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Cost-efficient payoffs under model ambiguity

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Abstract
Dybvig (1988a, 1988b) solves in a complete market setting the problem of finding a payoff that is cheapest possible in reaching a given target distribution ("cost-efficient payoff"). In the presence of ambiguity, the distribution of a payoff is, however, no longer known with certainty. We study the problem of finding the cheapest possible payoff whose worst-case distribution stochastically dominates a given target distribution ("robust cost-efficient payoff") and determine solutions under certain conditions. We study the link between "robust cost-efficiency" and the maxmin expected utility setting of Gilboa and Schmeidler (1989), as well as more generally in a possibly non-expected robust utility setting. Specifically, we show that solutions to maxmin robust expected utility are necessarily robust cost-efficient. We illustrate our study with examples involving uncertainty both on the drift and on the volatility of the risky asset.

Keywords Cost-efficient payoffs · Model ambiguity · Maxmin utility · Robust preferences · Drift and volatility uncertainty

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1 Introduction

In a (complete) market without ambiguity, Dybvig [23, 22] characterises optimal payoffs for agents having law-invariant increasing preferences (e.g. expected utility maximisers). His result is based on the observation that any optimal payoff X must be *cost-efficient* in the sense that there cannot exist another payoff with the same probability distribution that is strictly cheaper than X. He then derives, for a given target distribution of terminal wealth, the payoff that achieves this target distribution at the lowest possible cost (cost-efficient payoff). Optimal payoffs are thus driven by distributional constraints rather than appearing as a solution to some optimal expected utility problem. In this regard, Brennan and Solanki [11] note that "from a practical point of view, it may well prove easier for the investor to choose directly his optimal payoff function than it would be for him to communicate his utility function to a portfolio manager." Sharpe et al. [58] and Goldstein et al. [28] introduce a tool called the distribution builder, which makes it possible for investors to analyse distributions of terminal wealth and to choose their preferred one among alternatives with equal cost; see also Sharpe [57, Sect. 7.9] and Monin [44]. Moreover, the authors of [28] argue that such a tool makes it possible to better elicit investor's preferences.

Our main objective is to extend Dybvig's results when there is uncertainty on the real-world probability measure. Uncertainty has become a prime issue in many academic domains, from economics to environmental science and psychology. Model ambiguity refers to random phenomena or outcomes whose probabilities are themselves unknown. For instance, the random outcome of a coin toss is subject to model uncertainty when the probability of the coin showing either a head or a tail is not or is at most partially known. This notion of model ambiguity goes back to Knight [40, Chap. VII] and is therefore commonly referred to as Knightian uncertainty.

In the presence of ambiguity, the probability distribution of a payoff is not anymore determined. Thus looking for a minimum cost payoff with a given probability distribution is no longer possible. However, investors may still determine a desired distribution function that they would like to achieve "at least". In this paper, we look for a minimum cost payoff that dominates a target distribution for a chosen integral stochastic order under any plausible real-world probability distribution. Our contributions are three-fold. First, we solve this problem explicitly for a general stochastic ordering under certain assumptions. Solutions to this problem are called "robust cost-efficient." Second, we draw connections between such a minimum cost payoff and the problem of finding an optimal payoff under ambiguity for general sets of robust preferences. Third, we present a number of examples, including one on the robust payoff choice in the presence of volatility uncertainty.

Our results generalise the results on cost-efficiency given in Dybvig [23, 22], Cox and Leland [20] and Bernard et al. [5, 6]. When there is no ambiguity on the real-world probability, the robust cost efficient payoffs coincide with the cost-efficient payoffs studied in the literature. To derive our results, we build on the so-called quantile approach to solve the optimisation of a law-invariant increasing functional; see e.g. Schied [53], Carlier and Dana [12, 13, 14], Jin and Zhou [38], He and Zhou [36, 37], Bernard et al. [5], Xu [63] and Rüschendorf and Vanduffel [50].

Specifically, we consider a static setting and are able to address uncertainty about volatility. We show that under certain conditions, the solution to a general robust payoff maximisation problem is equal to the solution of a classical payoff maximisation problem under a *least favourable measure* $\mathbb{P}^*$ with respect to some stochastic ordering. This was already shown by Schied [54] for the case of robust expected utility theory and using first order stochastic dominance. Here, however, we show that these results extend to the case of more general preferences, focusing on the case of first order and second order stochastic dominance. To reach this conclusion, we make certain assumptions. Most notably, we assume the existence of a least favourable measure. This assumption was also made in [54] for the case of first order stochastic dominance, but in this paper, we deal also with the case of second order stochastic dominance (weaker assumption). We also assume that the pricing kernel is continuously distributed under this least favourable measure, as in e.g. [38, 36, 37, 5, 63], among many others. These assumptions are important to solve the robust cost-efficient problem.

Furthermore, we show that there is a natural correspondence between optimal payoffs in the maxmin utility setting of Gilboa and Schmeidler [27] with a concave increasing utility, and robust cost-efficient payoffs: for any robust cost-efficient payoff X^*, there is a utility function such that X^* solves the maxmin expected utility maximisation problem. We further show that the solution to a robust maximisation problem with respect to a general family of preferences is cost-efficient. This result implies that instead of solving a robust maximisation problem with respect to a general family of preferences, one could solve an expected utility maximisation problem under the single measure $\mathbb{P}^*$ for a suitable concave utility function.

The literature on optimal payoff choice under ambiguity includes the seminal setting of [27], that is, the so-called “maxmin expected utility,” which was later referred to as robust utility functional by Schied et al. [55]. Specifically, these authors characterise preferences that have a robust utility numerical representation $\min_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[u(X)]$ for some set of probabilities $\mathcal{P}$. Gundel [34] provides a dual characterisation of the solution for robust utility maximisation in both a complete and an incomplete market model. Klibanoff et al. [39] distinguish between subjective beliefs, i.e., the definition of the set of possible or plausible subjective probability measures, and ambiguity attitude, i.e., a characterisation of the agent’s behaviour toward ambiguity. Based on [39], Gollier [30] analyses the effect of ambiguity aversion on the demand for the uncertain asset in a payoff choice problem.

Schied [54] solves the maximisation problem of maxmin expected utility of [27] in a general complete market model with dynamic trading, provided there is a least favourable measure with respect to first order stochastic dominance. Specifically, he finds that the optimum for the maxmin utility setting of [27] can be derived in the standard expected utility setting under the least favourable measure. Schied [54] works with a complete market model and mainly in a static setting; dynamics only come into play when the martingale method is applied to the static solutions. A survey on robust preferences and robust payoff choice can be found in Schied et al. [55].

The paper is organised as follows. The robust cost-efficiency problem is described in Sect. 2. In Sect. 3, we solve the robust cost-efficiency problem, and we include two examples in a lognormal market with uncertainty on the drift and the volatility along with another example in a Lévy market in which the physical measure is obtained by the Esscher transform. In Sect. 4, we develop the correspondence between robust cost-efficient payoffs and strategies that solve a robust optimal payoff problem, including the maxmin utility setting of [27] as a special case. In Sect. 5, we show that the solution to a general robust optimal payoff problem can also be obtained as the solution to the maximisation of the maxmin utility setting of [27] for a well-chosen concave utility function. Section 6 concludes.

2 Problem statement

We assume a static market setting in which trading only takes place today and at the end of the planning horizon $T > 0$. There is a bank account earning the continuously compounded risk-free interest rate $r \in \mathbb{R}$. Let $\mathbb{R}_+ = [0, \infty)$. Let $S_T : \Omega \to \mathbb{R}_+$ represent the random value of a risky asset at maturity. We denote by $S_0 > 0$ its current value and by $\mathcal{F}$ the σ-algebra generated by S_T. Let $\mathcal{P}$ be a set of equivalent real-world probability measures on $(\Omega, \mathcal{F})$. The set $\mathcal{P}$ can be thought of as a collection of probability measures that the investor deems plausible for the market. We define the set of *payoffs* $\mathcal{X} = \{g(S_T) : g : \mathbb{R}_+ \to \mathbb{R}_+ \text{ measurable}, E_{\mathbb{Q}}[g(S_T)] < \infty\}$, where $\mathbb{Q}$ is a fixed pricing measure equivalent to all $\mathbb{P} \in \mathcal{P}$. A payoff X is also called a *contingent claim*. Furthermore, for any $X \in \mathcal{X}$, its price is given by $e^{-rT} E_{\mathbb{Q}}[X]$. By $F_X^{\mathbb{P}}$, we denote the cumulative distribution function of $X \in \mathcal{X}$ under $\mathbb{P} \in \mathcal{P}$.

Remark 2.1 As in Dybvig [22], we could use the discount factor $(1 + r)^{-T}$ for some $r \in (-1, \infty)$, based on compounded interest rates. However, working with exponential stock price models, it is more convenient and consistent with more recent literature to use the discount factor e^{-rT} for $r \in \mathbb{R}$.

Remark 2.2 Under the assumption that all call options $(S_T - K)^+$, $K \geq 0$, are traded, the market can be completed. This is shown by Ross [48] for discrete Ω and by Nachman [46, Corollary 6] for general static markets. Market completion by spanning call options as in [46] has been further developed by Madan and Milne [41], Bakshi and Madan [2], Carr and Wu [16] and many others. Rogge [47, Theorem 2] shows that n-period models are complete if a call-completeness condition is satisfied. Carr and Madan [15, Eq. (1)] provide an explicit replication strategy of a payoff $g(S_T)$ if g is twice continuously differentiable. See also Breeden and Litzenberger [10] for earlier results.

Consider an investor with a finite budget and planning horizon $T > 0$ who wishes to invest in the market while having ambiguous views on the real-world probability measure. How can she find her optimal investment strategy? As in Schied [54], she could maximise some robust expected utility à la Gilboa and Schmeidler [27]. The basic idea is then to look for a payoff that maximises the worst case expected utility, reflecting the idea that the investor aims to protect against the worst while hoping for the best. However, it seems easier for investors to specify the desired probability distribution of the terminal wealth rather than a utility function (see Brennan and Solanki [11], Sharpe et al. [58] and Goldstein et al. [28]).

As in [23, 58], Vrecko and Langer [61], Bernard et al. [5], we thus assume in this paper that the investor specifies a desired (cumulative) distribution function F_0 of future terminal wealth. Once the investor understands which distribution function F_0 is acceptable to her, the natural question arises as to how to find under ambiguity the cheapest payoff with a distribution function at maturity that is "at least as good" as F_0. This is the robust cost-efficiency problem formalised hereafter. In this regard, we need to recall the concept of integral stochastic ordering; see e.g. Denuit et al. [21, Sect. 3]. In this paper, we denote by $\mathbb{F}$ a set of measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$.

Definition 2.3 Let G and F be two distribution functions with support on $\mathbb{R}_+$. Then G dominates F in *integral stochastic ordering with respect to* $\mathbb{F}$, written as $F \preceq_{\mathbb{F}} G$, if

$$\int_{\mathbb{R}_+} f(x)dF \le \int_{\mathbb{R}_+} f(x)dG \qquad \text{for all } f \in \mathbb{F} \text{ such that the expectations are finite.}$$

Let $\mathbb{F}_{\mathrm{FSD}}$ denote the set of all nondecreasing functions from $\mathbb{R}_+$ to $\mathbb{R}$. The corresponding integral stochastic ordering is called *first order stochastic dominance* (FSD). Furthermore, let $\mathbb{F}_{\mathrm{SSD}}$ denote the set of all nondecreasing and concave functions from $\mathbb{R}_+$ to $\mathbb{R}$. The corresponding integral stochastic ordering is called *second order stochastic dominance* (SSD). It is well known that FSD reflects the common agreement of all investors with law-invariant increasing preferences, see Bernard et al. [6, Theorem 1], whereas SSD reflects the common agreement of those who have law-invariant increasing and diversification-loving preferences (risk-averse investors); see Bernard and Sturm [7, Corollary 2.6].

Problem 2.4 The $\mathbb{F}$-*robust cost-efficiency problem for a distribution function* F_0 is defined as

$$\inf_{X \in \mathcal{B}^{\mathbb{F}}_{F_0}} e^{-rT} E_{\mathbb{Q}}[X], \tag{2.1}$$

where $\mathcal{B}^{\mathbb{F}}_{F_0}$ denotes the class of admissible payoffs defined as

$$\mathcal{B}^{\mathbb{F}}_{F_0} = \{X \in \mathcal{X} : F_0 \preceq_{\mathbb{F}} F^{\mathbb{P}}_X, \forall \mathbb{P} \in \mathcal{P}\}.$$

A solution to (2.1) is called an $\mathbb{F}$-*robust cost-efficient payoff.*

As discussed above, the target distribution function of the investor is F_0. That is, we are interested in all payoffs that have a distribution function at maturity that is at least as good as F_0 under all plausible scenarios $\mathbb{P} \in \mathcal{P}$. For example, when $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$, we care about payoffs having distribution functions $F^{\mathbb{P}}_X$, $\mathbb{P} \in \mathcal{P}$, that dominate F_0 in FSD. In order not to "throw away investors' money", see Dybvig [23], we then aim to determine the cheapest among the payoffs in the admissible set $\mathcal{B}^{\mathbb{F}}_{F_0}$. In Theorem 3.1, we provide solutions to the $\mathbb{F}$-robust cost-efficiency problem (2.1) under regularity conditions on the set $\mathbb{F}$ and F_0.

Dybvig [23, 22] introduced the standard cost-efficiency problem without ambiguity on the set of physical measures, that is, when $\mathcal{P} = \{\mathbb{P}\}$. Specifically, for some fixed $\mathbb{P} \in \mathcal{P}$, the problem he considered reads as

$$\inf_{X \in \mathcal{A}^{\mathbb{P}}_{F_0}} e^{-rT} E_{\mathbb{Q}}[X], \tag{2.2}$$

where

$$\mathcal{A}^{\mathbb{P}}_{F_0} = \{X \in \mathcal{X} : F_0 = F^{\mathbb{P}}_X\}.$$

We refer to this problem as the *standard cost-efficiency problem*. Furthermore, we say that a payoff X that is distributed with $F^{\mathbb{P}}_X$ is $\mathbb{P}$-*cost-efficient* if X solves the standard cost-efficiency problem (2.2) under $\mathbb{P}$ with respect to $F_0 = F^{\mathbb{P}}_X$. By Bernard et al. [5], a payoff X is $\mathbb{P}$-cost-efficient if and only if X is nonincreasing in the state price $\xi^{\mathbb{P}} = e^{-rT} \frac{d\mathbb{Q}}{d\mathbb{P}}$ $\mathbb{P}$-a.s.; see also Schied [53, Proposition 2.5].

The standard cost-efficiency problem (2.2) has been solved in [23, 22]; see also Lemma B.1 in the Appendix. In Corollary 3.5, we show that if $\mathcal{P}$ is a singleton, the solution to the $\mathbb{F}$-robust cost-efficiency problem is unique and coincides with the solution to the standard cost-efficiency problem.

Remark 2.5 For a fixed $X \in \mathcal{X}$, in general, $F^{\mathbb{P}}_X$ cannot be equal to F_0 for all $\mathbb{P} \in \mathcal{P}$. Therefore, we replace the condition $F_0 = F^{\mathbb{P}}_X$ in the standard cost-efficiency problem with $F_0 \preceq_{\mathbb{F}} F^{\mathbb{P}}_X$ in the robust setting.

The next example anticipates Sects. 3.2.1 and 3.2.2 and is designed to help distinguish between the standard and the robust cost-efficiency problems. As in Embrechts and Hofert [24], we define for a nondecreasing function $T : \mathbb{R} \to \mathbb{R}$ the *generalised inverse* T^{-1} by

$$T^{-1}(y) = \inf\{x \in \mathbb{R} : T(x) \geq y\}, \qquad y \in \mathbb{R}.$$

Example 2.6 Assume the real-world distribution of S_T is lognormal. There are three investors: one investor assumes that the drift of S_T under the physical measure, denoted by $\mathbb{P}^{\mu_1}$, is equal to $\mu_1 > r$. Another investor assumes that the drift is given by $\mu_2 > \mu_1$ under the physical measure, denoted by $\mathbb{P}^{\mu_2}$. A third investor has ambiguity and assumes that the drift lies in the interval $[\mu_1, \mu_2]$, and thus considers the set $\mathcal{P} = \{\mathbb{P}^{\mu} : \mu \in [\mu_1, \mu_2]\}$ as the set of all plausible probability measures on $(\Omega, \mathcal{F})$. The cheapest payoffs to obtain a fixed target distribution function F_0 are well known for investors one and two and are given by

$$X_1^* := F_0^{-1}\big(F^{\mathbb{P}^{\mu_1}}_{S_T}(S_T)\big), \qquad X_2^* := F_0^{-1}\big(F^{\mathbb{P}^{\mu_2}}_{S_T}(S_T)\big),$$

respectively; see Bernard et al. [5, Proposition 3]. Within the set $\mathcal{P}$, $\mathbb{P}^{\mu_1}$ corresponds to a pessimistic view of the stock price behaviour, and we shall see in Sect. 3.2.1 that X_1^* is a solution to the cost-efficiency problem of the third investor if $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$. In the case in which $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$, X_1^* also solves the cost-efficiency problem of the third investor if additionally $F_0^{-1} \circ F_{S_T}^{\mathbb{P}^{\mu_1}}$ is concave; see Sect. 3.2.2. This example illustrates that the solution to the standard cost-efficiency problem for arbitrary $\mathbb{P} \in \mathcal{P}$ and the solution to the robust cost-efficiency problem do not coincide in general. Note that the case $\mu_1 < \mu_2 < r$ is economically less relevant. However, it can be shown that in this case, $X_1^* := F_0^{-1}(1 - F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T))$, $X_2^* := F_0^{-1}(1 - F_{S_T}^{\mathbb{P}^{\mu_2}}(S_T))$ and that the latter payoff solves the cost-efficiency problem of the third investor if $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$; see also Remark 3.6.

Remark 2.7 As in Rüschendorf and Wolf [51], we could also consider uncertainty on the target distribution function F_0. Specifically, it is assumed in [51] that the investor specifies finitely many acceptable distribution functions $F_0^1, \dots, F_0^N$. As all N distribution functions are acceptable to the investor, she could solve the robust cost-efficiency problem N times and buy the cheapest among the N solutions.

2.1 Assumptions

In order to solve the robust cost efficiency problem (2.1), we need some regularity conditions on the set $\mathbb{F}$ and on the target distribution F_0. In this regard, we define some concepts.

Recall first the concept of a *least favourable measure* introduced by Schied [54] for the case $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$. For $\mathbb{P} \in \mathcal{P}$, we define the corresponding *likelihood ratio* by $\ell^{\mathbb{P}} = \frac{d\mathbb{P}}{d\mathbb{Q}}$. We remark that the random variable $\frac{e^{-rT}}{\ell^{\mathbb{P}}}$ is also called *state price* because the price of a payoff $X \in \mathcal{X}$ can be expressed by

$$e^{-rT} E_{\mathbb{Q}}[X] = E_{\mathbb{P}}\left[\frac{e^{-rT}}{\ell^{\mathbb{P}}} X\right], \qquad \mathbb{P} \in \mathcal{P}.$$

Definition 2.8 A measure $\mathbb{P}^* \in \mathcal{P}$ with corresponding likelihood ratio $\ell^* := \frac{d\mathbb{P}^*}{d\mathbb{Q}}$ is called a *least favourable measure with respect to* $\mathbb{F}$ if $F_{\ell^*}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{\ell^*}^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$.

Definition 2.8 generalises [54, Definition 2.1] which assumed the existence of a least favourable measure with respect to $\mathbb{F}_{\mathrm{FSD}}$ to determine payoffs that solve the robust expected utility problem of Gilboa–Schmeidler [27]. We also need the following definition.

Definition 2.9 The set $\mathbb{F}$ is said to be *composition-consistent* if for $f, g \in \mathbb{F}$, also $f \circ g \in \mathbb{F}$.

Note that the sets $\mathbb{F}_{\mathrm{FSD}}$ and $\mathbb{F}_{\mathrm{SSD}}$ are composition-consistent. This follows from the fact that the composition of nondecreasing (resp. nondecreasing and concave) functions is again nondecreasing (resp. nondecreasing and concave).

The following result provides conditions that guarantee the existence of a least favourable measure and turns out to be very useful for applications.

Proposition 2.10 *Assume that* $\mathbb{F}$ *is composition-consistent. If* $F_{S_T}^{\mathbb{P}'} \preceq_{\mathbb{F}} F_{S_T}^{\mathbb{P}}$ *for some* $\mathbb{P}' \in \mathcal{P}$ *and all* $\mathbb{P} \in \mathcal{P}$, *and if* $\ell^{\mathbb{P}'} = f(S_T)$ *for some* $f \in \mathbb{F}$, *then* $\mathbb{P}'$ *is a least* *favourable measure with respect to* $\mathbb{F}$. *If, additionally,* S_T *is continuously distributed under* $\mathbb{P}'$ *and* f *is strictly increasing, then* $\ell^{\mathbb{P}'}$ *is continuously distributed under* $\mathbb{P}'$.

Proof Let $\mathbb{P}, \mathbb{P}' \in \mathcal{P}$. Let X be a payoff and $f \in \mathbb{F}$. Recall that $F_X^{\mathbb{P}'} \preceq_{\mathbb{F}} F_X^{\mathbb{P}}$ if and only if $E_{\mathbb{P}'}[g(X)] \leq E_{\mathbb{P}}[g(X)]$ for all $g \in \mathbb{F}$ such that the expectations are finite. Because $\mathbb{F}$ is composition-consistent, it follows that

$$F_X^{\mathbb{P}'} \preceq_{\mathbb{F}} F_X^{\mathbb{P}} \quad \Longrightarrow \quad F_{f(X)}^{\mathbb{P}'} \preceq_{\mathbb{F}} F_{f(X)}^{\mathbb{P}}. \tag{2.3}$$

The result $F_{\ell^{\mathbb{P}'}}^{\mathbb{P}'} \preceq_{\mathbb{F}} F_{\ell^{\mathbb{P}'}}^{\mathbb{P}}$ then follows by (2.3). Let f be strictly increasing. By [24], the generalised inverse f^{-1} of f is continuous on the range of f. Thus we have

$$\mathbb{P}'[\ell^{\mathbb{P}'} \leq x] = \mathbb{P}'[S_T \leq f^{-1}(x)] = F_{S_T}^{\mathbb{P}'}\big(f^{-1}(x)\big), \qquad x \in \mathbb{R},$$

and so $\ell^{\mathbb{P}'}$ is continuously distributed under $\mathbb{P}'$ since $F_{S_T}^{\mathbb{P}'}$ is continuous. □

We also need a definition that is, to the best of our knowledge, new to the literature.

Definition 2.11 The set $\mathbb{F}$ is called *cost-consistent* if for all $X, Y \in \mathcal{X}$ and all $\mathbb{P} \in \mathcal{P}$ such that X, Y are $\mathbb{P}$-cost-efficient, $F_X^{\mathbb{P}} \preceq_{\mathbb{F}} F_Y^{\mathbb{P}}$ implies $E_{\mathbb{Q}}[X] \leq E_{\mathbb{Q}}[Y]$ and additionally, $F_X^{\mathbb{P}} \neq F_Y^{\mathbb{P}}$ implies $E_{\mathbb{Q}}[X] < E_{\mathbb{Q}}[Y]$.

Proposition 2.12 *The set* $\mathbb{F}_{\mathrm{SSD}}$ *is cost-consistent. Moreover, if* $\mathbb{F}_{\mathrm{SSD}} \subseteq \mathbb{F}$, *then* $\mathbb{F}$ *is cost-consistent.*

Proof The cost-consistency of $\mathbb{F}_{\mathrm{SSD}}$ can be proved along the lines of Bernard et al. [8, proof of Lemma 2]. Furthermore, $F_X^{\mathbb{P}} \preceq_{\mathbb{F}} F_Y^{\mathbb{P}}$ implies $F_X^{\mathbb{P}} \preceq_{\mathbb{F}_{\mathrm{SSD}}} F_Y^{\mathbb{P}}$, which finishes the proof. □

As the set $\mathbb{F}_{\mathrm{SSD}}$ is contained in $\mathbb{F}_{\mathrm{FSD}}$, Proposition 2.12 implies that $\mathbb{F}_{\mathrm{FSD}}$ and $\mathbb{F}_{\mathrm{SSD}}$ are cost-consistent. We provide examples of sets $\mathbb{F}$ of functions that are not cost- or composition-consistent.

Example 2.13 Third order stochastic dominance is the integral stochastic ordering that arises from the set $\mathbb{F}_{\mathrm{TSD}}$ consisting of all functions $\mathbb{R} \to f : \mathbb{R}_+$ such that $f' > 0$, $f'' \leq 0$ and $f''' \geq 0$. The set $\mathbb{F}_{\mathrm{TSD}}$ is composition-consistent, but in general not cost-consistent; see Appendix A.

Example 2.14 Müller et al. [45] introduced the $(1+\gamma)$-stochastic dominance order for $\gamma \in (0,1)$, which lies between FSD and SSD ordering. The set induced by $(1+\gamma)$-stochastic dominance order is in general not composition-consistent, but is cost-consistent in light of Proposition 2.12.

Example 2.15 Rothschild and Stiglitz [49] introduced the concave stochastic order which is defined via the set of all concave (but not necessarily nondecreasing) functions. The concave stochastic order coincides with SSD if we compare two payoffs with the same mean; see Föllmer and Schied [25, Remark 2.63]. The set of all concave functions is cost-consistent, but not composition-consistent.

We now list a series of assumptions that we often use to derive our main results.

Assumption 2.16 The function F_0^{-1} is square-integrable, i.e., $\int_0^1 (F_0^{-1}(u))^2 du < \infty$, and we have $F_0(x) = 0$ for $x < 0$.

Assumption 2.17 The set $\mathbb{F}$ is composition-consistent and cost-consistent.

Assumption 2.18 The set $\mathcal{P}$ contains a least favourable measure with respect to $\mathbb{F}$. We denote this least favourable measure by $\mathbb{P}^*$.

Assumption 2.19 Denote by ℓ^* the likelihood ratio of the least favourable measure $\mathbb{P}^*$. We assume that $x \mapsto F_{\ell^*}^{\mathbb{P}^*}(x)$ is continuous and that $\frac{1}{\ell^*}$ has finite variance under $\mathbb{P}^*$.

Assumption 2.16 is technical and ensures that the robust cost-efficiency problem is well posed, i.e., $\mathcal{B}_{F_0}^{\mathbb{F}}$ is not empty; see Theorem 3.1.

Assumption 2.18 can also be found in Schied [54] for the case $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$. Note that when $\mathbb{F}$ becomes larger, Assumption 2.18 becomes stronger. Specifically, requiring a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ with respect to $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$ is more stringent than in the case $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$. In particular, Proposition 2.10 provides sufficient conditions for the existence of a least favourable measure $P^* \in \mathcal{P}$ with respect to $\mathbb{F}$. The condition in Assumption 2.19 that $x \mapsto F_{\ell^*}^{\mathbb{P}^*}(x)$ is continuous is also made in a setting without ambiguity in e.g. Jin and Zhou [38], He and Zhou [36, 37], Bernard et al. [5], Xu [63] among many others. It is a strong assumption in the sense that we essentially exclude discrete settings.

3 Robust cost-efficiency

3.1 Solution of the robust cost-efficiency problem

In the next result, we make the assumption that $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$. Note that this assumption is always true if $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$. The assumption is also true if $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$, provided that $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ is concave.

Theorem 3.1 *Suppose that Assumptions 2.16–2.19 hold and that $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$. Then the $\mathbb{F}$-robust cost-efficiency problem for F_0 has a $\mathbb{P}^*$-a.s. unique solution given by*

$$F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big).$$

Proof Recall that ℓ^* denotes the likelihood ratio that corresponds to $\mathbb{P}^*$. Let

$$X^* = F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big).$$

As $F_{\ell^*}^{\mathbb{P}^*}(\ell^*)$ is uniformly distributed under $\mathbb{P}^*$ by Assumption 2.19, it follows from Lemma B.1 that $F_{X^*}^{\mathbb{P}^*} = F_0$, and by Assumption 2.16, we have

$$E_{\mathbb{P}^*}\Big[\Big(F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big)\Big)^2\Big] = \int_0^1 \big(F_0^{-1}(u)\big)^2 du < \infty.$$

Assumption 2.19 thus implies that $E_{\mathbb{Q}}[X^*] < \infty$ because

$$E_{\mathbb{Q}}[X^*] = E_{\mathbb{P}^*}\left[\frac{1}{\ell^*} F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big)\right]$$

$$\leq \sqrt{E_{\mathbb{P}^*}\left[\frac{1}{(\ell^*)^2}\right] E_{\mathbb{P}^*}\left[\Big(F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big)\Big)^2\right]} < \infty.$$

Therefore $X^* \in \mathcal{X}$. By Assumptions 2.17 and 2.18 and as $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$, it follows from (2.3) that $F_0 = F_{X^*}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{X^*}^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$; hence $X^* \in \mathcal{B}_{F_0}^{\mathbb{F}}$. Let $Y \in \mathcal{B}_{F_0}^{\mathbb{F}}$ and define

$$Y^* = (F_Y^{\mathbb{P}^*})^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big).$$

Then Y^* is $\mathbb{P}^*$-cost-efficient for $F_Y^{\mathbb{P}^*}$ and we have $F_{X^*}^{\mathbb{P}^*} = F_0 \preceq_{\mathbb{F}} F_Y^{\mathbb{P}^*} = F_{Y^*}^{\mathbb{P}^*}$. By Assumption 2.17, $\mathbb{F}$ is cost-consistent, which implies that $E_{\mathbb{Q}}[X^*] \leq E_{\mathbb{Q}}[Y^*] \leq E_{\mathbb{Q}}[Y]$. Hence every admissible payoff is more expensive than X^*.

We now show uniqueness. If $\hat{X}$ is another solution to the robust cost-efficiency problem, then $F_0 \preceq_{\mathbb{F}} F_{\hat{X}}^{\mathbb{P}^*}$. If $F_{\hat{X}}^{\mathbb{P}^*} = F_0$ and $E_{\mathbb{Q}}[X^*] = E_{\mathbb{Q}}[\hat{X}]$, then $X^* = \hat{X}$ $\mathbb{P}^*$-a.s. by Lemma B.1, because the solution X^* corresponds to the solution of the standard $\mathbb{P}^*$-cost-efficiency problem for F_0, which has a unique solution. If $F_0 \neq F_{\hat{X}}^{\mathbb{P}^*}$, then $E_{\mathbb{Q}}[X^*] < E_{\mathbb{Q}}[\hat{X}]$ because $\mathbb{F}$ is cost-consistent. Hence X^* is the unique solution to the robust cost-efficiency problem. □

Remark 3.2 Instead of requiring that F_0^{-1} is square-integrable, the proof of Theorem 3.1 shows that it is sufficient to assume that $F_0^{-1}(F_{\ell^*}^{\mathbb{P}^*}(\ell^*))$ has a finite price.

Remark 3.3 Does ambiguity increase costs? Let the assumptions of Theorem 3.1 be in force. Let us compare two investors. Investor A has ambiguity and considers the set $\mathcal{P}$ as the set of possible real-world measures. Investor B has (e.g. based on a deep market analysis or insider knowledge) no ambiguity and knows that $\mathbb{P} \in \mathcal{P}$ is the true real-world measure. Both investors consider F_0 as the target distribution function. Investor A buys $X^* = F_0^{-1}(F_{\ell^*}^{\mathbb{P}^*}(\ell^*))$ according to Theorem 3.1, whereas investor B buys $X = F_0^{-1}(F_{\ell^{\mathbb{P}}}^{\mathbb{P}}(\ell^{\mathbb{P}}))$ (see Lemma B.1). As $X^* \in \mathcal{B}_{F_0}^{\mathbb{F}}$, it holds that $F_X^{\mathbb{P}} = F_0 \preceq_{\mathbb{F}} F_{X^*}^{\mathbb{P}}$. As the set $\mathbb{F}$ is cost-consistent, it follows that $E_{\mathbb{Q}}[X] \leq E_{\mathbb{Q}}[X^*]$. If we additionally have $F_X^{\mathbb{P}} \neq F_{X^*}^{\mathbb{P}}$, then it follows that $E_{\mathbb{Q}}[X] < E_{\mathbb{Q}}[X^*]$. In the robust setting, we end up with a payoff X^* whose distribution $F_{X^*}^{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$, dominates F_0 in stochastic ordering for all $\mathbb{P} \in \mathcal{P}$. Thus under ambiguity, the preferred payoff has a (strictly) higher price and the optimal robust choice X^* typically will not match the choice X without uncertainty.

Remark 3.4 The condition in Theorem 3.1 that the function $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ must be concave in the case where $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$ means that the target distribution function F_0 is required to be lighter-tailed than the distribution function $F_{\ell^*}^{\mathbb{P}^*}$. Specifically, $F_{\ell^*}^{\mathbb{P}^*}$ must dominate F_0 in the sense of the convex transform order (see Shaked and Shanthikumar [56, Definition 4.B.1]).

The next result shows that the standard and the robust cost-efficiency problem coincide in a setting without uncertainty. Note that we do not require $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$ as in Theorem 3.1.

Corollary 3.5 *Suppose that Assumptions 2.16 and 2.19 hold and that $\mathcal{P} = \{\mathbb{P}\}$ is a singleton. The solutions to the $\mathbb{F}$-robust cost-efficiency problem for F_0 and the standard cost-efficiency problem for F_0 are unique and identical.*

Proof Note that when $\mathcal{P} = \{\mathbb{P}\}$, then $\mathbb{P}$ is a least favourable measure. By Lemma B.1, $X^* = F_0^{-1}(F_{\ell^{\mathbb{P}}}^{\mathbb{P}}(\ell^{\mathbb{P}}))$ is the unique solution to the standard cost-efficiency problem. As in the proof of Theorem 3.1, one can show that $X^* \in \mathcal{X}$. Then $X^* \in \mathcal{B}_{F_0}^{\mathbb{F}}$ follows immediately because $F_{X^*}^{\mathbb{P}} = F_0$. As in the proof of Theorem 3.1, one can show that X^* is the only admissible payoff solving the $\mathbb{F}$-robust cost-efficiency problem. $\square$

In the following section, we illustrate Theorem 3.1 in a lognormal market setting with uncertainty on the drift *and* volatility, whereas in Sect. 3.3, we deal with a more general market setting.

3.2 Robust cost-efficient payoffs in lognormal markets

We assume that under the pricing measure $\mathbb{Q}$, S_T has a lognormal distribution with parameters $\log S_0 + (r - \frac{s^2}{2})T$ and $s\sqrt{T}$ with stock price $S_0 > 0$ today, interest rate $r \in \mathbb{R}$, time horizon $T > 0$ and volatility $s > 0$. Under $\mathbb{Q}$, S_T is lognormally distributed with density $f^{r,s}$, where for $m \in \mathbb{R}$ and $\varsigma > 0$, we define

$$f^{m,\varsigma}(x) = \frac{1}{x\varsigma\sqrt{T}\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \ln S_0 - (m - \frac{\varsigma^2}{2})T)^2}{2\varsigma^2 T}\right), \qquad x > 0. \quad (3.1)$$

3.2.1 Drift uncertainty: $\mathbb{F}_{\mathrm{FSD}}$-robust cost-efficient payoff

The real-world distribution function of S_T is assumed to be lognormal with parameters $\log S_0 + (\mu - \frac{s^2}{2})T$ and $s\sqrt{T}$, but there is uncertainty about the precise level of the drift parameter μ. In particular, the agent only expects the true drift parameter μ to lie in the interval $\mathcal{D}^{\mu_1} = \{\mu \in \mathbb{R} : \mu \geq \mu_1\}$ for $\mu_1 > r$, and thus she considers $\mathcal{P} = (\mathbb{P}^{\mu})_{\mu \in \mathcal{D}^{\mu_1}}$ as the set of all plausible probability measures on $(\Omega, \mathcal{F})$. Under $\mathbb{P}^{\mu}$, S_T is lognormal with density $f^{\mu,s}$. It follows that $F_{S_T}^{\mathbb{P}^*} \preceq_{\mathbb{F}_{\mathrm{FSD}}} F_{S_T}^{\mathbb{P}^{\mu}}$ for all $\mu \geq \mu_1$, where $\mathbb{P}^* := \mathbb{P}^{\mu_1}$. Let $h^{\mu,\varsigma}(x) = \frac{f^{\mu,\varsigma}(x)}{f^{r,s}(x)}$, $x > 0$. A straightforward computation shows that

$$h^{\mu,s}(x) = \left(\frac{x}{S_0}\right)^{\frac{(\mu - r)}{s^2}} \exp\left(\frac{r^2 - \mu^2 + s^2(\mu - r)}{2s^2} T\right), \qquad x > 0. \qquad (3.2)$$

Note that $\mathbb{P}^{\mu}[A] = \int_A h^{\mu,s}(S_T) d\mathbb{Q}$, $A \in \mathcal{F}$, which implies $\ell^{\mathbb{P}^{\mu_1}} = h^{\mu_1,s}(S_T)$. As $\mu_1 > r$, $\ell^{\mathbb{P}^{\mu_1}}$ is a strictly increasing function of S_T. Furthermore, $\frac{1}{\ell^{\mathbb{P}^{\mu_1}}}$ has finite variance. By Proposition 2.10, $\mathbb{P}^*$ is a least favourable measure with corresponding likelihood ratio $\ell^* := \ell^{\mathbb{P}^{\mu_1}}$, i.e., Assumptions 2.18 and 2.19 are satisfied. Theorem 3.1 shows that the $\mathbb{F}_{\mathrm{FSD}}$-robust cost-efficient payoff for a distribution function F_0 satisfying Assumption 2.16 is given by

$$X^* = F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big) = F_0^{-1}\big(F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T)\big).$$

The second equality follows from the increasingness of $\ell^{\mathbb{P}^{\mu_1}}$ in S_T. The agent thus chooses the optimal payoff as if she believes that the worst-case plausible value μ_1 for the drift parameter μ will materialise. This finding is consistent with the results obtained by Schied [54, Sect. 3.1] on the impact of drift uncertainty on optimal payoff choice in a Black–Scholes setting.

Remark 3.6 Let us consider $\widetilde{\mathcal{D}}^{\mu_1} = \{\mu \in \mathbb{R} : \mu \le \mu_1\}$ for $\mu_1 < r$ instead of $\mathcal{D}^{\mu_1}$. Then $\mathbb{P}^* = \mathbb{P}^{\mu_1}$ is a least favourable measure. This can be shown directly by observing that under $\mathbb{P}^{\mu}$ for $\mu \in \widetilde{\mathcal{D}}^{\mu_1}$, the likelihood ratio $\ell^{\mathbb{P}^*} = h^{\mu_1,s}(S_T)$ is decreasing in S_T and lognormally distributed. Further, we use the fact that a lognormal distribution G with parameters μ_G and $\sigma_G > 0$ dominates a lognormal distribution F with parameters μ_F and $\sigma_F > 0$ in FSD if and only if $\mu_G \ge \mu_F$ and $\sigma_G = \sigma_F$. By Theorem 3.1, the $\mathbb{F}_{\mathrm{FSD}}$-robust cost-efficient payoff for a distribution F_0 satisfying Assumption 2.16 is given by

$$X^* = F_0^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big) = F_0^{-1}\big(1 - F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T)\big).$$

The second equality follows from the decreasingness of $\ell^{\mathbb{P}^*}$ in S_T.

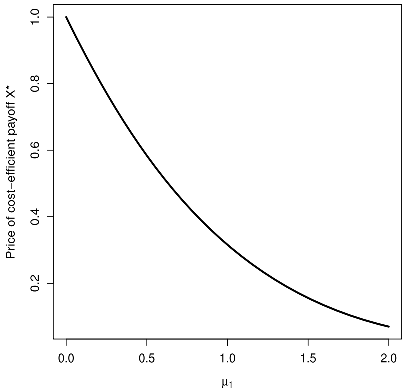
Example 3.7 We next consider the exponential distribution for the distribution function F_0, i.e., $F_0(x) = 1 - e^{-x}$, $x \ge 0$, which satisfies Assumption 2.16. Panel (A) of Fig. 1 displays the price of the robust cost-efficient payoff for varying levels of the parameter μ_1, which describes the ambiguity that the agent faces (consistently with Remark 3.3). The higher μ_1, the smaller the set $\mathcal{D}^{\mu_1}$, i.e., the lower the degree of ambiguity, and the cheaper X^*. In panel (B) of Fig. 1, we display for several values of μ_1 the robust cost-efficient payoff normalised by its initial price as a function of realisations s of S_T, i.e., we display the curve

$$s \mapsto \frac{F_0^{-1}(F_{S_T}^{\mathbb{P}^{\mu_1}}(s))}{\pi_{\mu_1}},$$

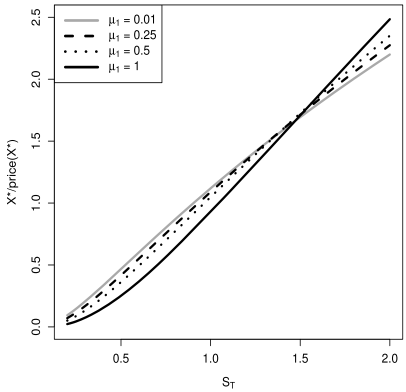
where $\pi_{\mu_1} = e^{-rT} E_{\mathbb{Q}}[F_0^{-1}(F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T))]$. We observe that the curve is flatter when μ_1 is smaller, i.e., more ambiguity gives rise to payoffs that reflect a higher degree of conservativeness.

3.2.2 Drift and volatility uncertainty: $\mathbb{F}_{\mathrm{SSD}}$-robust cost-efficient payoff

The real-world distribution function of S_T is again assumed to be lognormal with parameters $\log S_0 + (\mu - \frac{\sigma^2}{2})T$ and $\sigma\sqrt{T}$, but now the agent faces uncertainty about



(A)



(B)

Fig. 1 We use the parameters $S_0 = 1, r = 0, T = 1$ and $s = 0.9$. The reference distribution function is the exponential distribution function $F_0(x) = 1 - e^{-x}$: (A) price of the cost-efficient payoff X^* depending on the value of the ambiguity parameter μ_1; (B) cost-efficient payoff per unit of investment for various values of μ_1

the precise level of both parameters μ and σ. More precisely, the agent only expects the true parameters to lie within the cube

$$\mathcal{D}^{\mu_1,\mu_2,\sigma_1,s} = \{(\mu, \sigma) \subseteq \mathbb{R}^2 : \mu_1 \le \mu \le \mu_2, \sigma_1 \le \sigma \le s\}$$

for $r < \mu_1 < \mu_2$ and $0 < \sigma_1 \le s$, where we recall that s is the volatility under the risk-neutral measure $\mathbb{Q}$, and thus she considers $\mathcal{P} = (\mathbb{P}^{\mu,\sigma})_{(\mu,\sigma)\in\mathcal{D}^{\mu_1,\mu_2,\sigma_1,s}}$ as the set of all plausible probability measures on $(\Omega, \mathcal{F})$. Under $\mathbb{P}^{\mu,\sigma}$, S_T is lognormal with density $f^{\mu,\sigma}$ defined in (3.1). In this regard, note that while $r < \mu_1$ is a natural assumption, there is some empirical evidence for the hypothesis that the volatility under any physical measure is smaller than the volatility s under the pricing measure $\mathbb{Q}$, i.e., $\sigma \le s$; see Christensen and Prabhala [18, Table 1] and Christensen and Hansen [17, Table 1].

Remark 3.8 In contrast to the dynamic Black–Scholes model, in which the stock price S_T is also lognormally distributed, we work here in a static market setting. In a dynamic Black–Scholes framework where continuous trading is allowed at zero transaction cost, the absence of arbitrage opportunities implies that the volatility of the stock does not change when moving from the real-world measure to the risk-neutral measure, i.e., there is no uncertainty about the volatility in a *dynamic* Black–Scholes model. Here, however, we do not assume dynamic trading. Hence even when call option prices reflect a risk-neutral distribution for S_T that is lognormal, the agent may have a view on the real-world distribution that is different from a lognormal, and in particular may be unsure about the exact values for drift and volatility.

In the next result, we assume that

$$\frac{\mu_1 - r}{s^2} \in (0, 1]. \tag{3.3}$$

We shall see in the proof of Proposition 3.9 that the condition (3.3) helps to find a least favourable measure by applying Proposition 2.10: Under (3.3), the likelihood ratio in question is a *concave* function of S_T.

For example, the parameters $s \in [0.2, \infty)$ and $\mu_1 - r \in (0, 0.04]$ or $s \in [0.35, \infty)$ and $\mu_1 - r \in (0, 0.1]$ imply (3.3). In other words, there are economically reasonable environments such that (3.3) holds.

Proposition 3.9 *If $\frac{\mu_1 - r}{s^2} \in (0, 1]$, then $F_{S_T}^{\mathbb{P}^{\mu_1,s}} \preceq_{\mathbb{F}_{\mathrm{SSD}}} F_{S_T}^{\mathbb{P}^{\mu,\sigma}}$ for $(\mu, \sigma) \in \mathcal{D}^{\mu_1,\mu_2,\sigma_1,s}$, and Assumptions 2.18 and 2.19 are satisfied for the set $\mathbb{F}_{\mathrm{SSD}}$. The least favourable measure is $\mathbb{P}^* = \mathbb{P}^{\mu_1,s}$ with corresponding likelihood ratio $\ell^* = \ell^{\mathbb{P}^{\mu_1,s}}$. The $\mathbb{F}_{\mathrm{SSD}}$-robust cost-efficient payoff for a distribution function F_0 satisfying Assumption 2.16 and such that $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ is concave is then given by*

$$X^* := F_0^{-1}\big(F_{S_T}^{\mathbb{P}^{\mu_1,s}}(S_T)\big). \tag{3.4}$$

Proof For a lognormal distribution function F with parameters M and V, it holds that

$$\int_0^q F^{-1}(p)dp = e^{M+\frac{V^2}{2}} \Phi\big(\Phi^{-1}(q) - V\big), \qquad q \in (0, 1),$$

where Φ denotes the standard normal distribution function. It follows for $q \in (0, 1)$ and $\mu_1 \le \mu, \sigma \le s$ that

$$\begin{aligned}\int_0^q (F_{S_T}^{\mathbb{P}^{\mu_1,s}})^{-1}(p)dp &= e^{\log S_0 + \mu_1 T} \Phi\big(\Phi^{-1}(q) - s\sqrt{T}\big) \\ &\le e^{\log S_0 + \mu_1 T} \Phi\big(\Phi^{-1}(q) - \sigma\sqrt{T}\big).\end{aligned}$$

Hence $F_{S_T}^{\mathbb{P}^{\mu_1,s}} \preceq_{\mathbb{F}_{\mathrm{SSD}}} F_{S_T}^{\mathbb{P}^{\mu,\sigma}}$ for $(\mu, \sigma) \in \mathcal{D}^{\mu_1,\mu_2,\sigma_1,s}$. As in Sect. 3.2.1, let

$$\ell^{\mathbb{P}^{\mu_1,s}} = \frac{f^{\mu_1,s}(S_T)}{f^{r,s}(S_T)} = h^{\mu_1,s}(S_T).$$

Hence the likelihood ratio $\ell^{\mathbb{P}^{\mu_1,s}}$ is strictly increasing and concave in S_T if (3.3) is satisfied. By Proposition 2.10, Assumptions 2.18 and 2.19 are satisfied for the set $\mathbb{F}_{\mathrm{SSD}}$ with least favourable measure $\mathbb{P}^* = \mathbb{P}^{\mu_1,s}$ with likelihood ratio $\ell^* = \ell^{\mathbb{P}^{\mu_1,s}}$. As in Sect. 3.2.1, some simple calculations and Theorem 3.1 show that the robust cost-efficient payoff for the distribution function F_0 is given by (3.4). □

We now provide an example for F_0 that makes it possible to apply Proposition 3.9 to determine $\mathbb{F}_{\mathrm{SSD}}$-robust cost-efficient payoffs. In this regard, observe that $F_{\ell^*}^{\mathbb{P}^*}$ in Proposition 3.9 is the lognormal distribution with parameters $\frac{1}{2}\theta^2$ and θ for $\theta := \sqrt{T}\frac{\mu_1 - r}{s} > 0$.

Example 3.10 If F_0 is the lognormal distribution with parameters $M \in \mathbb{R}$ and $V > 0$, then $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ in Proposition 3.9 is concave if $V \le \theta$ because

$$(F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*})(x) = x^{\frac{V}{\theta}} \exp\bigg(-\frac{1}{2}\theta V + M\bigg), \qquad x > 0.$$

3.3 Robust cost-efficient payoffs in general markets using Esscher transform

Inspired by Corcuera et al. [19], fix $S_0 > 0$ and let $s > 0$ and Z be a payoff with mean zero and variance one. Under $\mathbb{Q}$, assume that Z has density $f_Z^{\mathbb{Q}}(x) > 0$, $x \in \mathbb{R}$, and model the future stock price at date T by

$$S_T = S_0 e^{(r+\omega)T + s\sqrt{T}Z},$$

where $\omega \in \mathbb{R}$ is a mean-correcting term, i.e., ω is chosen such that

$$e^{-rT} E_{\mathbb{Q}}[S_T] = S_0.$$

The density of $X = \log S_T$ under $\mathbb{Q}$ is

$$f_X^{\mathbb{Q}}(x) = \frac{1}{s\sqrt{T}} f_Z^{\mathbb{Q}}\left(\frac{x - \log S_0 - (r+\omega)T}{s\sqrt{T}}\right), \qquad x > 0.$$

The corresponding density of S_T under $\mathbb{Q}$ is denoted by $f_{S_T}^{\mathbb{Q}}$, and it holds that

$$f_{S_T}^{\mathbb{Q}}(x) = \frac{1}{x} f_X^{\mathbb{Q}}(\log x), \qquad x > 0.$$

Let $h^* > 0$ and $\mathcal{H} \subseteq [h^*, \infty)$ be a set containing h^* such that $E_{\mathbb{Q}}[(S_T)^h]$ exists for all $h \in \mathcal{H}$. Define a family of probability measures $\mathcal{P} = (\mathbb{P}^h)_{h\in\mathcal{H}}$ as follows: $\mathbb{P}^h$ is a measure such that X has density $f_X^{\mathbb{P}^h}$ under $\mathbb{P}^h$, where $f_X^{\mathbb{P}^h}$ is obtained from $f_X^{\mathbb{Q}}$ by applying the Esscher transform. The use of the Esscher transform can be supported by a utility maximising argument; see Gerber and Shiu [26]. More precisely, we define $\mathbb{P}^h$ such that

$$f_X^{\mathbb{P}^h}(x) = \frac{e^{hx} f_X^{\mathbb{Q}}(x)}{\int_{\mathbb{R}} e^{hy} f_X^{\mathbb{Q}}(y)dy} = \frac{e^{hx} f_X^{\mathbb{Q}}(x)}{E_{\mathbb{Q}}[(S_T)^h]}, \qquad x > 0.$$

It follows that

$$f_{S_T}^{\mathbb{P}^h}(x) = \frac{x^h}{x} \frac{f_X^{\mathbb{Q}}(\log x)}{E_{\mathbb{Q}}[(S_T)^h]}, \qquad x > 0.$$

The density $f_{S_T}^{\mathbb{P}^{h^*}}$ crosses $f_{S_T}^{\mathbb{P}^h}$ only once from above for $h^* < h$; hence by Denuit et al. [21, Property 3.3.32], it follows that

$$F_{S_T}^{\mathbb{P}^{h^*}} \preceq_{\mathbb{F}\mathrm{SD}} F_{S_T}^{\mathbb{P}^h} \quad\Longrightarrow\quad F_{S_T}^{\mathbb{P}^{h^*}} \preceq_{\mathbb{F}\mathrm{SSD}} F_{S_T}^{\mathbb{P}^h}, \qquad h \in \mathcal{H}.$$

For the likelihood ratio, it holds that

$$\ell^{\mathbb{P}^{h^*}} = \frac{f_{S_T}^{\mathbb{P}^{h^*}}(S_T)}{f_{S_T}^{\mathbb{Q}}(S_T)} = \frac{(S_T)^{h^*}}{E_{\mathbb{Q}}[(S_T)^{h^*}]},$$

which is strictly increasing in S_T as $h^* > 0$, and concave if $h^* \in (0, 1]$. We can apply Proposition 2.10 to show that Assumptions 2.18 and 2.19 are satisfied for the sets $\mathbb{F}_{\mathrm{FSD}}$ and $\mathbb{F}_{\mathrm{SSD}}$ with least favourable measure $\mathbb{P}^* = \mathbb{P}^{h^*}$ and corresponding likelihood ratio $\ell^* = \ell^{\mathbb{P}^{h^*}}$. We can use Theorem 3.1 to compute the cost-efficient payoff of a distribution function F_0.

4 Robust payoff selection

Gilboa and Schmeidler [27] provide axioms that justify a maxmin expected utility framework to make robust decisions when there is ambiguity on the probability measure $\mathbb{P}$, i.e., when $\mathcal{P}$ contains more than one element. In this framework, Schied [54] shows that when a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ with respect to the FSD ordering (i.e., the integral stochastic ordering induced by the set $\mathbb{F}_{\mathrm{FSD}}$ defined in Sect. 2) exists, an optimal payoff can be derived. In this section, we extend the work of [54] in two different ways. First, we account for preferences beyond expected utility. Specifically, we derive optimal payoffs for robust preferences that are in accord with expected utility theory, rank-dependent utility theory and Yaari's dual theory. Second, assuming the existence of a least favourable measure $\mathbb{P}^*$ with respect to a general integral stochastic ordering induced by some set $\mathbb{F}$, not necessarily identical to $\mathbb{F}_{\mathrm{FSD}}$, we derive the optimal payoff. Specifically, we derive optimal payoffs when a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ with respect to the SSD ordering exists (see Proposition 2.10 for a sufficient condition) and the target distribution F_0 is sufficiently light-tailed (see Remark 3.4).

4.1 Family-consistent preferences

A *preference* $\succcurlyeq$ is understood as a binary relation on the set $\mathcal{X}$ of payoffs. The interpretation is that $Y \succcurlyeq X$ if Y is preferred to X. A functional $W : \mathcal{X} \to \mathbb{R}$ is called a *representation of* $\succcurlyeq$ if $Y \succcurlyeq X$ if and only if $W(Y) \geq W(X)$. The functional W is also called a *utility functional*; see Schied et al. [55], He et al. [35] and Assa and Zimper [1]. In general, W may depend on the different measures $\mathbb{P} \in \mathcal{P}$ in a complicated way. In what follows, we denote a utility functional that depends solely on some $\mathbb{P} \in \mathcal{P}$ by $W_{\mathbb{P}}$.

Definition 4.1 Let $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of utility functionals. The utility functional $W_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$, is called $\mathbb{P}$*-law-invariant* if $F_X^{\mathbb{P}} = F_Y^{\mathbb{P}}$ implies that $W_{\mathbb{P}}(X) = W_{\mathbb{P}}(Y)$. The family $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is called *law-invariant* if each individual utility functional $W_{\mathbb{P}}$ is $\mathbb{P}$-law-invariant.

Example 4.2 A standard example of a $\mathbb{P}$-law-invariant utility functional is given by $W_{\mathbb{P}}(X) = E_{\mathbb{P}}[u(X)]$ for some increasing utility function u. In this case, the functional $W(X) = \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X)$ amounts to the worst-case expected utility, commonly called *robust expected utility*, which was introduced in [27]. It is also referred to as a *robust utility functional* in [55].

To the best of our knowledge, the next definition is new to the literature. It will be helpful in solving robust payoff choice problems.

Definition 4.3 Let $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of utility functionals. Let $\mathcal{Y} \subseteq \mathcal{X}$. The family of utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is called $\mathbb{F}$*-family-consistent on* $\mathcal{Y}$ *with respect to* $\mathbb{P}' \in \mathcal{P}$ if for all $Y \in \mathcal{Y}$, the inequality

$$F_Y^{\mathbb{P}'} \preceq_{\mathbb{F}} F_Y^{\mathbb{P}}, \qquad \mathbb{P} \in \mathcal{P}$$

implies that

$$W_{\mathbb{P}'}(Y) \leq W_{\mathbb{P}}(Y), \qquad \mathbb{P} \in \mathcal{P}.$$

$\mathbb{F}$-family-consistency of $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ on $\mathcal{Y}$ with respect to some $\mathbb{P}' \in \mathcal{P}$ has the following interpretation: If a measure $\mathbb{P}'$ yields the most pessimistic view of any payoff Y with respect to the stochastic ordering induced by some set $\mathbb{F}$, then the preference under that measure is the lowest as well.

Next, we discuss some examples. Let $\mathcal{Y} \subseteq \mathcal{X}$ be a set of payoffs and let $\mathcal{D}$ be the set of cumulative distribution functions induced by $\mathcal{Y}$, i.e.,

$$\mathcal{D} = \{F_Y^{\mathbb{P}} : Y \in \mathcal{Y}, \mathbb{P} \in \mathcal{P}\}.$$

Let us consider an agent taking into account a family of law-invariant utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$, i.e.,

$$W_{\mathbb{P}}(Y) = w(F_Y^{\mathbb{P}}), \qquad \mathbb{P} \in \mathcal{P}, \tag{4.1}$$

for some well-defined $w : \mathcal{D} \to \mathbb{R}$. If w respects the integral stochastic ordering, i.e.,

$$F \preceq_{\mathbb{F}} G \quad \Longrightarrow \quad w(F) \leq w(G), \qquad F, G \in \mathcal{D}, \tag{4.2}$$

then $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{F}$-family-consistent on $\mathcal{Y}$ with respect to all $\mathbb{P}' \in \mathcal{P}$. We provide some specific examples in the contexts of expected utility theory, Yaari's dual theory of choice and rank-dependent expected utility theory.

Example 4.4 Let $u : \mathbb{R}_+ \to \mathbb{R}$. Let $\phi : [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. For a given distribution function F, define

$$\begin{aligned}
w^{\mathrm{EUT}}(F) &= \int_{\mathbb{R}_+} u(x)dF(x), \\
w^{\mathrm{Yaari}}(F) &= \int_{\mathbb{R}_+} \phi\big(1 - F(x)\big)dx, \\
w^{\mathrm{RDEU}}(F) &= \int_{\mathbb{R}_+} u(x)d\Big(1 - \phi\big(1 - F(x)\big)\Big),
\end{aligned}$$

where we tacitly assume that all integrals exist. It is straightforward to show that when u and ϕ are nondecreasing, the family of utility functionals induced by w^{EUT},

w^{Yaari} or w^{RDEU} as in (4.1) is $\mathbb{F}_{\mathrm{FSD}}$-family-consistent on $\mathcal{Y}$ with respect to all $\mathbb{P}' \in \mathcal{P}$, where $\mathcal{Y}$ is restricted to contain random variables such that all relevant integrals exist. Furthermore, if u is strictly increasing and concave and ϕ is strictly increasing, continuously differentiable and convex, we obtain that such a family is $\mathbb{F}_{\mathrm{SSD}}$-family-consistent on $\mathcal{Y}$ with respect to all $\mathbb{P}' \in \mathcal{P}$; see Yaari [64], Wang and Young [62], He et al. [35] and Ryan [52].

Remark 4.5 One could allow the function w in (4.1) to depend on $\mathbb{P}$, i.e., define $W_{\mathbb{P}}(Y) = w_{\mathbb{P}}(F_Y^{\mathbb{P}})$ for some $w_{\mathbb{P}} : \mathcal{D} \to \mathbb{R}$, $\mathbb{P} \in \mathcal{P}$. The family of utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is then $\mathbb{F}$-family-consistent on $\mathcal{Y}$ with respect to some $\mathbb{P}' \in \mathcal{P}$ if both

$$w_{\mathbb{P}'}(F) \le w_{\mathbb{P}}(F), \qquad F \in \mathcal{D}, \mathbb{P} \in \mathcal{P}$$

and (4.2) hold. To see this, let $Y \in \mathcal{Y}$ with $F_Y^{\mathbb{P}'} \le F_Y^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$. It follows that

$$W_{\mathbb{P}'}(Y) = w_{\mathbb{P}'}(F_Y^{\mathbb{P}'}) \le w_{\mathbb{P}}(F_Y^{\mathbb{P}'}) \le w_{\mathbb{P}}(F_Y^{\mathbb{P}}) = W_{\mathbb{P}}(Y).$$

4.2 Optimal payoffs under robust preferences

Inspired by Gilboa and Schmeidler [27] and Schied [54], we consider the following problem.

Problem 4.6 Let $x_0 > 0$ be the initial wealth. Let $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of utility functionals. We consider the robust maximisation problem

$$\max_{X \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X), \tag{4.3}$$

where $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}} = \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{Y}^{x_0}_{W_{\mathbb{P}}}$ and

$$\mathcal{Y}^{x_0}_{W_{\mathbb{P}}} := \{X \in \mathcal{X} : W_{\mathbb{P}}[X] \in \mathbb{R}, e^{-rT} E_{\mathbb{Q}}[X] \le x_0\}, \qquad \mathbb{P} \in \mathcal{P}.$$

It turns out that under certain conditions, a solution to the robust optimisation problem (4.3) can be found as a solution to a *maximisation problem under a single measure* $\mathbb{P} \in \mathcal{P}$.

Problem 4.7 Let $x_0 > 0$ be the initial wealth. Let $W_{\mathbb{P}}$ for some $\mathbb{P} \in \mathcal{P}$ be a utility functional. We consider the *maximisation problem*

$$\max_{X \in \mathcal{Y}^{x_0}_{W_{\mathbb{P}}}} W_{\mathbb{P}}(X). \tag{4.4}$$

Under the assumption of the existence of a least favourable measure with respect to the FSD ordering, [54] showed that in order to solve the robust maximisation problem (4.3) for utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$, $W_{\mathbb{P}}(x) = E_{\mathbb{P}}[u(X)]$, it actually suffices to solve the single measure maximisation problem (4.4). The following result generalises this beyond the expected utility setting to a general law-invariant family of utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$. The result is illustrated in Sect. 4.3, where we consider a robust rank-dependent expected utility maximisation problem for an investor with ambiguity on the trend and/or volatility of the risky asset.

Theorem 4.8 *Let $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$. Under Assumptions 2.18 and 2.19, suppose $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is law-invariant and $\mathbb{F}_{\mathrm{FSD}}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^* \in \mathcal{P}$. Assume that the maximisation problem (4.4) under $\mathbb{P}^*$ has a solution $\tilde{X} \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$. Then it holds that*

$$\max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X).$$

Proof Let $h \in \mathbb{F}_{\mathrm{FSD}}$ be such that $h(\ell^*) \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$. Then by Assumption 2.18 and (2.3) and since $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{F}_{\mathrm{FSD}}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^*$, we have

$$W_{\mathbb{P}^*}\big(h(\ell^*)\big) \le \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}\big(h(\ell^*)\big). \tag{4.5}$$

Let

$$X^* = (F^{\mathbb{P}^*}_{\tilde{X}})^{-1}\big(F^{\mathbb{P}^*}_{\ell^*}(\ell^*)\big).$$

Then X^* solves the standard cost-efficiency problem for $F^{\mathbb{P}^*}_{\tilde{X}}$; so $E_{\mathbb{Q}}[X^*] \le E_{\mathbb{Q}}[\tilde{X}]$ and $F^{\mathbb{P}^*}_{\tilde{X}} = F^{\mathbb{P}^*}_{X^*}$, and thus the law-invariance of $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ yields $X^* \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$.

Moreover, X^* is a nondecreasing function of ℓ^*. It follows by (4.5) that

$$\begin{aligned}
\max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X) &= W_{\mathbb{P}^*}(\tilde{X}) \\
&= W_{\mathbb{P}^*}(X^*) \\
&\le \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X^*) \\
&\le \max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) \\
&\le \max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} W_{\mathbb{P}^*}(X) \\
&\le \max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X),
\end{aligned}$$

where the last inequality follows because $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}} \subseteq \mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}$. □

From Theorem 4.8, it follows immediately that solving robust preference maximisation problems may reduce to solving an optimisation problem under a single probability measure. The following example illustrates this consequence.

Example 4.9 Assume $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$ and that Assumptions 2.18 and 2.19 are satisfied. Let $W_{\mathbb{P}}(F) = w(F^{\mathbb{P}}_Y)$ as in (4.1), where $w \in \{w^{\mathrm{EUT}}, w^{\mathrm{Yaari}}, w^{\mathrm{RDEU}}\}$ as in Example 4.4. Assuming a solution to (4.4) under $\mathbb{P}^* \in \mathcal{P}$ exists, it follows that

$$\max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X).$$

The main assumption in Theorem 4.8 that is needed to solve the robust maximisation problem (4.3) in the case of a family of law-invariant utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is the existence of a least favourable measure $\mathbb{P}^*$ with respect to $\mathbb{F}_{\mathrm{FSD}}$. In the following result, we show that it is possible to weaken this assumption in that we only require existence of a least favourable measure $\mathbb{P}^*$ with respect to some $\mathbb{F} \subseteq \mathbb{F}_{\mathrm{FSD}}$, e.g. $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$. The theorem is illustrated in Sect. 4.3, where we consider a robust rank-dependent expected utility maximisation problem for an investor who faces ambiguity on expected return and volatility of the risky asset.

Theorem 4.10 *Consider a given set $\mathbb{F}$. Under Assumptions 2.18 and 2.19, suppose that the maximisation problem (4.4) under $\mathbb{P}^*$ has a solution $\tilde{X} \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$, which can $\mathbb{P}^*$-a.s. be expressed as $f(\ell^*)$ for some $f \in \mathbb{F}$. Further, assume that $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{F}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^*$. Then it holds that*

$$\max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X).$$

Proof Let $h \in \mathbb{F}$ be such that $h(\ell^*) \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$. Then (4.5) holds by the $\mathbb{F}$-family-consistency and (2.3). By assumption, $\tilde{X} = f(\ell^*)$ $\mathbb{P}^*$-a.s. for some $f \in \mathbb{F}$. Hence we obtain

$$\begin{aligned}
\max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X) &= W_{\mathbb{P}^*}(\tilde{X}) \\
&\le \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(\tilde{X}) \\
&\le \max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) \\
&\le \max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} W_{\mathbb{P}^*}(X) \\
&\le \max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X),
\end{aligned}$$

where the last inequality follows because $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}} \subseteq \mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}$. □

Note that in comparison to the statements in Theorem 4.8, the $W_{\mathbb{P}}$ in Theorem 4.10 need not be law-invariant. Moreover, as long as the solution can be expressed as a certain function of the likelihood ratio ℓ^*, the utility functionals need not be increasing, i.e., $X \le Y$ $\mathbb{P}$-a.s. need not imply $W_{\mathbb{P}}(X) \le W_{\mathbb{P}}(Y)$. As pointed out, Theorem 4.10 is applicable in particular for the case $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$. However, the requirement that $\tilde{X} \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ can be $\mathbb{P}^*$-a.s. expressed as $h(\ell^*)$ for some $h \in \mathbb{F}_{\mathrm{SSD}}$ is equivalent to h being increasing and concave. This property is difficult to verify ex ante. Hereafter, we show that if $W_{\mathbb{P}}$ is an expected utility, this condition translates into an easy-to-verify condition on the utility function. We formulate the following result.

Theorem 4.11 *Fix $\mathbb{P} \in \mathcal{P}$ with likelihood ratio $\ell^{\mathbb{P}}$. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a differentiable, concave and strictly increasing utility function such that u' is strictly decreasing. If the maximisation problem (4.4) under $\mathbb{P}$ has a solution, then the solution is a nondecreasing and concave function of $\ell^{\mathbb{P}}$ if and only if $\frac{1}{u'}$ is convex. If u is three times differentiable, $\frac{1}{u'}$ is convex if and only if*

$$a(x) \geq \frac{p(x)}{2}, \tag{4.6}$$

where $a(x) := -\frac{u''(x)}{u'(x)}$ is the absolute risk aversion and $p(x) := -\frac{u'''(x)}{u''(x)}$ the absolute prudence.

Proof By Bernard et al. [6, Lemma 2], the solution to (4.4) is unique and given by $(u')^{-1}(\frac{c_0}{\ell^{\mathbb{P}}})$ for some $c_0 > 0$. See also Merton [43] for a proof in a context in which the Inada conditions are satisfied. Note that $u' > 0$ and that $\frac{1}{u'}$ is strictly increasing. Observe that the inverse of $\frac{1}{u'}$ is $x \mapsto (u')^{-1}(\frac{1}{x})$, which is hence also strictly increasing. The inverse of a convex (concave) and strictly increasing function is concave (convex). For the second assertion, observe that $u'' < 0$ and that a function is convex on an open interval if and only if its second derivative is nonnegative. Then (4.6) follows immediately. □

Remark 4.12 Maggi et al. [42] have shown that $a(x) > p(x)$ if and only if the utility has increasing absolute risk aversion, which is somewhat unusual (it is typically assumed that agents have decreasing absolute risk aversion given that they become less risk averse as their wealth increases). But our condition (4.6) is not incompatible with decreasing absolute risk aversion due to the factor $\frac{1}{2}$.

Condition (4.6) has appeared several times in the literature. It has been found to play a role in the context of insurance models in Bourlès [9], but also appeared as a condition in the opening of a new asset market (see Gollier and Kimball [32]) when there is uncertainty on the size (see Gollier et al. [31]) or the probability of losses (see Gollier [29]) and under contingent auditing (see Sinclair-Desgagné and Gabel [59]). Further interpretation of this condition and in particular of the degree of concavity of the inverse of the marginal utility can be found in [9]. This condition also appears in Varian [60] in the context of payoff selection under ambiguity.

Example 4.13 As an illustration of Theorem 4.11, we provide two utility functions which are differentiable, concave and strictly increasing and such that the reciprocal of the marginal utility is convex:

- The exponential utility for risk-averse agents: $u : \mathbb{R}_+ \to \mathbb{R}, x \mapsto 1 - e^{-\lambda x}$ for $\lambda > 0$. It holds that $\frac{1}{u'(x)} = e^{\lambda x}$, which is strictly increasing and convex.
- CRRA utility: $u : \mathbb{R}_+ \to \mathbb{R}, x \mapsto \frac{x^{1-\eta}}{1-\eta}$ for $\eta > 1$. It holds that $\frac{1}{u'(x)} = x^{\eta}$, which is strictly increasing and convex.

4.3 Rank-dependent utility in lognormal markets

We now discuss some examples to illustrate Sect. 4.2 in a lognormal market setting with uncertainty on the drift and volatility. In particular, we explicitly solve a robust rank-dependent expected utility problem using Theorems 4.8 and 4.10. As in

Sect. 3.2.2, we assume that the real-world distribution of S_T is lognormal with parameters $\log S_0 + (\mu - \frac{\sigma^2}{2})T$ and $\sigma\sqrt{T}$ and that the investor has uncertainty on the trend and potentially also the volatility. She may expect the true parameters to lie within the cube, for $r < \mu_1 < \mu_2$ and $0 < \sigma_1 \leq s$,

$$\mathcal{D}^{\mu_1,\mu_2,\sigma_1,s} = \{(\mu,\sigma) \subseteq \mathbb{R}^2 : \mu_1 \leq \mu \leq \mu_2, \sigma_1 \leq \sigma \leq s\},$$

where s is the volatility under the pricing measure $\mathbb{Q}$. The investor thus considers $\mathcal{P} = (\mathbb{P}^{\mu,\sigma})_{(\mu,\sigma)\in\mathcal{D}^{\mu_1,\mu_2,\sigma_1,s}}$ as the set of all plausible probability measures on $(\Omega, \mathcal{F})$. Note that if $\sigma_1 = s$, the investor only faces drift ambiguity; otherwise, she considers ambiguity on both trend and volatility. Under $\mathbb{P}^{\mu,\sigma}$, S_T is lognormal with density $f^{\mu,\sigma}$ defined in (3.1). In the next example, Φ is the standard normal distribution function.

Example 4.14 Let $U(x) = \frac{x^{1-\eta}}{1-\eta}$, $x \geq 0$, for $\eta \in (0,1)$ be the CRRA utility function. Let $\gamma \in \mathbb{R}$ and let $w(u) = \Phi(\Phi^{-1}(u) + \gamma)$, $u \in [0,1]$, denote the so-called *Wang transform*, which is increasing concave if $\gamma > 0$ and increasing convex if $\gamma < 0$. Consider the payoff choice problem in which the investor maximises her expected rank-dependent utility, i.e.,

$$\max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^{\mu,\sigma}}}} \int_0^\infty U(x) d\Big(1 - w\big(1 - F_X^{\mathbb{P}^{\mu,\sigma}}(x)\big)\Big), \tag{4.7}$$

where $x_0 > 0$ is the initial wealth and $(\mu,\sigma) \in \mathcal{D}^{\mu_1,\mu_2,\sigma_1,s}$. Let $\theta := \sqrt{T}\frac{\mu-r}{\sigma}$. The solution to (4.7) is given by

$$X^*_{\mu,\sigma} := \begin{cases} \lambda^{-\frac{1}{\eta}} \exp(\frac{rT}{\eta} - \frac{1}{2}\frac{\gamma}{\eta}(\theta+\gamma))(\ell^{\mathbb{P}})^{\frac{\gamma}{\theta\eta}+\frac{1}{\eta}}, & \gamma > -\theta, \\ \lambda^{-\frac{1}{\eta}}, & \text{otherwise}, \end{cases} \tag{4.8}$$

where λ depends on η, γ and θ; see (4.10)–(4.12) below. The solution to the robust rank-dependent utility problem

$$\max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} \int_0^\infty U(x) d\Big(1 - w\big(1 - F_X^{\mathbb{P}}(x)\big)\Big) dx \tag{4.9}$$

is given by $X^*_{\mu_1,s}$ if there is no ambiguity on the volatility, i.e., when $\sigma_1 = s$. If there is ambiguity on the volatility, $\gamma < 0$ and $\frac{\mu_1 - r}{s^2} \in (0,1]$, then $X^*_{\mu_1,s}$ still solves (4.9).

Proof We first prove that $X^*_{\mu,\sigma}$ solves (4.7). Let $\mathbb{P} = \mathbb{P}^{\mu,\sigma}$. The state price $\xi^{\mathbb{P}} := \frac{e^{-rT}}{\ell^{\mathbb{P}}}$ is lognormally distributed with parameters $-rT - \frac{1}{2}\theta^2$ and $\theta > 0$; see (3.2). Hence as $\mu > r$, it holds that

$$F^{\mathbb{P}}_{\xi^{\mathbb{P}}}(x) = \mathbb{P}[\xi^{\mathbb{P}} \leq x] = \Phi\bigg(\frac{\log x + rT + \frac{1}{2}\theta^2}{\theta}\bigg), \qquad x > 0,$$

and

$$(F_{\xi^{\mathbb{P}}}^{\mathbb{P}})^{-1}(p) = \exp\left(\Phi^{-1}(p)\theta - rT - \frac{1}{2}\theta^2\right), \qquad p \in (0,1).$$

Let

$$H(z) = -\int_0^{w^{-1}(1-z)} (F_{\xi^{\mathbb{P}}}^{\mathbb{P}})^{-1}(t)dt, \qquad z \in [0,1].$$

The solution to the classical rank-utility problem (4.7) is well known (see for instance Xu [63, Theorem 4.1] or Rüschendorf and Vanduffel [50, Sect. 3.2]) and is given by

$$X^*_{\mu,\sigma} = (U')^{-1}\Big(\lambda \hat{H}'\Big(1 - w\big(F_{\xi^{\mathbb{P}}}^{\mathbb{P}}(\xi^{\mathbb{P}})\big)\Big)\Big),$$

where λ is determined by $E_{\mathbb{P}}[\xi^{\mathbb{P}} X^*_{\mu,\sigma}] = x_0$ and $\hat{H}$ is the concave envelope of H. Using $w'(u) = \frac{\Phi'(\Phi^{-1}(u)+\gamma)}{\Phi'(\Phi^{-1}(u))}$, we obtain after some calculations that

$$H'(z) = \frac{(F_{\xi^{\mathbb{P}}}^{\mathbb{P}})^{-1}(w^{-1}(1-z))}{w'(w^{-1}(1-z))} = \exp\left(\Phi^{-1}(1-z)(\gamma+\theta) - rT - \frac{1}{2}(\gamma+\theta)^2\right).$$

We distinguish two cases to find a more explicit expression for $X^*_{\mu,\sigma}$.

Case 1: $\gamma + \theta > 0$. Then H' is nonincreasing and hence H is concave and equal to $\hat{H}$. As $(U')^{-1}(y) = y^{-\frac{1}{\eta}}$, it is easy to see that

$$\begin{aligned} X^*_{\mu,\sigma} &= \lambda^{-\frac{1}{\eta}} \exp\left(-\frac{rT\gamma}{\theta\eta} - \frac{1}{2}\frac{\gamma}{\eta}(\theta+\gamma)\right)(\xi^{\mathbb{P}})^{-\frac{\gamma}{\theta\eta}-\frac{1}{\eta}} \\ &= \lambda^{-\frac{1}{\eta}} \exp\left(\frac{rT}{\eta} - \frac{1}{2}\frac{\gamma}{\eta}(\theta+\gamma)\right)(\ell^{\mathbb{P}})^{\frac{\gamma}{\theta\eta}+\frac{1}{\eta}}. \end{aligned}$$

If $1 - \frac{\gamma}{\theta\eta} - \frac{1}{\eta} = 0$, then $\xi^{\mathbb{P}} X^*_{\mu,\sigma}$ is constant and it holds that

$$\lambda = x_0^{-\eta} \exp\left(-\frac{rT\gamma}{\theta} - \frac{1}{2}\gamma(\theta+\gamma)\right). \tag{4.10}$$

Otherwise, $\xi^{\mathbb{P}} X^*_{\mu,\sigma}$ is lognormally distributed and it follows that

$$\lambda = x_0^{-\eta} \exp\left(rT(1-\eta) + \frac{1}{2}\Big(\theta^2(1-\eta) + \gamma^2 + \big(\theta\eta - (\gamma+\theta)\big)^2\Big)\right). \tag{4.11}$$

Case 2: $\gamma + \theta \le 0$. Then H is convex. Note that $H(0) = -1$ and $H(1) = 0$. As H is convex, the concave envelope $\hat{H}$ of H is given by $\hat{H}(x) = x - 1$. Then $\hat{H}' \equiv 1$ and

$$X^*_{\mu,\sigma} = (U')^{-1}(\lambda) = \lambda^{-\frac{1}{\eta}}.$$

Therefore $\xi^{\mathbb{P}} X^*_{\mu,\sigma} = \lambda^{-\frac{1}{\eta}} \xi^{\mathbb{P}}$, and hence

$$\lambda = x_0^{-\eta} e^{-rT\eta}. \tag{4.12}$$

Assume that there is no ambiguity on the volatility, i.e., $\sigma_1 = s$. Section 3.2.1 shows that the least favourable measure with respect to $\mathbb{F}_{\mathrm{FSD}}$ is given by $\mathbb{P}^* = \mathbb{P}^{\mu_1,s}$ with corresponding likelihood ratio $\ell^* = \ell^{\mathbb{P}^{\mu_1,s}}$. By Example 4.4, the utility functional in (4.9) is $\mathbb{F}_{\mathrm{FSD}}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^*$, and Theorem 4.8 shows that $X^*_{\mu,\sigma}$ solves the robust rank-dependent utility problem (4.9).

Finally, assume that there is ambiguity on the volatility. If $\frac{\mu_1 - r}{s^2} \in (0, 1]$, Proposition 3.9 shows that the least favourable measure with respect to $\mathbb{F}_{\mathrm{SSD}}$ is also $\mathbb{P}^*$. If $\gamma < 0$, then $X^*_{\mu,\sigma}$ is a concave and nondecreasing function of ℓ^*. By Example 4.4, the utility functional in (4.9) is $\mathbb{F}_{\mathrm{SSD}}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^*$. Apply Theorem 4.10 to show that also in this case, $X^*_{\mu,\sigma}$ solves the robust rank-dependent utility problem (4.9). □

To better understand the solution in Example 4.14, we "rationalise" the solution as in Bernard et al. [6], i.e., we show that the optimal investment strategy in the robust rank-dependent setting also solves an expected utility maximisation problem. Example 4.15 shows that the solution to the expected rank-dependent utility problems (4.7) and (4.9) involving a Wang transform with parameter γ and a CRRA utility function with parameter η can be rationalised by a CRRA utility with parameter $\frac{\eta\theta}{\gamma+\theta}$.

Example 4.15 Let $(\mu, \sigma) \in \mathcal{D}^{\mu_1,\mu_2,\sigma_1,s}$, $X^*_{\mu,\sigma}$, θ, x_0 and $\mathcal{Y}^{x_0}_{W_{\mathbb{P}^{\mu,\sigma}}}$ as in Example 4.14, and such that $\gamma > -\theta$ and $\eta\theta \neq \gamma + \theta$. Then $X^*_{\mu,\sigma}$ solves the expected utility maximisation problem

$$\max_{X \in \mathcal{Y}^{x_0}_{W_{\mathbb{P}^{\mu,\sigma}}}} \int_0^\infty u(x) dF_X^{\mathbb{P}^{\mu,\sigma}}(x)$$

for the utility function

$$u(x) = \frac{1}{1 - \frac{\eta\theta}{\gamma+\theta}} x^{1 - \frac{\eta\theta}{\gamma+\theta}}. \tag{4.13}$$

The function $u : \mathbb{R}_+ \to \mathbb{R}$ is nondecreasing and concave.

Proof Note that $\gamma > -\theta$ implies $\frac{\gamma}{\theta\eta} + \frac{1}{\eta} \neq 0$. Let $\mathbb{P} = \mathbb{P}^{\mu,\sigma}$ and $\xi^{\mathbb{P}} := \frac{e^{-rT}}{\ell^{\mathbb{P}}}$. As in [6], let $c > 0$ and define

$$\tilde{u}(x) = \int_c^x (F^{\mathbb{P}}_{\xi^{\mathbb{P}}})^{-1}\big(1 - F^{\mathbb{P}}_{X^*_{\mu,\sigma}}(y)\big) dy, \qquad x \in \mathbb{R}_+.$$

Because $X^*_{\mu,\sigma}$ is lognormally distributed, it follows that

$$(F^{\mathbb{P}}_{\xi^{\mathbb{P}}})^{-1}\big(1 - F^{\mathbb{P}}_{X^*_{\mu,\sigma}}(x)\big) = \kappa x^{-\frac{\eta\theta}{\gamma+\theta}},$$

where $\kappa > 0$ is a suitable constant. Thus

$$\tilde{u}(x) = \kappa \frac{\gamma + \theta}{\theta(1-\eta) + \gamma} \left(x^{-\frac{\eta\theta}{\gamma+\theta}+1} - c^{-\frac{\eta\theta}{\gamma+\theta}+1} \right).$$

In summary, as $\tilde{u}$ is only determined up to positive affine transformations, $X^*_{\mu,\sigma}$ solves the expected utility maximisation problem for the utility given in (4.13); see [6, Theorem 2]. □

5 Rationalising robust cost-efficient payoffs

When there is no ambiguity on the probability measure $\mathbb{P}$, there is a close relationship between cost-efficiency and payoff optimisation: For any cost-efficient payoff X, there exists a utility function u (unique up to a linear transformation) such that X also solves the expected utility maximisation problem (see [6]). In this section, we show that this result can be generalised to the robust setting developed previously in that robust cost-efficient payoffs can be rationalised in terms of the maxmin utility framework introduced in Gilboa and Schmeidler [27]. Specifically, we show – under the same assumptions as in Theorems 4.8 and 4.10 – that payoffs maximise a robust utility functional as in [27] if and only if they are robust cost-efficient.

In the following result, we distinguish two cases: (a) we deal with law-invariant utility functionals and FSD ordering and assume that the various (robust) maximisation problems have unique solutions, or (b) we deal with general utility functionals and stochastic ordering and do not require uniqueness of the solutions, but assume that the solution X^* of the various maximisation problems can be written as $X^* = f(\ell^*)$ for some $f \in \mathbb{F}$.

Theorem 5.1 *Assume $\mathbb{F}_{\mathrm{SSD}} \subseteq \mathbb{F} \subseteq \mathbb{F}_{\mathrm{FSD}}$. Let Assumptions 2.17–2.19 hold. Let $X^* \in \mathcal{X}$ be a payoff. Let $x_0 = e^{-rT} E_{\mathbb{Q}}[X^*] < \infty$. Let $c > 0$ be such that $F^{\mathbb{P}^*}_{X^*}(c) > 0$ for some $\mathbb{P}^* \in \mathcal{P}$. Let $\xi^* = \frac{e^{-rT}}{\ell^*}$ and define*

$$u(x) = \int_c^x F^{-1}_{\xi^*}\big(1 - F^{\mathbb{P}^*}_{X^*}(y)\big)dy, \qquad x \in \mathbb{R}_+. \tag{5.1}$$

Assume that $E_{\mathbb{P}}[u(X^*)] < \infty$ for all $\mathbb{P} \in \mathcal{P}$. We further assume that one of the following two conditions is satisfied:

(a) $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$.

(b) $X^* = f(\ell^*)$ $\mathbb{P}^*$*-a.s. for some* $f \in \mathbb{F}$ *and* $(F^{\mathbb{P}^*}_{X^*})^{-1} \circ F^{\mathbb{P}^*}_{\ell^*} \in \mathbb{F}$.

Then the following statements are equivalent:

i) *X^* is cost-efficient under $\mathbb{P}^*$.*

ii) *It holds that $X^* = (F^{\mathbb{P}^*}_{X^*})^{-1}(F^{\mathbb{P}^*}_{\ell^{\mathbb{P}^*}}(\ell^{\mathbb{P}^*}))$ $\mathbb{P}^*$-a.s.*

iii) *X^* is $\mathbb{P}^*$-a.s. nondecreasing in ℓ^*.*

iv) *X^* solves the $\mathbb{F}$-robust cost-efficiency problem for $F^{\mathbb{P}^*}_{X^*}$.*

v) *X^* solves the expected utility maximisation problem under $\mathbb{P}^*$ for the utility function u, i.e.,*

$$\max_{X\in\mathcal{Y}^{x_0}_{E_{\mathbb{P}^*}[u(\cdot)]}} E_{\mathbb{P}^*}[u(X)].$$

vi) *X^* solves the robust expected utility problem for the utility function u, i.e.,*

$$\max_{X\in\mathcal{Y}^{x_0}_{(E_{\mathbb{P}}[u(\cdot)])_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[u(X)],$$

and the solution is unique if condition (a) *is satisfied.*

vii) *There is a family of utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ that is $\mathbb{F}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^*$ such that $X^* \in \mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ and X^* is the solution to the maximisation problem under $\mathbb{P}^*$, i.e., to*

$$\max_{X\in\mathcal{Y}^{x_0}_{W_{\mathbb{P}^*}}} W_{\mathbb{P}^*}(X).$$

If (a) *holds, the solution is additionally unique and the family of utility functionals is law-invariant.*

viii) *There is a family of utility functionals $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ that is $\mathbb{F}$-family-consistent on $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ with respect to $\mathbb{P}^*$ such that X^* is the solution to the robust maximisation problem, i.e., to*

$$\max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X).$$

If (a) *holds, the solution is additionally unique and the family of utility functionals is law-invariant.*

Proof The equivalence between i), ii) and iii) follows from Lemma B.1. The equivalence between iv) and ii) follows from Theorem 3.1 and Remark 3.2. Note that $(F^{\mathbb{P}^*}_{X^*})^{-1}\circ F^{\mathbb{P}^*}_{\ell^*} \in \mathbb{F}_{\mathrm{FSD}}$ is always true. By [6, Theorem 3], i) implies v). By Lemma B.3 and [6, Lemma 3], v) implies i). By Example 4.4, v) implies vii) trivially; just define $W_{\mathbb{P}}(\cdot) = E_{\mathbb{P}^*}[u(\cdot)]$ for all $\mathbb{P} \in \mathcal{P}$. If (a) holds, v) implies vi) by Theorem 4.8 and Example 4.4 as $E_{\mathbb{P}}[u(X^*)] < \infty$ for all $\mathbb{P} \in \mathcal{P}$ by assumption. If (b) holds, v) implies vi) by Theorem 4.10. vi) implies viii) trivially. By Lemma B.3 and Lemma B.4, if (a) holds, vi) implies i) and vii) implies i) and viii) implies i). Note that if (b) holds, iii) is always true because $f \in \mathbb{F} \subseteq \mathbb{F}_{\mathrm{FSD}}$ is nondecreasing. □

Remark 5.2 Assuming that all functions in $\mathbb{F}$ are nondecreasing, i.e., that $\mathbb{F} \subseteq \mathbb{F}_{\mathrm{FSD}}$, is not really a restriction. Otherwise, there are two sure (constant) payoffs $x_0, y_0 \in \mathcal{X}$ such that $x_0 < y_0$, but the distribution of y_0 does not dominate the distribution of x_0 in the integral stochastic ordering with respect to $\mathbb{F}$.

Example 5.3 Assume $\mathbb{F} = \mathbb{F}_{\mathrm{SSD}}$. Consider the robust rank-dependent expected utility maximisation problem in Example 4.14 with the solution $X^*_{\mu_1,s}$ defined in (4.8). Let

$\mathbb{P}^* = \mathbb{P}^{\mu_1,s}$ and $\ell^* = \ell^{\mathbb{P}^*}$. With the help of the explicit expressions of $F^{\mathbb{P}^*}_{X^*_{\mu_1,s}}$ and $F^{\mathbb{P}^*}_{\ell^*}$ from the proofs of Examples 4.14 and 4.15, it is easy to see that (b) in Theorem 5.1 holds if $\gamma < 0$.

Let us start from viii) in Theorem 5.1. Equation (4.8) implies that the solution $X^*_{\mu_1,s}$ is a nondecreasing function of ℓ^*. Hence iii) in Theorem 5.1 is satisfied. As shown by Example 4.15, $X^*_{\mu_1,s}$ also solves an expected utility maximisation problem for the utility in (4.13), which illustrates v) in Theorem 5.1. One can easily verify that ii) in Theorem 5.1 is respected by $X^*_{\mu_1,s}$. Hence Theorem 3.1 implies that $X^*_{\mu_1,s}$ solves the robust $\mathbb{F}_{\mathrm{SSD}}$-cost-efficiency problem as stated in iv) in Theorem 5.1.

The utility functionals in Theorem 5.1 for case (b) need not be law-invariant or increasing, i.e., $X \le Y$ $\mathbb{P}$-a.s. need not imply $W_{\mathbb{P}}(X) \le W_{\mathbb{P}}(Y)$. We provide a simple example of such a utility functional.

Example 5.4 Define $X^* = f(\ell^*)$ for some $f \in \mathbb{F}$. Let $x_0 = E_{\mathbb{Q}}[X^*]$. For $\mathbb{P} \in \mathcal{P}$, define

$$W_{\mathbb{P}}(X) = \begin{cases} 1, & \text{if } X = X^* \ \mathbb{P}\text{-a.s.}, \\ 0, & \text{otherwise}, \end{cases}$$

which is trivially $\mathbb{F}$-family-consistent on $\mathcal{X}$ with respect to any $\mathbb{P}$. An agent with such a utility functional only likes X^* and neglects everything else. She is not law-invariant and does not prefer more to less. Someone interested only in the market portfolio or in the risk-free bond might have such a utility functional. The solution to the robust maximisation problem viii) is X^*, which is cost-efficient because X^* is nondecreasing in ℓ^*. A utility function for v) can be constructed as in (5.1).

6 Final remarks

In this paper, we assume that the agent has Knightian uncertainty. She is unsure about the precise physical measure describing the financial market and knows only that the true physical measure lies within a set $\mathcal{P}$ of probability measures. Given this ambiguity, it is no longer possible to target a payoff with a given distribution function. In particular, the close relation between payoffs that are the cheapest possible in reaching a target distribution function and the optimality thereof under law-invariant increasing preferences is a priori lost, as there is no consensus regarding what probability distribution to adopt. For this reason, we introduce the notion of a *robust cost-efficient payoff*.

For a given distribution function F_0, the robust cost-efficiency problem aims at finding the cheapest payoff whose distribution function dominates F_0 under all possible physical measures in some integral stochastic ordering. We solve this problem under some conditions (namely, when there exists a least favourable measure $\mathbb{P}^*$ and the integral stochastic ordering $\preceq_{\mathbb{F}}$ is cost-consistent). The solution is identical to the solution to the cost-efficiency problem without model ambiguity under the physical measure $\mathbb{P}^*$ and given in closed form. We are thus able to reduce the problem formulated in a robust setting to a problem formulated in a standard setting without model ambiguity.

Finally, we show that this notion of robust cost-efficiency plays a key role in optimal robust payoff selection and that a very general class of robust payoff selection problems (possibly in a nonexpected utility setting) can be reduced to the maxmin expected utility setting of [27] for a well-chosen concave utility function.

For this to hold, we make a relatively minor assumption on the family of utility functionals, namely, that it is family-consistent. To the best of our knowledge, family-consistency is new to the literature, and we provide several examples in the context of expected utility theory, Yaari's dual theory and rank-dependent utility theory.

We assume a static setting in which intermediate trading is not possible. While allowing dynamic rebalancing may make it possible to achieve higher levels for the objective at hand (e.g. robust expected utility), this possibility is only clear when there are no transaction costs, which is not realistic. In practice, transaction costs usually contain a fixed part, and hence dynamic trading can only occur a finite number of times since otherwise bankruptcy occurs. The study of optimal investments in the presence of fixed costs is not yet very well understood. Recently, Belak et al. [4] and Bayraktar et al. [3] provide optimal strategies in a Black–Scholes market without ambiguity and assuming expected utility. By contrast, our static setting makes it possible to deal with ambiguity and to address fairly general objectives.

Appendix A: Proof for Example 2.13

Lemma A.1 *Let F and G be two distribution functions. Then we have $G \preceq_{\mathbb{F}_{\mathrm{TSD}}} F$ if and only if*

$$\int_{-\infty}^{\eta}\int_{-\infty}^{\xi} F(x)dxd\xi \leq \int_{-\infty}^{\eta}\int_{-\infty}^{\xi} G(x)dxd\xi, \qquad \eta \in \mathbb{R}.$$

Proof See Gotoh and Konno [33, Theorem 2.2]. □

We are now ready to construct the counterexample stated in Example 2.13.

Proof of Example 2.13 Apply the chain rule to show that $\mathbb{F}_{\mathrm{TSD}}$ is composition-consistent. Next, we construct two distribution functions such that one dominates the other in TSD, but is cheaper.

Step 1: Define some market setting as in Sect. 3.2.1: Let $\mu_1 = 0.01$, $r = 0$, $T = 1$ and $s = 0.1$. Then $\frac{\mu_1 - r}{s^2} = 1$. Choose S_0 such that $\ell^* := \ell^{\mathbb{P}^{\mu_1}} = h^{\mu_1,s}(S_T) = S_T$, i.e., $\log S_0 = -0.0025$. Under $\mathbb{P}^* := \mathbb{P}^{\mu_1}$, the stock is lognormally distributed with parameters $\mu_1 - \frac{s^2}{2} + \log S_0 = 0.0025$ and s. Under $\mathbb{Q}$, the stock is also lognormally distributed with parameters $r - \frac{s^2}{2} + \log S_0 = -0.0075$ and s. Moreover, $\mathbb{P}^*$ is a least favourable measure with respect to the set $\mathbb{F}_{\mathrm{FSD}}$, and hence also with respect to $\mathbb{F}_{\mathrm{TSD}}$ because $\mathbb{F}_{\mathrm{TSD}} \subseteq \mathbb{F}_{\mathrm{FSD}}$.

Step 2: Define two distribution functions: Let

$$F(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases}$$

and for $p_0 \in (0, 1)$, let

$$G(x) = \begin{cases} 0, & x < 0, \\ p_0, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

So F is the uniform distribution function and G jumps at zero and at one. It follows that $F^{-1}(p) = p$ and that

$$G^{-1}(p) = \begin{cases} 0, & p \in (0, p_0], \\ 1, & p > p_0. \end{cases}$$

Step 3: Show that F dominates G in TSD: It holds for $\eta \in (0, 1)$ that

$$\int_{-\infty}^{\eta} \int_{-\infty}^{\xi} F(x) dx d\xi = \int_0^{\eta} \int_0^{\xi} x dx d\xi = \frac{1}{6}\eta^3$$

and that

$$\int_{-\infty}^{\eta} \int_{-\infty}^{\xi} G(x) dx d\xi = \int_0^{\eta} \int_0^{\xi} p_0 dx d\xi = \frac{1}{2} p_0 \eta^2.$$

Hence if $\frac{1}{6}\eta^3 \leq \frac{1}{2} p_0 \eta^2$ or, equivalently, $p_0 \geq \frac{1}{3}$, it follows that $G \preceq_{\mathbb{F}_{\mathrm{TSD}}} F$.

Step 4: Compute the lowest cost of both distribution functions: The cost-efficient payoff for F is

$$X_F = F^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big) = F_{S_T}^{\mathbb{P}^*}(S_T).$$

The lowest price of F can be computed numerically as

$$E_{\mathbb{Q}}[X_F] = \int_0^{\infty} F_{S_T}^{\mathbb{P}^*}(s) f_{S_T}^r(s) ds = 0.472.$$

The cost-efficient payoff for G is

$$X_G = G^{-1}\big(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\big) = \begin{cases} 1, & \text{if } S_T > (F_{S_T}^{\mathbb{P}^*})^{-1}(p_0), \\ 0, & \text{otherwise}. \end{cases}$$

Its price is

$$E_{\mathbb{Q}}[X_G] = \int_{(F_{S_T}^{\mathbb{P}^*})^{-1}(p_0)}^{\infty} f_{S_T}^r(s) ds = 1 - F_{S_T}^{\mathbb{Q}}\big((F_{S_T}^{\mathbb{P}^*})^{-1}(p_0)\big).$$

Under $\mathbb{P}^*$, X_F is uniformly distributed and X_G is a digital option. If $p_0 = \frac{1}{3}$, the lowest price for G is 0.63, which is greater than the lowest price to be paid for F. But in this case $G \preceq_{\mathbb{F}_{\mathrm{TSD}}} F$; hence TSD is not cost-consistent. □

Appendix B: Auxiliary results

Lemma B.1 *Fix $\mathbb{P} \in \mathcal{P}$. Let $\ell := \frac{d\mathbb{P}}{d\mathbb{Q}}$ be the Radon–Nikodým derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$. Assume that under $\mathbb{P}$, ℓ is continuously distributed and that $1/\ell$ has finite variance. Then there is a $\mathbb{P}$-a.s. unique optimiser to the standard cost-efficiency problem under the probability measure $\mathbb{P}$ given by*

$$X^* = F_0^{-1}\big(F^{\mathbb{P}}_{\ell^{\mathbb{P}}}(\ell)\big).$$

Moreover, X^* is left-continuous and nondecreasing $\mathbb{P}$-a.s.

Proof Let $\xi = \frac{e^{-rT}}{\ell}$. Then $1 - F^{\mathbb{P}}_{\xi}(\xi) = F^{\mathbb{P}}_{\ell^{\mathbb{P}}}(\ell)$. This claim follows both from Dybvig [23, 22] and from Bernard et al. [5, Corollary 2]. See also [53, Proposition 2.7] for the importance of the continuity assumption on ℓ in obtaining the uniqueness. □

Lemma B.2 *Fix $\mathbb{P} \in \mathcal{P}$. Let $\ell := \frac{d\mathbb{P}}{d\mathbb{Q}}$. Assume that ℓ is continuously distributed under $\mathbb{P}$. A payoff $X \in \mathcal{A}^{\mathbb{P}}_{F_0}$ is $\mathbb{P}$-cost-efficient if and only if it is nondecreasing in ℓ $\mathbb{P}$-a.s.*

Proof See [5, Corollary 2 and Proposition 2]. □

In the following two lemmas, we show that the solution of the single or robust maximisation problem is cost-efficient if it is unique.

Lemma B.3 *Let $\mathbb{P} \in \mathcal{P}$ with corresponding likelihood ratio $\ell^{\mathbb{P}}$. Assume that $\ell^{\mathbb{P}}$ is continuously distributed under $\mathbb{P}$ and that $W_{\mathbb{P}}$ is $\mathbb{P}$-law-invariant and $\tilde{X}$ is a $\mathbb{P}$-a.s. unique solution to the maximisation problem (4.4) under $\mathbb{P}$. Then $\tilde{X}$ is $\mathbb{P}$-cost-efficient.*

Proof Let

$$X^* = (F^{\mathbb{P}}_{\tilde{X}})^{-1}\big(F^{\mathbb{P}}_{\ell^{\mathbb{P}}}(\ell^{\mathbb{P}})\big).$$

Then X^* solves the standard cost-efficiency problem for $F^{\mathbb{P}}_{\tilde{X}}$ and therefore we have $E_{\mathbb{Q}}[X^*] \le E_{\mathbb{Q}}[\tilde{X}]$ and $F^{\mathbb{P}}_{\tilde{X}} = F^{\mathbb{P}}_{X^*}$. Hence by the law-invariance of $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$, it holds that $X^* \in \mathcal{Y}^{x_0}_{W_{\mathbb{P}}}$. It follows by law-invariance that

$$\max_{X \in \mathcal{Y}^{x_0}_{W_{\mathbb{P}}}} W_{\mathbb{P}}(X) = W_{\mathbb{P}}(\tilde{X}) = W_{\mathbb{P}}(X^*).$$

As $\tilde{X}$ is the unique solution, we must have $\tilde{X} = X^*$ $\mathbb{P}$-a.s., and so $\tilde{X}$ is cost-efficient. □

Lemma B.4 *Assume* $\mathbb{F} = \mathbb{F}_{\mathrm{FSD}}$. *Under Assumptions 2.17–2.19, suppose that the robust maximisation problem (4.3) has a unique solution* $\tilde{X}$ *and that* $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ *is law-invariant and* $\mathbb{F}_{\mathrm{FSD}}$*-family-consistent on* $\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}$ *with respect to* $\mathbb{P}^*$. *Then* $\tilde{X}$ *is* $\mathbb{P}^*$*-cost-efficient.*

Proof The proof is similar to the one for Lemma B.3. Let

$$X^* = (F^{\mathbb{P}^*}_{\tilde{X}})^{-1}\big(F^{\mathbb{P}^*}_{\ell^*}(\ell^*)\big).$$

It holds by law-invariance that

$$\max_{X\in\mathcal{Y}^{x_0}_{(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(\tilde{X}) = \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X^*). \qquad \square$$

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