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Profit and loss decomposition in continuous time and approximations

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Abstract
Financial institutions and insurance companies that analyse the evolution and sources of profits and losses often look at risk factors only at discrete reporting dates, ignoring the detailed paths. Continuous-time decompositions avoid this weakness and also make decompositions consistent across different reporting grids. We construct a large class of continuous-time decompositions from a rearranged version of Itô's formula, and uniquely identify a preferred decomposition from the axioms of exactness, symmetry and normalisation. This unique decomposition turns out to be a stochastic limit of recursive Shapley values, but it suffers from a curse of dimensionality as the number of risk factors increases. We develop an approximation that breaks this curse when the risk factors almost surely have no simultaneous jumps.

Keywords Profit and loss attribution · Sequential decompositions · Change analysis · Risk decomposition · Itô's formula

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1 Introduction

Profit and loss (P&L) attribution, also known as change analysis, has a long history in risk management. P&L attribution is the process of analysing the change between

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two valuation dates and explaining the evolution of the P&L by the movement of the sources (risk factors) between the two dates; see Candland and Lotz [4]. In other words, the change in the P&L over time is *decomposed* in terms of the different risk factors to explain how each factor *contributes* to the P&L. In the literature, there are many ways to obtain a P&L attribution. For example, consider a portfolio in EUR consisting of a long position in the S&P 500, Y for short. The P&L of such a portfolio is driven by two risk factors, namely Y and the USD/EUR exchange rate, X for short. To decompose the P&L over one year, we look for two random variables D^X and D^Y such that

$$X_1 Y_1 - X_0 Y_0 = D^X + D^Y.$$

The numbers D^X and D^Y are interpreted as the contributions of X and Y to the P&L. In the literature, we can find many desirable properties that a decomposition should possess; see Shubik [28], Friedman and Moulin [12] and Shorrocks [26] among many others. The authors argue that a decomposition should be *symmetric*, i.e., the contributions of the risk factors should be independent of the way in which the risk factors are labelled or ordered. These authors also require that the sum of all contributions equals the P&L; such decompositions are called *exact*. Further, Christiansen [6] argues that a decomposition should be *normalised*, i.e., if a risk factor remains constant, its contribution to the P&L should be zero. It is also desirable for a decomposition to consider the full path of each risk factor, i.e., to use all available information; see Mai [18] and Flaig and Junike [9].

A common method for creating decompositions is to sequentially update the risk factors one by one while "freezing" all other risk factors. This idea dates back at least to Oaxaca [20] and Blinder [3], who developed a *sequential updating* (SU) decomposition technique in a one-period setting. The SU decomposition is given by

$$D^X = X_1 Y_0 - X_0 Y_0, \qquad D^Y = X_1 Y_1 - X_1 Y_0,$$

when we update the risk factor X first. Alternatively, one may update Y first to obtain

$$D^X = X_1 Y_1 - X_0 Y_1, \qquad D^Y = X_0 Y_1 - X_0 Y_0.$$

Each SU decomposition is exact, but if there are d risk factors, there are $d!$ different updating orders and therefore $d!$ different SU decompositions. Candland and Lotz [4] call the one-period SU decomposition *waterfall* and apply it to P&L attribution. See Fortin et al. [10] for an overview on how the SU decomposition is used in various fields of economics.

The SU decomposition can also be defined in a multi-period setting by dividing the time horizon into subintervals and applying the SU decomposition recursively on each subinterval. Jetses and Christiansen [16] and Christiansen [6] analysed the limit of the SU decomposition when the mesh size of the time grid converges to zero. In the limit, the decomposition takes the whole path into account, and the limiting SU decomposition is called the *infinitesimal sequential updating* (ISU) decomposition. The ISU decomposition is independent of any time grid, which is helpful "to prevent inconsistencies when using conflicting subintervals for different purposes"; see Flaig and Junike [9, Sect. 1].

The *averaged sequential updating* (ASU) decomposition, also known as the *Shapley value*, is simply the arithmetic average of the $d!$ possible SU decompositions. It has many desirable properties; in particular, it is exact and symmetric. Shapley [27] introduces the ASU decomposition for cooperative games. Shubik [28] defines the ASU decomposition for cost functions. Sprumont [29] and Friedman and Moulin [12] provide an axiomatisation of the ASU decomposition for cost functions. Jetses and Christiansen [16] define the *infinitesimal averaged sequential updating* (IASU) decomposition as the average of the $d!$ possible ISU decompositions.

We now summarise our main contributions. In this paper, we start directly in a time-continuous setting. If the portfolio is a C^2-function of the risk factors and the latter have continuous paths, Itô's formula provides a natural additive decomposition of the portfolio. Our main contributions are as follows. In order to treat risk factors with jumps, we provide a rearranged version of Itô's formula and use it to define a large class of reasonable decompositions, which we call *Itô decompositions* and which include all $d!$ ISU and the IASU decompositions as special cases. We prove that there is a unique Itô decomposition (up to indistinguishability) that satisfies the three axioms of exactness, symmetry and normalisation. We show that it is indistinguishable from the IASU decomposition. We further show that the IASU decomposition can be interpreted as the limiting case of the ASU decomposition. Compared to Jetses and Christiansen [16], who assume that the covariations between the risk factors are zero, we use much weaker assumptions to prove the convergence of the SU/ASU decompositions to the ISU/IASU decompositions.

In summary, we propose to use the IASU decomposition to obtain a P&L attribution because it considers the whole paths of the risk factors and satisfies the axioms of exactness, symmetry and normalisation. However, in practical applications, the IASU decomposition has two drawbacks: a) similarly to the ASU decomposition, it suffers from the curse of dimensionality; b) the IASU decomposition is defined by stochastic integrals, which somehow must be approximated in practice. Naively approximating these integrals can lead to decompositions that are no longer exact. As another important contribution of this paper, we show that the IASU decomposition does not suffer from the curse of dimensionality if the risk factors do not have simultaneous jumps. In this case, the IASU decomposition is indistinguishable from the average of two (suitably selected) ISU decompositions. To avoid point b), we suggest approximating ISU/IASU by SU/ASU.

Up to now, most practitioners have applied an arbitrary SU decomposition in a one-period setting to obtain an annual P&L attribution; see Candland and Lotz [4]. Working with real market data, Flaig and Junike [9] empirically show that the SU decomposition depends significantly on the order or labelling of the risk factors, and that some SU decompositions may even ignore relevant risk factors, which may "lead to wrong trading and hedging decisions"; see Flaig and Junike [9, Sect. 1].

Our theoretical analysis suggests using the average of only two SU decompositions with a sufficiently fine time grid to obtain a P&L attribution, since such a decomposition is arbitrarily close to the IASU decomposition when the risk factors do not have simultaneous jumps. To obtain these two SU decompositions, define one SU decomposition in any order, e.g. alphabetically ascending, and another SU decomposition by the reverse order, e.g. alphabetically descending; see Theorem 3.10 for details.

Thus our analysis is highly relevant for practitioners: we recommend computing two SU decompositions instead of one and using a finer grid than just annual data to obtain a decomposition that is much closer to the IASU decomposition than a single SU decomposition. While the choice of the decomposition (the average of two SU decompositions) is theoretically justified, we have only numerical experiments available to estimate the time grid, and we recommend using monthly or weekly data.

Is there any other way to break the curse of dimensionality? Christiansen [6] proves that the ISU decomposition is symmetric if it is stable with respect to small perturbations in the empirical observation of the risk factors. In Appendix A.3, we show that the ISU decomposition of a simple product of two correlated Brownian motions is not stable. This shows that stability is a rather strong assumption.

There are other decomposition principles as well. There is the so-called *one-at-a-time* (OAT) decomposition, which is also known as *bump and reset*; see Candland and Lotz [4]. The OAT decomposition is closely related to the SU decomposition. It is symmetric, but in general not exact. Frei [11] analyses the limit of the OAT decomposition when the mesh size of the time grid converges to zero.

There are also completely different approaches. Fischer [8] uses a conditional expectations approach. Rosen and Saunders [24] use the Hoeffding method for a decomposition of credit risk portfolios. Frei [11] and Bielecki et al. [1] use the Euler principle for risk attribution. Ramlau-Hansen [23] and Norberg [19] decompose surplus in life insurance by heuristic integral representations, where the integrators are interpreted as the driving forces of change and determine the attribution. A similar idea is used in Schilling et al. [25] based on the martingale representation theorem.

This article is structured as follows. In Sect. 2, we establish some notation. In Sect. 3, we develop a rearranged version of Itô's formula and introduce the family of Itô decompositions. We show that the IASU decomposition is the only exact and symmetric Itô decomposition, and we break the curse of dimensionality of the IASU decomposition in Theorem 3.10. In Sect. 4, we prove that the IASU decomposition can be approximated by the ASU decomposition. In Sect. 5, we provide some numerical applications. Section 6 concludes.

2 Notation

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions, i.e., $\mathcal{F}_0$ contains all nullsets and $\mathbb{F}$ is right-continuous. Let $\mathcal{X}$ be the set of all real-valued $\mathbb{F}$-semimartingales. A so-called *risk basis* or *information basis* is a d-dimensional semimartingale $X \in \mathcal{X}^d$, and its d components are called *risk factors* or *sources of risk*. We denote the stopped semimartingale by $X^\sigma = (X^{1,\sigma}, \ldots, X^{d,\sigma})$ for a stopping time σ. Equality of random variables is understood in the almost sure sense, and equality of stochastic processes is understood up to indistinguishability. Let C^2 be the set of twice continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$. For $f \in C^2$ and $i, j = 1, \ldots, d$, we write f_i and f_{ij} for the partial derivatives $\partial_i f$ and $\partial_i \partial_j f$. By $x \wedge y$, we denote the minimum of two real numbers x and y. We call a map $F : \mathcal{X}^d \to \mathcal{X}$ *non-anticipative* if for any stopping time σ, it holds that for all

$X \in \mathcal{X}^d$,

$$F_t(X^\sigma) = F_{t\wedge\sigma}(X), \qquad t \geq 0. \tag{2.1}$$

Such a non-anticipative mapping depends only on the information up to time t, i.e., on X^t. By $\mathcal{M}$, we denote some subspace of all non-anticipative mappings. By $\mathcal{M}(C^2)$, we denote the space of functionals $F : \mathcal{X}^d \to \mathcal{X}$ such that $F(X) = (f(X_t))_{t\geq 0}$ for $X \in \mathcal{X}^d$ and some $f \in C^2$, which are clearly non-anticipative. By σ_d, we denote the set of all $d!$ permutations of $\{1, \ldots, d\}$. Let $\mathrm{id} \in \sigma_d$ be the identity. In a slight abuse of notation, we define for $\pi \in \sigma_d$,

$$\pi(x) = (x^{\pi(1)}, \ldots, x^{\pi(d)}), \qquad x \in \mathbb{R}^d,$$
$$\pi(X) = (X^{\pi(1)}, \ldots, X^{\pi(d)}), \qquad X \in \mathcal{X}^d.$$

For two one-dimensional semimartingales Z and Y and a càglàd process H, we denote by $\int_0^t H_s dZ_s := \int_{(0,t]} H_s dZ_s$ the stochastic integral. In particular, $\int_0^0 H_s dZ_s = 0$ by convention. We further set $Z_{0-} = 0$,

$$Z_{t-} = \lim_{\varepsilon \searrow 0} Z_{t-\varepsilon}, \qquad t > 0,$$
$$\Delta Z_t = Z_t - Z_{t-}, \qquad t \geq 0,$$
$$[Z, Y] = ZY - Z_0 Y_0 - \int_0^\cdot Z_{u-} dY_u - \int_0^\cdot Y_{u-} dZ_u,$$
$$[Z, Y]^c = [Z, Y] - \sum_{0<s\leq\cdot} \Delta Z_s \Delta Y_s.$$

We write $\xrightarrow{p}$ for the convergence in probability of a sequence of random variables. For $A \subseteq \{1, \ldots, d\}$, we define the projection

$$p_A : \mathbb{R}^d \to \mathbb{R}^d, \qquad x \mapsto \big(x^1 1_A(1), \ldots, x^d 1_A(d)\big),$$

where the function $1_A(h)$ is 1 if $h \in A$ and 0 otherwise.

3 Family of Itô decompositions

Similarly to Shorrocks [26], Christiansen [6], we define a *decomposition* as follows.

Definition 3.1 A map

$$\delta : \mathcal{M} \times \mathcal{X}^d \to \mathcal{X}^d, \qquad (F, X) \mapsto \big(\delta^1(F, X), \ldots, \delta^d(F, X)\big)$$

is called a *decomposition*.

We interpret $\delta^i_t(F, X)$ as the contribution of X^i to the profit and loss $F_t(X) - F_0(X)$ in $[0, t]$. We recall the following three axioms from the literature:

i) A decomposition is called *exact* if for all $F \in \mathcal{M}$ and $X \in \mathcal{X}^d$, it holds that

$$F(X) - F_0(X) = \delta^1(F, X) + \cdots + \delta^d(F, X).$$

ii) A decomposition is called *symmetric* if for all $\pi \in \sigma_d$, $F \in \mathcal{M}$ and $X \in \mathcal{X}^d$, it holds that

$$F(X) = F\big(\pi(X)\big) \qquad \Longrightarrow \qquad \delta^i(F, X) = \delta^{\pi^{-1}(i)}\big(F, \pi(X)\big).$$

iii) A decomposition is called *normalised* if for all $0 \le r < s < \infty$, $i = 1, \ldots, d$, $F \in \mathcal{M}$ and $X \in \mathcal{X}^d$, it holds that

$$\begin{aligned} &X^i \text{ is indistinguishable from a constant process on } (r, s] \\ &\Longrightarrow \delta^i(F, X) \text{ is indistinguishable from a constant process on } (r, s]. \end{aligned}$$

Axiom i) ensures that a decomposition is able to fully explain the P&L; see Shorrocks [26] and Christiansen [6]. Axiom ii) considers symmetric maps F and states that if F does not depend on the order or labelling of the risk factors, then neither does the decomposition. The symmetry axiom is motivated by the fact that $\delta^i(F, X)$ represents the contribution of X^i and that the term $\delta^{\pi^{-1}(i)}(F, \pi(X))$ also describes the contribution of

$$\big(\pi(X)\big)^{\pi^{-1}(i)} = \big(X^{\pi(1)}, \ldots, X^{\pi(d)}\big)^{\pi^{-1}(i)} = X^i.$$

The symmetry axiom has already been mentioned in similar form in Friedman and Moulin [12] and Shorrocks [26]. Finally, for axiom iii), if the risk factor X^i remains constant during $(r, s]$, it does not contribute to $F_s(X) - F_r(X)$, and so the contribution of X^i in $(r, s]$ should also be zero. This is exactly reflected by the normalisation axiom, taken from Christiansen [6].

Next, we indicate how Itô's formula helps to define decomposition principles. Let $f : \mathbb{R}^d \to \mathbb{R}$ be in C^2. For $i, j = 1, \ldots, d$, let

$$I^i := \int_0^\cdot f_i(X_{s-})dX^i_s, \qquad I^{ij} := \int_0^\cdot f_{ij}(X_{s-})d[X^i, X^j]^c_s, \tag{3.1}$$

$$S := \sum_{0<s\le\cdot} \bigg(f(X_s) - f(X_{s-}) - \sum_{i=1}^d f_i(X_{s-})\Delta X^i_s \bigg). \tag{3.2}$$

Itô's formula states that for $t \ge 0$, we have for any semimartingale $X \in \mathcal{X}^d$ that

$$f(X_t) - f(X_0) = \sum_{i=1}^d I^i_t + \frac{1}{2}\sum_{i=1}^d I^{ii}_t + \frac{1}{2}\sum_{\substack{i,j=1\\ i\ne j}}^d I^{ij}_t + S_t. \tag{3.3}$$

If we assume that X has continuous paths without interaction effects, i.e., $I^{ij} = 0$, $i \neq j$, and $S = 0$, then (3.3) provides a natural way to additively decompose the P&L $f(X_t) - f(X_0)$. Indeed, by the normalisation axiom, I^i and I^{ii} should be assigned to δ^i, which is interpreted as the contribution of X^i. To see this, assume that some δ^j depends on I^i or I^{ii} for $i \neq j$. Assume that X^j is constant everywhere. According to the normalisation axiom, we should then have $\delta^j = 0$. So δ^j must not depend on I^i or on I^{ii}.

However, how to handle the interaction effects I^{ij}, $i \neq j$, and the jump part S is not so obvious. Therefore, we provide in Proposition 3.3 a rearranged version of Itô's formula. Based on that result, we define the large family of *Itô decompositions* in Definition 3.4 and show in Sect. 3 that this family contains many well-known decomposition principles as special cases. Within the family of Itô decompositions, we identify a single decomposition that satisfies the axioms of exactness, symmetry and normalisation. For $A \subseteq \{1, \dots, d\}$, $i \in \{1, \dots, d\}$ and $s > 0$, define

$$Y_s^{i,A} := f\big(X_{s-} + p_A(\Delta X_s)\big) - f\big(X_{s-} + p_{A\setminus\{i\}}(\Delta X_s)\big) - f_i(X_{s-})\Delta X_s^i$$

and

$$S^{i,A}(X) := \sum_{0<s\leq \cdot} Y_s^{i,A}.$$

For $\pi \in \sigma_d$, define

$$S^{i,\pi}(X) := S^{i,\{j:\pi(j)\leq\pi(i)\}}(X). \tag{3.4}$$

To obtain $S^{i,\pi}(X)$, all time points s where X^i jumps are considered. All risk factors except X^i are fixed at s or $s-$, depending on the choice of π, and only X^i varies between $s-$ and s.

Lemma 3.2 *Fix $i \in \{1, \dots, d\}$, $X \in \mathcal{X}^d$ and $A \subseteq \{1, \dots, d\}$. If $i \in A$, then $S^{i,A}(X)$ is a semimartingale with a.s. paths of finite variation on compacts.*

Proof Fix $X \in \mathcal{X}^d$. Let N be a nullset such that $u \mapsto |X_u^i(\omega)|$, $i = 1, \dots, d$, is càdlàg for $\omega \in \Omega \setminus N$ and

$$\sum_{h,j=1}^{d} \sum_{0<s\leq t} |\Delta X_s^h(\omega)\Delta X_s^j(\omega)| < \infty, \qquad \omega \in \Omega \setminus N, t \geq 0. \tag{3.5}$$

Such an N exists as X is a semimartingale. Let $\omega \in \Omega \setminus N$ and $t \geq 0$. Let $M_\omega \subseteq \mathbb{R}^d$ be the closure of the set $\{X_u(\omega) : u \in [0, t]\}$, which is compact. The function f and its derivatives are continuous and reach a maximum and minimum on the convex hull of M_ω, which is compact by Carathéodory's theorem; see Grünbaum [14, Sect. 2.3]. Hence f and its derivatives are bounded on the convex hull of M_ω. For $s \in (0, t]$, we develop f around $X_{s-}(\omega)$ using a Taylor expansion. We have that

$$f\Big(X_{s-}(\omega) + p_A\big(\Delta X_s(\omega)\big)\Big) = f\big(X_{s-}(\omega)\big) + \sum_{h\in A} f_h\big(X_{s-}(\omega)\big)\Delta X_s^h(\omega) + R(\omega),$$

where $R(\omega)$ is the remainder term of the Taylor expansion, i.e., for some $\theta(\omega) \in [0, 1]$, it holds that

$$R(\omega) = \frac{1}{2} \sum_{h,j \in A} f_{hj}\Big(X_{s-}(\omega) + \theta(\omega) p_A\big(\Delta X_s(\omega)\big)\Big) \Delta X_s^h(\omega) \Delta X_s^j(\omega).$$

The term $f(X_{s-}(\omega)+p_{A\setminus\{i\}}(\Delta X_s(\omega)))$ can be treated similarly. Since $i \in A$, it holds for some $C(\omega) > 0$, which does not depend on s or $\theta(\omega)$, that

$$Y_s^{i,A} \leq C(\omega) \sum_{h,j \in A} |\Delta X_s^h(\omega) \Delta X_s^j(\omega)|.$$

It follows by (3.5) that

$$\sum_{0<s\leq t} |Y_s^{i,A}(\omega)| < \infty, \qquad \omega \in \Omega \setminus N. \tag{3.6}$$

Since t was arbitrary, (3.6) implies that $u \mapsto S_u^{i,A}(X)(\omega)$, $\omega \in \Omega \setminus N$, is càdlàg and of finite variation on compacts. Therefore $S^{i,A}(X)$ is a semimartingale. □

Proposition 3.3 *Let $\pi \in \sigma_d$, $f \in C^2$ and $X \in \mathcal{X}^d$. For all $t \geq 0$, it holds that*

$$f(X_t) - f(X_0) = \sum_{i=1}^d \Bigg(I_t^i + \frac{1}{2} I_t^{ii} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^d I_t^{ij} + S_t^{i,\pi} \Bigg),$$

where I^i and I^{ij} are defined in (3.1) and $S^{i,\pi}$ is defined in (3.4).

Proof Since the series telescopes, we have that

$$\begin{aligned} &f(X_s) - f(X_{s-}) \\ &= \sum_{i=1}^d f\big(X_{s-} + p_{\{j:\pi(j)\leq\pi(i)\}}(\Delta X_s)\big) - f\big(X_{s-} + p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big). \end{aligned}$$

By (3.6), it holds for any $t \geq 0$ that

$$\sum_{i=1}^d S_t^{i,\pi}(X) = \sum_{0<s\leq t} \sum_{i=1}^d Y_s^{i,\{j:\pi(j)\leq\pi(i)\}} = S_t, \tag{3.7}$$

where S is defined in (3.2). The claim then follows by Itô's formula. □

Definition 3.4 Let $\lambda_{ij} \in [0, 1]$ for $i, j = 1, \ldots, d$. Let $\mu_\pi \in [0, 1]$ for $\pi \in \sigma_d$. The decomposition

$$\delta^{\text{Itô}} : \mathcal{M}(C^2) \times \mathcal{X}^d \to \mathcal{X}^d, \qquad (F, X) \mapsto \big(\delta^{\text{Itô},1}(F, X), \ldots, \delta^{\text{Itô},d}(F, X)\big),$$

where

$$\delta^{\text{Itô},i}(F,X) = I^i + \frac{1}{2}I^{ii} + \sum_{\substack{j=1\\ j\neq i}}^{d} \lambda_{ij} I^{ij} + \sum_{\pi\in\sigma_d} \mu_\pi S^{i,\pi}(X), \qquad i = 1,\ldots,d,$$

is called *Itô decomposition with parameters* $(\lambda_{ij})_{i,j=1,\ldots,d}$ *and* $(\mu_\pi)_{\pi\in\sigma_d}$.

The definition of the Itô decomposition is motivated by Proposition 3.3 and the normalisation axiom. Below (3.3), we already argued that I^i and I^{ii} should be attributed to X^i in order to satisfy the normalisation axiom. If parts of the interaction effect I^{ij} were assigned to the contribution of X^h for $h \notin \{i,j\}$, the decomposition would no longer be normalised. Therefore only the risk factors X^i and X^j are assigned shares λ_{ij} and λ_{ji} of the interaction effect I^{ij}.

Note that $S^{i,\pi}(X)$ contains only jumps in the ith component. If $S^{i,\pi}(X)$ were assigned to the contribution of some X^j, $j \neq i$, the normalisation axiom would be violated if X^j is constant. Therefore $S^{i,\pi}$ should be assigned to the contribution of X^i. Since there are $d!$ different ways to decompose the jumps and violate neither the normalisation axiom nor the exactness axiom, we propose to assign to X^i a weighted average of all $S^{i,\pi}(X)$, $\pi \in \sigma_d$.

Remark 3.5 Since each Itô decomposition is linear in the first argument F, a portfolio can be decomposed by decomposing each individual instrument.

We recall some special members of the family of Itô decompositions, namely the IASU and the $d!$ different ISU decompositions, which were introduced in Jetses and Christiansen [16]. All Itô decompositions are normalised. We shall prove that the IASU decomposition is the only Itô decomposition that is exact and symmetric. We also see that the ISU decomposition is closely related to the IASU decomposition and that the IASU decomposition is the limiting case of the well-known ASU decomposition (also known as Shapley value), which is defined over a discrete time grid in Sect. 4.

Definition 3.6 The *IASU (infinitesimal averaged sequential updating) decomposition* $\delta^{\text{IASU}} : \mathcal{M}(C^2) \times \mathcal{X}^d \to \mathcal{X}^d$ is defined by

$$\delta^{\text{IASU},i}(F,X) = I^i + \frac{1}{2}\sum_{j=1}^{d} I^{ij} + \frac{1}{d!}\sum_{\pi\in\sigma_d} S^{i,\pi}(X), \qquad i = 1,\ldots,d.$$

Remark 3.7 The Itô decompositions are overparametrised: in view of (A.3) below in Lemma A.2, we can represent the IASU decomposition as

$$\delta^{\text{IASU},i}(F,X) = I^i + \frac{1}{2}\sum_{j=1}^{d} I^{ij} + \sum_{\substack{A\subseteq\{1,\ldots,d\}\\ i\in A}} S^{i,A}(X)\xi_{i,A},$$

where

$$\xi_{i,A} := \sum_{\substack{\pi \in \sigma_d \\ \{j:\pi(j)\leq\pi(i)\}=A}} \frac{1}{d!} = \frac{(|A|-1)!(d-|A|)!}{d!}. \tag{3.8}$$

So the computational effort to obtain $\delta^{\mathrm{IASU},i}$ can be reduced from $\mathcal{O}(d!)$ to $\mathcal{O}(2^{d-1})$ for $d \to \infty$.

Definition 3.8 Let $\pi \in \sigma_d$. The *ISU (infinitesimal sequential updating) decomposition* $\delta^{\mathrm{ISU},\pi} : \mathcal{M}(C^2) \times \mathcal{X}^d \to \mathcal{X}^d$ *with updating order* π is defined by

$$\delta^{\mathrm{ISU},i,\pi}(F,X) = I^i + \frac{1}{2}I^{ii} + \sum_{\substack{j=1 \\ \pi(j)<\pi(i)}}^{d} I^{ij} + S^{i,\pi}(X), \qquad i = 1,\dots,d.$$

Theorem 3.9 *Every Itô decomposition that is symmetric and exact is indistinguishable from the IASU decomposition. The IASU decomposition is related to the ISU decomposition by*

$$\delta^{\mathrm{IASU},i}(F,X) = \frac{1}{d!}\sum_{\pi\in\sigma_d} \delta^{\mathrm{ISU},i,\pi}(F,X), \qquad i = 1,\dots,d,\ X \in \mathcal{X}^d,\ F \in \mathcal{M}(C^2). \tag{3.9}$$

Proof First, we show that the IASU decomposition is exact and symmetric and satisfies (3.9). By Proposition 3.3, it follows that δ^{IASU} is an exact Itô decomposition. Use (A.4) to see that the IASU decomposition is symmetric. If $d = 1$, (3.9) is trivially true. Assume $d \geq 2$. Fix $i \in \{1,\dots,d\}$. Note that

$$\sum_{\pi\in\sigma_d} 1_{\{\pi(j)<\pi(i)\}} = \begin{cases} \frac{d!}{2}, & j \neq i, \\ 0, & j = i. \end{cases}$$

It follows that

$$\begin{aligned} \frac{1}{d!}\sum_{\pi\in\sigma_d}\sum_{\substack{j=1 \\ \pi(j)<\pi(i)}}^{d} I^{ij} &= \sum_{j=1}^{d} I^{ij}\frac{1}{d!}\sum_{\pi\in\sigma_d} 1_{\{\pi(j)<\pi(i)\}} \\ &= \frac{1}{2}I^{i1} + \dots + \frac{1}{2}I^{i(i-1)} + \frac{1}{2}I^{i(i+1)} + \dots + \frac{1}{2}I^{\mathrm{id}} \\ &= \frac{1}{2}\sum_{j\neq i} I^{ij}. \end{aligned} \tag{3.10}$$

Equation (3.10) implies (3.9).

Now we show that all exact and symmetric Itô decompositions are indistinguishable from the IASU decomposition. Let δ be a symmetric and exact Itô decomposition with parameters $(\lambda_{ij})_{i,j=1,\dots,d}$ and $(\mu_\pi)_{\pi\in\sigma_d}$. Since the Itô decomposition is over-parametrised, we use the alternative parametrisation according to (A.3). To prove that δ is indistinguishable from the IASU decomposition, we show that λ_{ij} and $\xi_{i,A,\mathrm{id}}$ are equal to the coefficients $\frac{1}{2}$ and $\xi_{i,A}$ defined in (3.8).

Suppose that $\lambda_{hk} \neq \frac{1}{2}$. Let $X \in \mathcal{X}^d$ have continuous paths with $X^i = 1, i \notin \{h,k\}$, and $[X^h, X^k] \neq 0$. Let $F(X) = \prod_{i=1}^d X^i$. Then $F(X) = F(\pi(X))$ for $\pi \in \sigma_d$. Note that $I^{kh} = I^{hk}$. As δ is exact, we have

$$\sum_{i=1}^d \delta^i(F, X) = I^h + I^k + \lambda_{hk} I^{hk} + \lambda_{kh} I^{kh} = F(X) - F_0(X) = I^h + I^k + I^{hk},$$

hence $\lambda_{kh} = 1 - \lambda_{hk} \neq \lambda_{hk}$. Let $\pi \in \sigma_d$ be such that $\pi^{-1}(h) = k$. Then we get

$$\delta^{\pi^{-1}(h)}\big(F, \pi(X)\big) = \delta^k\big(F, \pi(X)\big) = I^h + \lambda_{kh} I^{kh} \neq I^h + \lambda_{hk} I^{hk} = \delta^h(F, X).$$

This means that δ is not symmetric, which is a contradiction to our assumption. So we necessarily have $\lambda_{ij} = \frac{1}{2}, i, j = 1, \dots, d$.

Now let $a \in \{1, \dots, d\}$. Let $A, B \subseteq \{1, \dots, d\}$ with $|A| = |B| = a$ and $i \in A$, $j \in B$ for $i, j \in \{1, \dots, d\}$. Then there is a permutation $\eta \in \sigma_d$ such that $\eta^{-1}(A) = B$ and $j = \eta^{-1}(i)$. By (A.6), it follows that

$$\xi_{i,A,\mathrm{id}} = \xi_{j,B,\mathrm{id}}. \tag{3.11}$$

Let $A_1, \dots, A_d \subseteq \{1, \dots, d\}$ with $j \in A_j$ and $|A_j| = a$, $j = 1, \dots, d$. Since

$$\Big|\big\{A \subseteq \{1, \dots, d\} : j \in A, |A| = a\big\}\Big| = \binom{d-1}{a-1}, \tag{3.12}$$

we obtain by (A.7), (3.11) and (3.12) that

$$1 = \sum_{j=1}^d \sum_{\substack{A\subseteq\{1,\dots,d\}\\ |A|=a, j\in A}} \xi_{j,A,\mathrm{id}} = \sum_{j=1}^d \binom{d-1}{a-1} \xi_{j,A_j,\mathrm{id}} = d\binom{d-1}{a-1}\xi_{i,A,\mathrm{id}}$$

for $A \subseteq \{1, \dots, d\}$ with $i \in A$ and $|A| = a$. Therefore we can conclude that

$$\xi_{i,A,\mathrm{id}} = \frac{1}{d\binom{d-1}{|A|-1}} = \frac{(|A|-1)!(d-|A|)!}{d!}.$$

□

The next result shows that the curse of dimensionality of the IASU decomposition can be broken if there are no simultaneous jumps.

Theorem 3.10 *Let $X \in \mathcal{X}^d$ and $F \in \mathcal{M}(C^2)$. If $\Delta X^h \Delta X^j = 0$ for all $h, j \in \{1, \dots, d\}$ with $h \neq j$, then for any $\pi \in \sigma_d$ and $\pi' = d + 1 - \pi$,*

$$\delta^{\mathrm{IASU},i}(F, X) = \frac{1}{2}\big(\delta^{\mathrm{ISU},i,\pi}(F, X) + \delta^{\mathrm{ISU},i,\pi'}(F, X)\big), \qquad i = 1, \dots, d. \tag{3.13}$$

Proof Fix $0 < s < \infty$. If $\Delta X_s^i = 0$, we have

$$
\begin{aligned}
&f\big(X_{s-} + p_{\{j:\pi(j)\leq\pi(i)\}}(\Delta X_s)\big) - f\big(X_{s-} + p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big)\\
&= f\big(X_{s-} + p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big) - f\big(X_{s-} + p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big)\\
&= 0.
\end{aligned}
$$

If $\Delta X_s^i \neq 0$, we have $X_s^j = X_{s-}^j$ for all $j \neq i$ and hence

$$
\begin{aligned}
&f\big(X_{s-} + p_{\{j:\pi(j)\leq\pi(i)\}}(\Delta X_s)\big) - f\big(X_{s-} + p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big)\\
&= f(X_s) - f(X_{s-}).
\end{aligned}
$$

Hence for $\pi \in \sigma_d$ and $i = 1, \dots, d$, it holds that

$$
\begin{aligned}
\delta^{\mathrm{ISU},i,\pi} = I^i + \frac{1}{2} I^{ii} + \sum_{\substack{j=1\\ \pi(j)<\pi(i)}}^{d} I^{ij}\\
+ \sum_{\substack{0<s\leq\cdot\\ \Delta X_s^i\neq 0}} \big(f(X_s) - f(X_{s-}) - f_i(X_{s-})\Delta X_s^i\big).
\end{aligned}
\tag{3.14}
$$

Due to (3.9) and (3.10), we have that

$$
\begin{aligned}
\delta^{\mathrm{IASU},i}(F, X) = I^i + \frac{1}{2} \sum_{j=1}^{d} I^{ij}\\
+ \sum_{\substack{0<s\leq\cdot\\ \Delta X_s^i\neq 0}} \big(f(X_s) - f(X_{s-}) - f_i(X_{s-})\Delta X_s^i\big).
\end{aligned}
\tag{3.15}
$$

For $\pi \in \sigma_d$, let $\delta^{\mathrm{ISU},i,\pi}$ be the ISU decomposition with updating order π and define $\pi'(i) = d + 1 - \pi(i)$, $i = 1, \dots, d$. Note that

$$
\sum_{\substack{j=1\\ \pi(j)<\pi(i)}}^{d} + \sum_{\substack{j=1\\ \pi'(j)<\pi'(i)}}^{d} = \sum_{\substack{j=1\\ \pi(j)<\pi(i)}}^{d} + \sum_{\substack{j=1\\ \pi(j)>\pi(i)}}^{d} = \sum_{\substack{j=1\\ i\neq j}}^{d}. \tag{3.16}
$$

Equations (3.14)–(3.16) imply (3.13). □

Remark 3.11 Theorem 3.10 can be generalised to the case where some, but not all, risk factors have simultaneous jumps. For example, suppose $d = 3$ and $\Delta X^1 \Delta X^j = 0$, $j \in \{2, 3\}$, but possibly $\Delta X^2 \Delta X^3 \neq 0$. It is then easy to see that (3.13) still holds. Or, if $d = 4$ and $\Delta X^1 \Delta X^j = 0$, $j \in \{2, 3, 4\}$, the IASU decomposition can be written as a weighted average of four ISU decompositions instead of eight ISU decompositions, which would be necessary if all risk factors had simultaneous jumps.

Corollary 3.12 *Let $X \in \mathcal{X}^d$ and $F \in \mathcal{M}(C^2)$. If $[X^h, X^j] = 0$ for all $h, j \in \{1, \dots, d\}$ with $h \neq j$, then*

$$\delta^{\mathrm{IASU},i}(F, X) = \delta^{\mathrm{ISU},i,\pi}(F, X), \qquad i = 1, \dots, d,$$

where $\pi \in \sigma_d$ is arbitrary.

Proof The assumption $[X^i, X^j] = 0$ for $i \neq j$ implies $\Delta X^i \Delta X^j = \Delta[X^i, X^j] = 0$. Therefore $S^{i,\pi_1} = S^{i,\pi_2}$, $\pi_1, \pi_2 \in \sigma_d$; see the proof of Theorem 3.10. The assertion follows directly from Definitions 3.6 and 3.8. □

Example 3.13 How does the IASU decomposition deal with simultaneous jumps? Let $d = 2$ and assume that $X = (X^1, X^2)$ is a pure-jump semimartingale of finite variation. Then the IASU decomposition is given by

$$\delta^{\mathrm{IASU},1}(F, X) = \frac{1}{2} \sum_{0<s\leq \cdot} \Big(\big(f(X_s^1, X_{s-}^2) - f(X_{s-})\big) + \big(f(X_s) - f(X_{s-}^1, X_s^2)\big)\Big),$$

$$\delta^{\mathrm{IASU},2}(F, X) = \frac{1}{2} \sum_{0<s\leq \cdot} \Big(\big(f(X_s) - f(X_s^1, X_{s-}^2)\big) + \big(f(X_{s-}^1, X_s^2) - f(X_{s-})\big)\Big).$$

The latter formulas are averages of sequential updates from time $s-$ to time s.

Example 3.14 We decompose the P&L of the portfolio $P = X^1 X^2$ of a foreign stock, where X^1 is the foreign exchange rate and X^2 the stock price in the foreign currency. The instantaneous P&L of the foreign stock in domestic currency is given by

$$dP_t = X_{t-}^1 dX_t^2 + X_{t-}^2 dX_t^1 + d[X^1, X^2]_t,$$

i.e., it can be decomposed into the variation of the exchange rate, the variation of the stock price and interaction effects; compare with Mai [18]. The IASU decomposition equally distributes the interaction effect between $\delta^{\mathrm{IASU},1}$ and $\delta^{\mathrm{IASU},2}$. To see this, observe that

$$\begin{aligned}
\delta^{\mathrm{IASU},1}(F, X) &= \int_0^\cdot X_{s-}^2 dX_s^1 + \frac{1}{2}[X^1, X^2]^c \\
&\quad + \frac{1}{2} \sum_{0<s\leq \cdot} \big((X_s^1 X_{s-}^2 - X_{s-}^1 X_{s-}^2) + (X_s^1 X_s^2 - X_{s-}^1 X_s^2) \\
&\qquad - 2X_{s-}^2 (X_s^1 - X_{s-}^1)\big) \\
&= \int_0^\cdot X_{s-}^2 dX_s^1 + \frac{1}{2}[X^1, X^2],
\end{aligned}$$

where $F(X) = X^1 X^2$. For $\delta^{\mathrm{IASU},2}$, the reasoning is similar.

4 SU and ASU decompositions and their limits

The time-dynamic SU and ASU decompositions are defined on discrete time grids; see Jetses and Christiansen [16] and Christiansen [6]. A light introduction to the SU decomposition can be found in Candland and Lotz [4]. In this section, we recall the definitions of these decompositions and provide sufficient conditions such that the SU and the ASU decompositions converge to the ISU and IASU decompositions, respectively, as the mesh size of the discrete time grid converges to zero. We recall the following definition from Protter [21, Sect. II.5].

Definition 4.1 An infinite sequence of finite stopping times $0 = \sigma_0 < \sigma_1 < \cdots$ such that $\sup_k \sigma_k = \infty$ a.s. is called an *unbounded random partition*. A sequence $(\gamma_n)_{n\in\mathbb{N}}$ of unbounded random partitions $\gamma_n = \{0 = \sigma_0^n < \sigma_1^n < \cdots\}$ is said to *tend to the identity* if $\sup_k |\sigma_{k+1}^n - \sigma_k^n| \to 0$ a.s. for $n \to \infty$.

Definition 4.2 Let $\gamma = \{0 = \sigma_0 < \sigma_1 < \cdots\}$ be an unbounded random partition. The *SU (sequential updating) decomposition* $\delta^{\mathrm{SU},\pi,\gamma} : \mathcal{M} \times \mathcal{X}^d \to \mathcal{X}^d$ *with updating order* $\pi \in \sigma_d$ is defined by

$$\delta^{\mathrm{SU},i,\pi,\gamma}(F, X) = \sum_{\ell=0}^{\infty} \Big(F\big(X^{\sigma_\ell} + p_{\{j:\pi(j)\leq\pi(i)\}}(X^{\sigma_{\ell+1}} - X^{\sigma_\ell})\big) - F\big(X^{\sigma_\ell} + p_{\{j:\pi(j)<\pi(i)\}}(X^{\sigma_{\ell+1}} - X^{\sigma_\ell})\big)\Big). \tag{4.1}$$

In words, divide the time horizon $[0, t]$ into finitely many subintervals, and to obtain the contribution of X^i, fix all risk factors at the beginning σ_ℓ or the end $\sigma_{\ell+1}$ of each subinterval (depending on the updating order π) and allow only X^i to vary between σ_ℓ and $\sigma_{\ell+1}$.

Proposition 4.3 *The decomposition* $\delta^{\mathrm{SU},\pi,\gamma} : \mathcal{M} \times \mathcal{X}^d \to \mathcal{X}^d$ *is well defined by* (4.1) *and exact. The sum in* (4.1) *evaluated at* $t \in [0, \infty)$ *is a.s. finite.*

Proof Let $X \in \mathcal{X}^d$, $F \in \mathcal{M}$, $\pi \in \sigma_d$, $n \in \mathbb{N}$ and $t \geq 0$. Using (2.1) twice, we get

$$\delta_{t\wedge\sigma_n}^{\mathrm{SU},i,\pi,\gamma}(F, X) = \sum_{\ell=0}^{\infty} \Big(F_t\big(X^{\sigma_\ell\wedge\sigma_n} + p_{\{j:\pi(j)\leq\pi(i)\}}(X^{\sigma_{\ell+1}\wedge\sigma_n} - X^{\sigma_\ell\wedge\sigma_n})\big) - F_t\big(X^{\sigma_\ell\wedge\sigma_n} + p_{\{j:\pi(j)<\pi(i)\}}(X^{\sigma_{\ell+1}\wedge\sigma_n} - X^{\sigma_\ell\wedge\sigma_n})\big)\Big) \tag{4.2}$$

$$= \sum_{\ell=0}^{n-1} \Big(F_{t\wedge\sigma_n}\big(X^{\sigma_\ell} + p_{\{j:\pi(j)\leq\pi(i)\}}(X^{\sigma_{\ell+1}} - X^{\sigma_\ell})\big) - F_{t\wedge\sigma_n}\big(X^{\sigma_\ell} + p_{\{j:\pi(j)<\pi(i)\}}(X^{\sigma_{\ell+1}} - X^{\sigma_\ell})\big)\Big) \tag{4.3}$$

since all summands with $\ell \geq n$ on the right hand-side of (4.2) are equal to zero. By (4.3), for each n, the process $\delta^{\mathrm{SU},i,\pi,\gamma}(F, X)$ stopped at σ_n is a finite sum of semimartingales and hence a semimartingale. By Protter [21, Sect. II.2] and since $\sigma_n \to \infty$ a.s. for $n \to \infty$, the process $\delta^{\mathrm{SU},i,\pi,\gamma}(F, X)$ is a semimartingale and the decomposition $\delta^{\mathrm{SU},\pi,\gamma}$ is therefore well defined. The fact that $\sigma_n \to \infty$ a.s. implies that the sum in (4.1) evaluated at t is a.s. finite.

We show exactness. Let $x \in \mathbb{R}^d$. Since

$$p_{\{j:\pi(j)\le\pi(\pi^{-1}(d))\}}(x) = x \qquad \text{and} \qquad p_{\{j:\pi(j)<\pi(\pi^{-1}(1))\}}(x) = 0,$$

we have for any $t \in [0, \infty)$ and $n \in \mathbb{N}$ by (4.3) that

$$\begin{aligned}
&\sum_{i=1}^{d} \delta^{\mathrm{SU},i,\pi,\gamma}_{t\wedge\sigma_n}(F, X)\\
&= \sum_{\substack{i=1\\ i\neq\pi^{-1}(d)}}^{d} \sum_{\ell=0}^{n-1} F_{t\wedge\sigma_n}\big(X^{\sigma_\ell} + p_{\{j:\pi(j)\le\pi(i)\}}(X^{\sigma_{\ell+1}} - X^{\sigma_\ell})\big) + \sum_{\ell=0}^{n-1} F_{t\wedge\sigma_n}(X^{\sigma_{\ell+1}})\\
&\quad - \sum_{\substack{i=1\\ i\neq\pi^{-1}(1)}}^{d} \sum_{\ell=0}^{n-1} F_{t\wedge\sigma_n}\big(X^{\sigma_\ell} + p_{\{j:\pi(j)<\pi(i)\}}(X^{\sigma_{\ell+1}} - X^{\sigma_\ell})\big) - \sum_{\ell=0}^{n-1} F_{t\wedge\sigma_n}(X^{\sigma_\ell}).
\end{aligned}$$

For each $i \in \{1, \ldots, d\} \setminus \{\pi^{-1}(d)\}$, there is exactly one $k \in \{1, \ldots, d\} \setminus \{\pi^{-1}(1)\}$ such that

$$p_{\{j:\pi(j)\le\pi(i)\}}(x) = p_{\{j:\pi(j)<\pi(k)\}}(x),$$

since $\pi(k) = \pi(i) + 1$ if and only if $k = \pi^{-1}(\pi(i) + 1)$. Thus we get

$$\begin{aligned}
\sum_{i=1}^{d} \delta^{\mathrm{SU},i,\pi,\gamma}_{t\wedge\sigma_n}(F, X) &= \sum_{\ell=0}^{n-1} F_{t\wedge\sigma_n}(X^{\sigma_{\ell+1}}) - \sum_{\ell=0}^{n-1} F_{t\wedge\sigma_n}(X^{\sigma_\ell})\\
&= F_{t\wedge\sigma_n}(X^{\sigma_n}) - F_{t\wedge\sigma_n}(X^{\sigma_0})\\
&= F_{t\wedge\sigma_n}(X) - F_0(X).
\end{aligned}$$

Since t and n were arbitrary and $\sigma_n \to \infty$ a.s., the decomposition $\delta^{\mathrm{SU},\pi,\gamma}$ is exact. To see the last point, note that two processes with càdlàg paths are indistinguishable if they are modifications. □

Example 4.4 Assume $d = 2$. The SU decomposition with respect to γ defines $d! = 2$ decompositions, namely $\delta^{\mathrm{SU},\mathrm{id},\gamma}(F, X)$ and $\delta^{\mathrm{SU},\varrho,\gamma}(F, X)$ with $\varrho(1) = 2$ and $\varrho(2) = 1$, by

$$\delta^{\mathrm{SU},1,\mathrm{id},\gamma}(F, X) = \sum_{\ell=0}^{\infty} \big(F(X^{1,\sigma_{\ell+1}}, X^{2,\sigma_\ell}) - F(X^{1,\sigma_\ell}, X^{2,\sigma_\ell})\big),$$

$$\delta^{\mathrm{SU},2,\mathrm{id},\gamma}(F, X) = \sum_{\ell=0}^{\infty} \big(F(X^{1,\sigma_{\ell+1}}, X^{2,\sigma_{\ell+1}}) - F(X^{1,\sigma_{\ell+1}}, X^{2,\sigma_\ell})\big)$$

and

$$\delta^{\mathrm{SU},1,\varrho,\gamma}(F,X)=\sum_{\ell=0}^{\infty}\big(F(X^{1,\sigma_{\ell+1}},X^{2,\sigma_{\ell+1}})-F(X^{1,\sigma_{\ell}},X^{2,\sigma_{\ell+1}})\big),$$

$$\delta^{\mathrm{SU},2,\varrho,\gamma}(F,X)=\sum_{\ell=0}^{\infty}\big(F(X^{1,\sigma_{\ell}},X^{2,\sigma_{\ell+1}})-F(X^{1,\sigma_{\ell}},X^{2,\sigma_{\ell}})\big).$$

Definition 4.5 Let $\gamma=\{0=\sigma_0<\sigma_1<\cdots\}$ be an unbounded random partition. The *ASU (averaged sequential updating) decomposition* $\delta^{\mathrm{ASU},\gamma}:\mathcal{M}\times\mathcal{X}^d\to\mathcal{X}^d$ is defined by

$$\delta^{\mathrm{ASU},i,\gamma}(F,X)=\frac{1}{d!}\sum_{\pi\in\sigma_d}\delta^{\mathrm{SU},i,\pi,\gamma}(F,X),\qquad i=1,\dots,d.$$

Remark 4.6 As in Shorrocks [26], we observe that

$$\delta^{\mathrm{ASU},i,\gamma}(F,X)=\frac{1}{d!}\sum_{\pi\in\sigma_d}\delta^{\mathrm{SU},i,\pi,\gamma}(F,X)=\sum_{\substack{A\subseteq\{1,\dots,d\}\\ i\in A}}\delta^{\mathrm{SU},i,A,\gamma}(F,X)\xi_{i,A}$$

for $\xi_{i,A}$ defined in (3.8) and

$$\begin{aligned}&\delta^{\mathrm{SU},i,A,\gamma}(F,X)\\&:=\sum_{\ell=0}^{\infty}\Big(F\big(X^{\sigma_\ell}+p_A(X^{\sigma_{\ell+1}}-X^{\sigma_\ell})\big)-F\big(X^{\sigma_\ell}+p_{A\setminus\{i\}}(X^{\sigma_{\ell+1}}-X^{\sigma_\ell})\big)\Big).\end{aligned}$$

Thereby, the computational cost to obtain $\delta^{\mathrm{ASU},i,\gamma}$ can be reduced from $\mathcal{O}(d!)$ to $\mathcal{O}(2^{d-1})$.

Theorem 4.7 *Fix $\pi\in\sigma_d$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of unbounded random partitions tending to the identity. Let $F\in\mathcal{M}(C^2)$, $X\in\mathcal{X}^d$, $t\geq 0$ and $i\in\{1,\dots,d\}$. Then it holds for $n\to\infty$ that*

$$\delta_t^{\mathrm{SU},i,\pi,\gamma_n}(F,X)\xrightarrow{p}\delta_t^{\mathrm{ISU},i,\pi}(F,X),$$

$$\delta_t^{\mathrm{ASU},i,\gamma_n}(F,X)\xrightarrow{p}\delta_t^{\mathrm{IASU},i}(F,X).$$

Proof See Appendix A.2. □

The next example shows that the assumption $F\in\mathcal{M}(C^2)$ in Theorem 4.7 is important to ensure convergence.

Example 4.8 Let Z be a stochastic process with independent increments and $Z_0=0$. Suppose the jumps of Z only occur at fixed times $J=\{2-\ell^{-1}:\ell\in\mathbb{N}\}$, and for each $\ell \in \mathbb{N}$, the process jumps by $\pm\ell^{-1}$ with equal probability. The process Z stays constant between jumps. Then Z is a semimartingale; see Černý and Ruf [5]. Let

$$f(x^1, x^2) = |x^1 - x^2|$$

so that $f \notin C^2$. Let $(\gamma_n = \{0 = \sigma_0^n < \sigma_1^n < \cdots\})_{n\in\mathbb{N}}$ be a deterministic sequence of unbounded partitions tending to the identity such that γ_n contains the first n smallest elements of J, but the intersection with $(2-n^{-1}, 2]$ is empty. Assume that $X = (Z, Z)$. Then for $t \geq 2$, it follows that

$$\sum_{\ell=0}^{\infty} \big(f(X_t^{1,\sigma_{\ell+1}^n}, X_t^{2,\sigma_\ell^n}) - f(X_t^{1,\sigma_\ell^n}, X_t^{2,\sigma_\ell^n}) \big) = \sum_{\ell=1}^{n} \ell^{-1},$$

which is divergent for $n \to \infty$; so the SU decomposition does not converge for the map $F_t(X) := f(X_t), t \geq 0$.

How can the IASU decomposition be computed efficiently in practice? If we naively approximate the integrals in Definition 3.6 numerically, we may lose exactness of the decomposition, which is undesirable in many applications. Theorem 4.7 suggests using the ASU decomposition as an approximation of the IASU decomposition. However, this becomes computationally infeasible for moderately large d since the computational cost to obtain $\delta^{\mathrm{ASU},i,\gamma}$ scales like $\mathcal{O}(2^{d-1})$. The next result provides an elegant solution when there are no simultaneous jumps.

Definition 4.9 Let $\gamma = \{0 = \sigma_0 < \sigma_1 < \cdots\}$ be an unbounded random partition. The *2SU (average of two sequential updating) decomposition* $\delta^{2\mathrm{SU},\pi,\gamma} : \mathcal{M} \times \mathcal{X}^d \to \mathcal{X}^d$ *with updating order* $\pi \in \sigma_d$ is defined by

$$\delta^{2\mathrm{SU},i,\pi,\gamma}(F, X) = \frac{1}{2}\big(\delta^{\mathrm{SU},i,\pi,\gamma}(F, X) + \delta^{\mathrm{SU},i,\pi',\gamma}(F, X)\big), \qquad i = 1, \ldots, d,$$

where $\pi' = d + 1 - \pi$.

Corollary 4.10 *Fix* $\pi \in \sigma_d$ *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence of unbounded random partitions tending to the identity. Let* $F \in \mathcal{M}(C^2)$, $X \in \mathcal{X}^d$, $i \in \{1, \ldots, d\}$ *and* $t \geq 0$.

i) *If* $\Delta X^h \Delta X^j = 0$ *for all* $h, j \in \{1, \ldots, d\}$ *with* $h \neq j$, *then*

$$\delta_t^{2\mathrm{SU},i,\pi,\gamma_n}(F, X) \xrightarrow{p} \delta_t^{\mathrm{IASU},i}(F, X), \qquad n \to \infty.$$

ii) *If* $[X^h, X^j] = 0$ *for all* $h, j \in \{1, \ldots, d\}$ *with* $h \neq j$, *then*

$$\delta_t^{\mathrm{SU},i,\pi,\gamma_n}(F, X) \xrightarrow{p} \delta_t^{\mathrm{IASU},i}(F, X), \qquad n \to \infty.$$

Proof If $\Delta X^h \Delta X^j = 0$, $h \neq j$, Theorem 3.10 implies that

$$\delta^{\mathrm{IASU},i}(F, X) = \frac{1}{2}\big(\delta^{\mathrm{ISU},i,\pi}(F, X) + \delta^{\mathrm{ISU},i,\pi'}(F, X)\big),$$

$$\begin{tikzcd}
& \boxed{\delta^{\mathrm{ASU},i,\gamma}} \arrow[d, "p"] & \\
\boxed{\delta^{2\mathrm{SU},i,\pi,\gamma}} \arrow[r, "p"] & \boxed{\delta^{\mathrm{IASU},i}} & \boxed{\delta^{\mathrm{SU},i,\pi,\gamma}} \arrow[l, "p"'] \\
\text{if } \Delta X^h \Delta X^j = 0,\, h \neq j & & \text{if } [X^h, X^j] = 0,\, h \neq j
\end{tikzcd}$$

Fig. 1 Overview of discrete approximations of the IASU decomposition

which is the limit of $\delta^{2\mathrm{SU},i,\pi,\gamma_n}(F, X)$ by Theorem 4.7. If $[X^h, X^j] = 0$, $h \neq j$, apply Corollary 3.12 and Theorem 4.7. $\square$

In particular, the 2SU decomposition with arbitrary updating order π is exact and approximates the IASU decomposition when the risk factors do not have simultaneous jumps. In this case, the computationally expensive averaging to obtain the ASU decomposition can be omitted and the computational complexity to approximate $\delta^{\mathrm{IASU},i}$ decreases from $\mathcal{O}(2^{d-1})$ to $\mathcal{O}(1)$. Theorem 4.7 and Corollary 4.10 are also illustrated in Fig. 1.

Finally, we define the OAT decomposition. To obtain the contribution of X^i, all risk factors are fixed at the origin and only X^i is allowed to change from the beginning of a subinterval to the end of that subinterval.

Definition 4.11 Let $\gamma = \{0 = \sigma_0 < \sigma_1 < \cdots\}$ be an unbounded random partition. The *OAT (one-at-a-time) decomposition* $\delta^{\mathrm{OAT},\gamma} : \mathcal{M} \times \mathcal{X}^d \to \mathcal{X}^d$ is defined by

$$\begin{aligned}
&\delta^{\mathrm{OAT},i,\gamma}(F, X) \\
&= \sum_{\ell=0}^{\infty} \big(F(X^{1,\sigma_\ell}, \ldots, X^{i-1,\sigma_\ell}, X^{i,\sigma_{\ell+1}}, X^{i+1,\sigma_\ell}, \ldots, X^{d,\sigma_\ell}) - F(X^{\sigma_\ell}) \big).
\end{aligned}$$

Remark 4.12 The OAT decomposition is symmetric, but in general not exact. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of unbounded random partitions tending to the identity. For each $i \in \{1, \ldots, d\}$, choose a permutation $\pi_i \in \sigma_d$ such that $\pi_i(i) = 1$. Then $\delta^{\mathrm{OAT},i,\gamma_n}$ is indistinguishable from $\delta^{\mathrm{SU},i,\pi_i,\gamma_n}$. If $F \in \mathcal{M}(C^2)$, Theorem 4.7 gives for $t \geq 0$ that

$$\delta_t^{\mathrm{OAT},i,\gamma_n}(F, X) \xrightarrow{p} \delta_t^{\mathrm{ISU},i,\pi_i}(F, X), \qquad i = 1, \ldots, d,$$

for $n \to \infty$. Thus by Corollary 3.12, the three decompositions principles OAT, SU (with arbitrary order $\pi \in \sigma_d$) and ASU are asymptotically indistinguishable if there are no interaction effects.

5 Applications

Investment portfolios of financial institutions or insurance companies may include instruments such as stocks, plain vanilla or callable bonds, convertible bonds, inflationlinked bonds, contingent convertible bonds (CoCos), basket options, foreign exchange options and structured products. These instruments often depend on multiple risk factors such as different foreign exchange rates, interest rates for different maturities, credit spreads, inflation rate, some trigger activations for CoCos, multiple equities and time decay. Candland and Lotz [4] also considered defaults and rating changes as risk factors.

In order to obtain a P&L attribution of such instruments, we propose the IASU decomposition because it is exact, symmetric and normalised, and it takes into account the whole paths of the risk factors, i.e., uses all available information. The last point also avoids inconsistencies when reporting a P&L attribution for different time grids, e.g. on an annual, quarterly, monthly and weekly basis. The IASU decomposition involves a stochastic integral. To approximate the IASU decomposition, we propose the ASU or 2SU decomposition with a sufficiently fine time grid, as such an approximation is always an exact decomposition. The use of the 2SU decomposition is theoretically justified when the risk factors do not have simultaneous jumps.

In Sect. 5.1, we provide an exemplary decomposition of a plain vanilla call option with stochastic interest rates on a foreign stock. A change in the P&L of this option can be explained by movements in the stock, the yield curve, the foreign exchange rate and time decay. Thus there are $d = 4$ risk factors. We analyse the unexplained P&L of the OAT decomposition, the range of the SU and 2SU decompositions over all possible updating orders $\pi \in \sigma_d$ for different time grids, and the convergence of the ASU decomposition to the IASU decomposition.

Computing the ASU decomposition to approximate the IASU decomposition becomes infeasible when the number of risk factors d is moderately large; for example, a plain vanilla bond paying coupons may depend on d yield curves. A basket option may depend on d stocks. In practice, $d = 30$ is a common case for basket options; see Grzelak et al. [15]. In Sect. 5.2, we decompose a digital cash-or-nothing basket put option. We illustrate that it is impossible to obtain the ASU decomposition in reasonable time when $d = 30$, and we show how the 2SU decomposition is able to break the curse of dimensionality.

5.1 Decomposing a call option with stochastic interest rates

In this section, we allocate the P&L of the price of a plain vanilla European call option with strike K and maturity $T = 10$ with stochastic interest rates and foreign exchange exposure. The stock price S is given by a Black–Scholes model with constant volatility $\sigma_S > 0$ and with stochastic interest rates r. The dynamics under the risk-neutral measure are given by

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma_S S_t dB_t^S, \\ dr_t &= \kappa(\eta - r_t)dt + \sigma_r dB_t^r \end{aligned}$$

with constant volatility $\sigma_r > 0$, long-term mean $\eta \in \mathbb{R}$ and speed $\kappa > 0$ of mean-reversion. Under the physical measure, the stock has drift $\mu_S \in \mathbb{R}$, and the foreign exchange rate Y is assumed to follow a geometric Brownian motion with drift $\mu_Y \in \mathbb{R}$

and volatility $\sigma_Y > 0$ driven by the Brownian motion B^Y. The Brownian motions are assumed to have correlations

$$d\langle B^S, B^r\rangle_t = \rho_{Sr}dt, \qquad d\langle B^S, B^Y\rangle_t = \rho_{SY}dt, \qquad d\langle B^Y, B^r\rangle_t = \rho_{Yr}dt.$$

The time to maturity is denoted by $\tau(t) = T - t$. The price $p_{\text{call}}(t)$ at time t of the plain vanilla call option is given by a C^2-function $f : \mathbb{R}^d \to \mathbb{R}$, see Rabinovitch [22], i.e.,

$$p_{\text{call}}(t) = f\big(S_t, r_t, Y_t, \tau(t)\big) =: F_t(S, r, Y, \tau), \qquad t > 0,$$

with

$$f(s, r, y, \tau) = ys\Phi\big(d_+(s, r, \tau)\big) - yKP(r, \tau)\Phi\big(d_-(s, r, \tau)\big),$$

where Φ denotes the distribution function of a standard normal distribution and

$$\begin{aligned} d_\pm(s, r, \tau) &= \frac{1}{\sqrt{v(\tau)}}\left(\log\frac{s}{KP(r, \tau)} \pm \frac{1}{2}v(\tau)\right), \\ v(\tau) &= \sigma_S^2\tau + \sigma_r^2\frac{\tau - 2g_\kappa(\tau) + g_{2\kappa}(\tau)}{\kappa^2} - 2\rho_{Sr}\sigma_S\sigma_r\frac{\tau - g_\kappa(\tau)}{\kappa}, \\ g_\kappa(\tau) &= \frac{1 - e^{-\kappa\tau}}{\kappa}. \end{aligned}$$

The bond price $P(r, \tau)$ is given by

$$P(r, \tau) = A(\tau)e^{-g_\kappa(\tau)r},$$

where

$$A(\tau) = \exp\left(\left(\eta + \frac{\sigma_r^2\lambda}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right)\big(g_\kappa(\tau) - \tau\big) - \frac{1}{\kappa}\left(\frac{\sigma_r g_\kappa(\tau)}{2}\right)^2\right)$$

and λ denotes the market price of risk. For simplicity, we set the market price of risk to zero and hence assume that the dynamics of r under the physical and the risk-neutral measure are identical. Björk [2, Sect. 24.2] describes how to estimate the parameters for r from market data. We simulate 1000 paths of the stock, interest rate and foreign exchange rate under the physical measure over one year. For each path, we decompose the price of the call option at time $t = 1$ with respect to the $d = 4$ risk factors $X := (S, r, Y, \tau)$. We use the following parameters: $K = S_0 = 100$, $\mu_S = 0.05$, $\sigma_S = 0.4$, $Y_0 = 1.1$, $\mu_Y = 0$, $\sigma_Y = 0.05$, $r_0 = 0.08$, $\kappa = 0.1$, $\eta = 0.05$, $\sigma_r = 0.01$ and $\rho_{Sr} = -0.7$, $\rho_{SY} = -0.4$, $\rho_{Yr} = 0.7$.

Figure 2 shows the relative unexplained P&L of the OAT decomposition, i.e.,

$$\frac{|(F_1(X) - F_0(X)) - \sum_{i=1}^{d}\delta_1^{\text{OAT},i,\gamma}(F, X)|}{|F_1(X) - F_0(X)|}.$$

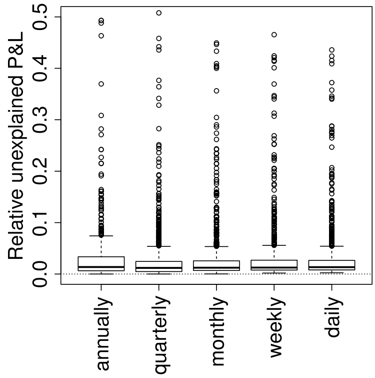

Fig. 2 Relative unexplained P&L for the OAT decomposition of a plain vanilla call option in a foreign currency at time $t = 1$ for different time grids

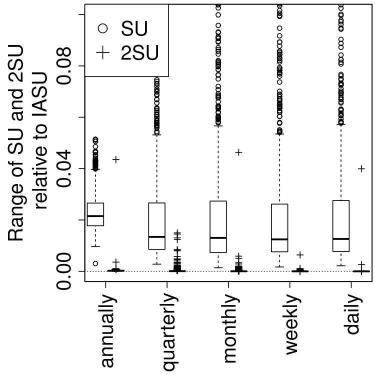

Fig. 3 Relative range of all SU and 2SU decompositions for the risk factor S

We use as time grids γ annual, quarterly, monthly, weekly and daily time steps. As observed in Flaig and Junike [9], we also see that the unexplained P&L of the OAT decomposition is significant for all time grids.

Figure 3 shows the relative range of the $d!$ SU decompositions for the risk factor S, i.e.,

$$\max_{\pi\in\sigma_d} \frac{\delta_1^{\mathrm{SU},1,\pi,\gamma}(F, X)}{\delta_1^{\mathrm{IASU},1}(F, X)} - \min_{\pi\in\sigma_d} \frac{\delta_1^{\mathrm{SU},1,\pi,\gamma}(F, X)}{\delta_1^{\mathrm{IASU},1}(F, X)},$$

and the relative range of the $\frac{d!}{2}$ 2SU decompositions for the risk factor S. The limiting IASU decomposition is approximated by an ASU decomposition with 10’000 time steps per year. We observe that the range is significant for the SU decompositions and insignificant for the 2SU decompositions.

The speed of convergence of the ASU to the IASU decomposition is illustrated in Fig. 4 for the risk factor S, i.e., we show the convergence

$$\frac{\delta_1^{\mathrm{ASU},1,\gamma}(F, X)}{\delta_1^{\mathrm{IASU},1}(F, X)} \longrightarrow 1 \qquad \text{as } \gamma \text{ tends to the identity.}$$

Figures 3 and 4 look similar for other risk factors.

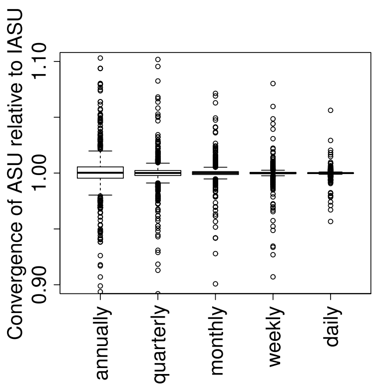

Fig. 4 Convergence of the ASU decomposition to the IASU decomposition for the risk factor S

In further numerical experiments, we calculate the relative difference between the ASU decomposition and the 2SU decompositions,

$$\left| \frac{\delta_1^{2\mathrm{SU},i,\pi,\gamma}(F, X) - \delta_1^{\mathrm{ASU},i,\gamma}(F, X)}{\delta_1^{\mathrm{IASU},i,\gamma}(F, X)} \right|,$$

over all risk factors $i \in \{1, \ldots, d\}$, time grids γ and updating orders $\pi \in \sigma_d$, and observe values of less than 0.6% in 95% of the simulations. In conclusion, we find that the ASU decomposition and the 2SU decompositions are strongly dependent on the time grid, but using monthly or weekly instead of annual time steps significantly reduces the deviation of the ASU and 2SU decompositions from the IASU decomposition.

5.2 Decomposing a basket option

In this section, we compare the computational cost of obtaining a one-year P&L attribution of a basket option using a naive SU decomposition with annual time grid to the computational cost of obtaining an ASU and a 2SU decomposition based on a monthly time grid, respectively. We consider d risk factors, namely time decay and $d-1$ different stocks. A digital cash-or-nothing basket put option pays \$1 at maturity T if $S_T^1 \leq K, \ldots, S_T^{d-1} \leq K$ and zero otherwise. The stock prices are given by a Black–Scholes model. We set the interest rate r to zero. We set $S_0^i = K = 100$, $i = 1, \ldots, d-1$ and $T = 2$. The price of the option at time $t \in [0, T)$ is equal to $\Phi(\log K, \ldots, \log K)$, where Φ is the distribution function of a $(d-1)$-dimensional normal distribution with location

$$\left(\log S_t^1 - \left(r - \frac{1}{2}\sigma^2\right)(T-t), \ldots, \log S_t^{d-1} - \left(r - \frac{1}{2}\sigma^2\right)(T-t) \right) \in \mathbb{R}^{d-1}$$

and covariance matrix $\Sigma(T-t)$, where we set $\sigma = 0.2$, $\rho = 0.5$ and

$$\Sigma_{ij} = \begin{cases} \sigma^2, & i = j, \\ \rho\sigma^2, & i \neq j. \end{cases}$$

Table 1 CPU time to compute the d contributions of the SU, ASU and 2SU decompositions of a basket option over one year using different time grids. The CPU time of Φ is obtained from a Monte Carlo simulation. The CPU times in brackets are estimated using the CPU time of Φ and the known complexities of the three decompositions

	Number of evaluations of Φ	$d = 4$	$d = 15$	$d = 30$
Evaluation of Φ	1	0.018 s	0.15 s	0.54 s
SU with annual grid	$d + 1$	0.09 s	2.4 s	16.7 s
2SU with monthly grid	$(12d + 1)2$	1.76 s	54.3 s	390 s
ASU with monthly grid	$(12d + 1)2^{(d-1)}$	7.06 s	123.6 hr	3318.7 yrs

Basket options are often priced by using Monte Carlo techniques; see Glasserman [13, Sect. 3.2.3]. For moderate dimensions, many basket options can also be priced by using faster Fourier techniques; see Eberlein et al. [7] and Junike and Stier [17]. We compute Φ by using a simple Monte Carlo simulation implemented in C++ with 100'000 simulations. The experiments are performed on a laptop with Intel i7-11850H processor and 32 GB RAM.

Table 1 shows the CPU time needed to obtain Φ for $d \in \{4, 15, 30\}$. We measure CPU times by averaging over 100 runs. Since in some cases, the arguments of Φ to obtain an SU decomposition with a certain update order π are the same for different contributions, we need to evaluate Φ only $dL + 1$ times, where L is the number of subintervals of $[0, T]$, to obtain the d individual contributions. For example, $(12d + 1)2$ and $(12d + 1)2^{d-1}$ evaluations of Φ are required for the 2SU and ASU decompositions with a monthly time grid.

Table 1 also shows the CPU time to compute the SU, ASU and 2SU decompositions. A naive SU decomposition based on an annual time grid is at most 24 times faster than a 2SU decomposition with a monthly time grid. The computational cost of the 2SU decomposition for each contribution is dimension-independent, except for the longer time required to evaluate Φ. Compared to the ASU decomposition, the 2SU decomposition is 2^{d-2} times faster. The ASU decomposition cannot be computed in reasonable time for $d \geq 30$.

Remark 5.1 To reduce the computational time, it is possible to compute the d contributions for the SU, 2SU and ASU decompositions in parallel, which would reduce the numerical effort by a factor of d. Furthermore, the sums for the SU, 2SU and ASU decompositions can also be parallelised. For example, for the 2SU decomposition, we need to perform $2(dL + 1)$ function evaluations to obtain all d contributions. If a function evaluation takes 0.54 s in $d = 30$ dimensions as in Table 1, the computation time for the 2SU decomposition with monthly time grid could be reduced from 390 s to about 0.54 s using 722 cores for parallelisation.

6 Conclusions

We showed that the IASU decomposition is the only (up to indistinguishability) exact and symmetric decomposition in the family of Itô decompositions, which is a large class of normalised decompositions based on a rearranged version of Itô's formula. This axiomatic result, together with the fact that the IASU decomposition is grid-independent and considers the full paths of the risk basis, makes it a decomposition of choice from a theoretical perspective. In practice, the calculation of the IASU decomposition comes with two challenges: it involves stochastic integrals that must be approximated, and the computational effort explodes as the number of risk factors increases.

We have shown that the IASU decomposition can be approximated by the ASU decomposition (which is always exact and symmetric) if we use a sufficiently fine time grid, but the ASU decomposition also suffers from the curse of dimensionality as the number of risk factors increases. For applications where different risk factors may have interactions, but almost surely do not have simultaneous jumps, we have shown that the IASU decomposition is indistinguishable from the average of two ISU decompositions, thus breaking the curse of dimensionality. Therefore, from a theoretical point of view, the 2SU decomposition with a sufficiently fine time grid is an appropriate approximation of the IASU decomposition.

Based on our own numerical experiments and the empirical analysis of Flaig and Junike [9], we recommend using monthly or even weekly instead of annual time steps.

The additional computational cost of our two recommendations is moderate, but the theoretical properties of the decomposition are dramatically improved.

Appendix

A.1 Auxiliary results

Lemma A.1 *Let $i, j \in \{1, \ldots, d\}$. Let $\pi, \eta \in \sigma_d$ and $x \in \mathbb{R}^d$. Then it holds that*

$$\eta^{-1}\Big(p_{\{j:\pi(j)\leq\pi(\eta^{-1}(i))\}}\big(\eta(x)\big)\Big) = p_{\{j:\pi(\eta^{-1}(j))\leq\pi(\eta^{-1}(i))\}}(x). \tag{A.1}$$

Proof Let $k \in \{j : \pi(j) \leq \pi(\eta^{-1}(i))\}$, which is equivalent to

$$\eta(k) \in \Big\{j : \pi\big(\eta^{-1}(j)\big) \leq \pi\big(\eta^{-1}(i)\big)\Big\}.$$

Since $(\eta^{-1}(x))_{\eta(k)} = x_k$ and $(\eta(x))_k = x_{\eta(k)}$, we obtain that

$$\begin{aligned}\Big(\eta^{-1}\Big(p_{\{j:\pi(j)\leq\pi(\eta^{-1}(i))\}}\big(\eta(x)\big)\Big)\Big)_{\eta(k)} &= \Big(p_{\{j:\pi(j)\leq\pi(\eta^{-1}(i))\}}\big(\eta(x)\big)\Big)_k \\ &= \big(p_{\{j:\pi(\eta^{-1}(j))\leq\pi(\eta^{-1}(i))\}}(x)\big)_{\eta(k)},\end{aligned}$$

which leads to (A.1). □

Lemma A.2 *Let $\eta \in \sigma_d$, $i \in \{1,\dots,d\}$, $X \in \mathcal{X}^d$, $F \in \mathcal{M}(C^2)$ and $(\mu_\pi)_{\pi\in\sigma_d} \subseteq [0,1]$. If $F(\eta(X)) = F(X)$, then it holds that*

$$\sum_{\pi\in\sigma_d} \mu_\pi S^{\eta^{-1}(i),\pi}\big(\eta(X)\big) = \sum_{\substack{A\subseteq\{1,\dots,d\}\\ i\in A}} S^{i,A}(X)\xi_{i,A,\eta}$$

with

$$\xi_{i,A,\eta} := \sum_{\substack{\pi\in\sigma_d\\ \{j:\pi(\eta^{-1}(j))\le\pi(\eta^{-1}(i))\}=A}} \mu_\pi. \tag{A.2}$$

In particular, for an Itô decomposition δ with parameters $(\lambda_{ij})_{i,j=1,\dots,d}$ and $(\mu_\pi)_{\pi\in\sigma_d}$, we have

$$\delta^i(F,X) = I^i + \frac{1}{2}I^{ii} + \sum_{\substack{j=1\\ j\ne i}}^{d} \lambda_{ij} I^{ij} + \sum_{\substack{A\subseteq\{1,\dots,d\}\\ i\in A}} S^{i,A}(X)\xi_{i,A,\mathrm{id}}. \tag{A.3}$$

Proof Let $\eta \in \sigma_d$ and $F_t(X) = f(X_t)$, $t \ge 0$, with $F(\eta(X)) = F(X)$ for $X \in \mathcal{X}^d$. Let $i \in \{1,\dots,d\}$. By (A.1), it holds for $s > 0$ that

$$\begin{aligned}
&f\Big(\eta(X_{s-}) + p_{\{j:\pi(j)\le\pi(\eta^{-1}(i))\}}\big(\eta(\Delta X_s)\big)\Big)\\
&= f\bigg(\eta\Big(X_{s-} + \eta^{-1}\Big(p_{\{j:\pi(j)\le\pi(\eta^{-1}(i))\}}\big(\eta(\Delta X_s)\big)\Big)\Big)\bigg)\\
&= f\Big(\eta\big(X_{s-} + p_{\{j:\pi(\eta^{-1}(j))\le\pi(\eta^{-1}(i))\}}(\Delta X_s)\big)\Big)\\
&= f\big(X_{s-} + p_{\{j:\pi(\eta^{-1}(j))\le\pi(\eta^{-1}(i))\}}(\Delta X_s)\big).
\end{aligned}$$

The last equality follows from the symmetry of f. Similarly, if we replace "$\le$" in (A.2) with "$<$", we get that

$$\begin{aligned}
&f\Big(\eta(X_{s-}) + p_{\{j:\pi(j)<\pi(\eta^{-1}(i))\}}\big(\eta(\Delta X_s)\big)\Big)\\
&= f\big(X_{s-} + p_{\{j:\pi(\eta^{-1}(j))<\pi(\eta^{-1}(i))\}}(\Delta X_s)\big).
\end{aligned}$$

Let $\eta \in \sigma_d$ and $f \in C^2$. If $f(x) = f(\eta(x))$, $x \in \mathbb{R}^d$, it is straightforward to see that for $x \in \mathbb{R}^d$, it holds that

$$f_i(x) = f_{\eta^{-1}(i)}\big(\eta(x)\big), \quad f_{ij}(x) = f_{\eta^{-1}(i)\eta^{-1}(j)}\big(\eta(x)\big), \quad \big(\eta(x)\big)_{\eta^{-1}(i)} = x_i. \tag{A.4}$$

Therefore it follows that

$$S^{\eta^{-1}(i),\pi}\big(\eta(X)\big) = S^{i,\pi\circ\eta^{-1}}(X). \tag{A.5}$$

Thus similarly to Shorrocks [26], for any re-ordering $\eta(X)$ of the risk basis X, we can conclude from (A.5) that

$$\begin{aligned}\sum_{\pi\in\sigma_d}\mu_\pi S^{\eta^{-1}(i),\pi}\big(\eta(X)\big) &= \sum_{\pi\in\sigma_d}\mu_\pi S^{i,\pi\circ\eta^{-1}}(X)\\ &= \sum_{\substack{A\subseteq\{1,\dots,d\}\\ i\in A}}\ \sum_{\substack{\pi\in\sigma_d\\ \{j:\pi(\eta^{-1}(j))\leq\pi(\eta^{-1}(i))\}=A}}\mu_\pi S^{i,\pi\circ\eta^{-1}}(X)\\ &= \sum_{\substack{A\subseteq\{1,\dots,d\}\\ i\in A}} S^{i,A}(X)\sum_{\substack{\pi\in\sigma_d\\ \{j:\pi(\eta^{-1}(j))\leq\pi(\eta^{-1}(i))\}=A}}\mu_\pi\\ &= \sum_{\substack{A\subseteq\{1,\dots,d\}\\ i\in A}} S^{i,A}(X)\xi_{i,A,\eta}.\end{aligned}$$

Equation (A.3) follows directly for $\eta = \text{id}$. □

Lemma A.3 *Let δ be an Itô decomposition with parameters $(\lambda_{ij})_{i,j=1,\dots,d}$ and $(\mu_\pi)_{\pi\in\sigma_d}$. Let $i\in\{1,\dots,d\}$. If δ is symmetric and exact, it follows that*

$$\xi_{i,A,\text{id}} = \xi_{\eta^{-1}(i),\eta^{-1}(A),\text{id}} \tag{A.6}$$

for any $\eta\in\sigma_d$, where $\xi_{i,A,\text{id}}$ is defined in (A.2) and $\eta(A) := \{\eta(j) : j\in A\}$. Furthermore, for any $a\in\{1,\dots,d\}$, it holds that

$$\sum_{j=1}^{d}\ \sum_{\substack{A\subseteq\{1,\dots,d\}\\ |A|=a,\, j\in A}}\xi_{j,A,\text{id}} = 1. \tag{A.7}$$

Proof First, we show (A.6). Let $A\subseteq\{1,\dots,d\}$ with $i\in A$. Let $\pi,\eta\in\sigma_d$. Because

$$\Big\{j:\pi\big(\eta^{-1}(j)\big)\leq\pi\big(\eta^{-1}(i)\big)\Big\} = A \quad\Longleftrightarrow\quad \Big\{j:\pi(j)\leq\pi\big(\eta^{-1}(i)\big)\Big\} = \eta^{-1}(A),$$

it holds that

$$\begin{aligned}\xi_{i,A,\eta} &= \sum_{\substack{\pi\in\sigma_d\\ \{j:\pi(\eta^{-1}(j))\leq\pi(\eta^{-1}(i))\}=A}}\mu_\pi\\ &= \sum_{\substack{\pi\in\sigma_d\\ \{j:\pi(j)\leq\pi(\eta^{-1}(i))\}=\eta^{-1}(A)}}\mu_\pi = \xi_{\eta^{-1}(i),\eta^{-1}(A),\text{id}}.\end{aligned}\tag{A.8}$$

Now let $f(x) = \prod_{j=1}^d x_j^2$ and $F_t(X) = f(X_t)$, $t\geq 0$, so that $F(X) = F(\pi(X))$, $\pi\in\sigma_d$. For $B\subseteq\{1,\dots,d\}$ with $i\in B$ and $t\geq 0$, let

$$X_t^j = \begin{cases} 1_{[1,\infty)}(t), & j\in B,\\ 1_{[0,1)}(t), & j\notin B.\end{cases}$$

Then it follows that

$$f\big(X_{1-} + p_A(\Delta X_1)\big) = \begin{cases} 1, & A = B, \\ 0, & A \neq B, \end{cases}$$

and therefore

$$S_1^{i,A}(X) = \begin{cases} 1, & A = B, \\ 0, & A \neq B, \end{cases}$$

for $A \subseteq \{1, \dots, d\}$ with $i \in A$. For $\eta \in \sigma_d$, it follows by Lemma A.2 that

$$\delta_1^{\eta^{-1}(i)}\big(F, \eta(X)\big) = \sum_{\substack{A \subseteq \{1,\dots,d\} \\ i \in A}} S_1^{i,A}(X) \xi_{i,A,\eta} = \xi_{i,B,\eta}.$$

Since δ is symmetric, we have by (A.8) that

$$\xi_{\eta^{-1}(i),\eta^{-1}(B),\mathrm{id}} = \xi_{i,B,\eta} = \delta_1^{\eta^{-1}(i)}\big(F, \eta(X)\big) = \delta_1^i(F, X) = \xi_{i,B,\mathrm{id}}.$$

Since B was arbitrary, we have shown (A.6).

Now we iteratively show (A.7). Let $X_t^j = 1_{[1,\infty)}(t)$, $t \geq 0$, $j = 1, \dots, d$, and let $f^a \in C^2$ be such that for $a \in \{1, \dots, d\}$,

$$f^a(x) = \begin{cases} 1, & \sum_{j=1}^d x_j = a, \\ 0, & \sum_{j=1}^d x_j \in (-\infty, a-1] \cup [a+1, \infty), \end{cases}$$

and $f_i^a(X) = 0$ if $\sum_{j=1}^d x_j \leq a - 1$, $i = 1, \dots, d$. Let $F_t^a(X) = f^a(X_t)$, $t \geq 0$. If $a = d$, then

$$S_1^{j,A}(X) = \begin{cases} 1, & |A| = a, \\ 0, & \text{otherwise}, \end{cases}$$

for $j = 1, \dots, d$ and $A \subseteq \{1, \dots, d\}$ with $j \in A$. By exactness and Lemma A.2, it follows that

$$\begin{aligned} 1 &= F_1^a(X) - F_0^a(X) \\ &= \sum_{j=1}^d \delta_1^j(F^a, X) \\ &= \sum_{j=1}^d \sum_{\substack{A \subseteq \{1,\dots,d\} \\ j \in A}} S_1^{j,A}(X) \xi_{j,A,\mathrm{id}} \\ &= \sum_{j=1}^d \sum_{\substack{A \subseteq \{1,\dots,d\} \\ |A| = d,\, j \in A}} \xi_{j,A,\mathrm{id}}. \end{aligned} \tag{A.9}$$

Now let $a = d - 1$; then

$$S_1^{j,A}(X) = \begin{cases} 1, & |A| = a, \\ -1, & |A| = a + 1, \\ 0, & \text{otherwise}, \end{cases}$$

for $A \subseteq \{1, \ldots, d\}$ with $j \in A$. Again by exactness, we have that

$$\begin{aligned} 0 &= F_1^a(X) - F_0^a(X) \\ &= \sum_{j=1}^{d} \delta_1^j(F^a, X) \\ &= \sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\ldots,d\} \\ j \in A}} S_1^{j,A}(X) \xi_{j,A,\mathrm{id}} \\ &= \sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\ldots,d\} \\ |A|=d-1,\, j \in A}} \xi_{j,A,\mathrm{id}} - \sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\ldots,d\} \\ |A|=d,\, j \in A}} \xi_{j,A,\mathrm{id}}. \end{aligned}$$

Using (A.9), we obtain that

$$\sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\ldots,d\} \\ |A|=d-1,\, j \in A}} \xi_{j,A,\mathrm{id}} = 1.$$

Iteratively for any $a \in \{1, \ldots, d\}$, it follows that

$$\sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\ldots,d\} \\ |A|=a,\, j \in A}} \xi_{j,A,\mathrm{id}} = 1.$$

□

A.2 Proof of Theorem 4.7

Let $t > 0$. Fix $i \in \{1, \ldots, d\}$ and some permutation π. Since $F \in \mathcal{M}(C^2)$, there is by definition an $f \in C^2$ with $F_t(X) = f(X_t)$, $t \geq 0$. We first show that $\delta_t^{\mathrm{SU},\pi,\gamma_n}(F, X) \xrightarrow{p} \delta_t^{\mathrm{ISU},\pi}(F, X)$ for $n \to \infty$. Let $\gamma_n = \{0 = \sigma_0^n < \sigma_1^n < \cdots\}$, $n \in \mathbb{N}$, be a sequence of unbounded random partitions tending to the identity. Let $\alpha > 0$ and define

$$\mathcal{A}_\alpha := \Big\{ s \in (0, t] : \max_{j=1,\ldots,d} |\Delta X_s^j| > \alpha \Big\},$$

the set of all time points in $[0, t]$ where at least one component of a path $u \mapsto X_u$ has a jump greater than α. The SU decomposition $\delta^{\mathrm{SU},i,\pi,\gamma_n}$ with respect to γ_n can be written as

$$\begin{aligned}\delta_t^{\mathrm{SU},i,\pi,\gamma_n}(F,X) &= \sum_{\ell\in\mathbb{A}_\alpha}\Big(f\big(X_t^{\sigma_\ell^n}+p_{\{j:\pi(j)\le\pi(i)\}}(X_t^{\sigma_{\ell+1}^n}-X_t^{\sigma_\ell^n})\big)\\
&\qquad - f\big(X_t^{\sigma_\ell^n}+p_{\{j:\pi(j)<\pi(i)\}}(X_t^{\sigma_{\ell+1}^n}-X_t^{\sigma_\ell^n})\big)\Big)\\
&\quad+\sum_{\ell\in\mathbb{A}_\alpha^c}\Big(f\big(X_t^{\sigma_\ell^n}+p_{\{j:\pi(j)\le\pi(i)\}}(X_t^{\sigma_{\ell+1}^n}-X_t^{\sigma_\ell^n})\big)\\
&\qquad - f\big(X_t^{\sigma_\ell^n}+p_{\{j:\pi(j)<\pi(i)\}}(X_t^{\sigma_{\ell+1}^n}-X_t^{\sigma_\ell^n})\big)\Big), \qquad (A.10)\end{aligned}$$

where $\mathbb{A}_\alpha=\{\ell\in\mathbb{N}_0:\mathcal{A}_\alpha\cap(\sigma_\ell^n,\sigma_{\ell+1}^n]\neq\emptyset\}$ and $\mathbb{A}_\alpha^c=\mathbb{N}_0\setminus\mathbb{A}_\alpha$. The first sum on the right-hand side of (A.10) converges a.s. for $n\to\infty$ to

$$\sum_{s\in\mathcal{A}_\alpha}\Big(f\big(X_{s-}+p_{\{j:\pi(j)\le\pi(i)\}}(\Delta X_s)\big)-f\big(X_{s-}+p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big)\Big). \qquad (A.11)$$

Using a Taylor expansion and the same arguments as in the proof of Itô's formula, one can show that the second sum on the right-hand side of (A.10) converges in probability for $n\to\infty$ to

$$\begin{aligned}I_t^i+\frac{1}{2}H_t^{ii}+\sum_{\pi(j)<\pi(i)}H_t^{ij}-\sum_{s\in\mathcal{A}_\alpha}\Big(f_i(X_{s-})\Delta X_s^i+\frac{1}{2}f_{ii}(X_{s-})(\Delta X_s^i)^2\\
+\sum_{\substack{j=1\\ \pi(j)<\pi(i)}}^{d}f_{ij}(X_{s-})\Delta X_s^i\Delta X_s^j\Big), \qquad (A.12)\end{aligned}$$

where $H^{ij}=\int_0^\cdot f_{ij}(X_{s-})d[X^i,X^j]_s$. The sum of (A.11) and (A.12) is

$$I_t^i+\frac{1}{2}H_t^{ii}+\sum_{\pi(j)<\pi(i)}H_t^{ij} \qquad (A.13)$$

$$\begin{aligned}&+\sum_{s\in\mathcal{A}_\alpha}\Big(f\big(X_{s-}+p_{\{j:\pi(j)\le\pi(i)\}}(\Delta X_s)\big)\\
&\qquad - f\big(X_{s-}+p_{\{j:\pi(j)<\pi(i)\}}(\Delta X_s)\big)-f_i(X_{s-})\Delta X_s^i\Big) \qquad (A.14)\end{aligned}$$

$$-\sum_{s\in\mathcal{A}_\alpha}\frac{1}{2}f_{ii}(X_{s-})(\Delta X_s^i)^2 \qquad (A.15)$$

$$-\sum_{s\in\mathcal{A}_\alpha}\sum_{\substack{j=1\\ \pi(j)<\pi(i)}}^{d}f_{ij}(X_{s-})\Delta X_s^i\Delta X_s^j. \qquad (A.16)$$

Since X is a semimartingale and because of Lemma 3.2, we can see that the sums (A.14)–(A.16) are absolutely convergent for $\alpha \to 0$ so that (A.13)–(A.16) converge for $\alpha \to 0$ to $\delta_t^{\mathrm{ISU},\pi}(F, X)$, using that

$$I^{ij} = H^{ij} - \sum_{0 \leq s \leq \cdot} f_{ij}(X_{s-}) \Delta X_s^i \Delta X_s^j.$$

By Theorem 3.9, we get $\delta_t^{\mathrm{ASU},\gamma_n}(F, X) \xrightarrow{p} \delta_t^{\mathrm{IASU},\pi}(F, X)$ for $n \to \infty$.

A.3 Stability

In this section, we use the notation of Christiansen [6]. Let $\tau_i : [0, \infty) \to [0, \infty)$ satisfy $\tau_i(t) \leq t$ for all $t \geq 0$ for $i = 1, 2$. The function $\tau(t) = (\tau_1(t), \tau_2(t))$ is called a *delay*. A delay is *phased* if there is an unbounded partition $(s_\ell)_{\ell \in \mathbb{N}}$ of $[0, \infty)$ with $\{0 = s_0 < s_1 < \cdots\}$ such that on each interval $(s_\ell, s_{\ell+1}]$, at most one component of τ is nonconstant. Let $(\tau^n)_{n \in \mathbb{N}}$ be a *refining sequence of delays that increase to the identity (rsdii)*, i.e.,

$$\tau_i^n([0, t]) \subseteq \tau_i^{n+1}([0, t]), \quad n \in \mathbb{N}, \qquad \text{and} \qquad \overline{\bigcup_{n \in \mathbb{N}} \tau_i^n([0, t])} = [0, t], \qquad i = 1, 2.$$

Let $\mathcal{T}$ be a set containing at least one phased rsdii. Let $X = (X^1, X^2)$ be a semimartingale and define

$$X \diamond \tau := (X^1 \circ \tau_1, X^2 \circ \tau_2), \qquad \tau \in \mathcal{T}.$$

Let

$$\mathbb{X} = \{X \diamond \tau : \tau \in \mathcal{T}\} \cup \{X\}.$$

Let $\mathbb{D}_0$ be the set of càdlàg processes starting in zero and let $\varrho : \mathbb{X} \to \mathbb{D}_0$. A mapping $\delta : \mathbb{X} \to \mathbb{D}_0^2$ is called a *decomposition scheme of* ϱ if $\varrho = \delta^1 + \delta^2$. The mapping δ assigns to each $Y \in \mathbb{X}$ a decomposition of $\varrho(Y)$. The ISU decomposition scheme is abbreviated δ^{ISU}. A decomposition scheme is called *stable at* X if

$$\delta_{t-}(X \diamond \tau^n) \xrightarrow{p} \delta_{t-}(X), \qquad n \to \infty,$$

at each $t > 0$ for all rsdii $(\tau^n)_{n \in \mathbb{N}} \subseteq \mathcal{T}$.

Proposition A.4 *Assume that $X = (X^1, X^2)$ with $X^1 = X^2 = B$ for a Brownian motion B. Let $\varrho(Y) = Y^1 Y^2$ be a simple product. Then there is a set $\mathcal{T}$ of continuous phased rsdii such that the ISU decomposition δ^{ISU} of ϱ is not stable at X.*

Proof Suppose that $\mathcal{T}$ contains a continuous phased rsdii $(\tau^n) = (\tau_1^n, \tau_2^n)$, $n \in \mathbb{N}$, with $\tau_1^n \leq \tau_2^n$, $n \in \mathbb{N}$. For a partition $(a_{\ell,i}^n, b_{\ell,i}^n]$, $\ell \in \mathbb{N}_0$, $i = 1, 2$ of $[0, \infty)$ such that $(\tau_j^n)_{j \neq i}$ is constant on $(a_{\ell,i}^n, b_{\ell,i}^n]$, let $\tau_1^n(a_{\ell,2}^n) = \tau_2^n(a_{\ell,2}^n)$, $n \in \mathbb{N}$, $\ell \in \mathbb{N}_0$. In addition, let $\mathcal{T}$ also contain $(\tilde{\tau}^n)_{n\in\mathbb{N}} = ((\tau_2^n, \tau_1^n))_{n\in\mathbb{N}}$. Since $\tau_2^n(a_{\ell,1}^n) = \tau_2^n(b_{\ell,1}^n) = \tau_1^n(b_{\ell,1}^n)$ and by the multidimensional Taylor theorem,

$$\begin{aligned}\delta_t^{\mathrm{ISU},1}(X \diamond \tau^n) &= \sum_\ell \Big(\varrho\big((X \diamond \tau^n)^{b_{\ell,1}^n \wedge t}\big) - \varrho\big((X \diamond \tau^n)^{a_{\ell,1}^n \wedge t}\big)\Big)\\ &= \sum_\ell \varrho^1\big((X \diamond \tau^n)^{a_{\ell,1}^n \wedge t}\big)\big(X^1_{\tau_1^n(b_{\ell,1}^n \wedge t)} - X^1_{\tau_1^n(a_{\ell,1}^n \wedge t)}\big).\end{aligned}$$

By the definitions of X^1, X^2 and ρ,

$$\begin{aligned}\delta_t^{\mathrm{ISU},1}(X \diamond \tau^n) &= \sum_\ell B_{\tau_2^n(a_{\ell,1}^n \wedge t)}(B_{\tau_1^n(b_{\ell,1}^n \wedge t)} - B_{\tau_1^n(a_{\ell,1}^n \wedge t)})\\ &= \sum_\ell B_{\tau_1^n(b_{\ell,1}^n \wedge t)}(B_{\tau_1^n(b_{\ell,1}^n \wedge t)} - B_{\tau_1^n(a_{\ell,1}^n \wedge t)})\\ &= \sum_\ell B_{t_\ell}(B_{t_\ell \wedge t} - B_{t_{\ell-1} \wedge t})\\ &= 2\sum_\ell \frac{(B_{t_\ell} + B_{t_{\ell-1}})}{2}(B_{t_\ell \wedge t} - B_{t_{\ell-1} \wedge t}) - \sum_\ell B_{t_{\ell-1}}(B_{t_\ell \wedge t} - B_{t_{\ell-1} \wedge t})\end{aligned}$$

for $t_\ell^n := \tau_1^n(b_{\ell,1}^n) = \tau_1^n(a_{\ell+1,1}^n) = \tau_2^n(b_{\ell-1,2}^n) = \tau_2^n(a_{\ell,2}^n)$. Let $\int_0^t B_s \circ dB_s$ denote the Stratonovich integral and $\int_0^t B_s dB_s$ the Itô integral. It holds that

$$\delta_t^{\mathrm{ISU},1}(X \diamond \tau^n) \xrightarrow{p} 2\int_0^t B_s \circ dB_s - \int_0^t B_s dB_s = \frac{1}{2}B_t^2 + \frac{1}{2}t$$

for $n \to \infty$. By the same arguments,

$$\begin{aligned}\delta_t^{\mathrm{ISU},1}(X \diamond \tilde{\tau}^n) &= \sum_\ell \Big(\varrho\big((X \diamond \tilde{\tau}^n)^{b_{\ell,2}^n \wedge t}\big) - \varrho\big((X \diamond \tilde{\tau}^n)^{a_{\ell,2}^n \wedge t}\big)\Big)\\ &= \sum_\ell \varrho^1\big((X \diamond \tilde{\tau}^n)^{a_{\ell,2}^n \wedge t}\big)(X^2_{\tau_2^n(b_{\ell,2}^n \wedge t)} - X^2_{\tau_2^n(a_{\ell,2}^n \wedge t)})\\ &= \sum_\ell B_{\tau_1^n(a_{\ell,2}^n \wedge t)}(B_{\tau_2^n(b_{\ell,2}^n \wedge t)} - B_{\tau_2^n(a_{\ell,2}^n \wedge t)})\\ &= \sum_\ell B_{\tau_2^n(a_{\ell,2}^n \wedge t)}(B_{\tau_2^n(b_{\ell,2}^n \wedge t)} - B_{\tau_2^n(a_{\ell,2}^n \wedge t)})\\ &= \sum_\ell B_{t_\ell}(B_{t_{\ell+1} \wedge t} - B_{t_\ell \wedge t})\\ &\xrightarrow{p} \int_0^t B_s dB_s\\ &= \frac{1}{2}B_t^2 - \frac{1}{2}t\end{aligned}$$

for $n \to \infty$. Therefore,

$$\text{p-lim}_{n\to\infty}\delta_t^{\mathrm{ISU},1}(X \diamond \tau^n) \neq \text{p-lim}_{n\to\infty}\delta_t^{\mathrm{ISU},i}(X \diamond \tilde{\tau}^n), \qquad i = 1, 2,$$

for $t > 0$, and hence the ISU decomposition of $\varrho(X)$ cannot be stable at X. □

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References

1. Bielecki, T.R., Cialenco, I., Pitera, M., Schmidt, T.: Fair estimation of capital risk allocation. Stat. Risk. Model. **37**, 1–24 (2020)
2. Björk, T.: Arbitrage Theory in Continuous Time, 3rd edn. Oxford University Press, London (2009)
3. Blinder, A.S.: Wage discrimination: reduced form and structural estimates. J. Hum. Resour. **8**, 436–455 (1973)
4. Candland, A., Lotz, C.: Profit and loss attribution. In: Cadoni, P. (ed.) Internal Models and Solvency II – from Regulation to Implementation, pp. 225–248. Risk Books, London (2014)
5. Černý, A., Ruf, J.: Pure-jump semimartingales. Bernoulli **27**, 2624–2648 (2021)
6. Christiansen, M.C.: On the decomposition of an insurer's profits and losses. Scand. Actuar. J. **1**, 1–20 (2022)
7. Eberlein, E., Glau, K., Papapantoleon, A.: Analysis of Fourier transform valuation formulas and applications. Appl. Math. Finance **17**, 211–240 (2010)
8. Fischer, T.: On the decomposition of risk in life insurance. Working Paper, Technische Universität Darmstadt (2004). Available online at https://www.ressources-actuarielles.net/C1256CFC001E6549/0/77BFD4ACB962F3EAC1256F62006261A5
9. Flaig, S., Junike, G.: Profit and loss attribution: an empirical study. Eur. Actuar. J. **14**, 1013–1019 (2024)
10. Fortin, N., Lemieux, T., Firpo, S.: Decomposition methods in economics. In: Ashenfelter, O., Card, D. (eds.) Handbook of Labor Economics, vol. 4, part A, pp. 1–102. Elsevier, Amsterdam (2011)
11. Frei, C.: A new approach to risk attribution and its application in credit risk analysis. Risks **8**(2), 65 (2020)
12. Friedman, E., Moulin, H.: Three methods to share joint costs or surplus. J. Econ. Theory **87**, 275–312 (1999)
13. Glasserman, P.: Monte Carlo Methods in Financial Engineering. Springer, New York (2004)
14. Grünbaum, B.: Convex Polytopes. Springer, New York (2013)

15. Grzelak, L.A., Jabłecki, J., Gątarek, D.: Efficient pricing and calibration of high-dimensional basket options. Int. J. Comput. Math. **101**, 865–888 (2023)
16. Jetses, J., Christiansen, M.C.: A general surplus decomposition principle in life insurance. Scand. Actuar. J. **2022**, 901–925 (2022)
17. Junike, G., Stier, H.: From characteristic functions to multivariate distribution functions and European option prices by the damped COS method. Preprint, (2024). Available online at https://arxiv.org/abs/2307.12843
18. Mai, J.F.: Performance attribution with respect to interest rates, FX, carry, and residual market risks. Preprint, (2023). Available online at https://arxiv.org/abs/2302.01010
19. Norberg, R.: A theory of bonus in life insurance. Finance Stoch. **3**, 373–390 (1999)
20. Oaxaca, R.: Male–female wage differentials in urban labor markets. Int. Econ. Rev. **14**, 693–709 (1973)
21. Protter, P.: Stochastic Differential Equations, 2nd edn. Springer, New York (2005)
22. Rabinovitch, R.: Pricing stock and bond options when the default-free rate is stochastic. J. Financ. Quant. Anal. **24**, 447–457 (1989)
23. Ramlau-Hansen, H.: Distribution of surplus in life insurance. ASTIN Bull. **21**, 57–71 (1991)
24. Rosen, D., Saunders, D.: Risk factor contributions in portfolio credit risk models. J. Bank. Finance **34**, 336–349 (2010)
25. Schilling, K., Bauer, D., Christiansen, M.C., Kling, A.: Decomposing dynamic risks into risk components. Manag. Sci. **66**, 5738–5756 (2020)
26. Shorrocks, A.F.: Decomposition procedures for distributional analysis: a unified framework based on the Shapley value. J. Econ. Inequal. **11**, 99–126 (2013)
27. Shapley, L.S.: A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games. Annals of Mathematical Studies, vol. 28, pp. 307–317. Princeton University Press, Princeton (1953)
28. Shubik, M.: Incentives, decentralized control, the assignment of joint costs and internal pricing. Manag. Sci. **8**, 325–343 (1962)
29. Sprumont, Y.: Ordinal cost sharing. J. Econ. Theory **81**, 126–162 (1998)

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