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Technology Adoption, Social Learning, and Economic Policy

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Technology Adoption, Social Learning, and Economic Policy*

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Abstract

We study a two-player dynamic investment model with information externalities and provide necessary and sufficient conditions for a unique switching equilibrium. When the public information is sufficiently high and a social planer therefore expects an investment boom, investments should be taxed. Conversely, any positive investment tax is suboptimally high if the public information is sufficiently unfavorable. We also show that an investment tax may *increase* overall investment activity.

JEL classification: D62, D83

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1 Introduction

We consider a two-player model with social learning. There is a state of the world drawn from a normal distribution. Both players get a normally distributed private signal about the realization of the state. They then simultaneously decide whether or not to act in period one. If a player acts, her payoff equals the state of the world. A player who has not acted in period one, observes the other player's period-one decision and then gets to reconsider her own choice in period two. Payoffs from acting late are discounted, and a player who doesn't act receives her outside option.

For an application, consider two firms—say regional monopolists—that produce output using a similar production technology but have distinct customer groups. Suppose a new technology of uncertain quality becomes available and both firms are considering whether or not to adopt it. Each firm investigates this novel technology and updates its belief about its profitability. In such a setting each firm can postpone her investment decision in order to learn from the other firm's decision: firms may engage in social learning. Our model investigates how this social learning affects investment behavior and optimal tax policy.

We also believe the second question to be of broader interest. A heavily debated question is whether investments should be discouraged when policymakers suspect too many people to invest. While focusing on a relatively simple two-player game, our model emphasizes a novel reason for why policymakers should tax investments in the presence of favorable public information even if all investors are better informed than the policymaker herself and are completely rational. Doing so improves the ability of investors to learn from each others' behaviors.

Throughout we focus on switching strategies in which a player invests in period one whenever she is sufficiently optimistic and not otherwise. The value of waiting in social learning environments depends on the behavior of others. In our game, whenever the other player's cutoff is sufficiently low, seeing him investing comes as no surprise. As an investment decision then contains little information, this makes waiting relatively undesirable. As the cutoff becomes higher, he will invest less often. An investment decision then reveals good private information which, in turn, makes waiting more desirable. Whenever this force is strong enough, multiple symmetric switching equilibria exist. Section 3 thus characterizes when there exists a unique symmetric switching equilibrium. In particular, if the variance of the prior distribution is high enough or the one of the private signal is low enough, equilibrium is unique. Similarly, if players are impatient enough or the prior mean is either sufficiently high or low, equilibrium is unique. Building on this characterization, Section 4 investigates the optimal investment tax/subsidy. Whenever the prior mean is sufficiently high, it is optimal to tax investments. Roughly speaking, if the prior mean is high, both firms are very likely to invest in the first period, which implies that the informational content of an investment decision is low. A tax, by making players less likely to invest in the first period, raises the informational content of the investment decision and thereby increases the positive information externality associated with any time-one investment. Conversely, any positive tax is suboptimally high if the prior mean is sufficiently low.

Section 5 elaborates on how an investment tax affects investment activity. Obviously, a tax reduces the payoff from investing. Ceteris paribus, this *direct effect* makes investing less attractive and thus tends to increase both the first- and the second-period equilibrium cutoffs. An increase in the investment tax, however, also raises the informational content of an investment: if —despite an investment tax—a player invests, this signals that her posterior is "very high" and not simply "high". This *indirect effect* decreases the second-period cutoff. Whenever this indirect effect dominates, there exist posterior realizations such that the overall investment activity is higher with a positive investment tax. Finally, in Section 6 we show that asymmetric equilibria in switching strategies can be ruled out in the presence of favorable public information. We also illustrate that the symmetric switching equilibrium is unique for any finite number of players when they have a Laplacian prior.

Social learning has been intensively studied in herding models in which the timing of players' decisions is exogenous.[1] Hendricks and Kovenock (1989) introduced a waiting game in which there are informational externalities and players decide on both whether and when to act. Although there are numerous papers considering different waiting games,[2] to the best of our knowledge only Gossner and Melissas (2006) and Levin and Peck (2008) study optimal taxation in such a game. Furthermore, no paper in the literature on social learning has systematically analyzed the relationship between prior public information and optimal taxation.

Most closely related to our model is the waiting game studied by Chamley (2004a) who nicely highlights and explains the possibility of multiple symmetric switching equilibria. In contrast to our paper, Chamley does not derive conditions under which the symmetric switching equilibrium is unique and, most importantly, does not investigate the optimal tax policy. Also, since we are interested in investigating the relationship between a policymaker's prior information and the optimal tax policy, we differ from Chamley by making the for our question more natural assumption that the state of the world is normally rather than binary distributed.

The main difference between our setup and the one in Levin and Peck (2008) is that they assume idiosyncratic investment costs. They show that following some histories observing more past investment activity can lead to a downward revision of players' posteriors. In large economies, it is optimal to either subsidize investment or not to interfere in the market. In a two-player economy, however, a small investment subsidy can lower welfare. Intuitively, a small subsidy can encourage some types with bad private information but low investment costs to invest, which can reduce the informational value of observing overall investment activity.[3]

[1]For an excellent overview, see Chamley (2004b).

[2]Waiting games have, among others, also been analyzed by Chamley and Gale (1994), Gul and Lundholm (1995), Zhang (1995), Choi (Section 4, 1997), Caplin and Leahy (1998), Frisell (2003), and Gunay (2008).

[3]In contrast to our paper, Levin and Peck do not provide conditions that guarantee uniqueness within the class of the symmetric switching strategies. In a simpler setup, Gossner and Melissas also showed that a small investment tax can—depending on the selected equilibrium—raise welfare.

In Angeletos and Pavan (2009) players receive a public and a private signal about the realization of some payoff-relevant variable.[4] Among other things, they argue that in the presence of positive payoff externalities, investment decisions may be too responsive to a player's private information (which increases the risk of miscoordination).[5] Since an investment subsidy that is increasing in the aggregate investment activity induces players to base their investment decisions more on the public instead of their private information, it is optimal to introduce such a subsidy scheme in their setup.

2 The Model

Two risk-neutral players have the possibility to invest in a risky project. Players can invest in two periods. If Player i invests at time one, she gets a monetary payoff of $\Theta - \tau$. Henceforth, we refer to $\Theta \in \mathbb{R}$ as the state of the world and $\tau \in \mathbb{R}$ as an investment tax ($\tau > 0$) or subsidy ($\tau < 0$). If player i invests at time two, she gets $\delta(\Theta - \tau)$, where $\delta \in (0, 1)$ denotes the common discount factor. Investments are irreversible. The state of the world Θ is randomly drawn from a normal distribution with mean $\bar{\Theta}$ and variance σ_Θ^2. Player i receives a normally distributed private signal s_i concerning Θ's realization. More precisely, we assume that $s_i = \Theta + \epsilon_i$, where ϵ_i is independently drawn from a normal distribution with zero mean and variance σ_ϵ^2.

The timing is as follows: At time zero, the government sets the investment tax τ. Thereafter, our waiting game starts with nature drawing the state of the world and all signals. After observing the investment tax τ and their private signals, players at

[4] Closely related is also Dasgupta (2007). He considers a two-period irreversible investment model with a continuum of players, exogenous observation noise, and positive network externalities. Dasgupta's paper focuses on how the ability to wait influences the extent of coordination failures in environments with positive network externalities and private information. He is not interested in the relationship between prior public information and optimal tax policy.

[5] This result contrasts nicely with Vives (1997) who showed that absent payoff externalities investment levels are not responsive enough to a player's private information since players do not internalize their information externalities.

period 1 simultaneously decide whether to invest or wait. At the beginning of period 2, players observe past investment choices. Any player who has not invested in period 1 then decides whether or not to invest in period 2. Finally, players receive their payoffs and the game ends.

We refer to the expected state of the world conditional on a player's signal as the player's time-one posterior mean, i.e. $\mu_i \equiv E(\Theta|s_i)$, and it is often useful to use time-one posterior means to describe equilibrium behavior. Throughout we mainly—though not exclusively—focus on equilibria in symmetric switching strategies. Player i is said to follow a switching strategy if she invests at time one whenever her time-one posterior mean exceeds a critical threshold value μ^c and refrains from investing otherwise. A pair of strategies is a symmetric switching equilibrium if, given that Player j follows a switching strategy with critical threshold μ^*, one has: (E1) it is strictly optimal for Player i to invest in period one if and only if $\mu_i > \mu^*$; and (E2) if Player i did not invest at time one, she does so at time two if and only if her expectation of Θ given μ_i and given the observed time-one decision of j exceeds τ. Below equilibrium more generally refers to Bayesian equilibrium.[6]

3 Existence and Uniqueness of Switching Equilibria

In this section, we analyze the waiting game for a given investment tax τ and derive necessary and sufficient conditions for existence and uniqueness of a symmetric switching equilibrium.

We first characterize properties of the best response to a switching strategy. To do so, it is useful to consider the expected payoff difference between investing early and delaying the investment decision. Let $\Delta(\mu_i, \mu_j^c)$ denote the difference between the gain of investing in period 1 and the gain of waiting as a function of Player i's posterior mean

[6] In our model players with sufficiently high (low) signals strictly prefer to invest (wait) at time one, independent of the other player's strategy. Hence, there are no off-the-equilibrium-path observations and players can always apply Bayes's rule so that any Bayesian equilibrium is consistent and sequentially rational.

μ_i under the assumption that Player j follows a switching strategy characterized by μ_j^c. Thus,

$$
\begin{aligned}
(1) \qquad \Delta(\mu_i, \mu_j^c) = \mu_i - \tau \quad &- \quad \delta \Pr(\mu_j > \mu_j^c | \mu_i) \ \max\{0, E(\Theta | \mu_i, \mu_j > \mu_j^c) - \tau\} \\
&- \quad \delta \Pr(\mu_j < \mu_j^c | \mu_i) \ \max\{0, E(\Theta | \mu_i, \mu_j < \mu_j^c) - \tau\}.
\end{aligned}
$$

If $\Delta > 0$ Player i prefers to invest, while if $\Delta < 0$ she prefers to wait. We first observe that a player who is more optimistic regarding the state of the world has a bigger incentive to invest early. Formally,

Lemma 1. *A player's incentive to invest early increases in her time-one posterior mean, i.e.*

$$\frac{\partial \Delta(\mu_i, \mu_j^c)}{\partial \mu_i} > 0, \forall \mu_j^c.$$

Lemma 1 states a common property of waiting games studied in the literature.[7] Intuitively, the higher i's time-one posterior mean, the higher the probability that it will be optimal for her to invest at time two. Due to discounting, this makes waiting less attractive.

Lemma 1 implies that there exists a unique time-one posterior mean at which Player i is indifferent between investing and waiting given that Player j follows a switching strategy characterized by μ_j^c. Formally, i's cutoff $\mu_i^I(\mu_j^c)$ is implicitly defined through the equation $\Delta(\mu_i^I, \mu_j^c) = 0$.[8]

Suppose $\mu_i > \tau$ and that i expects j to always wait so that $\mu_j^c = \infty$. Then, of course, j's waiting decision bears no informational content. Thus the difference between the gain of investing early and the gain of waiting and investing late is $\Delta(\mu_i, \infty) = (1-\delta)(\mu_i - \tau) > 0$. On the other hand, if $\mu_i < \tau$ and Player i expects Player j to always wait, Player i prefers not to invest. Hence, in this case i invests in the first period whenever her time-one posterior mean is greater than the tax rate τ and refrains otherwise. Using a similar reasoning, if Player i expects j to always invest, j's investment decision has no

[7] See for example Hendricks and Kovenock (1989) and Chamley (2004b, Lemma 6.1, p. 124).

[8] The superscript "I" stands for "indifferent".

informational content and thus $\mu_i^I(-\infty) = \mu_i^I(\infty) = \tau$. Furthermore, mere inspection of Equation 1 reveals that i's best response cutoff μ_i^I is continuous in μ_j^c. Lemma 1 thus implies that the cutoff μ^* characterizes a symmetric switching equilibrium if and only if $\mu_i^I(\mu^*) = \mu^*$, or equivalently, $\Delta(\mu^*, \mu^*) = 0$.[9] Graphically, μ^* is the point at which $\mu_i^I(\mu_j^c)$ crosses the 45-degree line (see Figure 5). Since $\mu_i^I(-\infty) = \mu_i^I(\infty) = \tau$, and since μ_i^I is continuous in μ_j^c, a symmetric switching equilibrium exists.

We now investigate which conditions guarantee uniqueness. First, observe that a player who is indifferent between investing and waiting must face a positive gain of investing. This implies that $\mu^* - \tau > 0$. Because $\mu^* < E(\Theta|\mu_i = \mu^*, \mu_j > \mu^*)$ a player with time-one posterior μ^* invests at time two after observing her fellow player investing. We next argue that if $\mu_i = \mu^*$, Player i does not invests in period two after observing that Player j waited, i.e. $E(\Theta|\mu_i = \mu^*, \mu_j < \mu^*) < \tau$. Given that j follows a switching strategy, observing him investing rather than waiting must make i more optimistic.[10] Hence, if i wants to invest after having observed that j waited, she must also want to invest after having observed that j invested. In such a case, she invests independent of j's time-one action. Her expected gain of waiting therefore is $\delta(\mu^* - \tau)$. However, she is then better off investing at time one and receiving an expected payoff of $\mu^* - \tau$.

Given this observation, we say that Player i receives "good news" when she observes j investing. Using that a cutoff type invests in period two only when receiving good news, $\Delta(\mu^*, \mu^*)$ simplifies to

$$(2) \qquad \Delta(\mu^*, \mu^*) = \mu^* - \tau - \delta \Pr(\mu_j > \mu^* | \mu_i = \mu^*) \Big[E(\Theta|\mu^*, \mu_j > \mu^*) - \tau \Big] = 0.$$

Our analysis below makes use of some intuitive and well-known properties of the normal distribution (see the Appendix for proofs). First, Player i's first-period posterior mean μ_i is a weighted average between her private signal s_i and the prior mean $\bar{\Theta}$. The more precise the prior information—i.e. the lower σ_Θ^2—the more weight Player i puts on

[9] It follows from Lemma 1 that E1 is satisfied when $\Delta(\mu^*, \mu^*) = 0$. E2 is satisfied as well as equation 1 prescribes Player i to make an optimal time-two choice.

[10] Formally, $E(\Theta|\mu^*, \mu_j > \mu^*) > E(\Theta|\mu^*, \mu_j < \mu^*)$.

the prior mean and the less weight she puts on her signal. Conversely, the more precise her private information—i.e. the lower σ_ϵ^2—the more she trusts her signal as opposed to the prior mean. In particular, this implies that if the variance of the prior is infinite, or if the variance of her signal is zero, her posterior mean is equal to her signal.

Second, Player i's expectation of Player j's posterior mean μ_j is a weighted average of her own posterior mean μ_i and the prior mean $\bar{\Theta}$. Intuitively, Player i believes that j's signal is distributed around her best guess of the true state of the world—i.e. her posterior mean. Player i, however, also realizes that Player j's posterior mean is a weighted average between j's signal and the prior mean, and therefore is likely to lie between i's posterior and the prior mean. Based on this, a key fact we use below is that if Player i's posterior mean increases by one unit, her expectation about j's posterior mean increases by *less* than one unit. Hence, for example, the further her posterior mean lies above the prior mean, the more likely i thinks that j is more pessimistic than herself. Closely related, if the signal is (nearly) perfect—i.e. the variance of the signal is (close to) zero—both players possess (almost) the same posterior. In that case Player i believes that she always (almost) lies in the "center of the world"—i.e. independent of her posterior there is a 50% chance of j being more optimistic than herself. A similar argument also applies with a completely uninformative prior—i.e. when the variance of the prior is infinite. In this case j puts zero weight on the prior mean when computing his posterior. As i believes j's signal to be distributed around her posterior mean, she also always believes to lie in the center of the world.

Third, conditional on having the cutoff posterior mean μ^*, the probability that j's posterior is greater than the cutoff is

$$\Pr(\mu_j > \mu^* | \mu_i = \mu^*) = 1 - F\left(\kappa_1 \left(\mu^* - \bar{\Theta}\right)\right), \tag{3}$$

where F denotes the cumulative distribution function of the standard normal and where κ_1 is a positive constant depending on the prior and signal variances. It follows from our second observation as well as the formula above that an increase in $\mu^* - \bar{\Theta}$ reduces the probability of j being more optimistic than the cutoff type i.

Fourth, we are interested in the cutoff type's expectation about the state of the world when waiting and receiving good news. In a symmetric switching equilibrium, Player i's expectation will be based on her own signal, the prior mean, and the fact that j invested and thus had a first-period posterior mean above the common cutoff μ^*. Here our distributional assumptions allow us to use known properties of the truncated normal distribution. Formally, in the Appendix we establish that

$$E(\Theta|\mu_i = \mu^*, \mu_j > \mu^*) = \mu^* + \kappa_2 h\left(\kappa_1\left(\mu^* - \bar{\Theta}\right)\right), \tag{4}$$

where κ_2 is a positive constant which (just as κ_1) depends on σ_Θ^2 and σ_ϵ^2, and where h represents the hazard rate of the standard normal distribution. Intuitively, Player i's second-period expectation is the first-period expectation about the state of the world plus an upward shift that depends on the cutoff, the prior mean, as well as—through the constants—the variance of signals and the prior. We have seen above that the cutoff type's probability of getting good news decreases in the cutoff μ^*. The above formula reveals that the impact of good news is also higher for higher cutoffs. Formally, this follows from the fact that the hazard rate of the standard normal distribution is increasing and thus, the upward shift is greater. The statistical intuition is as follows: Player i's belief of Player j's first-period posterior mean is normally distributed with—as we observed above—a mean that lies between i' posterior mean and the prior mean. As the cutoff increases, the expectation of Player j's posterior mean increases by less than the cutoff. Thus, if j invests he reveals that he lies in a higher quantile of this distribution. Since the expectation of a left-truncated normally distributed variable is increasing in the truncation point, the higher the cutoff the better the news for the cutoff type when observing j investing. Consider now the case in which the variance of the prior goes to infinity. As explained above, Player i then believes that she is in the "center of the world", i.e. there is, independent of her posterior, a 50 percent chance that j possesses a higher posterior than herself. This implies that the upward shift does not depend on the cutoff μ^*. Mathematically, in the Appendix we show that κ_1 tends to zero as the variance of the prior goes to infinity, while κ_2 converges to a positive constant. Thus in this special case the upward shift is independent of where the cutoff lies.

Using Equations 3 and 4, we rewrite the equilibrium condition 2 as

$$\underbrace{\mu^* - \tau}_{\text{gain of investing}} = \underbrace{\delta \underbrace{\left[1 - F\left(\kappa_1(\mu^* - \bar{\Theta})\right)\right]}_{\text{prob of good news}} \left[\mu^* - \tau + \underbrace{\kappa_2 h(\kappa_1(\mu^* - \bar{\Theta}))}_{\text{upward shift in beliefs}}\right]}_{\text{discounted gain of waiting}}.$$

As μ^* increases, there are two countervailing forces affecting the gain of waiting. On the one hand, the probability of getting good news decreases. On the other hand, as μ^* increases receiving good news leads to a greater upward shift in beliefs. Indeed the expected upwards shift $[1 - F(\cdot)]\kappa_2 h(\cdot) = \kappa_2 f(\cdot)$ and therefore is non-monotone and unimodal.[11] Rearranging by moving the linear terms in μ^* to the left-hand side and rewriting, yields

$$\mu^* - \tau = \kappa_2 \mathcal{X}\left(\kappa_1\left(\mu^* - \bar{\Theta}\right)\right), \text{ where } \mathcal{X}(\cdot) \equiv \frac{\delta f(\cdot)}{1 - \delta(1 - F(\cdot))}. \tag{5}$$

The left-hand side is linear in μ. The right-hand side is positive and goes to zero as μ goes to plus or minus infinity. Furthermore, Lemma 5 in the Appendix formally establishes many properties of our $\mathcal{X}$-function that are intuitive given that its numerator is the p.d.f. of a normally distributed random variable. In particular, we prove that $\mathcal{X}$ is unimodal, convex and increasing up to a critical value μ' and thereafter concave and increasing up to its mode $\hat{\mu}$. It is also easy to see that a unit increase in $\bar{\Theta}$ leads to a translation to the right of $\mathcal{X}$ by one unit. This property is easiest to check when $\bar{\Theta}$ increases from zero to one. In that case $\mathcal{X}(\kappa_1(0 - 0)) = \mathcal{X}(\kappa_1(1 - 1))$ as illustrated in Figure 1.

As Figure 2 illustrates, whenever the slope of $\kappa_2\mathcal{X}$ is greater than one, multiple equilibria can arise. Intuitively, a low posterior cutoff can be self-fulfilling since if $\underline{\mu}^*$ is "low" an agent's expected upward shift can be low, which makes waiting unattractive and thus a low posterior cutoff desirable. If agents, however, expect a higher cutoff the expected upward shift can be higher, making waiting in turn more attractive.

Recall that if the variance of the prior is (infinitely) large, i believes j's posterior mean to be equally likely to lie above or below hers—independent of her posterior mean.

[11] Throughout the paper, f denotes the p.d.f. of a standard normal distribution.

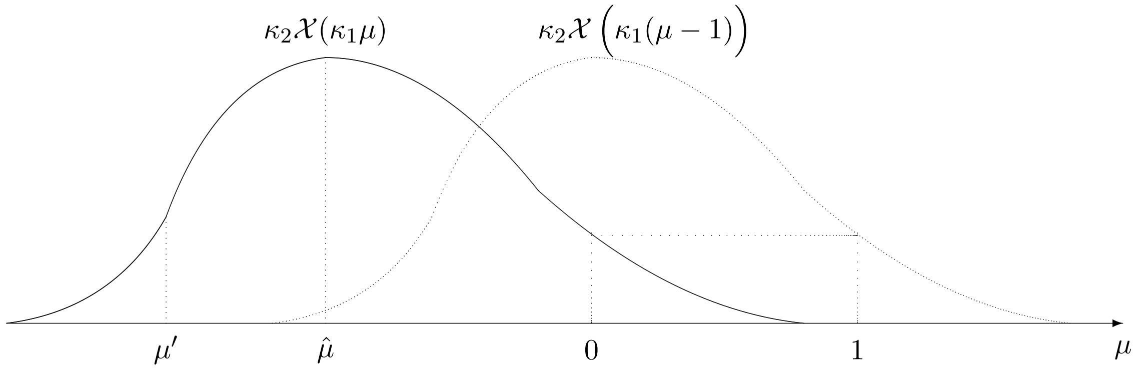

Figure 1: Shape of $\kappa_2\mathcal{X}(\kappa_1(\mu-\bar{\Theta}))$ for $\bar{\Theta}=0$ and $\bar{\Theta}=1$.

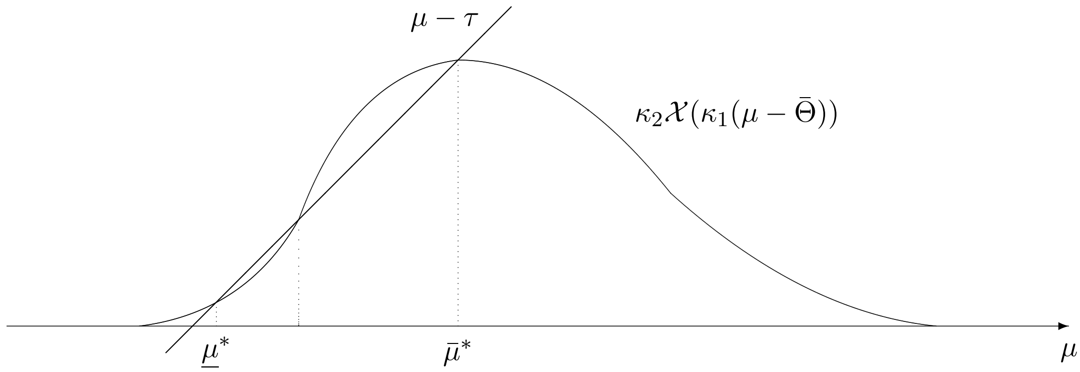

Figure 2: Three different equilibria.

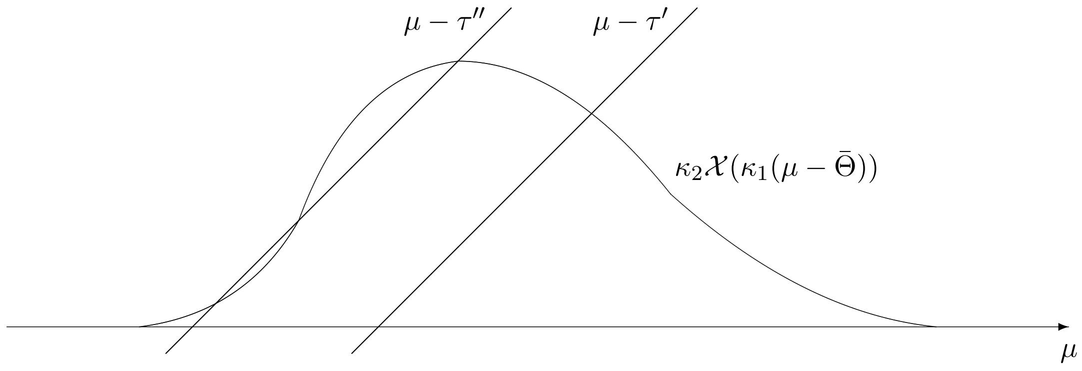

Figure 3: A change in τ may lead to multiple equilibria problems.

The cutoff type's expected upward shift in this case is thus independent of her posterior mean. Hence, as the variance of the prior becomes large, the expected upward shift tends towards a constant and therefore the slope of $\kappa_2\mathcal{X}$ tends to zero. But whenever the slope of $\kappa_2\mathcal{X}$ is less than one everywhere, Figure 2 implies that equilibrium must be unique. Thus, for a high enough variance of the prior, equilibrium is unique. Similarly, as the agent's signal becomes infinitely precise (i.e. as $\sigma_\epsilon^2 \to 0$) she believes that she is in the center of the world and the expected upward shift tends to a constant. Thus, equilibrium is also unique in this case. Furthermore, if the future becomes heavily discounted the gain of waiting and the slope of $\kappa_2\mathcal{X}$ tend to zero, and thus the unique equilibrium cutoff approaches τ in this case.

Of course, even if the maximal slope of $\kappa_2\mathcal{X}$ is greater than one, equilibrium may be unique. For example, if the gain of investing $\mu - \tau$ crosses the function $\kappa_2\mathcal{X}$ far enough in the right tail, equilibrium is unique. Similarly, if it crosses $\kappa_2\mathcal{X}$ where its slope is positive but sufficiently low, equilibrium will be unique. Whether it does so, however, depends on the tax rate as illustrated in Figure 3. In the Figure a decrease in the tax from τ' to τ'', leads to the existence of multiple equilibria. More generally, whenever the maximal slope of $\kappa_2\mathcal{X}$ is greater than one, there exists a tax rate τ for which the waiting game has multiple equilibria. Furthermore, we have argued above that a unit increase in $\bar{\Theta}$ leads to a translation by one unit to the right of $\kappa_2\mathcal{X}$. Hence, for any finite τ, one can reduce $\bar{\Theta}$ until the equilibrium condition 5 is satisfied in the right tail of $\kappa_2\mathcal{X}$. Similarly, for any finite τ we can increase $\bar{\Theta}$ until $\mu - \tau$ cuts $\kappa_2\mathcal{X}$ in its left tail. Thus for sufficiently high or sufficiently low $\bar{\Theta}$, equilibrium is unique. The insights present in this section are summarized below:

PROPOSITION 1. *If*

$$\frac{\partial \kappa_2 \mathcal{X}\left(\kappa_1(\mu - \bar{\Theta})\right)}{\partial \mu} \leq 1, \qquad \forall \mu \tag{6}$$

there exists a unique equilibrium. Furthermore, inequality 6 is satisfied if either:

1. $\sigma_\Theta^2 > (\sigma_\Theta^2)^c$ *for a given* $(\sigma_\Theta^2)^c < \infty$*, or*
2. $\sigma_\epsilon^2 < (\sigma_\epsilon^2)^c$ *for a given* $(\sigma_\epsilon^2)^c > 0$*, or*
3. $\delta \leq \delta^c$*, for a given* $\delta^c > 0$*.*

If inequality 6 is not satisfied, there exist values of $\bar{\Theta}$ *that support multiple equilibria. For any given tax rate* τ*, however, equilibrium is still unique if either*

4. $\bar{\theta} \leq \tau$*, or*
5. $\bar{\theta} \geq \bar{\theta}_u$ *for a given* $\bar{\theta}_u < \infty$*.*

4 Policy

It is easy to check that, from the planner's point of view, μ_i is normally distributed with mean $\bar{\Theta}$ and with some variance denoted by σ_μ^2.[12] Define $\underline{\mu}$ as the posterior mean which ensures that $E(\Theta|\underline{\mu}, \mu_j > \mu^*) = \tau$.[13] In equilibrium all types with a posterior mean

[12]In Section 3 we argued that i's posterior mean μ_i is a weighted average between her signal and the prior mean. Formally, $\mu_i = \alpha s_i + (1-\alpha)\bar{\Theta}$ (where $\alpha \in [0,1]$ depends on the prior and signal variances). By assumption $s_i = \Theta + \epsilon_i$, where $\Theta \sim N(\bar{\Theta}, \sigma_\Theta^2)$ and $\epsilon_i \sim N(0, \sigma_\epsilon^2)$. As ϵ_i is independent of Θ, from the planner's point of view $s_i \sim N(\bar{\Theta}, \sigma_\Theta^2 + \sigma_\epsilon^2)$. Hence, μ_i is the sum of a normally distributed random variable (multiplied by α) with mean $\bar{\Theta}$ and a constant (i.e. $(1-\alpha)\bar{\Theta}$). As is well known, this implies that $\mu_i \sim N(\bar{\Theta}, \sigma_\mu^2)$ where $\sigma_\mu^2 = \alpha^2(\sigma_\Theta^2 + \sigma_\epsilon^2)$.

[13]In the Appendix (see Lemma 6) we prove that for any first-period cutoff μ^*, there exists a unique $\underline{\mu}$. Intuitively, since observing player j investing makes a cutoff-type i more optimistic, it is obvious that $\underline{\mu} < \mu^*$. Formally, the result follows immediately from Lemma 3, which can also be found in the Appendix.

higher than μ^* invest at time one, all types with a posterior mean between $\underline{\mu}$ and μ^* invest at time two only if the other player invested at time one, and all types with a posterior mean lower than $\underline{\mu}$ never invest. Time-zero expected welfare W can thus be written as:

$$W = \int_{\mu^*}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_\mu}\right) d\mu_i + \delta \int_{\underline{\mu}}^{\mu^*} \Pr(\mu_j > \mu^* | \mu_i) E(\Theta | \mu_i, \mu_j > \mu^*) f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_\mu}\right) d\mu_i.$$

The first integral represents the weighted expected utility of all types that invest at time one. The second integral represents the weighted gross gain of waiting of all the inframarginal types. The social planner chooses τ to maximize W. Observe that τ does not enter directly into the welfare calculation. Implicitly, we assume that any tax collected by (or subsidy paid by) the government is costlessly redistributed to (or taxed from) other risk-neutral participants in the economy. Therefore, τ only affects welfare indirectly through its effect on μ^* and $\underline{\mu}$. We first observe that a social planner never sets an infinitely high tax or subsidy.

Lemma 2. *There exists $\underline{\tau}, \bar{\tau} \in \mathbb{R}$ such that for all $\tau \geq \bar{\tau}$ and for all $\tau \leq \underline{\tau}$, a planner can raise welfare by setting τ equal to zero.*

Intuitively, if the investment tax is "too" high, players with very optimistic beliefs postpone their investment plans. Similarly, if the investment subsidy is "too" high, players with very pessimistic beliefs are induced to invest. In both cases the planner can raise welfare by simply eliminating the investment tax (or subsidy). Throughout this section, we also assume that $\forall \tau \in [\underline{\tau}, \bar{\tau}]$ the equilibrium in symmetric switching strategies is unique. In the light of Proposition 1, this is equivalent to assuming that either inequality 6 is satisfied, or that the prior mean $\bar{\theta} \leq \underline{\tau}$, or that $\bar{\theta}$ is sufficiently high.

Differentiating the above welfare function, one has:

$$
\begin{aligned}
\frac{dW}{d\tau} &= -\frac{d\mu^*}{d\tau}\underbrace{\left[\mu^* - \delta\Pr(\mu_j > \mu^*|\mu^*)E(\Theta|\mu^*, \mu_j > \mu^*)\right]}_{\mu^*\text{'s gross gain of investing early}} f\left(\frac{\mu^* - \bar{\Theta}}{\sigma_\mu}\right) \\
&- \frac{d\underline{\mu}}{d\tau}\underbrace{\delta\left[\Pr(\mu_j > \mu^*|\underline{\mu})E(\Theta|\underline{\mu}, \mu_j > \mu^*)\right]}_{\underline{\mu}\text{'s gross gain of investing late}} f\left(\frac{\underline{\mu} - \bar{\Theta}}{\sigma_\mu}\right). \\
&+ \frac{d\mu^*}{d\tau}\underbrace{\delta\int_{\underline{\mu}}^{\mu^*}\frac{\partial}{\partial\mu^*}\left[\Pr(\mu_j > \mu^*|\mu_i)E(\Theta|\mu_i, \mu_j > \mu^*)\right] f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_\mu}\right) d\mu_i.}_{\text{weighted change in the inframarginal types' gross gain of waiting}}
\end{aligned}
$$

It follows from 2 that the first term between square brackets is equal to $\tau(1 - \delta\Pr(\mu_j > \mu^*|\mu^*))$. By definition of $\underline{\mu}$, $E(\Theta|\underline{\mu}, \mu_j > \mu^*) = \tau$. Hence, the above derivative can be written as:

$$
\begin{aligned}
\frac{dW}{d\tau} &= -\frac{d\mu^*}{d\tau}\left[1 - \delta\Pr(\mu_j > \mu^*|\mu^*)\right]\tau f\left(\frac{\mu^* - \bar{\Theta}}{\sigma_\mu}\right) \\
&- \frac{d\underline{\mu}}{d\tau}\delta\Pr(\mu_j > \mu^*|\underline{\mu})\tau f\left(\frac{\underline{\mu} - \bar{\Theta}}{\sigma_\mu}\right) \\
&+ \frac{d\mu^*}{d\tau}\delta\int_{\underline{\mu}}^{\mu^*}\frac{\partial}{\partial\mu^*}\left[\Pr(\mu_j > \mu^*|\mu_i)E(\Theta|\mu_i, \mu_j > \mu^*)\right] f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_\mu}\right) d\mu_i.
\end{aligned} \tag{7}
$$

An increase in τ represents a parallel shift of $\mu - \tau$ to the right (see Figure 3). As $\kappa_2\mathcal{X}$ is independent of τ, $d\mu^*/d\tau > 0$ whenever the symmetric switching equilibrium is unique. Intuitively, increasing the investment tax makes it less attractive both to invest early and to invest late. When deciding to wait, however, the first-period cutoff type only pays the tax upon receiving good news. Thus, increasing the tax raises the incentives to wait and therefore μ^* increases. In Lemma 7 we also show that $d\underline{\mu}/d\tau > 0$ when $\bar{\Theta}$ is higher than $\bar{\Theta}^{c_1}$ or lower than $\bar{\Theta}^{c_2}$. This is not implausible either: As an increase in τ reduces the net return from investing, the time-two marginal type must be more confident about its gross returns.[14]

[14] Observe, however, that this result is only valid when $\bar{\Theta}$ is either sufficiently high or sufficiently low. For "intermediate" values of $\bar{\Theta}$, $\underline{\mu}$ may *decrease* in τ. The intuition behind this counterintuitive result is explained in more detail below.

It follows from 7 and from our previous paragraph that whenever investments are subsidized (i.e. $\forall \tau \leq 0$), $dW/d\tau > 0$ if $\bar{\Theta} > \bar{\Theta}^{c_1}$ and if

$$(8) \qquad \forall \mu_i \in [\underline{\mu}, \mu^*], \qquad \frac{\partial}{\partial \mu^*}[\Pr(\mu_j > \mu^* | \mu_i) E(\Theta | \mu_i, \mu_j > \mu^*)] > 0.$$

In words, if the above conditions are satisfied, the social planner starting from any nonpositive tax rate can strictly increase welfare by raising the tax rate τ, implying that investments should be taxed (i.e. $\tau^* > 0$). It can be checked that i's gross gain of waiting is unimodal in μ^* for the same reason as $\mathcal{X}$ is unimodal in μ: if μ^* increases, the event $\mu_j > \mu^*$ becomes less likely, which, ceteris paribus, reduces i's gross gain of waiting. An increase in μ^*, however, also increases the upward shift in i's posterior mean when j invests, which, ceteris paribus, increases i's gross gain of waiting. Player i's gross gain of waiting is thus maximized at some intermediate value of μ. If μ^* lies below this value, the second effect dominates and i's gross gain of waiting is increasing in μ^*. The first effect dominates whenever μ^* lies above this value.

As explained in Section 3, j's posterior mean is a weighted average between his signal s_j and the prior mean $\bar{\Theta}$. Furthermore, i believes j's signal to be symmetrically distributed around her best guess of the state of the world, i.e. her posterior mean μ_i. Therefore, i's expectation of j's posterior mean (i.e. $E(\mu_j|\mu_i)$) lies between her posterior mean μ_i and the prior mean $\bar{\Theta}$. This implies that the more $\bar{\Theta}$ lies above μ^*, the more likely i believes j to invest. Formally, keeping μ^* fixed, $\Pr(\mu_j > \mu^* | \mu_i = \mu^*)$ is increasing in $\bar{\Theta}$. Therefore, the cutoff player's expected upward shift is small when $\bar{\Theta}$ is high and her gross gain of waiting is thus increasing in μ^*. As the inframarginal types's posteriors are even lower than μ^*, they also *all* agree that μ^* is too low and want the social planner to raise it via taxes.

A similar, though more subtle, argument also holds when the prior mean becomes very negative. The subtlety stems from the fact that $\underline{\mu}$ is decreasing in the prior mean: The lower the prior mean, the higher the upward shift in Player i's posterior mean.[15]

[15]In Lemma 4, which can be found in the Appendix, we prove that $E(\Theta|\mu_i, \mu_j > \mu^*) = \mu_i + \kappa_2 h(\frac{\mu^* - \alpha\mu_i - (1-\alpha)\bar{\Theta}}{\sigma_o})$, where σ_o is some positive constant. As in equation 4, the upward shift in Player

Hence, when the prior mean becomes very negative some types with a very negative (time-one) posterior mean may end up investing at time two. It is important to stress that those types invest at time two not because their time-one posterior mean was "not that low". Instead, those types invest because they experience a huge upward shift upon observing their rival investing at time one. Hence, types with very negative time-one posterior means want the social planner to choose τ to increase their expected upward shift to the largest possible extent. In particular, this implies that types with posterior means close to (but greater than) $\underline{\mu}$ think that μ^* is too low and want the social planner to raise it via taxes. Types with posterior means not close to $\underline{\mu}$, however, view things differently. Conditional upon getting good news, those types invest at time two, not so much because they experience a huge upward shift in their posterior beliefs, but rather because their time-one posterior means were not that low in the first place. When the prior mean is very negative, those types think that the probability of getting good news is too low. Hence, they want the social planner to reduce μ^* by subsidizing investments. When the prior mean becomes very negative, the social planner thus faces a tradeoff: she needs to weigh the benefit of a decrease in μ^* for the inframarginal types with not very low time-one posterior means against the losses of those types with time-one posterior means close to $\underline{\mu}$. As this exercise is analytically demanding, we have not been able to prove that a (strictly positive) subsidy is optimal when the prior mean is sufficiently low. Nevertheless, we establish that as the prior mean tends to minus infinity, the mass of types who prefer the social planner to tax investments tends to zero.[16] This result enables us to show that if $\tau > 0$, the social planner can raise welfare by reducing the investment tax when $\bar{\Theta}$ is below some critical value. This results also implies that in the limit (i.e. as $\bar{\Theta} \to -\infty$), $\tau^* \leq 0$.

Both results are summarized below:

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i's beliefs is thus captured by the term $\kappa_2 h(\cdot)$. This implies that the more negative μ_i or $\bar{\Theta}$, the higher the expected upward shift.

[16]For any finite prior mean, however, there always exist some types who prefer the social planner to tax investments.

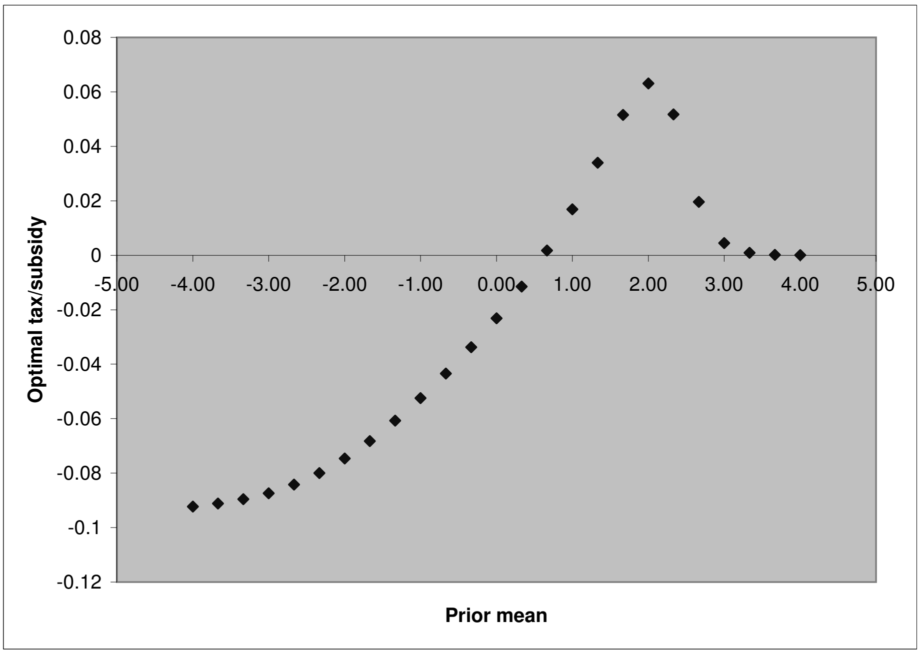

Figure 4: Optimal tax or subsidy as a function of the prior mean.

Proposition 2. *If the prior mean $\bar{\Theta}$ is sufficiently high, it is optimal for the social planer to tax investments ($\tau^* > 0$). Conversely, any positive investment tax ($\tau > 0$) is suboptimally high if $\bar{\Theta}$ lies below some threshold $\bar{\Theta}^c(\tau)$.*

Figure 4 shows τ^* (i.e. the optimal tax/subsidy) as a function of the prior mean when all the exogenous variances (i.e. σ_ϵ^2 and σ_Θ^2) are equal to one. For low values of the prior mean, the social planner should subsidize investment. Once the prior mean crosses some critical level, investments should be taxed.[17] As explained above, if the prior mean is sufficiently high the social planner increases the expected upward shift of all inframarginal types when taxing investment. Thus when stories about the high profitability of an investment opportunity abound, it is optimal to tax investments.

[17]In this example, the "subtlety" which we explained in our previous paragraph is thus irrelevant: the social planner subsidizes investments because with a low prior mean a large mass of inframarginal types fear that the other player will not invest. Thus, although there are always types who want the social planner to tax investments, these types' losses are overcompensated by the gains to inframarginal types with higher time-one posteriors.

Proposition 2 extends intuition about the insufficient use of private information derived in the original herding papers (see Banerjee (1992) and Bhikhchandani et al. (1992)) to an endogenous queue set-up. In an informational cascade, Player i gets say sufficiently good public information about the returns from investing, which arises when enough predecessors in a queue decide to invest, so that she follows the public information and invests even when possessing an unfavorable private signal. This investment decision is typically socially inefficient as it impedes subsequent movers to infer this player's signal from her action. A similar inefficiency also arises in our model: if the public information is very favorable (i.e. if $\bar{\Theta}$ is sufficiently high), an inefficiently high mass of types end up investing, making it harder for players who wait to confidently infer that the state of the world is indeed conducive to investing.

According to (perhaps recent) conventional wisdom, governments should not intervene in the presence of an investment bubble as one cannot ex ante know whether it is due to fundamentals (corresponding to the case in which $\Theta > 0$ in our model) or whether it is the result of incorrect beliefs of the private actors. Alan Greenspan, for example, wrote:

> "How do you draw the line between a healthy, exciting economic boom and a . . . bubble . . . ? ... After thinking a great deal about this, I decided that ... the Fed would not second-guess "hundreds of thousands of informed investors." Instead the Fed would position itself to protect the economy in the event of a crash." (Greenspan, 2008, pages 200-1)

Our model questions this rationale for non-intervention: even if policymakers in contrast to market participants receive no private signal about the state of the world, the policymakers' prior knowledge of $\bar{\Theta}$ can still be used to improve welfare.[18] In particular, in the presence of sufficiently favorable public information, investments should be taxed.

[18] Greenspan was primarily worried about the existence of an investment boom in the U.S. stockmarket, i.e. in a context in which prices supposedly aggregate information. As our model is void of any price mechanism, one might argue that Greenspan's quote does not really apply to our set-up. We feel, however, that (perhaps until recently) the vast majority of policy-makers would agree (or would have agreed) with Greenspan even in a fixed-price context.

More broadly, Proposition 2 is consistent with the idea that investment policy should be countercyclical: when $\bar{\Theta}$ is high (i.e. when expected investment activity is high), investments should be taxed, while if $\bar{\Theta} \to -\infty$ (i.e. when expected investment activity is zero) investment should not be taxed. Indeed, in the concrete example above an investment subsidy becomes optimal once the state of the world is sufficiently bad.

5 Is total investment activity decreasing in τ?

Denote by μ^0 and μ' the first-period equilibrium cutoffs respectively when $\tau = 0$ and when $\tau = \tau' > 0$. Similarly, denote by $\underline{\mu}^0$ and $\underline{\mu}'$, the second-period equilibrium cutoffs respectively when $\tau = 0$ and when $\tau = \tau'$. In our previous section, we argued that $d\mu^*/d\tau > 0$, i.e. the indifferent type at time one must possess a higher posterior to compensate for the increase in τ. Hence, $\mu^0 < \mu'$ and an increase in the tax rate τ cannot increase the number of time-one investments. Now consider the effect on the second-period cutoff. When the tax rises, it also reduces i's ex-post payoff of investing late—the direct effect. Ceteris paribus, this effect thus *increases* the second-period cutoff. The tax change, however, also favorably affects the upward shift in i's posterior mean when j invests—the indirect effect. Ceteris paribus, this decreases $\underline{\mu}$. Intuitively, if j—despite a higher investment tax—still invests, this means that his posterior mean lies above μ' and not μ^0. In words, if τ increases and j invests, this is "excellent" instead of "good" news. In the Appendix, we prove that—depending on the values of the parameters—either effect can dominate. If the direct effect dominates, (i.e. if $\underline{\mu}^0 < \underline{\mu}'$), total investment activity cannot increase when τ goes from zero to τ'. If the indirect effect dominates (i.e. if $\underline{\mu}' < \underline{\mu}^0$), total investment activity may increase after an increase in τ. This is easy to see: suppose (μ_1, μ_2) are such that $\mu_1 > \mu'$ and that $\mu_2 \in (\underline{\mu}', \underline{\mu}^0)$. Then, without the tax only player one invests, while with $\tau = \tau'$, both players end up investing. Observe that $d\underline{\mu}/d\tau$ may be negative despite our assumption that taxes are permanent. If the government were only to impose a period-one tax, the direct effect would disappear and trivially $\underline{\mu}' < \underline{\mu}^0$.

Consider now the extreme case in which $\delta = 0$. In this case, the first-period cutoff $\mu^* = \tau$. Hence, if the tax τ increases by $d\tau$ units, so does the first-period cutoff. When δ is very high, the positive effect of an increase in the tax on μ^* is also relatively "low". One reason is that for a higher δ, the future tax cost is discounted by less. Another, more subtle, reason is best understood in the limit equilibrium as $\delta \to 1$. As the (opportunity) cost of waiting is arbitrarily small, in equilibrium the gain of waiting must be close to zero. Hence, when getting bad news (i.e. when $\mu_j^c < \mu^*$) a first-period cutoff type must be indifferent between investing or not in period two.[19] Thus, one can equivalently think of her as investing in period two also after receiving bad news, which implies that she cannot escape the cost of taxation through waiting. Therefore, her incentives to wait are unaffected by the tax rate, which limits the extent to which μ^* increases. (Nevertheless, of course, in the limit equilibrium a first-period cutoff player must also be indifferent between investing and not investing when getting bad news, and an increase in the tax rate makes investing late less attractive; thus the first-period cutoff increases.) When $\delta = 0$ or when δ is close to one, a change in the tax τ does thus not have a "big" effect on μ^*.[20] As the effect on μ^* is relatively small, an increase in the tax τ does not increase the upward shift in i's posterior mean by "much" in these limit cases. This explains why in both limit cases the direct effect dominates. Our most important results are summarized below:

Proposition 3. *Suppose condition 6 is satisfied. Increasing the tax rate τ increases μ^* and, thus, weakly decreases first-period investment activity. Increasing the tax rate τ, however, can either increase or decrease $\underline{\mu}$. If $d\underline{\mu}/d\tau < 0$, an increase in the tax rate τ may increase overall investment activity. Nevertheless, if players are either sufficiently*

[19] Formally, this intuitive fact follows from equation 21 in the Appendix.

[20] When the prior mean is sufficiently high and when the discount factor takes "intermediate" values, a change in τ can have a much bigger effect on μ^*. Recall from the discussion of Figure 1 above that waiting decisions are strategic complements if the prior mean is sufficiently high. Hence, if—due to an increase in τ—Player j increases her cutoff level, this induces Player i to increase her cutoff by more than the tax increase.

impatient or sufficiently patient, increasing the tax rate increases $\underline{\mu}$ and, hence, also decreases overall investment activity.

6 Extensions

6.1 Asymmetric equilibria in switching strategies

We now provide a simple sufficient condition that rules out any asymmetric equilibrium in switching strategies.[21] Remember that μ_i^I denotes the value of μ_i such that i is indifferent between investing and waiting given that j invests if and only if her posterior exceeds μ_j^c. In the Appendix we prove that μ_i^I initially increases and then decreases in μ_j^c. The intuition behind this result is identical to the one that explains the unimodality of $\mathcal{X}$: if μ_j^c increases, this reduces the likelihood that j invests. Ceteris paribus, this reduces a player's gain of waiting and, thus, reduces μ_i^I. An increase in μ_j^c, however, also increases the upward shift in i's posterior mean when Player j invests. Ceteris paribus, this increases μ_i^I. For low values of μ_j^c, the second effect dominates, while the first one dominates once μ_j^c passes a critical threshold level. As mentioned in Section 3, the symmetric equilibrium cutoff μ^* can be graphically represented by the point in which $\mu_i^I(\mu_j^c)$ crosses the 45°-line. Suppose $\mu_i^I(\mu_j^c)$ crosses the 45°-line when $\frac{d\mu_i^I}{d\mu_j^c} > 0$. Asymmetric equilibria can then be ruled out as illustrated in Figure 5. In that Figure $(\mu_i^c)^{-1}\left(\mu_j^I\right)$ represents the inverse of $\mu_j^I(\mu_i^c)$. By symmetry of our set-up, $\mu_j^I(\mu_i^c)$ has the same shape as $\mu_i^I(\mu_j^c)$. Therefore, $(\mu_i^c)^{-1}\left(\mu_j^I\right)$ represents the rotated (over the 45°-line) image of $\mu_i^I(\mu_j^c)$. Using symmetry as well as that $\left.\frac{d\mu_i^I}{d\mu_j^c}\right|_{\mu_i=\mu^*} > 0$, and that $\mu_i^I(\mu_j^c)$ is unimodal, this implies that both functions cannot cross at a point that does not lie on the 45°-line. Hence, if $\left.\frac{d\mu_i^I}{d\mu_j^c}\right|_{\mu_i=\mu^*} > 0$, there exists no asymmetric equilibrium. If, additionally, condition 6 is satisfied, equilibrium is unique within the class of the switching strategies as illustrated in Figure 5. Furthermore, in the Appendix we also show that $\left.\frac{d\mu_i^I}{d\mu_j^c}\right|_{\mu_i=\mu^*} > 0$ is equivalent to $\left.\frac{\partial\mathcal{X}(\kappa_1(\mu-\bar{\Theta}))}{\partial\mu}\right|_{\mu=\mu^*} > 0$.

[21] In an asymmetric equilibrium in switching strategies player i invests at time one if and only if her posterior lies above μ_i^* (where $\mu_i^* \neq \mu_j^*$).

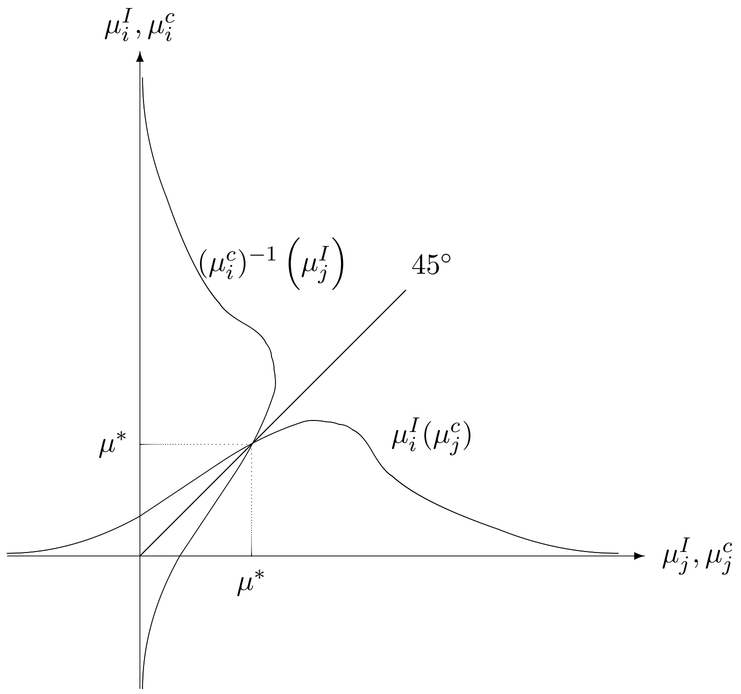

Figure 5: No asymmetric equilibria.

As illustrated in Figure 1, an increase in $\bar{\Theta}$ shifts $\mathcal{X}$ to the right. Hence, if $\bar{\Theta}$ is above some threshold level $\mathcal{X}$ is increasing at the equilibrium cutoff, which rules out asymmetric equilibria in switching strategies. To summarize:

PROPOSITION 4. *If* $\frac{\partial \mathcal{X}(\kappa_1(\mu-\bar{\Theta}))}{\partial \mu}$, *evaluated at the symmetric equilibrium, is positive, there exists no asymmetric equilibrium in switching strategies. Furthermore, for all potentially optimal tax levels, the condition is satisfied if the prior mean is sufficiently high.*

6.2 The N-player case

Consider the case in which players have a Laplacian prior. As argued in Section 3, Player i's posterior mean is then equal to her signal and she—independent of her signal—believes to be in the "center of the world", i.e. there is a 50%-chance that j possesses a higher signal than herself. As explained in Section 3, those beliefs yield a unique equilibrium in the two-player case. It is, however, easy to prove that those beliefs also yield a unique equilibrium within the class of the symmetric switching strategies for any number of players N.

To see this, let $\Delta(\mu_i, \mu^c)$ denote the difference between i's gain of investing and her gain of waiting given her posterior μ_i and given that all players invest at time one if and only if their posteriors exceed μ^c. Formally,

$$\Delta(\mu_i, \mu^c) = \mu_i - \tau - \delta \sum_{n=1}^{N-1} \Pr(n|\mu_i) \max\Big\{0, E(\Theta|\mu_i, n) - \tau\Big\}, \tag{9}$$

where $E(\Theta|\mu_i, n)$ denotes i's expectation of Θ given her time-one posterior mean μ_i and given that n players invested at time one.

Note that if $\mu_i < \tau$, i's expected payoff from investing at time one is negative. Hence, independent of the other players' strategies, she strictly prefers to wait in the first period. In particular, this implies that $\Delta(\mu_i, \mu^c = \mu_i)$ is negative for sufficiently low values of μ_i. Furthermore, a strategy in which i always waits is dominated as well. To see this, suppose that if i waits, she perfectly learns the state of the world. Player i's gain of waiting then equals $\delta \Pr(\Theta > 0|\mu_i)E(\Theta|\mu_i, \Theta > 0)$. Observe that for a high enough time-one posterior μ_i, $E(\Theta|\mu_i, \Theta > 0) \approx E(\Theta|\mu_i) = \mu_i$. Hence, there exists a $\bar{\mu}$ such that $\bar{\mu} = \delta \Pr(\Theta > 0|\bar{\mu})E(\Theta|\bar{\mu}, \Theta > 0)$. Thus for posteriors $\mu_i \geq \bar{\mu}$ Player i would not want to wait even if she would learn the state of the world perfectly when waiting. Hence, for high enough time-one posteriors Player i is strictly better off investing at time one—independent of the other players' strategies. This implies that $\Delta(\mu_i = \bar{\mu}, \mu^c = \bar{\mu}) > 0$.

Observe that trivially

$$\mu_i - \tau = \sum_{n=1}^{N-1} \Pr(n|\mu_i) \Big(E(\Theta|\mu_i, n) - \tau\Big).$$

The above equality allows us to rewrite $\Delta(\mu_i, \mu^c = \mu_i)$ as:

$$\begin{aligned} \Delta(\mu_i, \mu^c = \mu_i) &= (1-\delta) \sum_{n=1}^{N-1} \Pr(n|\mu_i) \max\Big\{0, E(\Theta|\mu_i, n) - \tau\Big\} \\ &\quad + \sum_{n=1}^{N-1} \Pr(n|\mu_i) \min\Big\{0, E(\Theta|\mu_i, n) - \tau\Big\}. \end{aligned} \tag{10}$$

As signals are independently drawn,

$$\Pr(n|\mu_i) = \frac{(N-1)!}{n!(N-n-1)!} \Pr(\mu_j > \mu_i|\mu_i)^n \Pr(\mu_j < \mu_i|\mu_i)^{N-n-1}.$$

As argued above, if players have a Laplacian prior i believes to be in the "center of the world", which implies that $\Pr(\mu_j > \mu_i|\mu_i) = \frac{1}{2}$. Hence, when players have a Laplacian prior $\Pr(n|\mu_i)$ is independent of μ_i. This insight, combined with the fact that $E(\Theta|\mu_i, n)$ is strictly increasing in μ_i, allows us to conclude that $\Delta(\mu_i, \mu^c = \mu_i)$ is strictly increasing in μ_i. It then follows from our previous paragraph that there exists a unique candidate equilibrium cutoff μ^* such that $\Delta(\mu^*, \mu^*) = 0$. In the Appendix we show that $\Delta(\mu_i, \mu^*) > 0 \Leftrightarrow \mu_i > \mu^*$, thereby establishing that this cutoff indeed characterizes a symmetric switching equilibrium. Summarizing:

PROPOSITION 5. *Suppose players have a Laplacian prior. For any finite number of players N, there exists a unique symmetric switching equilibrium.*

7 Final Remarks

We analyzed some policy implications of social learning when players are fully rational and have better information than the policymaker. Our model is particularly useful when public information is conducive to investing—which typically happens during "boom times". In this case, we establish that in the unique switching equilibrium investments should be taxed.

We haven chosen a two-player setup for our model. The general N player game is difficult to analyze. One "simple" alternative, however, would be to consider a model with a continuum of players. In that variation, for any given symmetric equilibrium cutoff, social learning would be perfect and hence a laissez-faire policy optimal. To circumvent this unrealistic feature, one needs to assume social learning to be imperfect. One possibility is to assume observational noise as in Chamley (2004a) or Dasgupta (2007). In such a setup, Player i's distribution about the other players' posterior means (i.e. $f(\mu_j|\mu_i)$) would still be computed in the same way as in our two-player model.

Therefore, if the prior mean is "very high" an inframarginal type expects—for "many" realizations of the state of the world—a large mass of players to invest at time one. As noisy observation of past investment behavior is then expected to reveal relatively little information about the realized state of the world, we conjecture that the inframarginal types then also prefer the social planner to raise the equilibrium cutoff via taxes. One drawback of such an approach, however, is that the observational noise is completely exogenously specified. An alternative assumption is that each player can only observe some (neighboring) players first-period decision.[22]

In our model information can only be transmitted through actions. As there are no payoff externalities, it is natural to ask why information cannot be transmitted through words instead. If players can fully exchange their private information via cheap talk, an efficient equilibrium of course exists. We feel, however, that this simple argument is misleading as communication—even where allowed and feasible—is often imperfect. Suppose, for example, that player one is asked to reveal her type to the other player(s) prior to the waiting game. As her signal is imperfect, she also wants to learn the other player(s)' signal(s). She therefore has an incentive to send the message which maximizes her gain of waiting. In an analysis of cheap talk, Gossner and Melissas (2006) have shown that this game may—depending on the values of the parameters—be characterized by a unique monotone equilibrium in which all types send the same message, i.e. information can only be revealed through actions. More generally, we believe the study of waiting games in the presence of imperfect communication to be an interesting avenue for future research.

Another noteworthy aspect of our model is that investment costs are exogenous.[23] In many applications in which policymakers are concerned about investment bubbles—such as stock market or housing market bubbles—one would expect investment costs to increase in the number of present and past investments. In an exogenous queue model

[22]Our model, for example, can be seen as a special case in which countably many players live on a circle and each player only observes her right-hand neighbor.

[23]See also the discussion in Footnote 18.

with a competitive market maker, Avery and Zemsky (1998) establish that herding does not occur with one dimensional information but does occur with multi-dimensional information. An interesting question for future research is how these results extend to endogenous queue setting and whether it is also optimal to tax investments during booms times in such a model.

For simplicity we assumed that players are fully rational. Eyster and Rabin (2009) nicely highlight some counterintuitive features of the rational learning model in an exogenous queue environment and propose a plausible alternative learning model. An interesting question is whether the introduction of inferentially naive and/or cursed players strengthens or qualifies our "taxation during booms" result in an endogenous queue environment.

Appendix

Definitions and Preliminaries

Throughout the appendix F, f, h, and r represent, respectively, the c.d.f., the p.d.f., the hazard rate $\left(\equiv \frac{f(\cdot)}{1-F(\cdot)}\right)$, and the reverse hazard rate $\left(\equiv \frac{f(\cdot)}{F(\cdot)}\right)$ of the standard normal distribution. We will also use the following notations: $\alpha \equiv \frac{\sigma_\Theta^2}{\sigma_\Theta^2+\sigma_\epsilon^2}$, $\beta \equiv \frac{\frac{2}{\sigma_\epsilon^2}}{\frac{1}{\sigma_\Theta^2}+\frac{2}{\sigma_\epsilon^2}}$, $\sigma_p^2 \equiv \frac{\sigma_\Theta^2\sigma_\epsilon^2}{\sigma_\Theta^2+\sigma_\epsilon^2}$, $\sigma_2^2 \equiv \sigma_p^2 + \sigma_\epsilon^2$, $\sigma_o^2 \equiv \alpha^2\sigma_2^2$, $\sigma_\mu^2 \equiv \alpha^2(\sigma_\Theta^2 + \sigma_\epsilon^2)$, $\kappa_1 \equiv \frac{1-\alpha}{\sigma_o}$, $\kappa_2 \equiv \frac{1}{2}\beta\sigma_2$, $x(\mu_j^c, \mu_i) \equiv \frac{\mu_j^c-\alpha\mu_i-(1-\alpha)\bar{\Theta}}{\sigma_o}$, $\mathcal{X}(\eta) \equiv \frac{\delta f(\eta)}{1-\delta(1-F(\eta))}$, $g(\mu) \equiv \mu - \tau - \kappa_2\mathcal{X}(\kappa_1(\mu - \bar{\Theta}))$, $\phi(\mu) \equiv \mu - \tau + \kappa_2 h(x(\mu^*, \mu))$, and $\tilde{\mu} \equiv (1-\alpha)\bar{\Theta} - \mu^*$.

In our set-up (see DeGroot (1984) for proofs) $\Theta|s_i \sim N(\mu_i, \sigma_p^2)$, where

$$\mu_i = \alpha s_i + (1-\alpha)\bar{\Theta}. \tag{11}$$

As ϵ_j is independent from Θ and ϵ_i, $s_j|s_i = \Theta|s_i + \epsilon_j$. As $\epsilon_j \sim N(0, \sigma_\epsilon^2)$, $s_j|s_i \sim N(\mu_i, \sigma_p^2 + \sigma_\epsilon^2)$. Furthermore, $\mu_j = \alpha s_j + (1-\alpha)\bar{\Theta}$, and thus,

$$\mu_j|s_i \sim N(\alpha\mu_i + (1-\alpha)\bar{\Theta}, \sigma_o^2). \tag{12}$$

Hence, $\Pr(\mu_j > \mu^*|\mu_i) = 1 - F\left(\frac{\mu^* - \alpha\mu_i - (1-\alpha)\bar{\Theta}}{\sigma_o}\right)$, and

$$\Pr(\mu_j > \mu^*|\mu_i = \mu^*) = 1 - F\left(\kappa_1\left(\mu^* - \bar{\Theta}\right)\right).$$

Proof of Lemma 1

We first state and prove the following lemma.

Lemma 3. *If signals and the state of the world are drawn from Normal distributions,*

$$1 > \frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1} > \frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_2^c} > 0,$$

and

$$\frac{\partial E(\Theta|\mu_1, \mu_2 < \mu_2^c)}{\partial \mu_1} > 0.$$

Proof: A well known statistical result (see DeGroot (1984) for a proof) is that if $\Theta \sim N(\bar{\Theta}, \sigma_\Theta^2)$ and if $\epsilon_i \sim N(0, \sigma_\epsilon^2)$, then $\Theta|s_1, s_2$ also tends to a normal and

$$E(\Theta|s_1, s_2) = \beta\frac{s_1 + s_2}{2} + (1-\beta)\bar{\Theta}. \tag{13}$$

We first tackle the case in which $\mu_2 > \mu_2^c$. It follows from 11 that $\mu_2 > \mu_2^c \Leftrightarrow s_2 > s_2^c \equiv \frac{\mu_2^c - (1-\alpha)\bar{\Theta}}{\alpha}$. One has,

$$\begin{aligned} E(\Theta|\mu_1, \mu_2 > \mu_2^c) &= \int [\beta\frac{s_1 + s_2}{2} + (1-\beta)\bar{\Theta}] f(s_2|s_1, s_2 \geq s_2^c) ds_2, \\ &= \frac{\beta}{2} s_1 + \frac{\beta}{2} E(s_2|s_1, s_2 > s_2^c) + (1-\beta)\overline{\Theta}. \end{aligned} \tag{14}$$

From the explanations provided after 11, we know that $s_2|s_1, s_2 > s_2^c$ is a left-truncated normal distribution with mean μ_1, variance σ_2^2 and truncation point s_2^c. Using Johnson et al. (1995) to calculate the expectation of a truncated normal variable, one has

$$E(s_2|s_1, s_2 > s_2^c) = \mu_1 + h\left(\frac{s_2^c - \mu_1}{\sigma_2}\right)\sigma_2. \tag{15}$$

Inserting 15 into 14 and differentiating, establishes that

$$\frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1} = \frac{\beta}{2}\left[\frac{1}{\alpha} + 1 - \left.\frac{\partial h(z)}{\partial z}\right|_{z = \frac{s_2^c - \mu_1}{\sigma_2}}\right]. \tag{16}$$

Inserting 15 into 14 and differentiating with respect to μ_2^c, one has

$$\frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_2^c} = \frac{\beta}{2\alpha} \left.\frac{\partial h(z)}{\partial z}\right|_{z=\frac{s_2^c-\mu_1}{\sigma_2}}. \tag{17}$$

As is well known (see, e.g. Greene (1993), Theorem 22.2), the slope of the hazard rate of a standard normal distribution, $h'(z) \in (0,1)\ \forall z$. This insight, combined with the fact that both α and β are positive, allows us to conclude that $\frac{\partial E(\Theta|\mu_1,\mu_2>\mu_2^c)}{\partial \mu_2^c} > 0$.

It follows from 16 and 17 that

$$\frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1} > \frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_2^c} \Leftrightarrow 1 > \left.\frac{\partial h(z)}{\partial z}\right|_{z=\frac{s_2^c-\mu_1}{\sigma_2}},$$

which is satisfied.

Observe that $\frac{\beta}{2}(\frac{1}{\alpha}+1) = 1$. This observation, combined with our earlier result that $h'(z) > 0$, allows us to conclude that $\frac{\partial E(\Theta|\mu_1,\mu_2>\mu_2^c)}{\partial \mu_1} < 1$.

We now tackle the case in which $\mu_2 < \mu_2^c$. As above,

$$E(\Theta|\mu_1, \mu_2 < \mu_2^c) = \frac{\beta}{2}s_1 + \frac{\beta}{2}E(s_2|\mu_1, \mu_2 < \mu_2^c) + (1-\beta)\overline{\Theta}. \tag{18}$$

From Johnson et al (1995), we know that

$$E(s_2|\mu_1, s_2 < s_2^c) = \mu_1 - r\left(\frac{s_2^c - \mu_1}{\sigma_2}\right)\sigma_2. \tag{19}$$

Inserting 19 into 18 and differentiating yields

$$\frac{\partial E(\Theta|\mu_1, \mu_2 < \mu_2^c)}{\partial \mu_1} = \frac{\beta}{2}\left[\frac{1}{\alpha} + 1 + \left.\frac{\partial r(z)}{\partial z}\right|_{z=\frac{s_2^c-\mu_1}{\sigma_2}}\right]. \tag{20}$$

It is well known (see, e.g. Greene (1993), Theorem 22.2) that $r'(\cdot) \in (-1,0)$. Hence, $\frac{\partial E(\Theta|\mu_1,\mu_2<\mu_2^c)}{\partial \mu_1}$ is positive. ■

Observe that for any finite μ_1 and μ_2^c, $E(\Theta|\mu_1, \mu_2 < \mu_2^c) < E(\Theta|\mu_1, \mu_2 > \mu_2^c)$. There are thus three possibilities:

(i) $E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau < E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau \leq 0$,

(ii) $E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau \leq 0 < E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau$, and

(iii) $0 < E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau < E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau$.

In case (i), $\Delta(\cdot) = \mu_1 - \tau$, which is increasing in μ_1.

In case (ii), $\Delta(\cdot) = \mu_1 - \tau - \delta \Pr(\mu_2 > \mu_2^c|\mu_1)[E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau]$. Observe that

$$\mu_1 - \tau = \Pr(\mu_2 > \mu_2^c|\mu_1)[E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau] + \Pr(\mu_2 < \mu_2^c|\mu_1)[E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau].$$

Inserting this last equality into $\Delta(\cdot)$, yields

$$\Delta(\cdot) = (1-\delta)\Pr(\mu_2 > \mu_2^c|\mu_1)[E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau] + \Pr(\mu_2 < \mu_2^c|\mu_1)[E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau]. \tag{21}$$

Differentiating this last expression of $\Delta(\cdot)$ yields:

$$\begin{aligned} \frac{\partial \Delta(\mu_1, \mu_2^c)}{\partial \mu_1} &= (1-\delta)\frac{\partial \Pr(\mu_2 > \mu_2^c|\mu_1)}{\partial \mu_1}[E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau] \\ &+ (1-\delta)\frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1}\Pr(\mu_2 > \mu_2^c|\mu_1) \\ &+ \frac{\partial \Pr(\mu_2 < \mu_2^c|\mu_1)}{\partial \mu_1}[E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau] \\ &+ \frac{\partial E(\Theta|\mu_1, \mu_2 < \mu_2^c)}{\partial \mu_1}\Pr(\mu_2 < \mu_2^c|\mu_1). \end{aligned} \tag{22}$$

In case (ii), $E(\Theta|\mu_1, \mu_2 > \mu_2^c) - \tau > 0$. As $\frac{\partial \Pr(\mu_2 > \mu_2^c|\mu_1)}{\partial \mu_1}$ is also positive, the first term of the RHS of 22 is positive. Moreover, from Lemma 3 we know that both $\frac{\partial E(\Theta|\mu_1, \mu_2 > \mu_2^c)}{\partial \mu_1}$ and $\frac{\partial E(\Theta|\mu_1, \mu_2 < \mu_2^c)}{\partial \mu_1}$ are positive. Hence, the second and the fourth term of the RHS of 22 are also positive. In case (ii), $E(\Theta|\mu_1, \mu_2 < \mu_2^c) - \tau \leq 0$. This assumption, combined with the fact that $\frac{\partial \Pr(\mu_2 < \mu_2^c|\mu_1)}{\partial \mu_1} < 0$, proves that the third term of the RHS of 22 is also positive.

Finally, in case (iii) $\Delta(\cdot) = (1-\delta)(\mu_1 - \tau)$, which is also increasing in μ_1. ■

Proof of Proposition 1

Recall that $\kappa_1 = \frac{1-\alpha}{\sigma_o}$, $\kappa_2 = \frac{1}{2}\beta\sigma_2$, $\sigma_2 = \sqrt{\sigma_p^2 + \sigma_\epsilon^2}$ and that $x(\mu_2^c, \mu_1) = \frac{\mu_2^c - \alpha\mu_1 - (1-\alpha)\bar{\Theta}}{\sigma_o}$. We first state and prove the following lemmas.

LEMMA 4. $E(\Theta|\mu_1, \mu_2 > \mu_2^c) = \mu_1 + \kappa_2 h(x(\mu_2^c, \mu_1))$.

Proof: It follows from 11 that $\mu_2 > \mu_2^c \Leftrightarrow s_2 > s_2^c$, where $s_2^c = \frac{\mu_2^c - (1-\alpha)\bar{\Theta}}{\alpha}$. Trivially, $E(\Theta|s_1, s_2) = E(\Theta|\mu_1, \mu_2)$. Hence, $E(\Theta|\mu_1, \mu_2 > \mu_2^c) = \int E(\Theta|s_1, s_2) f(s_2|s_1, s_2 >$

$s_2^c)ds_2$. Using 13, one has

$$(23) \qquad E(\Theta|s_1, s_2 > s_2^c) = \frac{\beta}{2}s_1 + (1-\beta)\bar{\Theta} + \frac{\beta}{2}\int s_2 f(s_2|s_1, s_2 > s_2^c)ds_2.$$

The integral in the equation above represents $E(s_2|s_1, s_2 > s_2^c)$. After 11, we argued that $s_2|s_1, s_2 > s_2^c$ is a left-truncated normal distribution with mean μ_1, variance σ_2^2 and truncation point s_2^c, and from Johnson et al. (1995), we know that

$$(24) \qquad E(s_2|s_1, s_2 > s_2^c) = \mu_1 + \frac{f(\frac{s_2^c - \mu_1}{\sigma_2})}{1 - F(\frac{s_2^c - \mu_1}{\sigma_2})}\sigma_2.$$

Replacing s_2^c by $\frac{\mu_2^c - (1-\alpha)\bar{\Theta}}{\alpha}$ and taking into account that $\sigma_o^2 = \alpha^2(\sigma_p^2 + \sigma_\epsilon^2) = \alpha^2\sigma_2^2$, allow us to rewrite 24 as $E(s_2|s_1, s_2 > s_2^c) = \mu_1 + h(x(\mu_2^c, \mu_1))\sigma_2$. Inserting this last equality into 23, and taking into account the fact that $\mu_1 = \alpha s_1 + (1-\alpha)\bar{\Theta}$, one has

$$E(\Theta|s_1, s_2 \geq s_2^c) = \frac{1}{2}\beta(1+\alpha)s_1 + \frac{1}{2}(2 - \beta(1+\alpha))\bar{\Theta} + \frac{1}{2}\beta h(x(\mu_2^c, \mu_1))\sigma_2.$$

It is easy to check that $\beta(1+\alpha) = 2\alpha$. Substituting this equality into our expectation proves the lemma. ■

Recall that

$$(25) \qquad \mathcal{X}(\eta) = \frac{\delta f(\eta)}{1 - \delta(1 - F(\eta))}.$$

LEMMA 5. *There exists a unique $\hat{\eta} < 0$ such that $\mathcal{X}(\hat{\eta}) = -\hat{\eta}$. $\mathcal{X}(\eta)$ increases until $\eta = \hat{\eta}$, after which it decreases. $\lim_{\eta \to -\infty} \mathcal{X}(\eta) = \lim_{\eta \to +\infty} \mathcal{X}(\eta) = 0$ and $\lim_{\eta \to -\infty} \mathcal{X}'(\eta) = \lim_{\eta \to +\infty} \mathcal{X}'(\eta) = 0$. $\mathcal{X}''(\eta) > 0$ if $\eta < \eta^m$ (where $\eta^m < \hat{\eta}$) and $\mathcal{X}''(\eta) < 0$ if $\eta \in (\eta^m, \hat{\eta})$. $\lim_{\delta \to 1} \hat{\eta} = -\infty$. $\lim_{\delta \to 0} \mathcal{X}'(\eta^m) = 0$ and $\lim_{\delta \to 1} \mathcal{X}'(\eta^m) = \infty$.*

Proof: Observe that $\mathcal{X}(\eta) > 0$ for $\delta > 0$. Hence, $\mathcal{X}(\eta) > -\eta$, $\forall \eta > 0$. Mere introspection of 25 reveals that for sufficiently low values of η, $\mathcal{X}(\eta) < -\eta$. By continuity, there exists at least one $\hat{\eta} < 0$ such that $\mathcal{X}(\hat{\eta}) = -\hat{\eta}$. Observe that the right hand side of the equality decreases in η and that

$$(26) \qquad \frac{\partial \mathcal{X}(\eta)}{\partial \eta} = \mathcal{X}'(\eta) = -\mathcal{X}(\eta)[\eta + \mathcal{X}(\eta)].$$

This slope is equal to zero if and only if $\mathcal{X}(\eta) = -\eta$. Hence, whenever $\mathcal{X}(\eta) = -\eta$, the right hand side of the equality strictly decreases in η, while its left hand side remains constant. As the slope of $\mathcal{X}(\eta)$ varies smoothly with changes in η, this implies that there is exactly one $\hat{\eta} < 0$ such that $\mathcal{X}(\hat{\eta}) = -\hat{\eta}$.

Note that if $\eta < \hat{\eta}$, $\mathcal{X}(\eta) < -\eta$, and $\mathcal{X}'(\eta) > 0$. Similarly, if $\eta > \hat{\eta}$, $\mathcal{X}'(\eta) < 0$. As the denominator of 25 is greater than $1 - \delta$ and as $\lim_{\eta \to +\infty} f(\eta) = \lim_{\eta \to -\infty} f(\eta) = 0$, one has: $\lim_{\eta \to -\infty} \mathcal{X}(\eta) = \lim_{\eta \to +\infty} \mathcal{X}(\eta) = 0$.

On the basis of 26, one has

$$\lim_{\eta \to \infty} \mathcal{X}'(\eta) = \lim_{\eta \to \infty} \mathcal{X}(\eta)(-\eta) - \lim_{\eta \to \infty} [\mathcal{X}(\eta)]^2.$$

Since $\lim_{\eta \to \infty} \mathcal{X}(\eta) = 0$, $\lim_{\eta \to \infty} [\mathcal{X}(\eta)]^2 = 0$. Observe also that

$$\mathcal{X}(\eta)(-\eta) = \frac{\delta f(\eta)(-\eta)}{1 - \delta(1 - F(\eta))} = \frac{\delta f'(\eta)}{1 - \delta(1 - F(\eta))}.$$

As $\lim_{\eta \to \infty} f'(\eta) = 0$ and as $\delta < 1$, $\lim_{\eta \to \infty} \mathcal{X}(\eta)(-\eta) = 0$. Hence, $\lim_{\eta \to \infty} \mathcal{X}'(\eta) = 0$. By the same reasoning, $\lim_{\eta \to -\infty} \mathcal{X}'(\eta)$ is zero.

Observe that

$$\mathcal{X}''(\eta) = -\eta \mathcal{X}'(\eta) - 2\mathcal{X}'(\eta)\mathcal{X}(\eta) - \mathcal{X}(\eta). \tag{27}$$

As $\lim_{\eta \to -\infty} \mathcal{X}(\eta) = \lim_{\eta \to -\infty} \mathcal{X}'(\eta) = 0$,

$$\lim_{\eta \to -\infty} \mathcal{X}''(\eta) = \lim_{\eta \to -\infty} -\eta \mathcal{X}'(\eta) \geq 0,$$

and for η sufficiently small $\mathcal{X}''(\eta) > 0$. As $\mathcal{X}'(\hat{\eta}) = 0$, it follows from 27 that $\mathcal{X}''(\hat{\eta}) < 0$. By continuity, there exists at least one $\eta^m \in (-\infty, \hat{\eta})$ such that $\mathcal{X}''(\eta^m) = 0$. Differentiating 27, and evaluating at the point $\eta = \eta^m$, one has

$$\mathcal{X}'''(\eta)\big|_{\eta = \eta^m} = -\mathcal{X}'(\eta^m) - 2\left(\mathcal{X}'(\eta^m)\right)^2 < 0,$$

where the inequality follows from the fact that $\mathcal{X}'(\eta^m) > 0$, as $\eta^m < \hat{\eta}$. We conclude that η^m is unique.

Recall that $\hat{\eta} < 0$. Suppose $\lim_{\delta\to 1} \mathcal{X}'(\hat{\eta}) = 0$ for some $\hat{\eta} \in (-\infty, 0)$. It follows from 26 that

$$\lim_{\delta\to 1} \mathcal{X}'(\hat{\eta}) = 0 \Leftrightarrow -\hat{\eta} = \lim_{\delta\to 1} \mathcal{X}(\hat{\eta}) = \frac{f(\hat{\eta})}{F(\hat{\eta})} = r(\hat{\eta}).$$

It is easy to check that $\frac{\partial r(\eta)}{\partial \eta} = -r(\eta)(r(\eta) + \eta)$. Hence, $r'(\hat{\eta}) = 0$. This, however, contradicts the fact that $\forall \eta \in (-\infty, \infty)$, $r'(\eta) < 0$ (see Greene, 1993, Theorem 22.2). Thus, $\lim_{\delta\to 1} \hat{\eta} = -\infty$.

Observe that $\lim_{\delta\to 0} \mathcal{X}(\eta) = 0\ \forall \eta$. Hence, $\lim_{\delta\to 0} \mathcal{X}'(\eta^m) = 0$.

As $\eta^m < \hat{\eta}$ and $\lim_{\delta\to 1} \hat{\eta} = -\infty$, $\lim_{\delta\to 1} \eta^m = -\infty$. Therefore, $\lim_{\delta\to 1} \mathcal{X}(\eta^m) = \frac{f(-\infty)}{F(-\infty)} = \infty$, where the last equality follows from l'Hôpital's rule. It follows from 27 that

$$\mathcal{X}''(\eta^m) = 0 \Leftrightarrow \eta^m = -\mathcal{X}(\eta^m) \left(\frac{1}{\mathcal{X}'(\eta^m)} + 2 \right). \tag{28}$$

Recall that $\mathcal{X}'(\eta^m) = -\mathcal{X}(\eta^m)[\eta^m + \mathcal{X}(\eta^m)]$. Replacing η^m on the right-hand side of this equality by the right-hand side of the last equality in 28, and rearranging, one has

$$\frac{[\mathcal{X}'(\eta^m)]^2}{1 + \mathcal{X}'(\eta^m)} = [\mathcal{X}(\eta^m)]^2. \tag{29}$$

As $\lim_{\delta\to 1} \mathcal{X}(\eta^m) = \infty$, $\lim_{\delta\to 1} [\mathcal{X}(\eta^m)]^2 = \infty$. Thus $\lim_{\delta\to 1} \frac{[\mathcal{X}'(\eta^m)]^2}{1+\mathcal{X}'(\eta^m)} = \infty$, which implies that $\lim_{\delta\to 1} \mathcal{X}'(\eta^m) = \infty$. ■

We are now ready to prove Proposition 1. Call LHS (RHS) the left-hand side (respectively right-hand side) of equation 5 after replacing μ^* by μ, and observe that

$$\frac{\partial LHS}{\partial \mu} = 1 \text{ and that } \frac{\partial RHS}{\partial \mu} = \kappa_1 \kappa_2 \mathcal{X}'(\kappa_1(\mu - \bar{\Theta})) = \frac{1-\alpha}{1+\alpha} \mathcal{X}'(\kappa_1(\mu - \bar{\Theta})),$$

where the last equality follows from the fact that $\frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$. From Lemma 5 we know that $\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta}))$ is maximal when $\kappa_1(\mu^* - \bar{\Theta}) = \eta^m$.[24] As $\mathcal{X}''(\cdot) > 0$ when $\kappa_1(\mu^* - \bar{\Theta}) < \eta^m$, as $\mathcal{X}''(\cdot) < 0$ when $\kappa_1(\mu^* - \bar{\Theta}) \in (\eta^m, \hat{\eta})$ and as $\mathcal{X}'(\cdot) < 0$ when $\kappa_1(\mu^* - \bar{\Theta}) > \hat{\eta}$, it follows that $\forall \bar{\Theta}$, there exists a unique equilibrium in symmetric switching strategies if and only if

$$\left.\frac{\partial RHS}{\partial \mu}\right|_{\mu=\mu^*=\frac{\eta^m}{\kappa_1}+\bar{\Theta}} \leq \left.\frac{\partial LHS}{\partial \mu}\right|_{\mu=\mu^*=\frac{\eta^m}{\kappa_1}+\bar{\Theta}} \Leftrightarrow \mathcal{X}'(\eta^m) \leq \frac{1+\alpha}{1-\alpha}.$$

[24] As a unit increase in $\bar{\Theta}$ leads to a translation of $\mathcal{X}(\cdot)$ to the right by one unit (as shown in Figure 1), it follows that there exists a unique $\bar{\Theta}$ such that $\kappa_1(\mu^* - \bar{\Theta}) = \eta^m$.

We know from Lemma 5 that $\lim_{\eta^m \to \infty} \mathcal{X}'(\eta^m) = \lim_{\eta^m \to -\infty} \mathcal{X}'(\eta^m) = 0$. Moreover, mere observation of 25 also reveals that $\mathcal{X}(\eta^m) < \infty$ if $\delta < 1$. Hence, $\mathcal{X}'(\eta^m) = -\mathcal{X}(\eta^m)(\eta^m + \mathcal{X}(\eta^m))$ is finite whenever $\delta < 1$. Observe that $\lim_{\sigma^2_\Theta \to \infty} \alpha = \lim_{\sigma^2_\epsilon \to 0} \alpha = 1$, which, combined with our finding that $\mathcal{X}'(\eta^m)$ is finite, implies that $\lim_{\sigma^2_\Theta \to \infty} \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m)$ $= \lim_{\sigma^2_\epsilon \to 0} \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m) = 0$. By continuity, there exists a $(\sigma^2_\Theta)^c < \infty$ ($(\sigma^2_\epsilon)^c > 0$) such that $\forall \sigma^2_\Theta > (\sigma^2_\Theta)^c$ ($\forall \sigma^2_\epsilon < (\sigma^2_\epsilon)^c$), $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m) \leq 1$. This establishes claims (1) and (2) of the proposition.

Since $\mathcal{X}(\eta^m) = 0$ when $\delta = 0$, one has $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m)\Big|_{\delta=0} = 0$. By continuity, there exists a $\delta^c \in (0,1]$ such that $\forall \delta \leq \delta^c$, $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\eta^m) \leq 1$. This establishes claim (3) of the proposition.

Recall that

$$g(\mu) = \mu - \tau - \kappa_2 \mathcal{X}(\kappa_1(\mu - \bar{\Theta})), \tag{30}$$

and observe that equilibrium condition 5 is equivalent to $g(\mu^*) = 0$. If $\mu < \tau$, $g(\mu) < 0$. Thus, $\mu^* > \tau$. Hence, if $\bar{\Theta} \leq \underline{\tau}$, $\kappa_1(\mu^* - \bar{\Theta}) > 0$, $\forall \tau$. It then follows from Lemma 5 that $\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) < 0$. This establishes claim (4) of the proposition.

Suppose that if player i waits, she perfectly learns the state of the world, which gives an upper bound on the value of learning. Player i's gain of waiting then equals $\delta \Pr(\Theta > \tau|\mu_i)E(\Theta|\mu_i, \Theta > \tau)$. Observe that for high enough a μ_i, $E(\Theta|\mu_i, \Theta > \tau) \approx E(\Theta|\mu_i) = \mu_i$. As $\delta < 1$, there exists a $\bar{\mu} < \infty$ such that $\bar{\mu} = \delta \Pr(\Theta > \tau|\bar{\mu})E(\Theta|\bar{\mu}, \Theta > \tau)$. If $\mu > \bar{\mu}$ player i strictly prefers to invest at time one. Hence, $\mu^* < \bar{\mu} < \infty$. As $\mu^* \in (\tau, \bar{\mu})$, $\kappa_1(\mu^* - \bar{\Theta}) \to -\infty$, as $\bar{\Theta} \to \infty$. It then follows from Lemma 5 that $\lim_{\bar{\Theta} \to \infty} \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) = 0$. By continuity, there exists a $\bar{\Theta}_u$ such that if $\bar{\Theta} \geq \bar{\Theta}_u$, $\frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) \leq 1$. This establishes claim (5) of the proposition.■

Proof of Lemma 2.

$W(\tau)$ denotes welfare when the investment tax equals τ. Note that $W(0)$ is bounded below by

$$\int_0^\infty \mu_i f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_{\mu_i}}\right) d\mu_i \equiv \underline{W}(0).$$

Intuitively, $\underline{W}(0)$ represents welfare when $\tau = 0$ and when players cannot engage in social learning (i.e. when players invest if their posterior mean exceed zero). Observe also that $W(\tau)$ is bounded above by the welfare that prevails when players observe the state of the world prior to the first investment date. In that case, welfare equals

$$\int_{\tau}^{\infty} \Theta f\left(\frac{\Theta - \bar{\Theta}}{\sigma_\Theta}\right) d\Theta \equiv \overline{W}(\tau),$$

which is decreasing in τ when $\tau > 0$. As $\overline{W}(0) > \underline{W}(0) > \overline{W}(\infty) = 0$, by continuity there exists a unique $\bar{\tau} > 0$ such that $\overline{W}(\bar{\tau}) = \underline{W}(0)$. As $W(\bar{\tau}) < \overline{W}(\bar{\tau}) = \underline{W}(0) < W(0)$, this implies that $\forall \tau \geq \bar{\tau}$, a social planner can increase welfare by setting the investment tax to zero.

Furthermore, as

$$\bar{\Theta} = \int_{-\infty}^{\infty} \mu_i f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_{\mu_i}}\right) d\mu_i < \underline{W}(0),$$

by an identical argument as above there exists a unique $\underline{\tau} < 0$ such that $\int_{\underline{\tau}}^{\infty} \Theta f\left(\frac{\Theta - \bar{\Theta}}{\sigma_\Theta}\right) d\Theta = \underline{W}(0)$. $\forall \tau \leq \underline{\tau}$, a social planner can increase welfare by setting the investment tax to zero. ■

Proof of Proposition 2.

We first state and prove the following lemmas.

Lemma 6. *For any first-period cutoff μ^*, there exists a unique-second period cutoff $\underline{\mu}$.*

Proof: Recall that

$$\phi(\mu) = \mu - \tau + \kappa_2 h\left(x(\mu^*, \mu)\right). \tag{31}$$

At the second-period cutoff $E(\Theta | \underline{\mu}, \mu_j > \mu^*) = \tau$, which is equivalent to $\phi(\underline{\mu}) = 0$. Differentiating and using that $\sigma_o = \alpha\sigma_2$ and that $\beta\sigma_2 = 2\kappa_2$, shows that

$$\frac{\partial \phi(\mu)}{\partial \mu} = 1 - \frac{\beta}{2} h'\left(x(\mu^*, \mu)\right) > 0$$

since $h' \in (0, 1)$ (see Theorem 22.2 (Greene, 1993)) and $\beta < 1$. Hence, if there exists a solution, it is unique. We are left to establish that a solution exists. First, observe that

$\lim_{\mu \to \infty} \phi(\mu) > 0$. Second, note that

$$\lim_{\mu \to -\infty} \left[\mu - \tau + \kappa_2 h\left(x(\mu^*, \mu)\right)\right] < 0,$$

is equivalent to

$$\lim_{\mu \to -\infty} \left[\frac{\mu - \tau + \kappa_2 h\left(x(\mu^*, \mu)\right)}{\mu}\right] > 0,$$

which by l'Hôpital's rule is equivalent to

$$\lim_{\mu \to -\infty} \left[1 - \frac{\beta}{2} h'\left(x(\mu^*, \mu)\right)\right] > 0.$$

Since $h' \in (0, 1)$ and $\beta < 1$ this holds, which establishes the existence of $\underline{\mu}$. ■

Lemma 7. *Suppose the symmetric switching equilibrium is unique. Then* $\frac{d\mu^*}{d\tau} > 0$. *Furthermore, there exists* $\bar{\Theta}^{c_1}, \bar{\Theta}^{c_2} \in \mathbb{R}$ *such that* $\frac{d\underline{\mu}}{d\tau} > 0$ *for all* $\bar{\Theta} \notin [\bar{\Theta}^{c_1}, \bar{\Theta}^{c_2}]$.

Proof: As stated above, equation 31 implicitly defines $\underline{\mu}$ as $\phi(\underline{\mu}) = 0$. It is important to realize however that the image of ϕ also depends on μ^*. A change in τ thus affects $\phi(\cdot)$ directly and indirectly through the induced change in μ^*. It therefore follows from the implicit function theorem that

$$\text{(32)} \qquad \frac{d\underline{\mu}}{d\tau} = \frac{-\frac{\partial \phi}{\partial \mu^*}\frac{d\mu^*}{d\tau} - \frac{\partial \phi}{\partial \tau}}{\frac{\partial \phi}{\partial \underline{\mu}}}.$$

Applying the implicit function theorem to equation 30, one has

$$\frac{d\mu^*}{d\tau} = -\frac{\frac{\partial g}{\partial \tau}}{\frac{\partial g}{\partial \mu^*}} = \frac{1}{1 - \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta}))}.$$

As the symmetric switching equilibrium is unique, the denominator is positive, which proves the first result stated in the lemma. After taking the appropriate partial derivatives of the ϕ-function, and taking into account the fact that $\sigma_o = \alpha\sigma_2$ and that $\frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$, one has

$$\text{(33)} \qquad \frac{d\underline{\mu}}{d\tau} = \frac{-\frac{1}{1+\alpha} h'\left(x(\mu^*, \underline{\mu})\right) \frac{1}{1 - \frac{1-\alpha}{1+\alpha}\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta}))} + 1}{1 - \frac{1}{2}\beta h'\left(x(\mu^*, \underline{\mu})\right)}.$$

Observe that the denominator is positive since $\beta < 1$ and $h' \in (0, 1)$. Hence $\frac{d\underline{\mu}}{d\tau} > 0$ if

$$1 + \alpha - (1 - \alpha)\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) > h'\left(x(\mu^*, \underline{\mu})\right).$$

Since $\mu^* \in (\tau, \bar{\mu})$ and both τ and $\bar{\Theta}$ are finite, Lemma 5 implies that $\lim_{\bar{\Theta}\to-\infty} \mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) = \lim_{\bar{\Theta}\to\infty} \mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) = 0$. Since $h' < 1$ the above inequality is therefore satisfied in both limit cases and the second statement of the lemma follows from the fact that $\underline{\mu}$ is continuous in $\bar{\Theta}$. ■

Recall that $\tilde{\mu} = (1-\alpha)\bar{\Theta} - \mu^*$ and that $x(\mu^*, \mu_i) = \frac{\mu^* - \alpha\mu_i - (1-\alpha)\bar{\Theta}}{\sigma_o}$.

LEMMA 8. *For any $\tau > 0$, there exists a $\bar{\Theta}^{c_3}(\tau)$ such that $\tilde{\mu} < \underline{\mu}$ for all $\forall\bar{\Theta} \le \bar{\Theta}^{c_3}(\tau)$.*

Proof: Rewriting $x(\mu^*, \tilde{\mu})$ using $\kappa_2 = \frac{1}{2}\beta\sigma_2$ and $\frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$ verifies that $\tilde{\mu} = -\kappa_2 x(\mu^*, \tilde{\mu})$. Furthermore, using the definition of ϕ in equation 31 and that $\underline{\mu}$ is implicitly defined through $\phi(\underline{\mu}) = 0$, one has $\underline{\mu} = \tau - \kappa_2 h(x(\mu^*, \underline{\mu}))$. Therefore,

$$\underline{\mu} - \tilde{\mu} = \tau - \kappa_2 \left(h(x(\mu^*, \underline{\mu})) - x(\mu^*, \tilde{\mu})\right). \tag{34}$$

Furthermore,

$$x(\mu^*, \tilde{\mu}) = x(\mu^*, \underline{\mu}) + \frac{\underline{\mu} - \tilde{\mu}}{\sigma_2}. \tag{35}$$

Inserting 35 into 34, and rearranging, yields

$$(\underline{\mu} - \tilde{\mu})\left(1 - \frac{1}{2}\beta\right) = \tau - \kappa_2\left(h(x(\mu^*, \underline{\mu})) - x(\mu^*, \underline{\mu})\right).$$

Recall that $h'(\eta) = h(\eta)[h(\eta) - \eta]$, that $h'(\eta) \in (0, 1)$, and that $\lim_{\eta\to\infty} h(\eta) = \infty$. Hence, $h(\eta) > \eta$ and $\lim_{\eta\to\infty}(h(\eta) - \eta) = 0$. Since $\underline{\mu} < \tau$, $\lim_{\bar{\Theta}\to-\infty} x(\mu^*, \underline{\mu}) = \infty$, which implies that

$$\lim_{\bar{\Theta}\to-\infty} -\kappa_2\left(h(x(\mu^*, \underline{\mu})) - x(\mu^*, \underline{\mu})\right) = 0.$$

As $1 - \frac{1}{2}\beta > 0$, this implies that $\lim_{\bar{\Theta}\to-\infty}(\underline{\mu} - \tilde{\mu}) = \tau / \left(1 - \frac{1}{2}\beta\right) > 0$. The lemma then follows from the fact that both $\underline{\mu}$ and $\tilde{\mu}$ are continuous in $\bar{\Theta}$. ■

We are now ready to prove the first statement of the proposition. Using $\sigma_o = \alpha\sigma_2$, equation 3, and Lemma 4, one has

$$\frac{\partial}{\partial\mu^*}[\Pr(\mu_j > \mu^*|\mu_i)E(\Theta|\mu_i, \mu_j > \mu^*)] = \frac{\partial}{\partial\mu^*}\left[(1 - F(x(\mu^*, \mu_i)))(\mu_i + \kappa_2 h(x(\mu^*, \mu_i)))\right]$$

$$= -f(x(\mu^*, \mu_i))\frac{1}{\alpha\sigma_2}(\mu_i + \kappa_2 h(x(\mu^*, \mu_i))) + \frac{\beta}{2\alpha}h'(x(\mu^*, \mu_i))(1 - F(x(\mu^*, \mu_i))).$$

Note that $h'(\eta)(1-F(\eta)) = f(\eta)(h(\eta)-\eta)$ and recall that $\kappa_2 = \frac{1}{2}\beta\sigma_2$ and that $1-\frac{\beta}{2} = \frac{\beta}{2\alpha}$. Hence,

(36)
$$\frac{\partial}{\partial\mu^*}[\Pr(\mu_j > \mu^*|\mu_i)E(\Theta|\mu_i, \mu_j > \mu^*)] = -f(x(\mu^*, \mu_i))\frac{\beta}{2\alpha^2\sigma_2}\left(\mu_i + \mu^* - (1-\alpha)\bar{\Theta}\right),$$

which implies that

$$\int_{\underline{\mu}}^{\mu^*} \frac{\partial}{\partial\mu^*}\left[\Pr(\mu_j > \mu^*|\mu_i)E(\Theta|\mu_i, \mu_j > \mu^*)\right] f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_\mu}\right) > 0, \tag{37}$$

if $(1-\alpha)\bar{\Theta} - \mu^* - \mu_i > 0$ for all $\mu_i \in [\underline{\mu}, \mu^*]$, which in turn holds if $\bar{\Theta} > \frac{2}{1-\alpha}\mu^*$. It follows from the proof of Proposition 1 that $\lim_{\bar{\Theta}\to\infty}\mu^* = \tau < \infty$. As μ^* is continuous in $\bar{\Theta}$, for sufficiently high $\bar{\Theta}$, 37 holds. Since by Lemma 7 μ^* and $\underline{\mu}$ are increasing in τ for sufficiently high $\bar{\Theta}$, inequality 37 implies that equation 7 is positive, i.e. that $\frac{dW}{d\tau} > 0$ for all $\tau \leq 0$. Hence for sufficiently high $\bar{\Theta}$ it is optimal to tax investment.

We now prove the second statement of the proposition. Fix $\tau' > 0$. Furthermore, observe that

$$\int_{\underline{\mu}}^{\mu^*} \frac{\partial}{\partial\mu^*}\left[\Pr(\mu_j > \mu^*|\mu_i)E(\Theta|\mu_i, \mu_j > \mu^*)\right] f\left(\frac{\mu_i - \bar{\Theta}}{\sigma_\mu}\right) < 0, \tag{38}$$

if $(1-\alpha)\bar{\Theta} - \mu^* - \mu_i < 0$ for all $\mu_i \in [\underline{\mu}, \mu^*]$, which in turn holds if $\tilde{\mu} < \underline{\mu}$. Because for sufficiently low $\bar{\Theta}$ Lemma 8 implies that $\tilde{\mu} < \underline{\mu}$ and Lemma 7 implies that both μ^* and $\underline{\mu}$ are increasing in τ, inspection of equation 7 shows that $\frac{dW}{d\tau}\mid_{\tau=\tau'} < 0$. Hence, any positive tax level is suboptimal for sufficiently low $\bar{\Theta}$. ■

Proof of Proposition 3

We first show that there exist values of our exogenous parameters such that equilibrium is unique and such that either $\frac{d\underline{\mu}}{d\tau} < 0$ or $\frac{d\underline{\mu}}{d\tau} > 0$ (Step 1). Next, we show that if δ is close to one or if $\delta = 0$, $\frac{d\mu^*}{d\tau} \leq 1$ (Step 2). Finally, we prove that the second step implies that $\frac{d\underline{\mu}}{d\tau} > 0$ (Step 3).

Step 1: It follows from 33 that $\frac{d\underline{\mu}}{d\tau} < 0$ and that equilibrium is unique if and only if

$$0 \leq 1 + \alpha - (1-\alpha)\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) < h'\left(x(\mu^*, \underline{\mu})\right). \tag{39}$$

(The first inequality ensures uniqueness while the second one ensures that $\frac{d\mu}{d\tau} < 0$.) From Lemma 5 we know that $\lim_{\delta\to 0} \mathcal{X}'(\cdot) = 0$. As $h'(\cdot) < 1$, this implies that the second inequality is not satisfied when δ is sufficiently low. Furthermore, Lemma 7 proves that the second inequality is also not satisfied when $\bar{\Theta}$ is either greater than $\bar{\Theta}^{c_1}$ or lower than $\bar{\Theta}^{c_2}$. From Lemma 5 we also know that $\lim_{\delta\to 1} \mathcal{X}'(\eta^m) = \infty$. As $\mathcal{X}'(\cdot)$ is continuous in $(\bar{\Theta}, \delta)$, and as $\lim_{\delta\to 0} \mathcal{X}'(\cdot) = 0$, there exists a $(\bar{\Theta}', \delta')$ such that $\mathcal{X}'(\kappa_1(\mu^* - \bar{\Theta})) = \frac{1+\alpha}{1-\alpha}$. As $h'(\cdot) > 0$, it then follows that inequality 39 is respected whenever $(\bar{\Theta}, \delta) = (\bar{\Theta}', \delta')$. As $h'(\cdot)$ is also continuous in $(\bar{\Theta}, \delta)$, there exist values of $(\bar{\Theta}, \delta)$ close to $(\bar{\Theta}', \delta')$, which ensure that the two inequalities of 39 are satisfied.

Step 2: In the body of the text, we explained why $\left.\frac{d\mu^*}{d\tau}\right|_{\delta=0} = 1$. We now show that $\lim_{\delta\to 1} \frac{d\mu^*}{d\tau} < 1$. It follows from equation 21 that

$$\lim_{\delta\to 1} E(\Theta|\mu^*, \mu_j < \mu_j^*) = \tau. \tag{40}$$

Inserting 19 into $E(\Theta|s_1, s_2 < s_2^c) = \frac{\beta}{2}s_1 + (1-\beta)\bar{\Theta} + \frac{\beta}{2}E(s_2|s_1, s_2 < s_2^c)$, taking into account that $\beta(1+\alpha) = 2\alpha$, and rearranging, one has

$$E(\Theta|\mu_1, \mu_2 < \mu_2^c) = \mu_1 - \kappa_2 r\left(x(\mu_2^c, \mu_1)\right). \tag{41}$$

It then follows from 40 and 41 that, for δ close to one,

$$\mu^* - \kappa_2 r(\kappa_1(\mu^* - \bar{\Theta})) = \tau.$$

Call LHS, the left-hand side of the above equality after replacing μ^* by μ. One has $\frac{\partial LHS}{\partial \mu} = 1 - \kappa_1\kappa_2 r'(\cdot) > 1$ as $r'(\cdot) < 0$. This implies that if the right-hand side of the equality above increases by $d\tau$, the above equality is satisfied only if μ^* increases by less than $d\tau$.

Step 3: Consider equation 32. As $\frac{\partial\phi}{\partial\mu^*} > 0$, it follows from Step 2 that if $\delta = 0$ or if $\delta \to 1$, $\frac{d\mu}{d\tau}$ is bounded below by $\frac{-\frac{\partial\phi}{\partial\mu^*} - \frac{\partial\phi}{\partial\tau}}{\frac{\partial\phi}{\partial\mu}}$. As argued below 33, the denominator of this lower bound is positive. Hence, $\frac{d\mu}{d\tau} > 0$, if the numerator is. Taking the appropriate derivatives of our ϕ-function, one has

$$-\frac{\partial\phi}{\partial\mu^*} - \frac{\partial\phi}{\partial\tau} > 0 \Leftrightarrow 1 + \alpha > h'\left(x(\mu^*, \underline{\mu})\right),$$

which is satisfied as $h'(\cdot) < 1$. As $\frac{d\mu^*}{d\tau}$ is continuous in δ, it then follows from equation 32 that $\frac{d\mu}{d\tau}$ remains strictly positive for δ sufficiently close to one or δ sufficiently close to zero.■

Proof of Proposition 4

We first show that $\mu_j^c - \alpha\mu_i^I(\mu_j^c)$ is increasing in μ_j^c (Step 1). Next, we show that this property implies that μ_i^I initially increases in μ_j^c and then decreases (Step 2). Finally, in Step 3, we prove that $\frac{d\mu_i^I}{d\mu_j^c}\Big|_{\mu_i=\mu_j^c=\mu^*} > 0 \Leftrightarrow \frac{\partial\mathcal{X}(\kappa_1(\mu-\bar{\Theta}))}{\partial\mu}\Big|_{\mu_i=\mu_j^c=\mu^*} > 0$. Those three steps, together with the explanations provided in the text, prove the first sentence of the proposition. The third step, combined with (i) the fact that $\mathcal{X}$ is initially increasing, (ii) that an increase in $\bar{\Theta}$ shifts $\mathcal{X}$ to the right (as illustrated in Figure 1), and (iii) that the optimal tax rate is finite (as stated in Lemma 2), implies the second sentence of the proposition.

Step 1: Observe that $\frac{d}{d\mu_j^c}[\mu_j^c - \alpha\mu_i^I(\mu_j^c)] > 0 \Leftrightarrow \frac{d\mu_i^I}{d\mu_j^c} < \frac{1}{\alpha}$. Furthermore, Lemma 4 and the explanations following 12 allow us to conclude that

$$\Delta(\mu_i^I, \mu_j^c) = \mu_i^I - \tau - \delta[1 - F(x(\mu_j^c, \mu_i^I))]\left[\mu_i^I + \kappa_2 h(x(\mu_j^c, \mu_i^I)) - \tau\right].$$

Recall that $x(\mu_j^c, \mu_i^I) = \frac{\mu_j^c - \alpha\mu_i^I - (1-\alpha)\bar{\Theta}}{\sigma_o}$. Define

$$\gamma(\mu_i^I, \mu_j^c) \equiv \mu_i^I - \tau - \kappa_2\mathcal{X}(x(\mu_j^c, \mu_i^I)) = \frac{\Delta(\mu_i^I, \mu_j^c)}{1 - \delta(1 - F(x(\mu_j^c, \mu_i^I)))}.$$

By definition of μ_i^I, $\Delta(\mu_i^I, \mu_j^c) = 0$ and hence $\gamma(\mu_i^I, \mu_j^c) = 0$. Thus, the implicit function theorem implies

$$\frac{d\mu_i^I}{d\mu_j^c} = -\frac{\frac{\partial\gamma}{\partial\mu_j^c}}{\frac{\partial\gamma}{\partial\mu_i^I}} = \frac{\kappa_2\frac{1}{\sigma_o}\mathcal{X}'(x(\mu_j^c, \mu_i^I))}{1 + \kappa_2\frac{\alpha}{\sigma_o}\mathcal{X}'(x(\mu_j^c, \mu_i^I))} < \frac{1}{\alpha}. \tag{42}$$

Step 2: It follows from Lemma 1 that $\frac{\partial\gamma}{\partial\mu_i}\Big|_{\mu_i=\mu_i^I} > 0$. The equalities presented in 42 then allow us to conclude that

$$\frac{d\mu_i^I}{d\mu_j^c} > 0 \quad \Leftrightarrow \quad \frac{\partial\gamma}{\partial\mu_j^c}\Big|_{\mu_i=\mu_i^I} < 0 \quad \Leftrightarrow \quad -\frac{1}{\sigma_o}\kappa_2\mathcal{X}'(x(\mu_j^c, \mu_i^I)) < 0 \quad \Leftrightarrow \quad \mathcal{X}'(x(\mu_j^c, \mu_i^I)) > 0.$$

From Step 1 we know that x is increasing in μ_j^c. It then follows from Lemma 5 that μ_i^I initially increases and then decreases in μ_j^c.

Step 3: It follows from 42 and from Step 2 that

$$\left.\frac{d\mu_i^I}{d\mu_j^c}\right|_{\mu_i=\mu_j^c=\mu^*} > 0 \quad \Leftrightarrow \quad \left.\frac{\partial \mathcal{X}(x(\mu_j^c,\mu_i^I))}{\partial x(\mu_j^c,\mu_i^I)}\right|_{\mu_i=\mu_j^c=\mu^*} > 0 \quad \Leftrightarrow \quad \left.\frac{\partial \mathcal{X}(\kappa_1(\mu-\bar{\Theta}))}{\partial \mu}\right|_{\mu_i=\mu_j^c=\mu^*} > 0,$$

where the last equivalence follows from the fact that $\kappa_1 > 0$. ■

Proof of Proposition 5

We first state and prove the following lemma.

LEMMA 9. *Suppose all players invest at time one if and only if their posterior is greater than μ^c. Then, $F(n|\mu)$ is increasing in μ in the sense of first order stochastic dominance.*

Proof: Suppose $\mu' > \mu$. Define $q \equiv \Pr(\mu_j \geq \mu^c|\mu)$ and $q' \equiv \Pr(\mu_j \geq \mu^c|\mu')$. As $\mu' > \mu$, $q' > q$. Observe that

$$\Pr(n|\mu) = C_{N-1}^n q^n (1-q)^{N-1-n},$$

$$\Pr(n|\mu') = C_{N-1}^n (q')^n (1-(q'))^{N-1-n}.$$

Observe also that $\Pr(0|\mu) = (1-q)^{N-1} > (1-q')^{N-1} = \Pr(0|\mu')$. We now show that $\exists\ a \in N^+$ such that $\forall n < a$, $\Pr(n|\mu) \geq \Pr(n|\mu')$ and $\forall n \geq a$, $\Pr(n|\mu) < \Pr(n|\mu')$. Suppose this were not true. As $\Pr(0|\mu) > \Pr(0|\mu')$, it follows then that $\exists\ q$, q' and nonnegative integers $z > y > x$ such that

$$q^z(1-q)^{N-1-z} > (q')^z(1-(q'))^{N-1-z}, \tag{43}$$

$$q^y(1-q)^{N-1-y} < (q')^y(1-(q'))^{N-1-y}, \tag{44}$$

$$q^x(1-q)^{N-1-x} > (q')^x(1-(q'))^{N-1-x}. \tag{45}$$

Taking logarithms, subtracting equation 44 from equation 43, and rewriting gives

$$\log q - \log q' > \log(1-q) - \log(1-q').$$

Taking logarithms, subtracting equation 44 from equation 45, and rewriting gives

$$\log q - \log q' < \log(1-q) - \log(1-q'),$$

a contradiction. Hence, $\Pr(n|\mu)$ and $\Pr(n|\mu')$ possess the single crossing property. As is well known, this proves the Lemma. ■

As $E(\Theta|\mu_i, n)$ is increasing in n, we can rewrite 10 as

$$\begin{aligned}\Delta(\mu_i, \mu^c) &= (1-\delta) \sum_{n \geq n_c} \Pr(n|\mu_i) \Big(E(\Theta|\mu_i, n) - \tau \Big) \\ &+ \sum_{n < n_c} \Pr(n|\mu_i) \Big(E(\Theta|\mu_i, n) - \tau \Big),\end{aligned}$$

where n_c is defined as the lowest integer such that $E(\Theta|\mu_i, n) \geq \tau$. If $E(\Theta|\mu_i, 0) \geq \tau$, $\Delta(\mu_i, \mu^c) = \mu_i - \tau$, which is strictly increasing in μ_i. A similar remark applies when $E(\Theta|\mu_i, N-1) < \tau$. Thus, suppose that $0 < n_c \leq N-1$. From Lemma 9 we know that an increase in μ_i puts more weight on higher realizations of n (and, thus, on higher values of $E(\Theta|\mu, n)$). This observation, combined with the fact that $E(\Theta|\mu_i, n)$ is strictly increasing in μ_i, proves that $\Delta(\mu_i, \mu^c)$, is strictly increasing in μ_i. Hence, any symmetric candidate equilibrium cutoff μ^* for which $\Delta(\mu^*, \mu^*) = 0$ gives rise to a symmetric switching equilibrium. We have established in the text that there is a unique such candidate equilibrium, which completes the proof of Proposition 5.■

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