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Abstract

We study multi-object auctions where agents have private and additive valuations for heterogeneous objects. We focus on the revenue properties of a class of dominant strategy mechanisms where a weight is assigned to each partition of objects. The weights influence the probability with which partitions are chosen in the mechanism. This class contains efficient auctions, pure bundling auctions, mixed bundling auctions, auctions with reserve prices and auctions with prepackaged bundles. For any number of objects and bidders, both the pure bundling auction and separate, efficient auctions for the single objects are revenue-inferior to an auction that involves mixed bundling.

1 Introduction

Very little is known about the revenue maximizing auction for multiple, heterogeneous objects, and we doubt that the problem is analytically tractable. It is well known that reserve prices may increase expected revenue by inefficiently withholding the objects. An important and different insight is that an optimal auction will not necessarily allocate the objects to the buyers who value them most[1]. Palfrey (1983) introduced the study of *bundling* to the auction literature, and showed that a *pure bundling auction* is revenue

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[1]This, in principle, leaves scope for a resale market. As we want to focus on the role of bundling, we assume here that the resale-market is absent.

superior to separate (and efficient) auctions if there are only two bidders[2]. Under pure bundling, a single bidder gets all objects if she has the highest value for the entire bundle, although other bidders attach higher values to parts of the bundle. Under some assumptions on the distribution of valuations, Chakraborty (1999) showed that there is a critical number of bidders, below which bundling outperforms separate auctions, and above which separate auctions outperform the bundling auction. This analysis suggests that pure bundling may increase revenue by increasing competition (i.e., by "importing" demand from one object to another), but that it may also decrease revenue by creating inefficiencies. The logic behind Chakraborty's result is that, with many bidders, the inefficiency effect outweighs the competition effect.

The main purpose of this paper is to construct and analyze the properties of a class of *mixed bundling auctions* that improve upon both pure bundling and upon separate, efficient auctions. Moreover, we offer a methodology that can be used to identify the most promising bundling strategy.

In our model, valuations for bundles are equal to the sum of the valuations for the included objects, so that bundling is solely driven by strategic considerations of the auctioneer rather than by technological features of the demand functions (such as complementarities).

The constructed mechanisms assign to each partition of objects among the agents a partition-specific weight. For each realization of values, the implemented partition is the one that maximizes the sum of the bidders' values and the assigned weight. Thus, our mechanisms can be seen as weighted Vickrey-Clarke-Groves mechanisms, and it is a dominant strategy for bidders to reveal their true values. For example, assigning a positive weight to all partitions where some bundle of objects is not sold corresponds to introducing a reserve price on this bundle. Assigning a large positive weight to all partitions where the objects are sold to a single buyer corresponds to a pure bundling auction, while a smaller weight on such partitions will sometimes allow for separate sales, thus yielding mixed bundling.

The advantages of mixed bundling have been previously pointed out in the literature on monopolistic pricing. One might think that the optimal pricing strategy for a monopolist selling two goods (for which customers' valuations are known to be additive) is to price each good at the price a single-product monopolist would optimally charge. Adams and Yellen (1976) show (via examples) that this is not necessarily the case. McAfee et al. (1989) generalize the non-optimality of additive pricing to arbitrarily distributed

[2] Jehiel and Moldovanu (2001) exploit Palfrey's results from the perspective of the revenue equivalence theorem.

valuations, and suggest a class of situations in which the optimal prices include a discount for bundles - this is referred to as mixed bundling by Adams and Yellen[3]. Manelli and Vincent (2001) determine necessary conditions for the optimal monopolistic prices for each bundle of objects, and study how non-deterministic sale mechanisms may improve upon deterministic ones. Armstrong (1996) considers a more general monopoly model where objects are divisible, and he looks for optimal tariffs. His main result is that the seller will always want to withhold sales from some low-type buyers, no matter how profitable selling some quantity to them might be. We can apply Armstrong's argument to our auction model in order to show that the revenue of both pure and mixed bundling auctions can be increased by introducing reserve prices. Rochet and Choné (1998) further analyze the multi-product monopoly problem (by relaxing some of the assumptions used by Armstrong), and show how bunching (i.e. different types get the same allocation) is a robust feature of the optimal mechanism.

Since monopoly pricing can be viewed as an auction design with a unique potential bidder, one contribution of this paper is to generalize the idea of mixed bundling from monopoly pricing to the more involved theory of auctions. For example, the proof of Proposition 4 parallels an insight obtained by McAfee et al. (1989).

Most of the auction literature (and, in particular, that literature searching for optimal mechanisms) has studied models where each bidder's information can be represented by a one-dimensional signal. Most prominently, Myerson (1981) and Riley and Samuelson (1981) find the revenue-maximizing auction[4] (in the class of Bayes-Nash implementable allocation rules) for the private values, one-object case. If bidders are ex-ante symmetric, the optimal auction is a second-price auction with a reserve price. Hence this mechanism is implementable in dominant strategies, and, as mentioned above, belongs to the class of mechanisms discussed here.

Models with one-dimensional signals are of limited use for studies of auctions for multiple, heterogeneous objects. Jehiel, Moldovanu and Stacchetti (1999) have shown that the main difficulty with multi-dimensional signals is that incentive compatibility constraints[5] yield a complex integrability condition on the vector of interim expected probabilities with which buyers get

[3] In other cases the optimal prices contain a surcharge for the bundle instead of a discount. This raises the problem of deterring customers from buying the objects separately in order to evade the surcharge.

[4] Bulow and Roberts (1989) illustrate the parallels between optimal auction design and monopolistic pricing for one object.

[5] These constraints reduce to a monotonicity condition when private information is one-dimensional.

the various bundles[6]. Thus, under this constraint the maximization problem is very hard - this is the main obstacle in front of a characterization of the revenue-maximizing multi-object auction. Instead of trying to characterize such an auction, we focus here on a large class of mechanisms that includes pure and mixed bundling, as well as separate, efficient auctions. A major practical advantage is that revenue maximization in this class reduces to an optimization problem over vectors of real numbers (the weights attached to the partitions) that can be numerically solved by any personal computer.

Armstrong (2000) and Avery and Hendershott (2000) study 2-object auctions in discrete settings. In particular, Armstrong finds that the revenue maximizing auction is efficient. The discrepancy to our results stems from Armstrong's assumption that valuations are binary, i.e. they can either be "high" or "low", but not "intermediate". This leaves no scope for mixed bundling, as analyzed in this paper[7].

As the focus of this paper is to identify and describe a tractable way of improving auction revenues rather than to characterize the optimal auction, we do not consider here how Bayes-Nash implementation might improve revenues over dominant strategy implementation. Mookerjee and Reichelstein (1992) describe a class of mechanism design problems (including the quest for a revenue-maximizing one object auction) where using the stronger requirement of dominant strategies is without loss of generality. Their result does not hold for general models with multidimensional signals, and very little is known about this issue in such contexts.

The paper is organized as follows: Section 2 lays out the model, and introduces the weighted Clarke-Groves-Vickrey mechanisms. In Section 3 we first calculate derivatives of revenue and expected revenue with respect to the weights attached to each partition of objects (Theorem 1). In Subsection 3.1 we use this Theorem to show how to improve upon pure bundling and prepackaged auctions (Proposition 1). In Subsection 3.2 we use the Theorem in order to show how to improve upon separate, efficient auctions (Proposition 2). This results also identifies the most promising directions for increasing the weights on partitions around the efficient auction (where all weights are zero). The results in this Subsection depend on a condition on the distribution of values that requires symmetry among agents and independence across objects. Proposition 3 in Subsection 3.3 combines the above insights, and

[6] These authors also translate the integrability constraint into a geometric condition about the boundaries of the regions in the type spaces where various allocations are chosen. The condition says that the vector representing the "jump" in the allocation must be perpendicular to the boundary where the jump occurs.

[7] Armstrong notes that when the type spaces increases in size, the revenue-maximizing auction should not be expected to remain efficient.

shows that a simple mixed bundling auction improves both upon the pure bundling auction and upon the efficient auction. This Subsection ends with a numerical example where we solve for the optimal level of mixed bundling. Proposition 4 in Subsection 3.4 shows that the revenue enhancing effect of mixed bundling is robust to the introduction of (optimal) reserve prices. Section 3.5 contains an example showing that the optimal weighted Vickrey-Clarke-Groves mechanism may be asymmetric even in symmetric situations. Section 4 concludes.

2 The Model

A seller s owns a set $\mathcal{K}$ of K heterogeneous objects. There is a set $\mathcal{N} = \{1, .., i, j, .., N\}$ of potential buyers (whom we also call bidders, or agents). Each bidder i privately knows her valuation v_k^i for of each object k. For the seller, we assume $v_k^s = 0$. We assume that agent i's valuation for a subset $P_i \subseteq \mathcal{K}$ of objects is given by[8] $v_{P_i}^i := \sum_{k \in P_i} v_k^i$, and that her utility function is quasi-linear in money. Thus, a bidder receiving bundle P_i for a payment of T has a utility of $\sum_{k \in P_i} v_k^i - T$.

We assume that v_k^i can take values on an interval $\left[\underline{v}_k, \overline{v}^k\right] \subseteq \mathbb{R}_0^+$, and that $v = (v_k^i)_{i,k}$ is distributed according to a continuous, strictly positive density function $f : \Omega \to R_0^+$ where $\Omega = \prod_{k \in \mathcal{K}} \left[\underline{v}_k, \overline{v}^k\right]^N$. The K-dimensional type of agent i is denoted by $v^i \in \prod_{k \in \mathcal{K}} \left[\underline{v}_k, \overline{v}^k\right]$, and $v^{-i} \in \prod_{k \in \mathcal{K}} \left[\underline{v}_k, \overline{v}^k\right]^{N-1}$ denotes the $K(N-1)$-dimensional vector of all but bidder i's valuations.

The outcome of an auction is a labeled partition $P = (P_s, P_1, ..., P_N) \in \mathcal{P}$ of objects and a vector of transfers $(t^i)_{i \in \{1...N\}} \in \mathbb{R}^N$. The interpretation is that bidder $i \in \{1, ..., N\}$ receives bundle P_i, the seller keeps the bundle P_s, and bidder i makes a payment of t^i to the seller. $\mathcal{P}_{bun} \subset \mathcal{P}$ denotes the subset of pure bundling allocations, i.e. $P \in \mathcal{P}_{bun} \Rightarrow \exists i \in \mathcal{N}$, such that $P_i = \mathcal{K}$.

We study deterministic, dominant strategy, direct revelation mechanisms $(g, t) : \Omega \to \mathcal{P} \times R^N$ that specify for each vector of valuations v an allocation $g(v)$ and payments from each agent $t^i(v)$. By payoff equivalence (see e.g. Green and Laffont (1979) , and Holmstrom (1979)), the payments in any dominant strategy mechanism are, up to a constant, pinned down by the allocation rule g.

Two such mechanisms immediately come to mind: separate Vickrey auctions (one for each object), and a Vickrey auction for the bundle of all the

[8]This model immediately extends to any kind of complementarities or substitutabilities between objects. We focus on additive valuations in order to single out the strategic effect of bundling.

objects:

1) In the separate auctions every object goes to the bidder who values it most: $g(v) = \arg\max_{P \in \mathcal{P}} \left\{ \sum_{j \in \mathcal{N}} v^j_{P_j} \right\}$. This allocation rule is implemented in dominant strategies by the well-known second-price transfer rule which charges bidder i the highest rejected bid on each object she receives.

2) In the (pure) bundling auction the set of all objects goes to the bidder who values highest the entire lot: $g(v) = \arg\max_{P \in \mathcal{P}_{bun}} \left\{ \sum_{j \in \mathcal{N}} v^j_{P_j} \right\}$. Charging the successful bidder the highest rejected bid on the entire lot makes this mechanism strategy-proof.

2.1 General λ-auctions

We now define a large class of auctions that encompasses many well-known formats, including the two mentioned above:

Definition 1 *For a given real vector $\lambda = \{\lambda_P\}_{P \in \mathcal{P}}$ let*

$$g_\lambda(v) := \arg\max_{P \in \mathcal{P}} \left\{ \sum_{j \in \mathcal{N}} v^j_{P_j} + \lambda_P \right\} \tag{1}$$

and

$$g_\lambda^{-i}(v) := \arg\max_{P \in \mathcal{P}} \left\{ \sum_{j \neq i} v^j_{P_j} + \underline{v}^i_{P_i} + \lambda_P \right\}$$

*The **λ-auction** (g_λ, t_λ) is defined by the allocation rule*[9] $g = g_\lambda(v)$ *, and by the transfer rule*

$$t^i_\lambda(v) = -\left(\sum_{j \neq i} v^j_{g_\lambda(v)_j} + \lambda_{g_\lambda(v)} \right) + \tau^i_\lambda(v) \tag{2}$$

where $\tau^i_\lambda(v) = \left(\sum_{j \neq i} v^j_{g_\lambda^{-i}(v)_j} \right) + \underline{v}^i_{g_\lambda^{-i}(v)_i} + \lambda_{g_\lambda^{-i}(v)}$.

To interpret the above definition, we suggest a conceptual "trick": Consider an additional, fictitious agent who does not hold private information and cannot receive objects, but who attaches a value of λ_P to allocation

[9] To circumvent the problem of tie-breaks in cases where argmax sets are not singletons, we assume throughout the paper that ties are broken according to a fixed order on $\mathcal{P}$. The tie-breaking rule does not matter given our assumption of an atomless distribution.

P. The λ-auction is then the Vickrey-Clarke-Groves mechanism for the enlarged set of agents. The precise value of $\tau_{\lambda}^{i}\left(v^{-i}\right)$ - which is irrelevant for the dominance argument - is determined so that $\underline{v}^{i}$, the lowest type of bidder i, always receives zero utility in the truth-telling equilibrium of the λ-auction, in analogy to the Clarke mechanism. Note also that a bidder who receives no objects makes no payments[10].

Together with payoff-equivalence, the above observations immediately yield:

Lemma 1 *1) Truth-telling is a dominant strategy in an λ-auction. 2) Ex-post participation constraints are satisfied, and the lowest type of a bidder has a utility of zero in the truth-telling equilibrium. 3) Given the allocation rule g_{λ}, the transfer rule t_{λ} is the unique one with the above properties. Thus, t_{λ} is the highest transfer rule implementing g_{λ}.*

We now show how several well-known auction formats fit into the class of λ-auctions.

Example 1

- *Separate second-price auctions for the K objects with reserve price r_k for object k are represented by the λ-auction where $\lambda_P = \sum_{k \in P_s} r_k$, i.e. the weight on an allocation is the sum of the reservation prices over all the objects that are not sold.*

- *The pure bundling auction is represented by the λ-auction where $\lambda_P = \sum_k \left(\bar{v}^k - \underline{v}_k\right)$ for pure bundling allocations $P \subset \mathcal{P}_{bun}$, and $\lambda_P = 0$ else. This ensures that* $\arg\max_{P \in \mathcal{P}} \left\{\sum_{j \in \mathcal{N}} v_{P_j}^{j} + \lambda_P\right\} \in \mathcal{P}_{bun}$ *for all possible realizations of v.*

- *More generally, consider any partition $Q = \{Q_1, .., Q_\tau\}$ of $\mathcal{K}$. A prepackaged auction with partition Q is represented by the λ-auction where $\lambda_P = \sum_k \left(\bar{v}^k - \underline{v}_k\right)$ if the allocation P is compatible with the partition Q (i.e. each sub-bundle Q_μ is allocated to one bidder[11]) , and $\lambda_P = 0$ else.*

- *Pure bundling with reserve price r on the bundle is represented by the λ-auction where $\lambda_P = \sum_k \left(\bar{v}^k - \underline{v}_k\right)$ for $P \in \mathcal{P}_{bun}$, $\lambda_P = \sum_k \left(\bar{v}^k - \underline{v}_k\right) + r$ for P with $P_s = \mathcal{K}$, and $\lambda_P = 0$ else.*

[10] This is because in this case $g_{\lambda}^{-i}(v) = g_{\lambda}(v)$ and $\underline{v}^{i}_{g_{\lambda}^{-i}(v)_i} = 0$, as i receives no object.

[11] This means: $\forall \mu \in \{1, ..., \tau\}, \exists i \in \mathcal{N} : Q_\mu \subseteq P_i$.

Weighted VCG mechanisms have been used by Roberts (1979) in order to characterize all dominant-strategy (not necessarily efficient) implementable choice rules in a general social choice setting. Note that, however large, the class of choice rules implemented by λ-auctions is, it is a strict subclass of dominant-strategy implementable choice rules[12]. It is not even true that all anonymous allocation mechanisms are λ-auctions[13]. We focus on $\lambda-$auctions since this class is analytically tractable, and since, as illustrated above, it includes well known auction formats and their "convex combinations".

3 Optimal Bundling

In this Section we study how expected revenue depends on the choice of the λ parameters. The main working horse for the subsequent results is Theorem 1 which computes partial derivatives of revenue with respect to λ. We then apply this result in order to show how to improve the revenue from the pure bundling auction by making it slightly less inefficient, and how to improve upon the revenue of efficient auctions by rendering them slightly less efficient. After looking at these extreme cases, we focus on mixed-bundling auctions, a one-dimensional subclass of λ-auctions, which generalizes the idea of a discount on a bundle of objects in monopolistic mixed bundling.

Let us first introduce some notation. Given fixed, truthfully reported

[12] In contrast to Roberts (1979), there is no simple characterization of all dominant strategy implementable choice rules in our framework. For example allocating goods only to bidder i, according to an arbitrary function of bidder j's signal is implementable. The reason is, basically, the absence of allocative externalities. If bidders' preferences were allowed to depend on the entire profile of allocations (as opposed to their allocation only), Roberts' result would apply and the set of allocation rules implementable in dominant strategy would be the set of affine maximizers (i.e., the same as λ-auctions except that the weight given to the various bidders may be different). See Bickchandani et al. (2004), Gui et al. (2004) and Lavi et al. (2004) for recent research on the study of dominant strategy implementation in externality-free environments.

[13] Consider the following 1-object allocation mechanism: bidder i receives the good for a payment of $t^i = 2v^j$ if $v^i > 2v^j$; bidder j receives the good for a payment of $t^j = 2v^i$ if $v^j > 2v^i$; the seller keeps the object in all other cases. This mechanism is easily seen to be strategy-proof but it is not a λ-auction. We are grateful to a referee that offered this counter-example.

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valuations v in the λ-auction, denote by

$$\begin{aligned} R_\lambda(v) &= \sum_i t^i_\lambda(v), \\ S_\lambda(v) &= \sum_i v^i_{g_\lambda(v)_i}, \\ BS_\lambda(v) &= S_\lambda(v) - R_\lambda(v) = \sum_i \left(v^i_{g_\lambda(v)_i} - t^i_\lambda(v) \right) \end{aligned}$$

realized revenue, social surplus and bidder surplus, respectively. Also, denote by $R(\lambda) = \mathbb{E}_v[R_\lambda(v)]$, $S(\lambda) = \mathbb{E}_v[S_\lambda(v)]$, and $BS(\lambda) = \mathbb{E}_v[BS_\lambda(v)]$ expected revenue, expected social surplus, and expected bidder surplus, respectively.

Theorem 1 *1. Fix v and $P \in \mathcal{P}$. Let $\mathbb{I}$ be the indicator function, and let $P'' = \arg\max_{P' \neq P} \left\{ \sum_{j \in \mathcal{N}} v^j_{P'_j} + \lambda_{P'} \right\}$. The revenue $R_\lambda(v)$ is a piecewise linear function of λ_P , and has a single discontinuity of $(\sum_j v^j_{P_j} - \sum_j v^j_{P''_j})$ at $\lambda^*_P := \left(\sum_{j \in \mathcal{N}} v^j_{P''_j} \right) + \lambda_{P''} - \sum_{j \in \mathcal{N}} v^j_{P_j}$. Whenever differentiable, its derivative is given by:*

$$\frac{\partial}{\partial \lambda_P} R_\lambda(v) = \sum_i \left(-\mathbb{I}_{\{(\lambda,v): g_\lambda(v) = P\}} + \mathbb{I}_{\left\{(\lambda,v): g^{-i}_\lambda(v) = P\right\}} \right) \tag{3}$$

2. Expected revenue is a differentiable function of λ_P with partial derivative given by:

$$\frac{\partial R}{\partial \lambda_P}(\lambda) = \frac{\partial S}{\partial \lambda_P}(\lambda) + \sum_{i \in \mathcal{N}} \left(-\Pr(g_\lambda(v) = P) + \Pr\left(g^{-i}_\lambda(v) = P\right) \right) \tag{4}$$

Proof. 1) The revenue effect of changing λ can be decomposed into a surplus generation effect, and a surplus extraction effect :

$$R_\lambda(v) = S_\lambda(v) - BS_\lambda(v).$$

Social surplus depends on λ only via the chosen allocation $g_\lambda(v)$. Therefore $S_\lambda(v)$ is a step function of λ_P with a discontinuity of $(\sum_{j \in \mathcal{N}} v^j_{P_j} - \sum_{j \in \mathcal{N}} v^j_{P''_j})$ at the point λ^*_P where the allocation switches from P'' to P, i.e., at

$$\lambda^*_P := \sum_{j \in \mathcal{N}} v^j_{P''_j} + \lambda_{P''} - \sum_{j \in \mathcal{N}} v^j_{P_j}.$$

Thus, its derivative is zero whenever it is well-defined.

Bidder i's surplus is given by:

$$\begin{aligned} v^i_{g_\lambda(v)_i} - t^i_\lambda(v) &= \sum v^j_{g_\lambda(v)_j} + \lambda_{g_\lambda(v)} - \tau^i_\lambda(v) \\ &= \sum v^j_{g_\lambda(v)_j} + \lambda_{g_\lambda(v)} - \sum_{j \neq i} v^j_{g^{-i}_\lambda(v)_j} - \underline{v}^i_{g^{-i}_\lambda(v)_i} - \lambda_{g^{-i}_\lambda(v)}. \end{aligned}$$

The bidders' surplus is given by:

$$BS_\lambda(v) = \sum_i \left[\max_{P \in \mathcal{P}} \left\{ \sum_{j \in \mathcal{N}} v^j_{P_j} + \lambda_P \right\} - \max_P \left\{ \sum_{j \neq i} v^j_{P_j} + \underline{v}^i_{P_i} + \lambda_P \right\} \right]$$

Thus, $BS_\lambda(v)$ is the difference of maxima of continuous functions. As such, it is continuous and piecewise linear in λ_P. Its derivative is given by:

$$\frac{\partial}{\partial \lambda_P} BS_\lambda(v) = \sum_i \left(\mathbb{I}_{\{(\lambda,v): g_\lambda(v)=P\}} - \mathbb{I}_{\left\{(\lambda,v): g^{-i}_\lambda(v)=P\right\}} \right) \tag{5}$$

The claim follows by combining the two insights above.

2) Again, we compute the revenue effect as the difference of the effects on total surplus, and on bidder surplus. The latter is computed by integration of equation 5 with respect to v:

$$\begin{aligned} \frac{\partial}{\partial \lambda_P} BS(\lambda) &= \frac{\partial}{\partial \lambda_P} \mathbb{E}_v \left[BS_\lambda(v) \right] \\ &= \mathbb{E}_v \left[\frac{\partial}{\partial \lambda_P} BS_\lambda(v) \right] \\ &= \mathbb{E}_v \left[\sum_i \mathbb{I}_{\{(\lambda,v): g_\lambda(v)=P\}} - \mathbb{I}_{\left\{(\lambda,v): g^{-i}_\lambda(v)=P\right\}} \right] \\ &= \sum_i \left(\Pr\left(g_\lambda(v) = P\right) - \Pr\left(g^{-i}_\lambda(v) = P\right) \right) \end{aligned}$$

For the second equation, we used Lebesgue's theorem and the Lipschitz-continuity of $BS_\lambda(v)$ in λ in order to exchange the order of differentiation and integration.

It remains to show that, unlike realized social surplus, expected social surplus is differentiable as a function of λ_P, i.e. that $\frac{\partial S}{\partial \lambda_P}(\lambda)$ exists. To do so, denote by

$$\partial_\lambda(P, P') := \left\{ v \in \Omega : P, P' \in \arg\max_{P \in \mathcal{P}} \left(\sum_{j \in \mathcal{N}} v^j_{P_j} + \lambda_P \right) \right\}$$

the boundary in the type space between areas where g_λ chooses allocations P and P', respectively, and by $d^{NK-1}v$ the $NK-1$ dimensional Lebesgue measure. With this notation, we compute $\frac{\partial S}{\partial \lambda_P}(\lambda)$ by applying Leibniz's rule to integrate the discontinuity of realized social surplus $S_\lambda(v)$ over all valuations at which the discontinuity occurs:

$$\frac{\partial S}{\partial \lambda_P}(\lambda) = \sum_{P' \in \mathcal{P}} \int_{\partial_\lambda(P,P')} \left(\sum_{j \in \mathcal{N}} v^j_{P_j} - \sum_{j \in \mathcal{N}} v^j_{P'_j} \right) f(v)\, d^{NK-1}v$$

Given the continuity of f , this expression is well defined for all λ.

Note that $\left(\sum_{j\in\mathcal{N}} v^j_{P_j} - \sum_{j\in\mathcal{N}} v^j_{P'_j}\right)$ is constant, and equal to $(\lambda_{P'} - \lambda_P)$ on the set $\partial_\lambda(P,P')$. Let $|\partial_\lambda(P,P')| = \int_{\partial_\lambda(P,P')} f(v)\, d^{NK-1}v$ be the measure of $\partial_\lambda(P,P')$. Since $\frac{\partial}{\partial \lambda_P} \Pr(g_\lambda = P') = -|\partial_\lambda(P,P')|$, we obtain that

$$\frac{\partial S}{\partial \lambda_P}(\lambda) = \sum_{P' \in \mathcal{P}} (\lambda_{P'} - \lambda_P) |\partial_\lambda(P,P')| = -\sum_{P' \in \mathcal{P}} (\lambda_{P'} - \lambda_P) \frac{\partial}{\partial \lambda_P} \Pr(g_\lambda = P')$$

We will apply this equation in Proposition 4 where we analyze the effect of mixed bundling on auctions with (optimal) reserve prices. ■

Recall the interpretation of the λ-auction as the efficient auction for society including a virtual agent whose preferences are determined by λ. Changing these preferences affects the externality that agent i imposes on the virtual agent, and this translates into a change of agent i's payment. For a fixed allocation $g_\lambda(v)$, the marginal externality is given by the term under the summation in equation (3). When the allocation $g_\lambda(v)$ is affected by the change in λ, the aggregated externality on the virtual agent that results from changing the allocation from P'' to P equals the discontinuity in aggregated payments of the real agents:

$$\lambda_{P''} - \lambda^*_P := \sum_j v^j_{P_j} - \sum_j v^j_{P''_j}.$$

For an illustration, consider a single-good auctioneer setting a reserve price, and recall that a reserve price corresponds to a weight on the allocation where the auctioneer keeps the object. To determine the optimal reserve price, the auctioneer is trading off the marginal loss of efficiency due to missing the opportunity to sell, against the marginal gain of additional revenue. For any realized v, the efficiency loss is captured by the discontinuity of $R_\lambda(v)$. The marginal revenue from agent i is captured by the term under the summation in equation (3): it equals 1 for realizations of v in which only the valuation of agent i exceeds the reserve price. In expectation, the efficiency loss is captured by the first term of equation (4) , while the expected marginal revenue from i equals agent i's expected externality on the virtual agent, represented by the term under summation in equation (4).

3.1 Improving upon pure bundling and pre-packaged auctions

We now apply Theorem 1 to the pure bundling auction, and we show how to increase revenue by allowing separate sales for extreme realizations. Recall that the pure bundling auction is represented by the λ-auction where $\lambda_P = \sum_k \left(\bar{v}^k - \underline{v}_k\right)$ for $P \subset \mathcal{P}_{bun}$, and $\lambda_P = 0$ else. Decreasing the parameter λ_P for $P \in \mathcal{P}_{bun}$ will allow allocations other than pure bundling to be chosen.

Definition 2 *The c-bundling auction (g_c, t_c) is the λ-auction with:*

$$\lambda_P := \begin{cases} c & \text{if } P \in \mathcal{P}_{bun} \\ 0 & \text{else.} \end{cases}$$

We denote by $R_c(v)$ the revenue of a c-bundling auction for a given realization of v, and by $R(c)$ the expected revenue. Let $c^* := \inf\{c : g_c(\Omega) \subseteq \mathcal{P}_{bun}\}$ be the smallest value of c for which the c-bundling auction is the pure bundling auction[14]*.*

Proposition 1 *1. For sufficiently small $\varepsilon > 0$, and for any realization of v, the revenue of the $(c^*-\varepsilon)$- bundling auction is higher than the revenue of the pure bundling auction: $R_{c^*-\varepsilon}(v) \geq R_{c^*}(v)$, with strict inequality for v such that $g_{c^*-\varepsilon}(v) \notin \mathcal{P}_{bun}$. In particular, $R(c^*-\varepsilon) > R(c^*)$ for generic distribution functions.*

2. The revenue of a pre-packaged auction with partition $\mathcal{Q}$ is increased by uniformly lowering the parameter λ_P on all allocations P compatible with $\mathcal{Q}$ below the corresponding threshold value $c^*_{\mathcal{Q}}$.

Proof. 1) Consider first v with $g_{c^*-\varepsilon}(v) \in \mathcal{P}_{bun}$. By part 1 of Theorem 1 we get that

$$\begin{aligned} R_{c^*}(v) - R_{c^*-\varepsilon}(v) &= \int_{c^*-\varepsilon}^{c^*} \frac{\partial}{\partial c} R_c(v)\, dc = \int_{c^*-\varepsilon}^{c^*} \sum_{P \in \mathcal{P}_{bun}} \frac{\partial}{\partial \lambda_P} R_c(v)\, dc \\ &= \int_{c^*-\varepsilon}^{c^*} \sum_{P \in \mathcal{P}_{bun}} \sum_i \left(-\mathbb{I}_{\{(c,v):g_c(v)=P\}} + \mathbb{I}_{\left\{(\lambda,v):g_c^{-i}(v)=P\right\}} \right) dc \\ &= \sum_i \int_{c^*-\varepsilon}^{c^*} \left(-\mathbb{I}_{\{(c,v):g_c(v)\in\mathcal{P}_{bun}\}} + \mathbb{I}_{\left\{(\lambda,v):g_c^{-i}(v)\in\mathcal{P}_{bun}\right\}} \right) dc \end{aligned}$$

[14] We assume that ties in the c^*-bundling auction are broken in favor of pure bundling allocations.

As we assumed $g_{c^*-\varepsilon}(v) \in \mathcal{P}_{bun}$ we have, a fortiori, that $g_c(v) \in \mathcal{P}_{bun}$ for all $c \in [c^* - \varepsilon; c^*]$, yielding that the integrand is non-positive and the result follows.

Consider now v with $g_{c^*-\varepsilon}(v) \notin \mathcal{P}_{bun}$. Denote by

$$\widetilde{c}(v) := \inf \{c \in [c^* - \varepsilon; c^*] : g_c(v) \in \mathcal{P}_{bun}\}$$

the lowest value of c for which a pure bundling allocation is chosen at realization v. By Theorem 1 the discontinuity of $R_c(v)$ at $c = \widetilde{c}(v)$ equals exactly $-\widetilde{c}(v)$. We get:

$$\begin{aligned} & R_{c^*}(v) - R_{c^*-\varepsilon}(v) \\ = \; & \sum_i \int_{c^*-\varepsilon}^{c^*} \left(-\mathbb{I}_{\{(c,v):g_c(v)\in\mathcal{P}_{bun}\}} + \mathbb{I}_{\left\{(\lambda,v):g_c^{-i}(v)\in\mathcal{P}_{bun}\right\}} \right) dc - \widetilde{c}(v) \end{aligned}$$

As the first term is bounded above by $N\varepsilon$, this expression is strictly negative for small enough ε.

2) The pre-packaged auction with partition $\mathcal{Q}$ can be interpreted as a set of separate pure-bundling auctions for each of the packages Q_μ. Thus, the result follows by applying part 1 to each of these separate pure-bundling auctions. ■

The above argument can be equally used to show that a pure bundling auction with a reserve price can be improved upon by allowing for separate sales in extreme cases.

3.2 Improving upon the efficient auction

By definition, the social surplus $S(\lambda)$ is maximized at the efficient auction where $\lambda = 0$. Therefore, $\nabla S(0) = 0$. Together with Theorem 1, this observation yields:

Corollary 2

$$\frac{\partial R}{\partial \lambda_P}(0) = \sum_{i\in\mathcal{N}} \left(-\Pr(g_0(v) = P) + \Pr\left(g_0^{-i}(v) = P\right)\right)$$

Thus, a necessary condition for the efficient auction to maximize revenue in the class of $(\lambda_P)_{P\in\mathcal{P}}$-auctions is :

$$\forall P \in \mathcal{P}, \; \sum_{i\in\mathcal{N}} \left(-\Pr(g_0(v) = P) + \Pr\left(g_0^{-i}(v) = P\right)\right) = 0 \qquad (6)$$

This is a highly non-generic condition on the density function governing the distribution of the signals v. In other words, by introducing a small λ_P for some P one can almost always improve upon the efficient auction[15].

We now add some more structure to the distributions of valuations in order to better identify the most promising directions in which to improve on the efficient auction. Specifically, we make the following symmetry assumption that will be used throughout the subsection:

Condition 3 (SAIO) *Valuations $v = (v_k^i)_{i,k}$ are distributed symmetrically across agents and independently across objects. Formally, the density function f satisfies: 1) $f(..., v^i, ..., v^j, ...) = f(..., v^j, ..., v^i, ...)$, and 2) $f(v) = \prod_{k \in \mathcal{K}} f_k(v_k)$, where $f_k : [\underline{v}_k, \overline{v}^k]^N \to R_0^+$ is the marginal density with respect to object k.*

Proposition 2 *Assume that condition SAIO is met, and denote by $\#P := \#\{i \in \mathcal{N} : P_i \neq \emptyset\}$, the number of bidders receiving some objects in allocation P. We have*

$$\begin{aligned} \frac{\partial R}{\partial \lambda_P}(0) &= \frac{N^{K-1}(N - \#P) - (N-1)^K}{N^{K-1}(N-1)^K}, \text{ for } P \text{ with } P_s = \emptyset \qquad (7) \\ \frac{\partial R}{\partial \lambda_P}(0) &= 0, \text{ for } P \text{ with } P_s \neq \emptyset \end{aligned}$$

In particular, introducing a bundling parameter $\lambda_P = c$ for all $P \in \mathcal{P}_{bun}$ leads to an increase in expected revenue versus the expected revenue in the efficient auction.

Proof. Consider first P with $P_s = \emptyset$. As the density function f is assumed to be symmetrical across bidders, the probability $\Pr(k \in g_0(v)_i)$ of bidder i having the highest valuation for some object k equals $\frac{1}{N}$. As valuations are assumed to be independent across objects, the probability of an allocation P being efficient is obtained by multiplying this constant probability over all objects: $\Pr(g_0(v) = P) = \frac{1}{N^K}$. To determine $\Pr\left(g_0^{-i}(v) = P\right)$ we apply the same logic, and we obtain $\Pr\left(g_0^{-i}(v) = P\right) = \frac{1}{(N-1)^K}$ if $P_i = \emptyset$, and $\Pr\left(g_0^{-i}(v) = P\right) = 0$ else.

[15] The FOC need not be sufficient for a global maximum, as $R(\lambda)$ need not be a concave function of λ. Subsection 3.5 contains an example showing that the best λ-auction may be asymmetric even in symmetric situations, thereby illustrating the potential non-concavity of $R(\lambda)$.

Using Corollary 2, we obtain:

$$\begin{aligned}\frac{\partial R}{\partial \lambda_P}(0) &= \sum_{i\in\mathcal{N}}[-\Pr(g_0(v)=P)+\Pr\left(g_0^{-i}(v)=P\right)] \\ &= -\frac{N}{N^K}+\frac{N-\#P}{(N-1)^K}=\frac{N^{K-1}(N-\#P)-(N-1)^K}{N^{K-1}(N-1)^K}\end{aligned}$$

For P with $P_s \neq \emptyset$ the result follows from noticing that $\Pr(g_0(v)=P) = \Pr\left(g_0^{-i}(v)=P\right) = 0$. Introducing a small reserve price has an expected marginal effect of zero as both the size of the effect, and the probability of the realization of the case are marginal[16].

The second part of the Proposition follows by observing that $\#P = 1$ for $P \in \mathcal{P}_{bun}$. Thus, we obtain

$$\begin{aligned}\frac{\partial R}{\partial c}(0) &= \sum_{P\in\mathcal{P}_{bun}}\frac{\partial R}{\partial \lambda_P}(0)=N\frac{N^{K-1}(N-1)-(N-1)^K}{N^{K-1}(N-1)^K} \\ &= N(N-1)\left(\frac{N^{K-1}-(N-1)^{K-1}}{N^{K-1}(N-1)^K}\right)>0.\end{aligned}$$

■

Remark 1 *Note that* $\frac{\partial R}{\partial \lambda_P}(0)$ *is decreasing in* $\#P$, *positive for* $\#P = 1$, *negative for* $\#P = \min\{N, K\}$ *and positive for every fixed* $\#P \leq \min\{N, K\}-1$ *as either* N *or* K *tends towards infinity. Thus, in the space of parameter combinations* $\lambda = (\lambda_P)_{P\in\mathcal{P}}$ *increasing* λ_P *locally around the efficient auction has the highest positive impact for* $P \in \mathcal{P}_{bun}$. *Raising* λ_P *for partitions* P *that allocate objects to all bidders,* $\#P = N$, *or that sell the objects as singletons,* $\#P = K$, *decreases expected revenue.*

Remark 2 *To broaden our focus let us consider raising* λ_P *for all allocations* P *that are derived from a partition* $\mathcal{Q}$: $Q_1 \uplus ... \uplus Q_\tau$, *i.e. for all* P *such that* $\forall \mu \in \{1, ..., \tau\}, \exists i : Q_\mu = P_i$. *By definition, precisely* τ *agents receive objects in each of these allocations. To compute the number of such* P, *observe that*

[16] It is worth bearing in mind, that the partial derivatives at the efficient auction need not be a good indicator for the impact of significantly increasing λ_P. It is well known that the introduction of a "large" reserve price can have a large impact on expected revenue.

Note also that the optimal one-object auction for heterogeneous bidders as in Myerson (1981) need not be an λ-auction. This is because, in general, it sets different reserve prices for different agents, which a λ-auction can not do.

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there are N possibilities to allocate bundle Q_1, $N-1$ possibilities to allocate bundle Q_2, and so on. Thus, we get

$$\sum_{P \text{ derived from } \mathcal{Q}} \frac{\partial R}{\partial \lambda_P}(0) = \frac{N!}{(N-\tau)!} \frac{N^{K-1}(N-\tau)-(N-1)^K}{N^{K-1}(N-1)^K}$$

The revenue impact only depends on τ, the number of sub-bundles in the partition, but not on whether $\mathcal{Q}$ divides up the objects evenly or not. Similarly to equation 7 , the obtained expression is positive for small τ and becomes negative for τ close to $\min\{N, K\}$. But now it is not strictly decreasing in τ. Therefore the largest impact comes from raising λ_P for all P with an intermediate τ. For example, in a setting with 5 bidders and 3 objects, we obtain that, optimally, $\tau = 2$. Thus, the partition $\mathcal{Q}$ should contain two sub-bundles of one and two objects, respectively.

We note in the next example that, without condition SAIO, $\frac{\partial R}{\partial \lambda_P}(0)$ may be negative even for P with $\#P = 1$:

Example 2 *Let $\left[\underline{v}_k, \overline{v}^k\right] = [0,1]$ and assume that v^i are i.i.d. but that there is correlation between objects. Specifically let v^i be distributed on $[0,1]^K$ in a way that puts probability $\frac{1}{2}$ uniformly on $\left[0, \frac{1}{2}\right]^K$ and probability $\frac{1}{2}$ on the "upper diagonal" $\left\{(x, ..., x) \in [0,1]^K : x \geq \frac{1}{2}\right\}$. Here $\frac{\partial R}{\partial \lambda_P}(0) < 0$ for $P \in \mathcal{P}_{bun}$.*

Proof. Let $P \in \mathcal{P}_{bun}$. Due to symmetry between agents we have that

$$\frac{\partial R}{\partial \lambda_P}(0) = \frac{1}{N} \sum_{i \in \mathcal{N}} \left(\Pr\left(g_0^{-i}(v) \in \mathcal{P}_{bun}\right) - \Pr\left(g_0(v) \in \mathcal{P}_{bun}\right)\right)$$

Let us first evaluate $\Pr\left(g_0(v) \in \mathcal{P}_{bun}\right)$: With probability $1 - \left(\frac{1}{2}\right)^N$ there is at least one agent i with signal v^i on the upper diagonal, in which case there is a bidder who has the highest valuation for all objects. With probability $\left(\frac{1}{2}\right)^N$ this is not the case, and the probability of there being an overall winner is $\left(\frac{1}{N}\right)^{K-1}$. Thus

$$\Pr\left(g_0(v) \in \mathcal{P}_{bun}\right) = 1 - \left(\frac{1}{2}\right)^N + \left(\frac{1}{2}\right)^N \left(\frac{1}{N}\right)^{K-1} \tag{8}$$

Analogously, we get that

$$\Pr\left(g_0^{-i}(v) \in \mathcal{P}_{bun}\right) = 1 - \left(\frac{1}{2}\right)^{N-1} + \left(\frac{1}{2}\right)^{N-1} \left(\frac{1}{N-1}\right)^{K-1}$$

As $[1 - \left(\frac{1}{2}\right)^N + \left(\frac{1}{2}\right)^N \left(\frac{1}{N}\right)^{K-1}]$ is increasing in N, we obtain that

$$\frac{\partial R}{\partial \lambda_P}(0) = \frac{1}{N} \sum_{i \in \mathcal{N}} \left(\Pr\left(g_0^{-i}(v) \in \mathcal{P}_{bun}\right) \ - \Pr\left(g_0(v) \in \mathcal{P}_{bun}\right) \right) < 0$$

■

3.3 Mixed Bundling Auctions

So far we have made use of local arguments to show that neither pure bundling nor separate auctions can be optimal (under SAIO). In this subsection, we focus on the mixed bundling auctions as introduced in Definition 2. This class is parametrized by a parameter c that acts as a surcharge on the payment of a bidder who receives some, but not all, of the objects (or, equivalently, as a discount on the payment of a bidder who receives the entire lot). By definition, this class encompasses the efficient auction, represented by the c-bundling auction with $c = 0$, and the pure bundling auction, represented by the c-bundling auction with $c = c^* = \inf\{c : g_c(\Omega) \subseteq \mathcal{P}_{bun}\}$. As a corollary to Propositions 1 and 2, we get:

Proposition 3 *Assume that condition SAIO is met. Then there exists $c \in (0, c^*)$ such that the $c-$bundling auction achieves a higher expected revenue than both the pure bundling auction and the efficient auction.*

Proof. Expected revenue is a continuous function of c. Thus, it must have a maximum on the closed interval $[0, c^*]$. Proposition 1 showed the maximum cannot be at $c = c^*$,while Proposition 2 showed that the maximum cannot be at $c = 0$. ■

To understand the comparative statics for c in the interior of $[0, c^*]$, consider the case of two bidders i, j. Then the formula for payments reduces to:

$$t_c^i(v) = \begin{cases} v_{\mathcal{K}}^j & \text{if } P^i = \mathcal{K}, \\ v_{P_i}^j + c & \text{if } \emptyset \neq P^i \subsetneq \mathcal{K}, \\ 0 & \text{if } P^i = \emptyset. \end{cases} \tag{9}$$

Increasing the bundling parameter by ε increases revenue by 2ε for valuations such that the objects are sold separately. It decreases revenue discontinuously by c for valuations such that the $\varepsilon-$increase in c leads to a bundling allocation being chosen instead of the efficient one. Figure 1 shows the realized revenue $R_c(v)$ for some fixed values of v as a function of c.

Proposition 2 stated that introducing mixed bundling to the efficient auction increases revenue (since the externality on the virtual bidder is added as

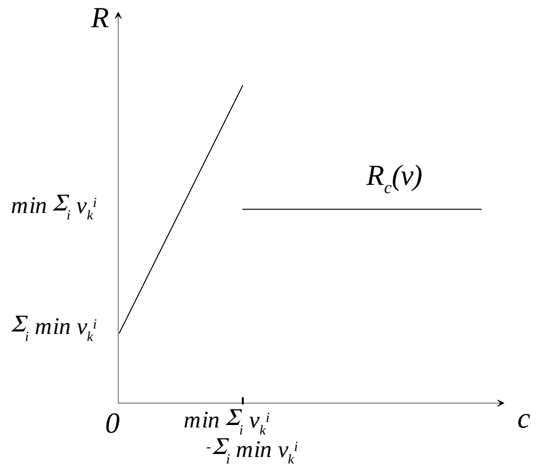

Figure 1: Realized Revenue $R_c(v)$ as a function of c (fixing v such that bundling is inefficient).

a surcharge on bidders' payments). This is reflected in the positive slope of $R_c(v)$ at $c = 0$. Proposition 1 stated that making the pure bundling auction slightly less inefficient by lowering c also increases revenue. This is reflected in the upward jump in $R_c(v)$ when c moves from right to left.

We now analytically compute the optimal bundling parameter in a simple example.

Example 3 *There are two bidders i, j competing for two objects A, B. Valuations are uniformly and independently distributed on $[0, 1]$. The revenue-maximizing bundling parameter is $c = \frac{1}{3}$.*

Proof. By Theorem 1 we know that $\frac{dR(c)}{dc} = \frac{dS(c)}{dc} + 2\Pr(g_c(v) \notin \mathcal{P}_{bun})$. As the marginal loss in expected social welfare is the inefficiency c times the marginal probability that this inefficiency is incurred, $\frac{d}{dc}\Pr(g_c(v) \in \mathcal{P}_{bun})$, we get that:

$$\frac{dR(c)}{dc} = -c\frac{d}{dc}\Pr(g_c(v) \in \mathcal{P}_{bun}) + 2\Pr(g_c(v) \notin \mathcal{P}_{bun})$$

Note that $\Pr(g_c(v) \notin \mathcal{P}_{bun}) = 2\Pr(g_c(v) = [\emptyset, \{A\}, \{B\}])$. Only for $\{v_A^j < 1 - c, v_B^j > c\}$ there are v^i such that $g_c(v) = [\emptyset, \{A\}, \{B\}]$. The area of this region of i's valuations is given by $\left(1 - \left(v_A^j + c\right)\right)\left(v_B^j - c\right)$, as is illustrated in figure 2. This yields:

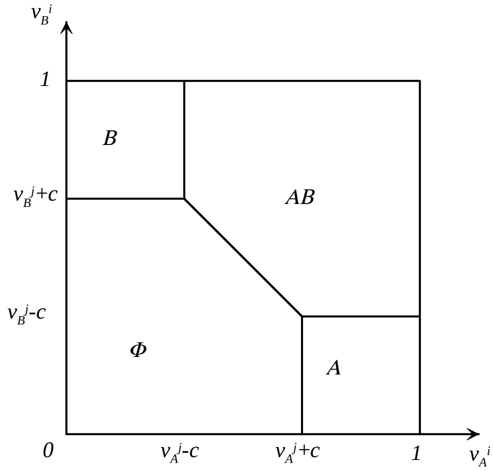

Figure 2: i's allocation $(\emptyset, A, B$ or $AB)$ as a function of v^i on a slice through the 4-dimensional type space for fixed v_A^j, v_B^j.

$$\Pr(g_c(v) \notin \mathcal{P}_{bun}) = 2\int_c^1 \int_0^{1-c} \left(1 - \left(v_A^j + c\right)\right)\left(v_B^j - c\right) dv_A^j dv_B^j = \frac{1}{2}(1-c)^4$$

Thus, we get:

$$\frac{dR(c)}{dc} = (1-c)^4 - 2c(1-c)^3$$

Note that $\frac{dR(c)}{dc} = 0$ for $c = \frac{1}{3}$. To calculate expected revenue, we use the fact that $R(0) = \frac{2}{3}$ (twice the expectation of the second-order statistic of two random variables distributed uniformly on $[0, 1]$) , and we get

$$R(c) = \frac{2}{3} + c - 3c^2 + 4c^3 - \frac{5}{2}c^4 + \frac{3}{5}c^5.$$

Thus, we obtain $R(0) = \frac{2}{3}$, $R(c^*) = R(1) = \frac{23}{30} \approx 0.766$, and $R(\frac{1}{3}) = \frac{637}{810} \approx 0.786$. The improvement caused by mixed bundling seems small in this particular example. But, of course, it can get quite large if valuations are increased. A meaningful calibration can only be done within the context of specific applications. ■

3.4 Reserve prices

Reserve prices are known to increase revenue (see Myerson, 1981). Propositions 1 and 2 show that, when the seller cannot retain the objects, neither separate nor pure bundling auctions can be optimal. Could it be that separate auctions with well adjusted reserve prices are optimal? The general formula for marginal revenue in Theorem 1 (equation 4) suggested that this would require strict conditions on the distribution on valuations.

In this subsection we go beyond the above insight by showing that, analogously to the efficient auction, separate auctions with optimal reserve prices can be improved upon by the introduction of mixed bundling if condition SAIO is met. For example, in the setting of Example 3, optimal separate auctions (with a reserve price of 0.5 on each object) yield a revenue of $R(0,(0.5,0.5)) = 0.833$, while a mixed bundling auction with a bundling parameter of 0.3 and reserve prices of 0.6 on each object yield a revenue of $R(0.3,(0.6,0.6)) = 0.870$.

We denote by $(g_{c,r}, t_{c,r})$ the c-bundling auction with a vector of reserve prices $r = (r_1, ..., r_K)$, i.e, the λ-auction[17] with:

$$\lambda_P := \begin{cases} c & \text{if } P \in \mathcal{P}_{bun}, \\ \sum_{k \in P_s} r_k & \text{else.} \end{cases}$$

Denote the expected revenue (resp. surplus, resp. bidders surplus) of the above auction by $R(c,r)$ (resp. $S(c,r), BS(c,r)$).

Proposition 4 *Assume that condition SAIO is met. If* $\frac{\partial R}{\partial r_k}(0,r) = 0$ *for some* k *with* $r_k \in (\underline{v}_k; \overline{v}_k)$*, then it holds that* $\frac{\partial R}{\partial c}(0,r) > 0$*. In particular,* $\frac{\partial R}{\partial c}(0,r^*) > 0$ *for optimal reserve prices* r^*.

Proof. Splitting up $\frac{\partial R}{\partial r_k}(0,r) = 0$ into effects on social surplus and bidders surplus, respectively, we get that

$$\begin{aligned} 0 &= \frac{\partial S}{\partial r_k}(0,r) - \frac{\partial BS}{\partial r_k}(0,r) \\ &= r_k \frac{\partial}{\partial r_k} \Pr\left(k \notin g_{0,r}(v)_s\right) + \Pr\left(k \notin g_{0,r}(v)_s\right) \\ &= \sum_i \left(r_k \frac{\partial}{\partial r_k} \Pr\left(k \in g_{0,r}(v)_i\right) + \Pr\left(k \in g_{0,r}(v)_i\right) \right) \end{aligned}$$

Due to symmetry among bidders, this equation must hold for every i. Using independence between objects, we get that

$$\begin{aligned} 0 &= r_k \frac{\partial}{\partial r_k} \Pr\left(k \in g_{0,r}(v)_i \,|\, l \in g_{0,r}(v)_i, \forall l \neq k\right) \\ &\quad + \Pr\left(k \in g_{0,r}(v)_i \,|\, l \in g_{0,r}(v)_i, \forall l \neq k\right), \end{aligned}$$

[17] Note that c acts as a discount on the reserve prices if a bidder purchases the full bundle. This is somewhat arbitrary, as we could also define a (c,r)-auction by setting $\lambda_P = \sum_{k \in P_s} r_k + c$ for $P = (\mathcal{K}, \emptyset, ..., \emptyset)$.

Multiplying by $\Pr\left(l \in g_{0,r}(v)_i, \forall l \neq k\right)$, and noting that
$\frac{\partial}{\partial r_k}\Pr\left(l \in g_{0,r}(v)_i, \forall l \neq k\right) = 0$ yields:

$$r_k \frac{\partial}{\partial r_k}\Pr\left(g_{0,r}(v)_i = \mathcal{K}\right) + \Pr\left(g_{0,r}(v)_i = \mathcal{K}\right) = 0$$

Denote by K_i the allocation where agent i receives all the objects and by $K_i\left(\widehat{k}\right)$ the allocation where agent i receives all objects besides object k, which stays with the seller. Recall from the proof of Theorem 1 that $\frac{\partial}{\partial \lambda_P}\Pr\left(g_\lambda = P'\right) = -\left|\partial_\lambda\left(P, P'\right)\right|$. Thus we get

$$0 = r_k \sum_{P \text{ with } k \in P_s} -\left|\partial_{0,r}\left(P, \mathcal{K}_i\right)\right| + \Pr\left(g_{0,r}(v) = \mathcal{K}_i\right)$$

Note that the dimension of $\partial_{0,r}\left(P, \mathcal{K}_i\right)$ - the boundary between the subsets of valuations where the allocations P and K_i are chosen by $g_{0,r}$ - is lower than $NK - 1$. This implies $\left|\partial_{0,r}\left(P, \mathcal{K}_i\right)\right| = 0$ for all $P \neq K_i\left(\widehat{k}\right)$ which, in turn, yields:

$$-r_k\left|\partial_{0,r}\left(\mathcal{K}_i, (\widehat{k})\mathcal{K}_i\right)\right| + \Pr\left(g_{0,r}(v) = \mathcal{K}_i\right) = 0 \tag{10}$$

To calculate $-\frac{\partial BS}{\partial c}(0, r)$, we first note that, by the proof of Theorem 1, we have:

$$\begin{aligned}
\frac{\partial BS}{\partial \lambda_{\mathcal{K}_i}}(0, r) &= \sum_j \left(\Pr\left(g_{0,r}(v) = \mathcal{K}_i\right) - \Pr\left(g_{0,r}^{-j}(v) = \mathcal{K}_i\right)\right) \\
&< \Pr\left(g_{0,r}(v) = \mathcal{K}_i\right) - \Pr\left(g_{0,r}^{-i}(v) = \mathcal{K}_i\right) \\
&= \Pr\left(g_{0,r}(v) = \mathcal{K}_i\right)
\end{aligned}$$

The inequality stems from the fact that $g_{0,r}(v) = K_i$ which ensures that $g_{0,r}^{-j}(v) = K_i$ for $j \neq i$. We thus obtain:

$$-\frac{\partial BS}{\partial c}(0, r) = -\sum_i \frac{\partial BS}{\partial \lambda_{\mathcal{K}_i}}(0, r) > -\sum_i \Pr\left(g_{0,r}(v) = \mathcal{K}_i\right)$$

Reassembling, we get

$$\begin{aligned}
\frac{\partial R}{\partial c}(0, r) &= \frac{\partial S}{\partial c}(0, r) - \frac{\partial BS}{\partial c}(0, r) \\
&> \sum_i \left(r_k\left|\partial_{0,r}\left(\mathcal{K}_i\left(\widehat{k}\right), \mathcal{K}_i\right)\right| - \Pr\left(g_{0,r}(v) = \mathcal{K}_i\right)\right) \\
&= 0
\end{aligned}$$

where the last equality follows from equation 10. ■

The intuition for Proposition 4 is as follows: At the optimal reserve price, a slight reduction in the reserve price leads to a marginal gain in social surplus that is exactly offset by a gain in bidder surplus. The introduction of the bundling parameter yields an additional marginal gain in social surplus (due to additional sales of the other objects). This insight is similar to the one appearing in McAfee et al. (1989)'s study of a monopoly problem. In addition, the introduction of the bundling parameter induces here a decrease in bidders surplus due to the surcharge that bidder i must pay whenever some other bidder j would get all objects if i were not present (this extra effect, that also played a role in Proposition 2, does not arise in the monopoly problem. The two effects together yields the result of Proposition 4.

Proposition 4 makes a local argument around the optimal separate auctions. It does not characterize the optimal combination of reserve prices and mixed bundling, nor does it give an estimate of the amount to be won by mixed bundling.

3.5 Asymmetric λ-auctions

An interesting question is whether in a symmetric setting the optimal λ-auction is symmetric in the sense that it treats the bidders anonymously. The following example[18] illustrates that this need not be the case.

Example 4 *There are two bidders competing for one object. Valuations are independently distributed across bidders, and take two values*[19]*: $\overline{v}$ with probability p and $\underline{v}$ with probability $1-p$, where $\overline{v} > \underline{v}$. The only parameter in a symmetric single-object λ-auction is the reserve price: setting a low reserve price yields expected revenue of $p^2\overline{v} + (1-p^2)\underline{v}$, while setting a high reserve price yields $p(2-p)\overline{v}$. We now construct an asymmetric λ-auction that outperforms both of these, and thus all, symmetric λ-auctions.*

Let $\lambda_i^\varepsilon = \varepsilon$ and $\lambda_j^\varepsilon = \overline{v} - \underline{v}$ where i (j) refers here to the allocation where i (j) receives the object. For small values of ε, the allocation and the payments in the λ^ε-auction are given by the following table:

	$v^j = \underline{v}$	$v^j = \overline{v}$
$v^i = \overline{v}$	i wins; $t^i = -\varepsilon + \underline{v} + (\overline{v} - \underline{v})$	j wins; $t^j = -(\overline{v} - \underline{v}) + \overline{v} + \varepsilon$
$v^i = \underline{v}$	j wins; $t^j = \underline{v}$	j wins; $t^j = \underline{v}$

.

[18] The same example may be used to illustrate that adding asymmetric bidder-specific weights (see Roberts, 1979) may also increase revenue.

[19] The same conclusions will hold for distributions with continuous density approaching the discrete one considered here.

Thus, expected revenue is given by

$$
\begin{aligned}
& p(1-p)(\overline{v}-\varepsilon)+[1-p(1-p)]\underline{v}+p^2\varepsilon \\
= \;& p(1-p)\overline{v}+[1-p(1-p)]\underline{v}+p\,(2p-1)\,\varepsilon
\end{aligned}
$$

This expression exceeds $p^2\overline{v}+(1-p^2)\underline{v}$ *for* $p>\frac{1}{2}$ *and exceeds* $p(2-p)\overline{v}$ *for large values of* $\underline{v}$*. Thus, whenever* $p>\frac{1}{2}$ *and* $\underline{v}$ *is sufficiently large, the asymmetric* λ^{ε}*-auction generates more revenue than the optimal symmetric* λ*-auction.*

4 Conclusion

We have identified the role of mixed bundling for raising revenue in a multi-object auction without a resale market. We have shown that it is neither in the self-interest of the auctioneer to sell the objects in separate, efficient auctions, nor to force the sale of the entire bundle of objects to a single bidder.

The direct mechanisms used in this paper (which are theoretically easy to define and work with) can be seen as Vickrey-Clarke-Groves mechanisms for a society with an additional "virtual" agent, who receives no objects, but gets utility when bundling allocations are being chosen. Thus, even though the real agents' valuations are assumed to be additive, the virtual agent has "preferences" that imply complementarity among objects. An interesting question is whether one can implement mixed bundling via more "intuitive" bidding mechanisms (e.g., an ascending auction). The answer to this question is not entirely clear since complementarities are known to distort other agents' incentives in ascending-bid auctions (see for example Ausubel and Milgrom, 2002).

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