ and $BY = Y$. □

Combining the two lemmas with linearity, we see that the condition (4) holds for all X, Y and Z. This finally completes the proof of the theorem. □

Example 37. Consider the situation of Theorem 25, where an S^1-family of Born structures is constructed from a hypersymplectic structure. In that case, the underlying Künneth structures all have for their Künneth connection ∇^K the Levi–Civita connection ∇^g of the hypersymplectic metric. Moreover, the almost complex structure $\tilde{J}$ is independent of the parameter $\theta \in S^1$. Therefore, the connection ∇ constructed in Proposition 31 by averaging ∇^K under the conjugation with the almost complex structure is also independent of θ. Whenever $\tilde{J}$ is integrable, this is the Born connection ∇^B, which is then the same for all Born structures in the S^1-family.

Finally we note that in the special case where the Künneth connection ∇^K commutes with B, the connection ∇ we defined equals the Künneth connection. Therefore, Theorem 34 has the following consequence.

Corollary 38. *For an integrable Born structure with B or, equivalently, J, parallel with respect to* ∇^K, *we have* $\nabla^c = \nabla^g = \nabla^K = \nabla^B$.

In this special situation ∇^B is torsion-free in the usual sense. In general, the torsion of ∇^B measures the failure of ∇^K to commute with B. This is the content of the following proposition.

Proposition 39. *Let* (g, h, ω) *be an integrable Born structure, and* T^B *the torsion tensor of the Born connection* ∇^B. *Then* $T^B(X, Y)$ *vanishes if* X *and* Y *are in the same eigenbundle of* B.

If $BX = X$ *and* $BY = -Y$, *then*

$$\begin{aligned} T^B(X, Y) &= -\frac{1}{2}\left(\nabla^K_X Y + B\nabla^K_X Y\right) + \frac{1}{2}\left(\nabla^K_Y X - B\nabla^K_Y X\right) \\ &= -\pi_+(\nabla^K_X Y) + \pi_-(\nabla^K_Y X) , \end{aligned}$$

where $\pi_\pm$ *are the projections to the* (± 1)*-eigenbundles of* B.

In particular, T^B *vanishes identically if and only if* ∇^K *commutes with* B.

Proof. Using Theorem 34, we shall calculate with the formula from Proposition 31.

Assume first that $BX = X$ and $BY = Y$. Then

$$\begin{aligned} T^B(X, Y) &= \nabla^B_X Y - \nabla^B_Y X - [X, Y] \\ &= \frac{1}{2}\left(\nabla^K_X Y + B\nabla^K_X BY\right) - \frac{1}{2}\left(\nabla^K_Y X + B\nabla^K_Y BX\right) - [X, Y] \\ &= \frac{1}{2}\left(\nabla^K_X Y + B\nabla^K_X Y\right) - \frac{1}{2}\left(\nabla^K_Y X + B\nabla^K_Y X\right) - [X, Y] \\ &= \pi_+(\nabla^K_X Y) - \pi_+(\nabla^K_Y X) - \pi_+([X, Y]) \\ &= \pi_+(\nabla^K_X Y - \nabla^K_Y X - [X, Y]) \\ &= \pi_+(0) = 0 , \end{aligned}$$

where we have used $B[X, Y] = [X, Y]$ by the assumptions on X and Y and the integrability of B, and the torsion-freeness of ∇^K in the integrable case. The same argument works if $BX = -X$ and $BY = -Y$.

Next assume that $BX = X$ and $BY = -Y$. The torsion-freeness of ∇^K implies

$$\begin{aligned} T^B(X, Y) &= \frac{1}{2}\left(\nabla^K_X Y + B\nabla^K_X BY\right) - \frac{1}{2}\left(\nabla^K_Y X + B\nabla^K_Y BX\right) - [X, Y] \\ &= \frac{1}{2}\left(\nabla^K_X Y - B\nabla^K_X Y\right) - \frac{1}{2}\left(\nabla^K_Y X + B\nabla^K_Y X\right) - \nabla^K_X Y + \nabla^K_Y X \\ &= -\frac{1}{2}\left(\nabla^K_X Y + B\nabla^K_X Y\right) + \frac{1}{2}\left(\nabla^K_Y X - B\nabla^K_Y X\right) \\ &= -\pi_+(\nabla^K_X Y) + \pi_-(\nabla^K_Y X) , \end{aligned}$$

as claimed. □

5. Examples of Integrable Born Structures

In this section we want to provide some examples of integrable Born structures on closed manifolds that go beyond the rather obvious ones we have seen in Examples 18 and 19.

Given an integrable Born structure

$$\begin{array}{ccc} & g & \\ {}^{A}\swarrow & & \searrow^{B} \\ \omega & \xrightarrow{-J} & h \end{array} \quad ,$$

we know that (h, J) is a pseudo-Kähler structure with Kähler form ω, and (ω, L_+, L_-) is a Künneth structure. Therefore, we will look for examples in classes of manifolds for which it is known that one or both of these structures occur.

A very tractable class of manifolds consists of nilmanifolds. Note that for a simply connected nilpotent Lie group G admitting a lattice Γ any left-invariant integrable Born structure descends to the compact nilmanifold $\Gamma \setminus G$. Therefore, to investigate whether a nilmanifold is left-invariant Born, it suffices to work at the level of the Lie algebra.

The definition of a Born structure can be transcribed to Lie algebras in the obvious way. We will call a Born structure on a Lie algebra $\mathfrak{g}$ *integrable*, if its 2-form is closed under the Chevalley–Eilenberg differential, the endomorphism J has vanishing Nijenhuis tensor and the eigenspaces of A are subalgebras of $\mathfrak{g}$. Furthermore, a Lie algebra admitting an integrable Born structure will be called *Born*.

Integrable Born structures on a Lie algebra yield left-invariant integrable Born structures on the corresponding Lie group. Since all of the nilpotent Lie algebras that we consider have a basis with rational structure constants, the corresponding Lie groups admit lattices and therefore, the integrable Born structures we find give rise to examples of compact Born manifolds.

A *compatible pair* on a Lie algebra $\mathfrak{g}$ is a symplectic form Ω on $\mathfrak{g}$ together with a complex structure J on $\mathfrak{g}$, such that $\Omega(JX, JY) = \Omega(X, Y)$ and, therefore, $h(\cdot, \cdot) := \Omega(\cdot, J\cdot)$ defines a pseudo-Kähler metric on $\mathfrak{g}$. By Sect. 3.2, the existence of a compatible pair on a Lie algebra is necessary for the existence of an integrable Born structure on the respective Lie algebra.

5.1. Dimension 4. According to [17, Section 9.4], the only non-Abelian 4-dimensional Lie algebra admitting a Künneth structure is $\mathfrak{nil}_3 \oplus \mathbb{R}$, the Lie algebra of $Nil^3 \times \mathbb{R}$. This has a basis $\{e_1, e_2, e_3, e_4\}$ with the only non-zero bracket relation $[e_1, e_2] = e_3$.

This Lie algebra not only has Künneth structures, it even has a hypersymplectic structure, so that we can apply the construction of Sect. 3.4 to it.

Theorem 40. *There is an S^1-family of integrable Born structures on* $\mathfrak{nil}_3 \oplus \mathbb{R}$.

Proof. It is proved in [17, Section 9.5] that the symplectic forms

$$\begin{aligned} \alpha &= \alpha_{14} - \alpha_{23} \\ \beta &= -\alpha_{13} - \alpha_{24} \\ \omega &= -\alpha_{13} + \alpha_{24} \end{aligned}$$

define a hypersymplectic structure

$$\begin{array}{ccc} & \omega & \\ {}^{A}\swarrow & & \searrow^{B} \\ \alpha & \xrightarrow{J} & \beta \end{array}$$

on $\mathfrak{nil}_3 \oplus \mathbb{R}$. Here, the $\{\alpha_i\}$ denote the dual basis of $\{e_i\}$ and we use the abbreviation $\alpha_{ij} = \alpha_i \wedge \alpha_j$. The recursion operators of the hypersymplectic structure act on the basis $\{e_i\}$ by

$$\begin{aligned} Ae_1 &= e_2, \quad Ae_2 = e_1, \quad Ae_3 = -e_4, \quad Ae_4 = -e_3 \\ Be_1 &= e_1, \quad Be_2 = -e_2, \quad Be_3 = e_3, \quad Be_4 = -e_4, \\ Je_1 &= e_2, \quad Je_2 = -e_1, \quad Je_3 = -e_4, \quad Je_4 = -e_3. \end{aligned}$$

It follows that

$$B^*\alpha_1 = \alpha_1, \quad B^*\alpha_2 = -\alpha_2, \quad B^*\alpha_3 = \alpha_3, \quad B^*\alpha_4 = -\alpha_4.$$

Therefore, the hypersymplectic metric $g(\cdot,\cdot) = \alpha(\cdot, B\cdot)$ can be expressed in terms of the dual basis $\{\alpha_i\}$ as

$$\begin{aligned} g &= \big(\alpha_1 \otimes B^*\alpha_4 - \alpha_4 \otimes B^*\alpha_1\big) - \big(\alpha_2 \otimes B^*\alpha_3 - \alpha_3 \otimes B^*\alpha_2\big) \\ &= -\left(\alpha_1 \otimes \alpha_4 + \alpha_4 \otimes \alpha_1 + \alpha_2 \otimes \alpha_3 + \alpha_3 \otimes \alpha_2\right). \end{aligned}$$

Let $\tilde{J}$ be the endomorphism defined by

$$\tilde{J}e_1 = e_2, \quad \tilde{J}e_2 = -e_1, \quad \tilde{J}e_3 = e_4, \quad \tilde{J}e_4 = -e_3.$$

Then $\tilde{J}^2 = -\mathrm{Id}$ by construction and $\tilde{J}$ anti-commutes with A and B. Since the only non-vanishing bracket relation of $\mathfrak{nil}_3 \oplus \mathbb{R}$ is $[e_1, e_2] = e_3$ and e_1 and e_2 are mapped to each other under $\tilde{J}$, the only component of the Nijenhuis tensor of $\tilde{J}$ that does not vanish trivially is $N_{\tilde{J}}(e_1, e_2)$. For this, we observe

$$\begin{aligned} N_{\tilde{J}}(e_1, e_2) &= [\tilde{J}e_1, \tilde{J}e_2] - [e_1, e_2] - \tilde{J}\left([\tilde{J}e_1, e_2] + [e_1, \tilde{J}e_2]\right) \\ &= -[e_2, e_1] - [e_1, e_2] - \tilde{J}\left([e_2, e_2] - [e_1, e_1]\right) \\ &= 0. \end{aligned}$$

Moreover, we have

$$\tilde{J}^*\alpha_1 = -\alpha_2, \quad \tilde{J}^*\alpha_2 = \alpha_1, \quad \tilde{J}^*\alpha_3 = -\alpha_4, \quad \tilde{J}^*\alpha_4 = \alpha_3$$

and it follows that

$$\begin{aligned} \tilde{J}^*g &= -\left\{\tilde{J}^*\alpha_1 \otimes \tilde{J}^*\alpha_4 + \tilde{J}^*\alpha_4 \otimes \tilde{J}^*\alpha_1 + \tilde{J}^*\alpha_2 \otimes \tilde{J}^*\alpha_3 + \tilde{J}^*\alpha_3 \otimes \tilde{J}^*\alpha_2\right\} \\ &= -\left\{-\alpha_2 \otimes \alpha_3 + \alpha_3 \otimes (-\alpha_2) + \alpha_1 \otimes (-\alpha_4) - \alpha_4 \otimes \alpha_1\right\} \\ &= -g. \end{aligned}$$

Therefore, by Theorem 25 we obtain an S^1-family of integrable Born structures

$$\begin{array}{ccc} & g & \\ {}^{I_\theta}\swarrow & & \searrow^{\tilde{B}_\theta} \\ \beta_\theta & \xrightarrow{-\tilde{J}} & h_\theta \end{array},$$

where

$$\beta_\theta = -\sin(\theta)\alpha + \cos(\theta)\beta, \quad I_\theta = \cos(\theta)A + \sin(\theta)B$$

and $\tilde{B}_\theta = \tilde{J} I_\theta$. □

Thus the Lie group $Nil^3 \times \mathbb{R}$ carries left-invariant integrable Born structures that descend to all the associated nilmanifolds. These nilmanfolds are exactly the non-trivial principal T^2-bundles over T^2.

It was shown in [17, Theorem 10.27] that the infra-nilmanifolds of $Nil^3 \times \mathbb{R}$ also carry Künneth structures, although they are not hypersymplectic since the hypersymplectic structure does not descend. These infra-nilmanifolds cannot carry integrable Born structures since they are not complex manifolds. The genuine infra-nilmanifolds of type $Nil^3 \times \mathbb{R}$ have even first Betti number $b_1 = 2$. It is a fact that was first known from Kodaira's classification of compact complex surfaces and later proved directly by Buchdahl [5] and Lamari [23] that compact complex surfaces with even first Betti number are Kähler. However, since infra-nilmanifolds are finitely covered by nilmanifolds, such a Kähler structure would lift to these nilmanifolds, which is impossible, since they have first Betti number 3 and so cannot be Kähler.

It was shown in [17, Example 9.46] that the Lie group $Sol^3 \times \mathbb{R}$, where Sol^3 is the solvable non-nilpotent Thurston geometry, admits left-invariant Künneth structures. These cannot be upgraded to integrable Born structures because again the closed manifolds $\Gamma \setminus (Sol^3 \times \mathbb{R})$ carrying such a geometry have even first Betti number, and so would be Kähler by the result mentioned above. The lattice Γ would then be a solvable Kähler group that is not virtually nilpotent. Such Kähler groups cannot exist be a result of Delzant [8].

5.2. Dimension 6. The 6-dimensional nilpotent Lie algebras carrying Künneth structures were classified in [16], see also the summary in [17, Section 9.4.3]. The upshot is that there are 15 different non-Abelian Lie algebras that do have this structure. Similarly, the 6-dimensional nilpotent Lie algebras carrying pseudo-Kähler structures were classified by Cordero, Fernández and Ugarte [6], who found 13 non-Abelian cases. These 13 Lie algebras all have Künneth structures by the result of [16], and so they are candidates for having integrable Born structures. The two examples from [16] which have Künneth structures but do not have pseudo-Kähler structures do not have to be considered.

We will show now that some of the pseudo-Kähler Lie algebras from [6] actually have integrable Born structures, whereas at least one of them does not. We do not decide this question in all cases, leaving the completion of the classification for future research, cf. the subsequent paper [15].

Example 41. The Lie algebra denoted $\mathfrak{h}_{15}$ in [6] cannot have an integrable Born structure, since it was shown by Andrada [1] that it does not admit a complex product structure; cf. Remark 22.

Example 42. The Lie algebra denoted by $\mathfrak{h}_8$ in [6] is the direct sum $\mathfrak{nil}_3 \oplus \mathbb{R}^3$. Since we have seen that $\mathfrak{nil}_3 \oplus \mathbb{R}$ has integrable Born structures, the same is true in this case, since we can just sum the structures with the standard Born structure on $\mathbb{R}^2$ from Example 17.

For more complicated examples we use the following:

Lemma 43. *Let J be an almost complex structure with Nijenhuis tensor N_J. If $N_J(X, Y)$ vanishes, so do $N_J(JX, JY)$, $N_J(JX, Y)$ and $N_J(X, JY)$.*

Proof. This follows from

$$\begin{aligned} N_J(JX, JY) =& [J^2X, J^2Y] - [JX, JY] - J\left([J^2X, JY] + [JX, J^2Y]\right) \\ =& - N_J(X, Y) \\ N_J(JX, Y) =& [J^2X, Y] - [JX, Y] - J\left([J^2X, Y] + [JX, JY]\right) \\ =& - J \cdot N_J(X, Y). \end{aligned}$$

□

Example 44. The Lie algebra denoted by $\mathfrak{h}_4$ in [6] is not a direct sum of Lie algebras of lower dimension. It has a basis $e_1, \ldots, e_6$ for which the only non-zero commutators are $[e_1, e_2] = -e_5$ and $[e_1, e_4] = [e_2, e_3] = -e_6$. We shall work with the dual 1-forms α_i. They are closed for $i \leq 4$ and satisfy $\mathrm{d}\alpha_5 = \alpha_1 \wedge \alpha_2$ and $\mathrm{d}\alpha_6 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3$. We will abbreviate $\alpha_{ij} = \alpha_i \wedge \alpha_j$ and $\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k$.

The 2-form $\omega = \alpha_{13} + \alpha_{26} + \alpha_{45}$ on $\mathfrak{h}_4$ is non-degenerate and since

$$\mathrm{d}\omega = -\alpha_{214} - \alpha_{223} - \alpha_{412} = 0$$

it is also closed and therefore symplectic.

The complementary subspaces

$$\mathfrak{g}_+ := \langle e_1, e_2, e_5 \rangle, \quad \mathfrak{g}_- := \langle e_3, e_4, e_6 \rangle$$

of $\mathfrak{h}_4$ are Lagrangian for ω. Moreover, since $[e_1, e_2] = -e_5$, they are also integrable. Therefore, $(\omega, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Künneth structure.

The endomorphism J defined by

$$Je_1 = -2e_3, \quad Je_2 = -e_4, \quad Je_3 = \frac{1}{2}e_1, \quad Je_4 = e_2, \quad Je_5 = e_6, \quad Je_6 = -e_5.$$

squares to $-Id$. To see that it is integrable, we compute

$$\begin{aligned} N_J(e_1, e_2) =& [-2e_3, -e_4] - [e_1, e_2] - J\,([-2e_3, e_2] + [e_1, -e_4]) \\ =& e_5 + Je_6 = 0\,. \end{aligned}$$

By Lemma 43 this implies that also $N_J(e_1, e_4) = N_J(e_2, e_3) = N_J(e_3, e_4) = 0$. Moreover, since $N_J(e_1, Je_1) = 0 = N_J(e_2, Je_2)$, we find $N_J(e_1, e_3) = 0 = N_J(e_2, e_4)$. Furthermore, since brackets with e_5 and e_6 vanish in $\mathfrak{h}_4$, the remaining components of the Nijenhuis tensor vanish trivially.

We will now show that J is compatible with the Künneth structure $(\omega, \mathfrak{g}_+, \mathfrak{g}_-)$. The subalgebras $\mathfrak{g}_\pm$ are clearly interchanged by J. It remains to show that ω is compatible with J. For this, we observe that J acts on the dual basis as

$$J^*\alpha_1 = \frac{1}{2}\alpha_3, \quad J^*\alpha_2 = \alpha_4, \quad J^*\alpha_3 = -2\alpha_1, \quad J^*\alpha_4 = -\alpha_2, \quad J^*\alpha_5 = -\alpha_6, \quad J^*\alpha_6 = \alpha_5.$$

It follows that

$$J^*\omega = -\alpha_{31} + \alpha_{45} + \alpha_{26} = \omega$$

and, hence, (ω, J) is a compatible pair. We conclude that $(\omega, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Künneth structure compatible with the complex structure J and by the discussion in Sect. 3.3 this defines an integrable Born structure.

Example 45. The Lie algebra $\mathfrak{h}_9$ in [6] is three step nilpotent and again, cannot be written as a direct sum of Lie algebras of lower dimension. It has a basis $e_1, \ldots, e_6$ for which the only non-zero commutators are $[e_1, e_2] = -e_4$, $[e_1, e_4] = -e_6$ and $[e_2, e_5] = -e_6$. As before, we are working with the dual 1-forms α_i which satisfy $\mathrm{d}\alpha_i = 0$ for $i \leq 4$, $\mathrm{d}\alpha_5 = \alpha_{12}$ and $\mathrm{d}\alpha_6 = \alpha_{14} + \alpha_{25}$.

The non-degenerate 2-form $\omega = \alpha_{13} + 4\alpha_{26} - 4\alpha_{45}$ on $\mathfrak{h}_9$ is closed:

$$\mathrm{d}\omega = -4\alpha_{214} - 4\alpha_{223} + 4\alpha_{412} = 0$$

and hence symplectic.

Moreover, the abelian subalgebras

$$\mathfrak{g}_+ = \langle e_1, e_5, e_6 \rangle, \quad \mathfrak{g}_- = \langle e_2, e_3, e_4 \rangle$$

of $\mathfrak{h}_9$ are Lagrangian for ω and therefore, $(\omega, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Künneth structure.

Furthermore, we consider the endomorphism J of $\mathfrak{h}_9$ defined by

$$Je_1 = -e_2, \quad Je_2 = e_1, \quad Je_3 = -\frac{1}{4}e_6, \quad Je_4 = -e_5, \quad Je_5 = e_4, \quad Je_6 = 4e_3.$$

Clearly, $J^2 = -\mathrm{Id}$. Moreover, we find

$$\begin{aligned} N_J(e_1, e_3) &= [-e_2, -\frac{1}{4}e_6] - [e_1, e_3] - J\left([-e_2, e_3] + [e_1, -\frac{1}{4}e_6]\right) = 0 \\ N_J(e_1, e_4) &= [-e_2, -e_5] - [e_1, e_4] - J\left([-e_2, e_4] + [e_1, -e_5]\right) \\ &= -e_6 + e_6 = 0 \\ N_J(e_3, e_4) &= [-\frac{1}{4}e_6, -e_5] - [e_3, e_4] - J\left([e_3, -e_5] + [-\frac{1}{4}e_6, e_4]\right) = 0. \end{aligned}$$

Using Lemma 43 it follows that the remaining components of the Nijenhuis tensor also vanish and, hence, J is integrable.

The complex structure J clearly interchanges the subalgebras $\mathfrak{g}_\pm$. To see that J is compatible with ω we observe that

$$J^*\alpha_1 = \alpha_2, \quad J^*\alpha_2 = -\alpha_1, \quad J^*\alpha_3 = 4\alpha_6, \quad J^*\alpha_4 = \alpha_5, \quad J^*\alpha_5 = -\alpha_4, \quad J^*\alpha_6 = -\frac{1}{4}\alpha_3.$$

This yields

$$J^*\omega = 4\alpha_{26} + \alpha_{13} + 4\alpha_{54} = \omega.$$

Hence, the Künneth structure $(\omega, \mathfrak{g}_+, \mathfrak{g}_-)$ is compatible with the complex structure J and following the discussion in Sect. 3.3 this defines an integrable Born structure on $\mathfrak{h}_9$.

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