Statistics and Probability Letters 222 (2025) 110395

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Precise quantile function estimation from the characteristic function

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ARTICLE INFO

MSC:
65T40
62-08
60E10

Keywords:
Quantile function
Numerical inversion
Characteristic function

ABSTRACT

We provide theoretical error bounds for the accurate numerical computation of the quantile function given the characteristic function of a continuous random variable. We show theoretically and empirically that the numerical error of the quantile function is typically several orders of magnitude larger than the numerical error of the cumulative distribution function for probabilities close to zero or one. We introduce the COS method for computing the quantile function. This method converges exponentially when the density is smooth and has semi-heavy tails and all parameters necessary to tune the COS method are given explicitly. Finally, we numerically test our theoretical results on the normal-inverse Gaussian and the tempered stable distributions.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable with density f, cumulative distribution function (CDF) F, quantile function (QF) F^{-1} and characteristic function (CF) φ. In this research, we are interested in the precise numerical computation of the QF, provided that the CF is given in closed form. QFs are used, for example, in Monte Carlo simulations to generate random numbers using the fact that X and $F^{-1}(U)$ are equal in distribution where U is uniformly distributed. For the tempered stable (TS) distribution, for example, "neither the density function nor specific random number generators are available", see Schoutens (2003, Sec. 8.4.3, p. 111). Therefore, a precise approximation of the QF of the TS distribution is useful for random number generation. Applications of the QF in statistics are discussed in Gilchrist (2000).

Typically, one first calculates F using the Gil-Pelaez formula and then inverts F numerically. The Gil-Pelaez formula is stated, for example, in Hughett (1998, Corollary 2) and Abate and Whitt (1992, Eq. (3.6)) and reads

$$F(y) = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{\varphi(u)}{2i\pi u} e^{-iuy} du, \quad y \in \mathbb{R}, \quad \text{and} \quad F(y) = \frac{2}{\pi} \int_{0}^{\infty} \Re\{\varphi(u)\} \frac{\sin(yu)}{u} du, \quad y \geq 0 \tag{1}$$

for CDFs with full support and support on the positive reals, respectively. The integrals in (1) must be solved numerically. In this research we use the COS method, which is introduced in Section 2, to approximate F from φ since all parameters necessary to tune the COS method can be obtained directly from φ, and the COS method converges exponentially, provided f is smooth and has semi-heavy tails. Furthermore, the COS method makes it possible to approximate F and f simultaneously, which will be helpful in estimating the error on the QF. When f has heavy tails, e.g., when f belongs to the stable law, other Fourier inversion methods — such as the Gil-Pelaez formula or the Carr–Madan formula (see Carr and Madan (1999)) – are numerically more efficient, see Junike (2024). A robust version of the COS method for unbounded functions can be found in Wang (2017). A direct link between φ and F^{-1} via non-linear integro-differential equations is given in Shaw and McCabe (2009). Suppose H is a numerical approximation of F, in the sense that

$$\sup_{y \in \mathbb{R}} |F(y) - H(y)| \leq \varepsilon, \tag{2}$$

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https://doi.org/10.1016/j.spl.2025.110395
Received 25 January 2025; Received in revised form 18 February 2025; Accepted 19 February 2025
Available online 28 February 2025
0167-7152/

given some predefined error tolerance $\varepsilon > 0$. Depending on the exact Fourier technique and the numerical integration method, H depends on parameters such as truncation range, number of terms, step size and so on. In the case of the Gil-Pelaez formula and the COS method, bounds on these parameters are given explicitly, see Abate and Whitt (1992) and Junike (2024). That is, given ε, it is well known how to construct H such that Inequality (2) holds. Let $p \in (0,1)$ and $\delta > 0$. To the best of our knowledge, however, it is not known how to choose ε such that Inequality (2) implies

$$|F^{-1}(p) - H^{-1}_{\mathrm{Num}}(p)| < \delta, \tag{3}$$

where H^{-1} is the (exact) inverse of H and H^{-1}_{Num} is the approximation of H^{-1} by a numerical inversion technique. In our main Theorem 2.3 we show that the error between F^{-1} and H^{-1}_{Num} depends linearly on the error between H^{-1} and H^{-1}_{Num}, linearly on the error between F and H and reciprocally on the derivative $h := H'$. In particular, in the tails, h tends to zero, i.e., we show theoretically that for p close to zero or close to one, ε must be several orders of magnitude smaller than δ to ensure that Inequality (2) implies Inequality (3).

We confirm the theoretical results by numerical experiments on distributions with support on $(-\infty,\infty)$ and $(0,\infty)$. In particular, we consider the normal distribution, the normal-inverse Gaussian (NIG) distribution (see Barndorff-Nielsen (1978, 1997)) and the TS distribution (see Hougaard (1986)).

The CDF and the QF of the NIG and the TS distributions are not known in closed form. The density of the NIG distribution can be expressed in terms of the modified Bessel function of the third kind, but for the density of the TS distribution only an infinite series representation is known.

This letter is organized as follows: in Section 2 we discuss our main results: how the numerical error on the CDF propagates to the QF and we introduce the COS method. In Section 3 we present numerical experiments confirming the theoretical results.

2. The COS method and QF by CF

Let F be a differentiable CDF and $H : \mathbb{R} \to \mathbb{R}$ be a differentiable function, which serves as an approximation of F. Let $f = F'$ and $h = H'$. We make the following assumptions:

Assumption A1. There is an interval $(\alpha,\beta) \subset \overline{\mathbb{R}}$ such that $f(x) > 0$ if and only if $x \in (\alpha,\beta)$.

Assumption A2. There is an interval $(a,b) \subset \overline{\mathbb{R}}$ such that $h(x) > 0$ if and only if $x \in (a,b)$.

These intervals can be chosen to be open, since densities are only almost surely unique. We then have that F and H are bijective maps from (α,β) to $(0,1)$ and (a,b) to $(0,1)$, respectively. The inverse function of F and H are denoted by F^{-1} and H^{-1}. We introduce another error by numerically inverting H. We denote the approximation of H^{-1} by H^{-1}_{Num}. The next theorem explains how the bias between F and H and H^{-1} and H^{-1}_{Num} propagates when F^{-1} is approximated by H^{-1}_{Num}.

Theorem 2.1. *Assume Assumptions A1 and A2 hold. Let $p \in (0,1)$ and $\varepsilon > 0$ with $0 < p \pm \varepsilon < 1$. Assume $\sup_{y\in\mathbb{R}} |F(y) - H(y)| \le \varepsilon$ and $|H^{-1}(p) - H^{-1}_{Num}(p)| \le \varepsilon$. Let $y = H^{-1}_{Num}(p)$. Then it holds for some $c \in [-\varepsilon,\varepsilon]$ that $h(y+c) > 0$ and*

$$|F^{-1}(p) - H^{-1}_{Num}(p)| \le \frac{2\varepsilon}{h(y+c)} + \varepsilon + o(\varepsilon). \tag{4}$$

Proof. We use

$$|F^{-1}(p) - H^{-1}_{\mathrm{Num}}(p)| \le |F^{-1}(p) - H^{-1}(p)| + |H^{-1}(p) - H^{-1}_{\mathrm{Num}}(p)|. \tag{5}$$

The second term at the right-hand side of Inequality (5) is less or equal than ε by assumption, which also implies that there is a $c \in [-\varepsilon,\varepsilon]$ with $H^{-1}(p) = y + c$. By Assumption A1, it holds that $h(y+c) > 0$. We analyze the first term at the right-hand side of Inequality (5): We have for $\tilde{y} := H^{-1}(p+\varepsilon)$ that $F(\tilde{y}) \ge H(\tilde{y}) - \varepsilon = p$. So, $H^{-1}(p+\varepsilon) \ge F^{-1}(p)$. Similarly, $F^{-1}(p) \ge H^{-1}(p-\varepsilon)$ holds. By the monotonicity of H^{-1} it holds that $H^{-1}(p+\varepsilon) \ge H^{-1}(p) \ge H^{-1}(p-\varepsilon)$. Therefore,

$$|H^{-1}(p) - F^{-1}(p)| \le H^{-1}(p+\varepsilon) - H^{-1}(p-\varepsilon).$$

Next, we use the inverse function rule to conclude that

$$H^{-1}(p \pm \varepsilon) = H^{-1}(p) \pm \frac{\varepsilon}{h\big(H^{-1}(p)\big)} + o(\varepsilon).$$

Hence, $H^{-1}(p+\varepsilon) - H^{-1}(p-\varepsilon) = \frac{2\varepsilon}{h(y+c)} + o(\varepsilon)$, which completes the proof. □

In order to apply Theorem 2.1, we have to compute H such that the absolute difference between H and F is small. In this research we use the COS method to obtain H from the CF φ, however, there are other Fourier inversion techniques to obtain H from φ, e.g., the Gil-Pelaez formula.

To apply the COS method we assume that f has semi-heavy or lighter tails, which implies that the COS method converges exponentially, see Junike (2024). Formally, we make the following assumption:

Assumption A3. For constants $C_1, C_2, L_0 > 0$ we assume that $|f(x)| \leq C_1 \exp(-C_2|x|), \quad |x| \geq L_0$.

Example 2.2. The Generalized Hyperbolic distribution has semi-heavy tails and support on $(-\infty,\infty)$. It contains many other distributions as special cases, e.g. the NIG, the Variance Gamma and the Hyperbolic distribution, see Schoutens (2003, Sec. 5.3.11). The TS distribution has semi-heavy tails and support on $(0,\infty)$. It includes the normal-inverse Gaussian and the Gamma distribution as special cases. Densities with heavy tails that do not meet Assumption A3 are, for example, the Pareto and stable distributions.

Next, we introduce the COS method, see Fang and Oosterlee (2009). Let $(\alpha,\beta) \subset \overline{\mathbb{R}}$ such that Assumption A1 is satisfied. Let $(a,b) \subset \mathbb{R}$ be a large but finite interval with $(a,b) \subset (\alpha,\beta)$. Let $N \in \mathbb{N}$ be large enough. Since only φ is given, we approximate f as follows: first we truncate f, then the truncated density is approximated by a classical Fourier-Cosine series, i.e.,

$$f(x) \approx f(x)1_{(a,b)}(x) \approx \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos\left(k\pi\frac{x-a}{b-a}\right) \approx \frac{c_0}{2} + \sum_{k=1}^{N} c_k \cos\left(k\pi\frac{x-a}{b-a}\right) =: h_{\text{COS}}(x).$$

The coefficients a_k are defined and approximated as follows:

$$\begin{aligned} a_k &:= \frac{2}{b-a}\int_a^b f(x)\cos\left(k\pi\frac{x-a}{b-a}\right)dx \\ &\approx \frac{2}{b-a}\int_\alpha^\beta f(x)\cos\left(k\pi\frac{x-a}{b-a}\right)dx \\ &= \frac{2}{b-a}\Re\left\{\varphi\left(\frac{k\pi}{b-a}\right)\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} =: c_k. \end{aligned}$$

For two real numbers x and y we denote by $x \wedge y := \min(x,y)$ and by $x \vee y := \max(x,y)$. Given h_{COS}, we approximate $F(y) = \int_{-\infty}^{y} f(x)dx$ by zero for $y \leq a$ and for $y > a$ by

$$F(y) \approx \int_a^{y\wedge b} h_{\text{COS}}(x)dx = \frac{c_0}{2}(y \wedge b - a) + \sum_{k=1}^{N} c_k \frac{b-a}{k\pi}\sin\left(k\pi\frac{y\wedge b - a}{b-a}\right) =: H_{\text{COS}}(y).$$

Since $f(x) > 0$, observe that $h_{\text{COS}}(x) > 0$ for $x \in (a,b)$ and N large enough if f is continuous and piecewise continuously differentiable on (α,β) since then the Fourier-Cosine series converges pointwise. Further, $\int_{-\infty}^{\infty} h_{\text{COS}}(x)dx = \frac{c_0}{2}(b-a) = 1$. So, h_{COS} is a density and H_{COS} is a CDF, which is bijective as a map from (a,b) to $(0,1)$. In particular, h_{COS} satisfies Assumption A2. The next Theorem gives conditions on (a,b) such that $\sup_{y\in\mathbb{R}} |H_{\text{COS}}(y) - F(y)| \leq \varepsilon$, which is an essential assumption in Theorem 2.1.

Theorem 2.3. *Assume f is a bounded density satisfying Assumption A3. Let $\varepsilon > 0$ be small enough. Let $N \in \mathbb{N}$ be large enough. Let $n \in \mathbb{N}$ be even and set $\mu := E[X]$ and*

$$\ell := \sqrt[n]{\frac{2E[(X-\mu)^n]}{\varepsilon}}, \quad a := (\mu - \ell) \vee \alpha, \quad b := (\mu + \ell) \wedge \beta.$$

It then follows that $\sup_{y\in\mathbb{R}} |H_{COS}(y) - F(y)| \leq \varepsilon$.

Proof. The inequality $|H_{\text{COS}}(y) - F(y)| < \varepsilon$ for all $y \in \mathbb{R}$ follows as in Junike and Pankrashkin (2022, Corollary 9) using Markov's inequality and the fact that f has semi-heavy tails. □

In the following Remarks, we provide more details on how to implement Theorems 2.1 and 2.3.

Remark 2.4. In practical applications, we suggest replacing the right-hand side in Inequality (4) by $\frac{2\varepsilon}{\min\{h(y\pm\varepsilon)\}} + \varepsilon$. Observe that $E[(X-\mu)^n] = \frac{1}{i^n}\frac{\partial}{\partial u^n}\psi(u)|_{u=0}$, where ψ is the CF of $X - \mu$, i.e., $\psi(u) = \varphi(u)e^{-iu\mu}$. So, we need only obtain the nth-derivative of ψ to compute $E[(X-\mu)^n]$. Junike and Pankrashkin (2022) suggest choosing $n \in \{4,6,8\}$. In our experiments, we set $n = 8$.

Remark 2.5. We suggest a root-finding algorithm, e.g., Newton's method, the secant method, the fixed point iteration method or the bisection method, to invert H. The bisection method has the advantage of providing a full error control, i.e., we are able to compute H_{Num}^{-1} such that $|H^{-1}(p) - H_{\text{Num}}^{-1}(p)| \leq \varepsilon$ holds. The method repeatedly bisects the interval (a,b) by selecting the subinterval in which the function $H(\cdot) - p$ changes its sign until the bisected interval has a length less than ε.

Remark 2.6. Let $\delta > 0$ and $p \in (0,1)$ be given. Suppose we need $|H_{\text{Num}}^{-1}(p) - F^{-1}(p)| \leq \delta$. How should we choose the error tolerance ε for the CDF? This is a tricky question: the choice of ε affects the truncation range (a,b) and thus h and H_{Num}^{-1}. However, the right-hand side of Inequality (4) also depends on h. We suggest starting with a reasonable error tolerance ε. Then, compute h and H_{Num}^{-1} and check if the inequality $\frac{2\varepsilon}{\min\{h(y\pm\varepsilon)\}} + \varepsilon \leq \delta$ is satisfied. If it is not satisfied, reduce ε successively until the inequality $\frac{2\varepsilon}{\min\{h(y\pm\varepsilon)\}} + \varepsilon \leq \delta$ holds. Then we can be sure that $|H_{\text{Num}}^{-1}(p) - F^{-1}(p)| \leq \delta$.

Table 1
QF of the distributions TS, N(0,1) and NIG. The parameters a, b and N of the COS method for NIG and N(0,1) are obtained as described in Section 2. In the case of the TS distribution, we set $N = 50$.

F	ε	$b-a$	N	p	$y := H_{\text{Num}}^{-1}(p)$	$F(y)$	$\lvert H(y)-F(y)\rvert$	$\lvert H_{\text{Num}}^{-1}(p)-F^{-1}(p)\rvert$	$h(y-\varepsilon)\wedge h(y+\varepsilon)$	RHS of (4)
TS	0.005	10.2	50	0.01	0.01492	0.01475	0.00013	0.00481	0.980	0.02
TS	0.005	10.2	50	0.1	0.10444	0.10370	0.00055	0.00370	1.000	0.02
TS	0.005	10.2	50	0.25	0.24370	0.24619	0.00039	0.00362	1.056	0.01
TS	0.005	10.2	50	0.75	0.70127	0.75166	0.00077	0.00172	0.940	0.02
TS	0.005	10.2	50	0.9	0.98974	0.89978	0.00084	0.00156	0.142	0.08
TS	0.005	10.2	50	0.99	3.36711	0.98996	0.00007	0.00504	0.009	1.14
N(0,1)	0.005	7.6	12	0.75	0.67617	0.75053	0.00000	0.00168	0.316	0.04
N(0,1)	0.005	7.6	12	0.9	1.28214	0.90010	0.00000	0.00059	0.174	0.06
N(0,1)	0.005	7.6	12	0.99	2.32411	0.98994	0.00000	0.00224	0.026	0.38
NIG	0.005	11.9	79	0.75	0.53675	0.74896	0.00000	0.00284	0.365	0.03
NIG	0.005	11.9	79	0.9	1.14023	0.90019	0.00000	0.00124	0.153	0.07
NIG	0.005	11.9	79	0.99	2.70116	0.98999	0.00000	0.00073	0.014	0.73
NIG	0.0005	15.8	114	0.99	2.70203	0.99000	0.00000	0.00014	0.014	0.07

Remark 2.7. Let $L := \frac{b-a}{2}$ and $s \in \mathbb{N}$ be odd. If f is $s+1$ times differentiable with bounded derivatives and the derivatives also have semi-heavy tails, then the number of terms in Theorem 2.3 can be determined by

$$N \geq \left(\frac{1}{\pi}\int_0^\infty u^{s+1}|\varphi(u)|du\right)^{\frac{1}{s}} \times \left(\frac{2^{s+\frac{5}{2}}L^{s+2}}{s\pi^{s+1}}\frac{12}{\varepsilon}\right)^{\frac{1}{s}},$$

see Junike (2024, Eq. (3.8)). The last integral can be solved numerically using standard techniques, e.g., Gauss–Laguerre quadrature, and in some cases (e.g., normal and NIG distributions) it is given explicitly. Junike (2024) suggests $s \in \{19, \dots, 39\}$. In our experiments, we set $s = 39$. There is also an implicit way to find N without additional smoothness assumptions on f, but it requires that $\int_{-\infty}^\infty |\varphi(u)|^2 du$ is given exactly, see Junike and Stier (2024, Corollary 3.12).

3. Numerical experiments

In our numerical experiments, we consider three random variables, X_1, X_2 and X_3, and compute their quantiles via Theorems 2.1 and 2.3. By X_1 we denote the standard normal random variable, abbreviated as N(0,1), which has mean 0, variance 1, skewness 0 and kurtosis 3.

X_2 is a TS distributed random variable with parameters $c > 0$, $d \geq 0$ and $\kappa \in (0,1)$, which has the characteristic function $u \mapsto \exp\left(cd - c\left(d^{\frac{1}{\kappa}} - 2iu\right)^{\kappa}\right)$. We set $c = d = 1$ and $\kappa = \frac{3}{4}$. The random variable X_2 has support on $(0,\infty)$, mean 1.5, variance 0.75, skewness 2.89 and kurtosis 18.

X_3 follows a NIG distribution with parameters $\gamma > 0$, $\theta \in (-\gamma, \gamma)$ and $\nu > 0$, which is defined as a normal variance-mean mixture where the mixing density is the inverse Gaussian distribution. The random variable X_3 has characteristic function $u \mapsto \exp\left(-\nu\left(\sqrt{\gamma^2 - (\theta + iu)^2} - \sqrt{\gamma^2 - \theta^2}\right)\right)$. We set $\nu = \gamma = 1$ and $\theta = 0$. Then X_3 has support on $\mathbb{R}$ and has mean 0, variance 1, skewness 0 and kurtosis 6, i.e., much heavier tails than the normal distribution.

Remark 3.1. On the computation of reference values: We compute reference values for Table 1 for F by the COS method using (a, b) as in Theorem 2.3 with $\varepsilon = 10^{-9}$. We set $N = 10^7$. We confirm the reference values by the Gil-Palaez formula, see (1). The values for the CDF using the COS method and the Gil-Palaez formula agree up to 12 digits. We then apply the bisection method for numerical inversion with error tolerance $\varepsilon = 10^{-9}$ to obtain a reference value for F^{-1}. Theorem 2.1 ensures that the reference value for F^{-1} and the true QF coincide up to 6 digits. In the case of the normal distribution, we double check that the reference values for F and F^{-1} agree with the known closed form solutions up to 9 digits.

Table 1 shows the parameters a, b and N of the COS method for different error tolerances of ε for the three distributions. For different probabilities $p \in (0,1)$, we compute $y := H_{\text{Num}}^{-1}(p)$, $F(y)$, $|H(y) - F(y)|$, $|H_{\text{Num}}^{-1}(p) - F^{-1}(p)|$ and $h(y)$. We observe that the right-hand side (RHS) of Inequality (4) is always satisfied and to some extent overestimates the true error on the QF, since the inequalities in the proofs of Theorems 2.1 and 2.3 are not sharp for the three distributions.

We provide an example how Theorem 2.3 can be used to choose the error tolerance ε for the NIG CDF to approximate the NIG QF arbitrarily closely. (Without calculating any reference values). Suppose $p = 0.99$ and $\delta = 0.1$ is the error tolerance for the QF. First, set $\varepsilon_1 := 0.005$ and obtain $a(\varepsilon_1) = -5.9$, $b(\varepsilon_1) = 5.9$ and $N(\varepsilon_1) = 79$ as described in Section 2. From these parameters, compute H_1 and h_1. By numerical inversion using the bisection method, we get $y_1 := H_{\text{Num},1}^{-1}(p) = 2.7$. We see that the RHS of Inequality (4) is (approximately) equal to $\frac{2\varepsilon_1}{\min\{h_1(y_1 \pm \varepsilon_1)\}} + \varepsilon_1 = 0.73 > \delta$. So, ε_1 is too large. In the next step, we set $\varepsilon_2 := 0.0005 \approx \frac{\delta}{\frac{2}{\min\{h_1(y_1 \pm \varepsilon_1)\}} + 1}$ and obtain $a(\varepsilon_2) = -7.9$, $b(\varepsilon_2) = 7.9$ and $N(\varepsilon_2) = 114$. We observe in Table 1 that the RHS of Inequality (4) is now satisfied, and Theorem 2.3 ensures that $|H_{\text{Num},2}^{-1}(p) - F^{-1}(p)| < \delta$.

Finally, a word about computational time. Note that the formulas for a, b and N in Section 2 do not depend on y or p and need to be computed only once. The COS method must evaluate the CF N times, which is extremely fast since φ is given in closed form. For example, using the R software and vectorized code on an Intel i7-10750H CPU computing the CDF of the TS distribution with $N = 50$ takes on average 11 μs.

Data availability

No data was used for the research described in the article.

References

Abate, J., Whitt, W., 1992. The Fourier-series method for inverting transforms of probability distributions. Queueing Syst. 10, 5–87.
Barndorff-Nielsen, O., 1978. Hyperbolic distributions and distributions on hyperbolae. Scand. J. Stat. 151–157.
Barndorff-Nielsen, O.E., 1997. Normal inverse Gaussian distributions and stochastic volatility modelling. Scand. J. Stat. 24 (1), 1–13.
Carr, P., Madan, D., 1999. Option valuation using the fast Fourier transform. J. Comput. Finance 2 (4), 61–73.
Fang, F., Oosterlee, C.W., 2009. A novel pricing method for European options based on Fourier-cosine series expansions. SIAM J. Sci. Comput. 31 (2), 826–848.
Gilchrist, W., 2000. Statistical Modelling with Quantile Functions. Chapman and Hall/CRC.
Hougaard, P., 1986. Survival models for heterogeneous populations derived from stable distributions. Biometrika 73 (2), 387–396.
Hughett, P., 1998. Error bounds for numerical inversion of a probability characteristic function. SIAM J. Numer. Anal. 35 (4), 1368–1392.
Junike, G., 2024. On the number of terms in the COS method for European option pricing. Numer. Math. 156 (2), 533–564.
Junike, G., Pankrashkin, K., 2022. Precise option pricing by the COS method–How to choose the truncation range. Appl. Math. Comput. 421, 126935.
Junike, G., Stier, H., 2024. The multidimensional COS method for option pricing. arXiv preprint arXiv:2307.12843.
Schoutens, W., 2003. Lévy Processes in Finance: Pricing Financial Derivatives. Wiley Online Library.
Shaw, W.T., McCabe, J., 2009. Monte Carlo sampling given a characteristic function: quantile mechanics in momentum space. arXiv preprint arXiv:0903.1592.
Wang, C., 2017. Pricing European Options by Stable Fourier-Cosine Series Expansions. arXiv preprint arXiv:1701.00886.