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A Characterization of the Conditions for Optimal Auction with Resale

by Tymofiy Mylovanov and Thomas Tröger*

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Abstract

Zheng has proposed a seller-optimal auction for (asymmetric) independent-private-value environments where inter-bidder resale is possible. Zheng's construction requires novel conditions—Resale Monotonicity, Transitivity, and Invariance—on the bidders' value distribution profile. The only known examples of distribution profiles satisfying these conditions in environments with three or more bidders are uniform distributions. Our characterization result shows that Zheng's conditions, while being strong, are satisfied by many non-uniform distribution profiles. A crucial step in our analysis is to show that Invariance implies Resale Monotonicity and Transitivity.

Keywords: independent private values, optimal auction, resale, inverse virtual valuation function

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1 Introduction

Zheng (2002) makes an important contribution to the theory of mechanism design with limited commitment.[1] Starting from the observation that "much of the auction design literature makes the unrealistic assumption that winning bidders cannot attempt to resell the good to losing bidders," Zheng proposes an alternative auction design that takes into account the inability of the initial seller to prohibit resale. He considers a sequential mechanism selection game where each current owner of the good chooses her sales mechanism knowing that the winner of today's mechanism will herself choose a sales mechanism that is optimal given that the next winner will choose an optimal sales mechanism, and so on.

Zheng (2002) establishes conditions on the profile of the distributions (c.d.f.s) of the bidders' values such that the sequential mechanism selection game has an equilibrium where the initial seller obtains the same profit as when she can prohibit resale. Zheng's conditions have five parts: Hazard Rate (HR), Uniform Bias (UB), Resale Monotonicity (RM), Transitivity (TR), and Invariance (IV). The first two are straightforward: HR essentially requires smoothness of the bidders' c.d.f.s, and that every c.d.f. has a weakly increasing hazard rate, while UB requires that the bidders can be ranked according to their c.d.f.s' supports and hazard rates, where bidder 1 is the one who has the smallest support and the largest hazard rate.

The novel conditions RM, TR, and IV are more difficult—the only known examples of c.d.f. profiles satisfying these conditions in environments with three or more bidders are uniform distributions (Zheng, 2002, Example 3). However, as Zheng (2002, pp. 2213–2216) explains, the novel conditions are crucial for his construction. Hence, it is important to understand more generally which c.d.f. profiles satisfy the conditions. A second motivation for investigating the conditions is that related conditions can be expected to come up in future work building on Zheng (2002). For example, Lebrun (2005) considers the design of personalized entry fees in a second-price auction with resale. Under a condition similar to RM, Lebrun constructs an equilibrium in mixed strategies that implements the same

[1] Other recent contributions include Bester-Strausz (2001), Skreta (forthcoming), Calzolari-Pavan (forthcoming).

allocation as Zheng.[2]

We provide a characterization of the set of c.d.f. profiles satisfying Zheng's conditions. It turns out that the conditions, while being strong, are satisfied by many non-uniform c.d.f. profiles. Hence, Zheng's result applies more generally than one may have thought before.

Technically, our crucial insight is that the conditions relate the bidders' c.d.f.s at points where the bidders tie with their virtual valuations. Hence, by reformulating the conditions in terms of inverse virtual valuation functions, we obtain a more transparent form of the conditions.

In environments with two bidders, RM is the crucial condition because TR and IV are empty. RM is equivalent to a differential inequality in terms of inverse virtual valuation functions (Proposition 1), from which all two-bidder c.d.f. profiles satisfying Zheng's conditions can be computed.

In environments with $n \geq 3$ bidders, condition IV is equivalent to a set of differential equations in terms of inverse virtual valuation functions, supplemented with a lower bound requirement for derivatives (Proposition 2). We show that IV implies RM and TR (Propositions 3 and 4). Furthermore, for any c.d.f. of bidder n that has a weakly decreasing density, and any profile of nested supports, there exists a unique profile of c.d.f.s for bidders 2 to $n-1$ such that IV is satisfied, while bidder 1's c.d.f. remains unrestricted; we provide explicit formulae for the inverse virtual valuation functions of bidders 2 to $n-1$ (Proposition 5). Using these formulae, one can compute all profiles of c.d.f.s satisfying Zheng's conditions. In particular, if the highest possible valuation is the same for all bidders, then the c.d.f.s for bidders 2 to $n-1$ are affine transformations of bidder n's c.d.f. (Corollary 1).

2 Results

We reiterate only those aspects of Zheng's model that are needed to state his assumptions. Consider an independent-private-value auction environment with $n \geq 2$ bidders. The distri-

[2]Lebrun (2005) relaxes UB. As long as UB is satisfied, his condition (2005, Corollary 7) is essentially equivalent to RM (cf. footnote 4). He also explains (2005, Appendix 9) that his condition is necessary within a restricted class of allocation rules. Conditions TR and IV play no role because Lebrun's analysis is restricted to two bidders.

bution (c.d.f.) for the valuation of bidder $i = 1, \ldots, n$ is denoted F_i with support T_i.

Assumption 1 (HR) of Zheng consists of standard elements and needs no further discussion.

Assumption 1 (Hazard Rate) *For each player i, the support T_i of F_i is convex and bounded from below. If T_i is a non-degenerate interval, the density function f_i is positive and continuous on T_i and differentiable in its interior, and $(1 - F_i(t_i))/f_i(t_i)$ is a weakly decreasing function of t_i on T_i.*

We add the assumptions that for all i, the support T_i is non-degenerate and bounded, the derivative f_i' exists at the boundary of T_i, and f_i' is continuous on T_i. Let $\underline{t}_i = \min T_i$ and $\bar{t}_i = \max T_i$. Define the hazard rate $\lambda_i(t_i) = f_i(t_i)/(1 - F_i(t_i))$ for all t_i such that $F_i(t_i) < 1$.

The virtual valuation functions V_i $(i = 1, \ldots, n)$ are defined by $V_i(t_i) = t_i - (1 - F_i(t_i))/f_i(t_i)$ $(t_i \in T_i)$. Given the above assumptions, the derivative V_i' exists and is continuous and ≥ 1. Moreover,

$$V_i(T_i) = [V_i(\underline{t}_i), \bar{t}_i] \quad (i = 1, \ldots, n). \tag{1}$$

The inverse virtual valuation function V_i^{-1} is well-defined on $V_i(T_i)$. The derivative $(V_i^{-1})'$ is continuous and

$$\forall v_i \in V_i(T_i): \ (V_i^{-1})'(v) \in (0, 1]. \tag{2}$$

A straightforward computation shows that

$$f_i \text{ weakly decreasing} \ \Leftrightarrow \ (V_i^{-1})' \geq 1/2. \tag{3}$$

Because (see, e.g., Krishna (2002, p. 255)) $f_i(t) = \lambda_i(t) \exp(-\int_{\underline{t}_i}^{t} \lambda_i(t')\mathrm{d}t')$ and $\lambda_i(t) = 1/(t - V_i(t))$ for all $t \in [\underline{t}_i, \bar{t}_i)$,

$$f_i(V_i^{-1}(v)) \quad = \quad \frac{1}{V_i^{-1}(v) - v} e^{-\int_{\underline{t}_i}^{V_i^{-1}(v)} \frac{1}{t' - V_i(t')}\mathrm{d}t'} \qquad (v \in [V_i(\underline{t}_i), \bar{t}_i)). \tag{4}$$

Similarly,

$$F_i(V_i^{-1}(v)) \quad = \quad 1 - e^{-\int_{\underline{t}_i}^{V_i^{-1}(v)} \frac{1}{t' - V_i(t')}\mathrm{d}t'} \qquad (v \in [V_i(\underline{t}_i), \bar{t}_i))). \tag{5}$$

Using (4) and (5), we can determine the marginal effect of a change of a bidder's virtual valuation on her logarithmic hazard rate and logarithmic density,

$$\frac{\mathrm{d}}{\mathrm{d}v}\ln\lambda_i(V_i^{-1}(v)) = \frac{1-(V_i^{-1})'(v)}{V_i^{-1}(v)-v} \quad (v\in[V_i(\underline{t}_i),\bar{t}_i)), \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\ln f_i(V_i^{-1}(v)) = \frac{1-2(V_i^{-1})'(v)}{V_i^{-1}(v)-v} \quad (v\in[V_i(\underline{t}_i),\bar{t}_i)). \tag{7}$$

Assumption 2 (UB) of Zheng states that the bidders $i = 1,\dots,n$ can be ranked in terms of the support T_i and of the virtual valuation function V_i. Observe that Assumption 2 is equivalent to hazard rate dominance if $T_1 = \dots = T_n$.[3]

Assumption 2 (Uniform Bias) *For all $i, j = 1,\dots,n$, if $i < j$ then $T_i \subseteq T_j$ and $V_i(x) \geq V_j(x)$ for all $x \in T_i$.*

By (1) and UB,

$$\forall i,j=1,\dots,n: \quad \text{if } i<j \text{ then } V_i(T_i)\subseteq V_j(T_j). \tag{8}$$

For $i < j$, let $\nu_{ij}(t_i) = V_j^{-1}(V_i(t_i))$. Zheng defines functions $\beta_{ij} : T_i \to T_j$ implicitly by

$$F_j(\beta_{ij}(t_i)) = F_j(\nu_{ij}(t_i)) + (\nu_{ij}(t_i)-t_i)f_j(\nu_{ij}(t_i)). \tag{9}$$

The β_{ij} functions play a central role in Zheng's equilibrium construction.[4] In particular, he requires Assumption 3 (RM).

Assumption 3 (Resale Monotonicity) *For all $i, j = 1,\dots,n$, if $i < j$ then β_{ij} is weakly increasing.*

Simplifying equation (9) and solving for β_{ij} yields

[3]Hazard rate dominance is a stronger requirement than stochastic dominance and a weaker requirement than likelihood ratio dominance (see, e.g., Krishna (2002, Appendix B)).

[4] Observe that the functions ν_{ij}, and hence the functions β_{ij}, are, in general, well-defined only if UB holds. Lebrun (2005) relaxes UB and extends the definition of β_{ij}. Accordingly, he obtains a generalized version of RM.

Lemma 1 *For all $i, j = 1, \ldots, n$, if $i < j$ then for all $t_i \in T_i$,*

$$\beta_{ij}(t_i) = F_j^{-1}\left(1 - \frac{f_j(\nu_{ij}(t_i))}{\lambda_i(t_i)}\right) \quad \text{if} \quad t_i < \bar{t}_i,$$

and $\beta_{ij}(\bar{t}_i) = \bar{t}_j$.

Proof. Dividing (9) by $f_j(\nu_{ij}(t_i))$ and using the definition of V_j yields that (9) is equivalent to

$$\forall t_i \in [\underline{t}_i, \bar{t}_i] : \ \frac{F_j(\beta_{ij}(t_i)) - 1}{f_j(\nu_{ij}(t_i))} \underbrace{-V_j(\nu_{ij}(t_i)) + t_i}_{=-V_i(t_i)+t_i} = 0. \tag{10}$$

The fact that $\beta_{ij}(\bar{t}_i) = \bar{t}_j$ follows because $-V_i(\bar{t}_i) + \bar{t}_i = 0$. The proof is completed by noting that $-V_i(t_i) + t_i = 1/\lambda_i(t_i)$ for all $t_i < \bar{t}_i$. *QED*

From Lemma 1 one sees that RM is satisfied if and only if, for all $i < j$,

$$\frac{\lambda_i(t_i)}{f_j(\nu_{ij}(t_i))} \quad \text{is weakly increasing for all } t_i \in [\underline{t}_i, \bar{t}_i). \tag{11}$$

The reason it is difficult to see which c.d.f. profiles satisfy RM is that the expression $f_j(\nu_{ij}(t_i))$ in (11) depends on bidder i's c.d.f. as well as bidder j's c.d.f.. If, however, (11) is reformulated in terms of virtual valuations, and the logarithm is taken, then quantities referring to bidder i become additively separated from quantities referring to bidder j. This yields a characterization of RM in terms of differential inequalities (12) involving inverse virtual valuation functions.

Proposition 1 *Suppose that HR and UB hold. Then RM holds if and only if, for all $i < j$,*

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i) : \ \frac{1 - (V_i^{-1})'(v)}{V_i^{-1}(v) - v} \geq \frac{1 - 2(V_j^{-1})'(v)}{V_j^{-1}(v) - v}. \tag{12}$$

Proof. By Lemma 1, β_{ij} is weakly increasing if and only if (11) holds. Substitution of variables $t_i = V_i^{-1}(v)$ yields that β_{ij} is weakly increasing if and only if

$$\frac{\lambda_i(V_i^{-1}(v))}{f_j(V_j^{-1}(v))} \quad \text{is weakly increasing for all } v \in [V_i(\underline{t}_i), \bar{t}_i).$$

Taking the logarithm yields that β_{ij} is weakly increasing if and only if

$$\ln \lambda_i(V_i^{-1}(v)) - \ln f_j(V_j^{-1}(v)) \quad \text{is weakly increasing for all } v \in [V_i(\underline{t}_i), \bar{t}_i).$$

Because a continuously differentiable function is weakly increasing if and only if its derivative is non-negative, β_{ij} is weakly increasing if and only if

$$\frac{\mathrm{d}}{\mathrm{d}v}\ln\lambda_i(V_i^{-1}(v)) \geq \frac{\mathrm{d}}{\mathrm{d}v}\ln f_j(V_j^{-1}(v)) \qquad (v \in [V_i(\underline{t}_i), \overline{t}_i)).$$

Using (6) and (7), the proof is complete. *QED*

It is not possible to simplify (12) by using additional properties of virtual valuation functions, because there are essentially no additional properties: any continuously differentiable function defined on an interval $[\underline{t}_i, \check{t}]$ ($\check{t} < \overline{t}_i$) with derivative not smaller than 1 and values below the identity function can be extended to the virtual valuation function of some c.d.f. F_i satisfying HR (to see this, use, e.g., Krishna, 2002, p. 255).

In environments with two bidders, the conditions assumed in Zheng (2002) are HR, UB, and RM. Proposition 1 can then be used to compute all c.d.f. profiles satisfying Zheng's assumptions: for any given c.d.f. of bidder $n = j = 2$, (12) provides a linear differential inequality for the c.d.f. of bidder $i = 1$. Observe that RM is satisfied whenever the c.d.f. of bidder $n = 2$ has a weakly decreasing density. In this case, the left-hand side of (12) with $i = 1$ is ≥ 0 by (2), and the right-hand side of (12) with $j = 2$ is ≤ 0 by (3).[5]

For environments with $n \geq 3$ bidders, Zheng (2002) makes the additional assumptions Transitivity (TR) and Invariance (IV). Let $\beta_{ij}^{-1}(t_j) = \inf\{t_i \in T_i \mid \beta_{ij}(t_i) \geq t_j\}$ for $t_j \leq \overline{t}_j$.

Assumption 4 (Transitivity) *If bidder i is ranked before bidder j and j is ranked before bidder k ($i < j < k$), then for any t_j less than or equal to the supremum of the range of β_{ij}, $\beta_{ik}(\beta_{ij}^{-1}(t_j)) \geq V_k^{-1}(V_j(t_j))$.*

[5]Things are less straightforward if bidder 2's density is not weakly increasing. The right-hand side of (12) is then > 0 for some $v = \check{v}$. Condition RM can still hold (for example, when both bidders have the same c.d.f. $F_1 = F_2$). However, one can always find bidder-1 c.d.f.s (with the same support as the bidder-2 c.d.f.) such that RM is violated. The proof works by constructing bidder 1's c.d.f. such that the left-hand side of (12) equals 0 at $v = \check{v}$ (Mylovanov and Tröger, 2005).

Assumption 5 (Invariance)

For all $w = 1, \ldots, n$, *and* $i, j > w$, *if* $t_i \le \beta_{wi}(t_w)$ *and* $t_j \le \beta_{wj}(t_w)$,

then $V_i(t_i) \ge (\text{resp.} =) V_j(t_j)$ *implies* $f_i(\nu_{wi}(t_w))/f_i(t_i) \ge (\text{resp.} =) f_j(\nu_{wj}(t_w))/f_j(t_j)$.[6]

The following result characterizes IV in terms of a set of differential equations and inequalities involving inverse virtual valuation functions. The proof begins by reformulating IV in terms of virtual valuations (Lemma 2). From this reformulation one sees (13) that IV requires the expressions (7) to be identical for all bidders except bidder 1. The inequality part in the definition of IV is captured in (14), which by (3) is equivalent to the requirement that all c.d.f.s except bidder 1's have a weakly decreasing density.

Proposition 2 *Suppose that* $n \ge 3$ *and HR and UB hold. Then IV holds if and only if for all* $j > i \ge 2$,

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i) : \quad \frac{1 - 2(V_i^{-1})'(v)}{V_i^{-1}(v) - v} = \frac{1 - 2(V_j^{-1})'(v)}{V_j^{-1}(v) - v}, \tag{13}$$

$$(V_i^{-1})' \ge 1/2. \tag{14}$$

The proof of Proposition 2 relies on

Lemma 2 *Suppose that* $n \ge 3$ *and HR and UB hold. Then IV holds if and only if for* $i, j \ge 2$ *there exist constants* $c_{ij} > 0$ *such that*

$$\forall v_i \in V_i(T_i),\ v_j \in V_j(T_j) : \quad v_i \ge (\text{resp.} =) v_j \Rightarrow \frac{f_i(V_i^{-1}(v_i))}{f_j(V_j^{-1}(v_j))} \le (\text{resp.} =) c_{ij}. \tag{15}$$

Proof of Lemma 2. "only if": Define $t_i = V_i^{-1}(v_i)$ and $t_j = V_j^{-1}(v_j)$. By Lemma 1, $t_i \le \bar{t}_i = \beta_{1i}(\bar{t}_1)$, and $t_j \le \bar{t}_j = \beta_{1j}(\bar{t}_1)$. Hence, using IV with $w = 1$ and $t_w = \bar{t}_1$,

$$\frac{f_i(V_i^{-1}(v_i))}{f_j(V_j^{-1}(v_j))} = \frac{f_i(t_i)}{f_j(t_j)} \le (\text{resp.} =) \frac{f_i(\nu_{1i}(\bar{t}_1))}{f_j(\nu_{1j}(\bar{t}_1))} =: c_{ij}.$$

"if": Consider $i, j > w \ge 1$ and $t_i \le \beta_{wi}(t_w)$, $t_j \le \beta_{wj}(t_w)$ such that $V_i(t_i) \ge (\text{resp.} =) V_j(t_j)$. Using (15) with $v_i = v_j = V_w(t_w)$,

$$c_{ij} = \frac{f_i(V_i^{-1}(V_w(t_w)))}{f_j(V_j^{-1}(V_w(t_w)))} = \frac{f_i(\nu_{wi}(t_w))}{f_j(\nu_{wj}(t_w))}. \tag{16}$$

[6] Zheng's paper contains a typo in Assumption 5 that is corrected here. He requires that ">" implies ">", but this is not needed and obviously is not meant because it would be violated by his own Example 3.

Using (15) with $v_i = V_i(t_i)$ and $v_j = V_j(t_j)$,

$$\frac{f_i(t_i)}{f_j(t_j)} \le (\text{resp.} =) c_{ij} \overset{(16)}{=} \frac{f_i(\nu_{wi}(t_w))}{f_j(\nu_{wj}(t_w))}.$$

This completes the proof. *QED*

Proof of Proposition 2. "only if": Let $j > i \ge 2$. Taking the logarithm on the r.h.s. of (15), there exist constants C_{ij} such that for all $v_i \in V_i(T_i)$ and $v_j \in V_j(T_j)$,

$$v_i \ge (\text{resp.} =) v_j \;\; \Rightarrow \;\; \ln(f_i(V_i^{-1}(v_i))) \le (\text{resp.} =) \ln(f_j(V_j^{-1}(v_j))) + C_{ij}. \tag{17}$$

Using (17) with $v = v_i = v_j$,

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i) : \; \ln(f_i(V_i^{-1}(v))) = \ln(f_j(V_j^{-1}(v))) + C_{ij}. \tag{18}$$

Taking derivatives in (18) and using (7), we obtain (13).

Using (15) with $i = j \ge 2$ and $v_i = v_j$, one sees that $c_{ii} = 1$. Hence, (15) implies that $f_i(V_i^{-1}(\cdot))$ is weakly decreasing, which implies (14) by (3).

"if": By (13) and (7), there exist constants C_{ij} such that (18) holds for $j > i \ge 2$. Applying the exponential function to (18) yields

$$\forall v \in [V_i(\underline{t}_i), \bar{t}_i) : \; \frac{f_i(V_i^{-1}(v))}{f_j(V_j^{-1}(v))} = c_{ij}, \tag{19}$$

where we define $c_{ij} = \exp C_{ij}$. For all $i \ge 2$, define $c_{ii} = 1$. For all $i > j \ge 2$, define $c_{ij} = 1/c_{ji}$. By (19), for all $i, j \ge 2$,

$$\forall v \in V_i(T_i) \cap V_j(T_j) : \; \frac{f_i(V_i^{-1}(v))}{f_j(V_j^{-1}(v))} = c_{ij}. \tag{20}$$

Using (14) and (3), $f_i(V_i^{-1}(\cdot))$ and $f_j(V_j^{-1}(\cdot))$ are weakly decreasing. Together with (20) this implies (15), and IV follows by Lemma 2. *QED*

From Propositions 1 and 2 it is easy to see that IV implies RM.[7]

[7] Alternatively, Proposition 3 can be obtained directly from Lemma 1. Using the definition of IV with $w = 1$, $t_w = \bar{t}_1$, and $t_j = \nu_{ij}(t_i)$, one sees that $f_j(\nu_{ij}(t_i))/f_i(t_i)$ is independent of t_i for all $j > i \ge 2$. Hence, $f_j(\nu_{ij}(t_i))/\lambda_i(t_i) = (f_j(\nu_{ij}(t_i))/f_i(t_i))(1 - F_i(t_i))$ is strictly decreasing in t_i, which implies that β_{ij} is weakly increasing by Lemma 1. Using the definition of IV with $w = 1$, $t_w = \bar{t}_1$, and $i = j \ge 2$, one sees that f_j is weakly decreasing, hence $f_j(\nu_{1j}(t_1))/\lambda_1(t_1)$ is weakly decreasing in t_1 by HR, which implies that β_{1j} is weakly increasing by Lemma 1.

Proposition 3 *Suppose that $n \geq 3$ and HR, UB, and IV hold. Then RM is satisfied.*

Proof. By Proposition 2, for all $j > i \geq 2$ and all $v \in [V_i(\underline{t}_i), \bar{t}_i)$,

$$\frac{1-(V_i^{-1})'(v)}{V_i^{-1}(v)-v} \overset{(2)}{\geq} \frac{1-2(V_i^{-1})'(v)}{V_i^{-1}(v)-v} = \frac{1-2(V_j^{-1})'(v)}{V_j^{-1}(v)-v},$$

which shows (12) for all $j > i \geq 2$. For all $j \geq 2$,

$$\frac{1-(V_1^{-1})'(v)}{V_1^{-1}(v)-v} \overset{(2)}{\geq} 0 \overset{(14)}{\geq} \frac{1-2(V_j^{-1})'(v)}{V_j^{-1}(v)-v} \quad (v \in [V_1(\underline{t}_1), \bar{t}_1)),$$

showing (12) for all $j > i = 1$. *QED*

The next result shows that IV implies TR. The proof relies on auxiliary functions

$$y_{li}(v) = \int_v^{V_l \beta_{il} V_i^{-1}(v)} \frac{(V_l^{-1})'(w)}{V_l^{-1}(w)-w} \mathrm{d}w. \quad (l > i,\ v \in V_i(T_i)). \tag{21}$$

From UB it follows that $y_{li}(v)$ is weakly increasing in l.[8] From UB and IV it follows (24) that the integrand in (21) is weakly decreasing in l. Hence, the area of integration must be weakly increasing in l, which implies TR, where we use a restatement of TR in terms of virtual valuations (22).

Proposition 4 *Suppose that $n \geq 3$ and HR, UB, and IV hold. Then TR is satisfied.*

Proof. To show TR, it is sufficient that, for all $i < j < k$ and $v \in V_i(T_i)$,

$$V_k \beta_{ik} V_i^{-1}(v) \geq V_j \beta_{ij} V_i^{-1}(v). \tag{22}$$

(To see this, set $v = V_i(\beta_{ij}^{-1}(t_j))$, and use that β_{ij} is weakly increasing by Proposition 3.)

By Lemma 1 with $t_i = V_i^{-1}(v)$, we have $1 - F_l \beta_{il}(V_i^{-1}(v)) = f_l(V_l^{-1}(v))(V_i^{-1}(v) - v)$ for all $l > i$. Hence,

$$1 - F_l V_l^{-1}(V_l \beta_{il} V_i^{-1}(v)) = f_l(V_l^{-1}(v))(V_i^{-1}(v) - v).$$

[8]The intuition is as follows. Using (5), $e^{-y_{li}(v)} = F_l(\beta_{il} V_i^{-1}(v)) - F_l(V_l^{-1}(v))$ equals the probability that bidder i, if her virtual valuation equals v and if no other bidders are involved, wins against bidder l and subsequently resells to l, according to Zheng's equilibrium construction. UB implies that this probability is weakly decreasing in l. Thus, y_{li} is weakly increasing in l.

Hence, using (4) and (5),

$$e^{-\int_{\underline{t}_l}^{\beta_{il}V_i^{-1}(v)} \frac{1}{t'-V_l(t')}\mathrm{d}t'} = \frac{V_i^{-1}(v)-v}{V_l^{-1}(v)-v} e^{-\int_{\underline{t}_l}^{V_l^{-1}(v)} \frac{1}{t'-V_l(t')}\mathrm{d}t'}.$$

Rearranging yields

$$\int_{V_l^{-1}(v)}^{\beta_{il}V_i^{-1}(v)} \frac{1}{t'-V_l(t')}\mathrm{d}t' = \ln \frac{V_l^{-1}(v)-v}{V_i^{-1}(v)-v}.$$

Substitution of variables $w = V_l(t')$ yields

$$y_{li}(v) \overset{(21)}{=} \int_{v}^{V_l\beta_{il}V_i^{-1}(v)} \frac{(V_l^{-1})'(w)}{V_l^{-1}(w)-w}\mathrm{d}w = \ln \frac{V_l^{-1}(v)-v}{V_i^{-1}(v)-v}.$$

Hence, using UB,

$$y_{ki}(v) \geq y_{ji}(v). \tag{23}$$

Using the characterization of IV in Proposition 2,

$$\frac{2(V_k^{-1})'(v)}{V_k^{-1}(v)-v} - \frac{2(V_j^{-1})'(v)}{V_j^{-1}(v)-v} = \frac{1}{V_k^{-1}(v)-v} - \frac{1}{V_j^{-1}(v)-v} \overset{\text{UB}}{\leq} 0. \tag{24}$$

Suppose that (22) fails. Then,

$$y_{ki}(v) < \int_{v}^{V_j\beta_{ij}V_i^{-1}(v)} \frac{(V_k^{-1})'(w)}{V_k^{-1}(w)-w}\mathrm{d}w \overset{(24)}{\leq} y_{ji}(v), \tag{25}$$

in contradiction with (23). *QED*

Propositions 2, 3, and 4 suggest a procedure to construct systematically all c.d.f. profiles that satisfy Zheng's assumptions if $n \geq 3$. One begins with an arbitrary profile of nested supports for the bidders' c.d.f.s, and with any c.d.f. for bidder n that is consistent with HR and satisfies $(V_n^{-1})' \geq 1/2$ (this inequality is necessary for IV, see (14)). Then one solves the differential equations (13) with $j = n$ to compute inverse virtual valuation functions for bidders 2 to $n-1$ (the solutions will depend on the chosen supports). The corresponding c.d.f.s can be calculated from (5). Finally, one chooses for bidder 1 any c.d.f. that is consistent with HR and UB. Proposition 5 shows that this procedure works.

The following result provides a complete characterization of the set of c.d.f. profiles satisfying HR, UB, RM, TR, and IV. In particular, we give the solutions (26) to the differential equations (13).

Proposition 5 *Let $n \geq 3$. Let $T_1 \subseteq \ldots \subseteq T_n$ be any compact intervals. Let F_n be any c.d.f. that is consistent with HR and satisfies $(V_n^{-1})' \geq 1/2$.*

Then there exists a unique profile $(F_2, \ldots, F_{n-1})$ such that HR, UB, RM, TR, and IV hold for $(F_1, \ldots, F_n)$, where F_1 is any c.d.f. that is consistent with HR and UB.

For all $j > i \geq 2$ and $v \in [V_i(\underline{t}_i), \overline{t}_i]$,

$$V_i^{-1}(v) = V_j^{-1}(v) - \sqrt{(V_j^{-1}(v) - v)(V_j^{-1}(\overline{t}_i) - \overline{t}_i)}\; e^{-\frac{1}{2}\int_{V_j^{-1}(v)}^{V_j^{-1}(\overline{t}_i)} \frac{1}{t' - V_j(t')} dt'}. \qquad (26)$$

The proof of Proposition 5 relies on

Lemma 3 *Let $l \in \{2, \ldots, n-1\}$. Let F_{l+1} denote a c.d.f. that is consistent with HR and has a weakly decreasing density. Let V_{l+1} denote the corresponding virtual valuation function.*

Then, for any interval $[\underline{t}, \overline{t}] \subseteq [V_{l+1}(\underline{t}_{l+1}), \overline{t}_{l+1}]$, there exists a unique continuously differentiable function g on $[\underline{t}, \overline{t}]$ such that

$$g \leq V_{l+1}^{-1}, \qquad (27)$$

$$g(\overline{t}) = \overline{t}, \qquad (28)$$

and

$$\forall v \in [\underline{t}, \overline{t}) : \; \frac{2g'(v) - 1}{g(v) - v} = \frac{2(V_{l+1}^{-1})'(v) - 1}{V_{l+1}^{-1}(v) - v} =: h_{l+1}(v). \qquad (29)$$

Proof of Lemma 3. Suppose that $\overline{t} < \overline{t}_{l+1}$. Because h_{l+1} is continuous at $v = \overline{t}$, standard results for differential equations (see, e.g., Walter (1998, p. 62)) show the existence of a unique g satisfying (28) and (29). It remains to show (27). From (29),

$$\forall v \in [\underline{t}, \overline{t}) : \text{ if } g(v) > V_{l+1}^{-1}(v) \text{ then } g'(v) \geq (V_{l+1}^{-1})'(v).$$

Hence, if $g(\hat{v}) > V_{l+1}^{-1}(\hat{v})$ for some $\hat{v} < \overline{t}$, then $g(\overline{t}) > V_{l+1}^{-1}(\overline{t})$. On the other hand, $V_{l+1}^{-1}(\overline{t}) \geq \overline{t} = g(\overline{t})$ by definition of V_{l+1}, a contradiction. Thus, (27).

Now suppose that $\overline{t} = \overline{t}_{l+1}$. Because $h_{l+1}(v) \to \infty$ as $v \to \overline{t}$, standard uniqueness results for differential equations do not apply. However, $g = V_{l+1}^{-1}$ obviously satisfies (27), (28), and (29). Let $g = k$ denote another function satisfying the same conditions.

Multiplying (29) by $g(v) - v$ and subtracting the resulting expression with $g = k$ from the resulting expression with $g = V_{l+1}^{-1}$ yields the homogeneous linear equation $2m'(v) = m(v)h_{l+1}(v)$ for $m := V_{l+1}^{-1} - k$. Hence,

$$k(v) = \alpha \, e^{\int_{\underline{t}}^{v} \frac{h_{l+1}(w)}{2} \mathrm{d}w} + V_{l+1}^{-1}(v), \tag{30}$$

for some $\alpha \in \mathbb{R}$. Because $k \leq V_{l+1}^{-1}$ by (27), we have $\alpha \leq 0$. By (30),

$$k'(v) = \alpha \, \frac{h_{l+1}(v)}{2} \, e^{\int_{\underline{t}}^{v} \frac{h_{l+1}(w)}{2} \mathrm{d}w} + (V_{l+1}^{-1})'(v). \tag{31}$$

By (3), $h_{l+1}(v) \geq 0$. Hence, (31) implies $k'(v) \leq (V_{l+1}^{-1})'(v)$. Together with $k(\bar{t}) = \bar{t} = V_{l+1}^{-1}(\bar{t})$ this implies $k \geq V_{l+1}^{-1}$. Hence, $k = V_{l+1}^{-1}$. *QED*

Proof of Proposition 5. By Propositions 3 and 4, we can ignore RM and TR throughout the proof.

"Existence": We show the existence of $F_2, \ldots, F_{n-1}$ by proving inductively, for all $l < n$, claim

> (*l) There exists a profile $F_{l+1}, \ldots, F_n$ that is consistent with HR and UB, (13) holds for all $j > i \geq l + 1$, and $(V_i^{-1})' \geq 1/2$ for all $i \geq l + 1$.

Claim (*$(n-1)$) holds by assumption. Suppose that (*l) holds for some $l \in \{2, \ldots, n-1\}$. Let V_{l+1} denote the virtual valuation function for F_{l+1}. By Lemma 3, there exists a continuously differentiable function g on $[V_{l+1}(\underline{t}_{l+1}), \bar{t}_l]$ such that (27), (28), and (29) hold with $\underline{t} = V_{l+1}(\underline{t}_{l+1})$ and $\bar{t} = \bar{t}_l$.

Recall from (2) and (3) that

$$\forall v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l] : \; \frac{1}{2} \leq (V_{l+1}^{-1})'(v) \leq 1. \tag{32}$$

Consider

$$A = \arg \min_{v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l]} g(v) - v.$$

If $\hat{v} < \bar{t}_l$ and $\hat{v} \in A$, then $g'(\hat{v}) - 1 \geq 0$ from the first-order conditions, hence $g(\hat{v}) - \hat{v} > 0$ by (29) and (32), in contradiction with (28). We conclude that $A = \{\bar{t}_l\}$. Hence,

$$\forall v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l) : \; g(v) > v. \tag{33}$$

From (29), (32), and (33),

$$\forall v \in [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l] : \ g'(v) \geq \frac{1}{2}. \tag{34}$$

In particular, we can define the inverse

$$g^{-1} : \ [g(V_{l+1}(\underline{t}_{l+1})), \bar{t}_l] \to [V_{l+1}(\underline{t}_{l+1}), \bar{t}_l]. \tag{35}$$

From (27) we obtain $g(V_{l+1}(\underline{t}_{l+1})) \leq \underline{t}_{l+1} \leq \underline{t}_l$. Hence, g^{-1} exists on $[\underline{t}_l, \bar{t}_l]$. Moreover, by (33), $t' - g^{-1}(t') > 0$ for all $t' \in [\underline{t}_l, \bar{t}_l)$. Thus, we can define

$$F_l(t) = \begin{cases} 1 - e^{-\int_{\underline{t}_l}^{t} \frac{1}{t' - g^{-1}(t')} \mathrm{d}t'} & \text{if } t \in [\underline{t}_l, \bar{t}_l), \\ 1 & \text{if } t = \bar{t}_l. \end{cases} \tag{36}$$

The function F_l is continuous at $\bar{t}_l$ because, for all $t < \bar{t}_l$,

$$\begin{aligned} \int_{\underline{t}_l}^{t} \frac{1}{t' - g^{-1}(t')} \mathrm{d}t' &= \int_{g^{-1}(\underline{t}_l)}^{g^{-1}(t)} \frac{g'(w)}{g(w) - w} \mathrm{d}w \\ &> \int_{g^{-1}(\underline{t}_l)}^{g^{-1}(t)} \frac{g'(w) - 1}{g(w) - w} \mathrm{d}w \\ &= \ln(t - g^{-1}(t)) - \ln(\underline{t}_l - g^{-1}(\underline{t}_l)) \\ &\quad \to_{t \to \bar{t}_l} \infty \quad \text{because } \bar{t}_l - g^{-1}(\bar{t}_l) = 0. \end{aligned}$$

From (36) we obtain on $[\underline{t}_l, \bar{t}_l)$ the continuously differentiable density

$$f_l(t) = \frac{1}{t - g^{-1}(t)} e^{-\int_{\underline{t}_l}^{t} \frac{1}{t' - g^{-1}(t')} \mathrm{d}t'} \quad (t \in [\underline{t}_l, \bar{t}_l)). \tag{37}$$

Using (4) with $i = l + 1$ and (37), equation (29) implies that the derivative $(\ln f_l(g(v)))' = \ln(f_{l+1}(V_{l+1}^{-1}(v)))'$ for all $v \in [g^{-1}(\underline{t}_l), \bar{t}_l)$. Hence,

$$\exists c > 0 \ \forall v \in [g^{-1}(\underline{t}_l), \bar{t}_l) : \ f_l(g(v)) = c f_{l+1}(V_{l+1}^{-1}(v)). \tag{38}$$

Because $f_{l+1}(V_{l+1}^{-1}(g^{-1}(\cdot)))$ is continuously differentiable on $[\underline{t}_l, \bar{t}_l]$, (38) shows that f_l extends continuously differentiable to the point $t = \bar{t}_l$.

Define $V_l = g^{-1}|_{T_l}$. Using (27), equation (29) implies $g' \leq (V_{l+1}^{-1})'$. Hence, $g' \leq 1$ by (32). Hence, $V_l' \geq 1$, which implies that $(1 - F_l(\cdot))/f_l(\cdot)$ is weakly decreasing. In summary, F_l

is consistent with HR. By (34), $(V_l^{-1})' \geq 1/2$. By (27), F_l is consistent with UB. By (29), equation (13) holds with $i = l$ and $j = l + 1$. By induction, (13) holds for all $j \geq i = l$. This completes the proof of claim $(*(j-1))$.

"Uniqueness": Let $l \in \{2, \ldots, n-1\}$. Consider a c.d.f. F_{l+1} that is consistent with HR. Let V_{l+1} denote the corresponding virtual valuation function. Consider two c.d.f.s F and $\check{F}$ for bidder l with support T_l that are consistent with HR, UB, and IV. Denote by V and $\check{V}$ the corresponding virtual valuation functions. Without loss of generality, $V(\underline{t}_l) \leq \check{V}(\underline{t}_l)$. By Proposition 2 and the uniqueness statement in Lemma 3,

$$\forall v \in [\check{V}(\underline{t}_l), \bar{t}_l] : \ V^{-1}(v) = \check{V}^{-1}(v). \tag{39}$$

Applying (39) at $v = \check{V}(\underline{t}_l)$ yields $V^{-1}(\check{V}(\underline{t}_l)) = \check{V}^{-1}(\check{V}(\underline{t}_l)) = \underline{t}_l$, hence $\check{V}(\underline{t}_l) = V(\underline{t}_l)$. This together with (39) implies $V = \check{V}$ and thus $F = \check{F}$, completing the uniqueness proof.

It remains to verify (26). Let g denote a function on $[V_i(\underline{t}_i), \bar{t}_i]$ that equals the r.h.s. of (26). It is straightforward to check that g satisfies (27), (28), and (29) with $l = j - 1$, $\underline{t} = V_i(\underline{t}_i)$, and $\bar{t} = \bar{t}_i$. Hence, $g = V_i^{-1}$ on $[V_i(\underline{t}_i), \bar{t}_i]$ by Proposition 2 and the uniqueness statement in Lemma 3. *QED*

It is interesting to contrast Proposition 5 with Zheng (2002, Example 3), where it is shown that Zheng's conditions are satisfied if every bidder's c.d.f. is uniform (on a possibly different interval for each bidder). Proposition 5 reveals that if the c.d.f. for bidder n is uniform, then the conditions are satisfied if and only if the c.d.f.s for bidders 2 to $n-1$ are uniform as well (with nested supports), while bidder 1 may have any c.d.f. that is consistent with HR and UB.

The corollary below describes the c.d.f. profiles satisfying Zheng's conditions if the largest possible valuation is the same for the bidders 2 to n.

Corollary 1 *Let $n \geq 3$. Suppose that HR holds and $\bar{t}_2 = \ldots = \bar{t}_n$.*

Then UB, RM, TR, and IV are satisfied if and only if (i) $T_1 \subseteq T_2$ and $\underline{t}_2 \geq \ldots \geq \underline{t}_n$, (ii) $V_1(t_1) \geq V_2(t_1)$ for all $t_1 \in T_1$, (iii) the density f_n is weakly decreasing, and (iv)

$$\forall i \geq 2 : \ F_i(t) = \frac{F_n(t) - F_n(\underline{t}_i)}{1 - F_n(\underline{t}_i)} \quad (t \in T_i). \tag{40}$$

In general, it can take up to n subsequent sale transactions on Zheng's (2002) equilibrium path until the final allocation is obtained. Corollary 1 implies that there will be at most two transactions if the largest possible valuation is the same for the bidders 2 to n. From (40) it follows that whenever bidder $i \geq 2$ has a larger valuation than bidder $j \geq 2$, then i's virtual valuation is also larger than j's. Given this, there will be no resale trading between bidders 2 to n on the equilibrium path—the final allocation is obtained by the initial auction or by a resale transaction from bidder 1 to one of the other bidders if bidder 1 wins the initial auction.

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