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## The Tail of the Stationary Distribution of a Random Coefficient AR(q) Model

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# The tail of the stationary distribution of a random coefficient AR( $q$ ) model \*

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## Abstract

We investigate a stationary random coefficient autoregressive process. Using renewal type arguments tailor-made for such processes we show that the stationary distribution has a power-law tail. When the model is normal, we show that the model is in distribution equivalent to an autoregressive process with ARCH errors. Hence we obtain the tail behaviour of any such model of arbitrary order.

**Key words:** ARCH model, autoregressive model, geometric ergodicity, heteroscedastic model, random coefficient autoregressive process, random recurrence equation, regular variation, renewal theorem for Markov chains, strong mixing.

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# 1 Introduction

We consider the following random coefficient autoregressive model.

$$y_n = \alpha_1(n)y_{n-1} + \cdots + \alpha_q(n)y_{n-q} + \xi_n, \quad n \in \mathbb{N}, \quad (1.1)$$

with random variables  $\alpha_i(n) = a_i + \sigma_i \eta_i(n)$ , where  $a_i \in \mathbb{R}$  and  $\sigma_i \geq 0$  for  $i = 1, \dots, q$ . Set

$$\alpha(n) = (\alpha_1(n), \dots, \alpha_q(n))', \quad \eta(n) = (\eta_1(n), \dots, \eta_q(n))',$$

where  $'$  denotes the transposition operation. We suppose that the sequences of coefficient vectors  $(\eta(n))_{n \in \mathbb{N}}$  and noise variables  $(\xi_n)_{n \in \mathbb{N}}$  are both iid with  $\mathbf{E}\xi_n = \mathbf{E}\eta_i(n) = 0$  and  $\mathbf{E}\xi_n^2 = \mathbf{E}\eta_i^2(n) = 1$  for  $i = 1, \dots, q$ .

Questions of interest concern the existence of a stationary version of the process  $(y_n)_{n \in \mathbb{N}}$ , represented by a random variable  $y_\infty$ , whose tail behaviour

$$\mathbf{P}(y_\infty > t) \quad \text{as} \quad t \rightarrow \infty \quad (1.2)$$

is of prime interest. This is in particular the first step for an investigation of the extremal behaviour of the corresponding stationary process, which will be done in a forthcoming paper. Stationarity of (1.1) is guaranteed by condition  $\mathbf{D}_0$  below; see e.g. Nicholls and Quinns [18]. We pursue in this paper the tail behaviour of the limit variable given in (1.2). To this end we embed  $(y_n)_{n \in \mathbb{N}}$  into a multivariate set-up.

Setting  $Y_n = (y_n, \dots, y_{n-q+1})'$ , the multivariate process  $(Y_n)_{n \in \mathbb{N}}$  can be considered in the much wider context of random recurrence equations of the type

$$Y_n = A_n Y_{n-1} + \zeta_n, \quad n \in \mathbb{N}, \quad (1.3)$$

where  $((A_n, \zeta_n))$  is an iid sequence, the  $A_n$  are iid random  $(q \times q)$ -matrices and the  $\zeta_n$  are iid  $q$ -dimensional vectors. Moreover, for every  $n$ , the vector  $Y_{n-1}$  is independent of  $(A_n, \zeta_n)$ .

Such equations play an important role in many applications as e.g. in queueing; see Brandt, Franken and Lisek [2] and in financial time series; see Engle [8]. See also Diaconis and Freedman [4] for an interesting review article with a wealth of examples.

In the one-dimensional case ( $q = 1$ ) Goldie [10] has solved the problem in a very elegant way and found the tail behaviour (1.2). For the multivariate model ( $q > 1$ ) one can show (see, for example, Kesten [13] and Le Page [19]) that for the stationary random variable  $Y$ , the function  $\mathbf{P}(x'Y > t)$  is asymptotically equivalent to a renewal function, that is

$$\mathbf{P}(x'Y > t) \sim G(x, t) = \mathbf{E}_x \sum_{i=0}^{\infty} g(x_n, t - v_n), \quad t \rightarrow \infty, \quad (1.4)$$

where  $\sim$  means that the quotient of both sides tends to 1. Note that, if we set  $x' = (1, 0, \dots, 0)$ , then we obtain again (1.2). Here  $g(\cdot, \cdot)$  is some continuous function satisfying condition (4.1) below.

In model (1.1), we have  $\zeta_n = (\xi_n, 0, \dots, 0)'$  and

$$A_n = \begin{pmatrix} \alpha_1(n) & \cdots & \alpha_q(n) \\ I_{q-1} & & 0 \end{pmatrix}, \quad n \in \mathbb{N}. \quad (1.5)$$

Here  $I_{q-1}$  is the identity matrix of order  $q - 1$ .

Standard conditions for the existence of a stationary solution to (1.3) are given in Kesten [14] (see also Goldie and Maller [11]) and require that

$$\mathbf{E} \log^+ |A_1| < \infty \quad \text{and} \quad \mathbf{E} \log^+ |\zeta_1| < \infty \quad (1.6)$$

and that the top Lyapunov exponent

$$\tilde{\gamma} = \inf\{n^{-1} \mathbf{E} \log |A_1 \cdots A_n| : n \in \mathbb{N}\} < 0. \quad (1.7)$$

In our case, conditions (1.6) are satisfied. Moreover, we can replace (1.7) by the simpler condition

**D<sub>0</sub>**) The eigenvalues of the matrix

$$\mathbf{E} A_1 \otimes A_1 \quad (1.8)$$

have moduli less than one, where  $\otimes$  denotes the Kronecker product of matrices.

In the context of model (1.1) the processes  $(x_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  are defined as

$$x_0 = x \in S, \quad x_n = \frac{A_n x_{n-1}}{|A_n x_{n-1}|} = \frac{A_n \cdots A_1 x}{|A_n \cdots A_1 x|}, \quad n \in \mathbb{N}, \quad (1.9)$$

and

$$v_0 = 0, \quad v_n = \sum_{i=1}^n u_i = \log |A_n \cdots A_1 x|, \quad \text{with } u_n = \log |A_n x_{n-1}|, \quad n \in \mathbb{N}. \quad (1.10)$$

Here  $|\cdot|$  is any norm in  $\mathbb{R}^q$  and  $|A| = \sup_{|x|=1} |Ax|$  is the corresponding operator norm.

Since GARCH models are commonly used as volatility models, modelling the (positive) standard deviation of a financial time series, Kesten's work could be applied to such models; see e.g. Diebolt and Guegan [5]. Kesten [13, 14] proved and applied a Key Renewal Theorem for function (1.4) under certain conditions on the function  $g$ , the Markov chain  $(x_n)_{n \geq 0}$  and the stochastic process  $(v_n)_{n \geq 0}$ ; a special case being the random recurrence model (1.3) with  $P(A_n > 0) = 1$  and condition (4.1) below. By completely different, namely point process methods, Davis, Mikosch and Basrak [3] show that models (1.5)

– again with positive matrices  $A_n$  – have regularly varying tails. Some special examples have been worked out as ARCH(1) and GARCH(1,1); see Goldie [10], de Haan et al. [12] and Mikosch and Starica [17]. The random coefficient model (1.1), however, does not necessarily satisfy the positivity condition on the matrices  $A_n$ ; see Section 2 for examples.

Consequently, we derived a new Key Renewal Theorem in the spirit of Kesten’s results, but tailor-made for Markov chains with compact state space, general matrices  $A_n$  and functions  $g$  satisfying condition (4.1); see Klüppelberg and Pergamenchtchikov [15], Theorem 2.1. We apply this theorem to the random coefficient model (1.1).

The paper is organised as follows. Our main results are stated in Section 2. We give weak conditions implying a power-law tail for the stationary distribution of the random coefficient model (1.1). For the Gaussian model (all random coefficients and noise variables are Gaussian) we show that model (1.1) is in distribution equivalent to an autoregressive model with ARCH errors of the same order as the random coefficient model. Since the limit variable of the random recurrence model (1.3) is obtained by iteration, products of random matrices have to be investigated. This is done in Section 3. In Section 4 we check the sufficient conditions and apply the Key Renewal Theorem from [15] to model (1.1). Some auxiliary results are summarized in the Appendix.

## 2 Main results

Our first result concerns stationarity of the multivariate process  $(Y_n)_{n \in \mathbb{N}}$  given by (1.3).

**Theorem 2.1.** *Under the condition  $\mathbf{D}_0$ , if the iid random variables  $\xi_n$  have a positive density on  $\mathbb{R}$ , the process  $Y_n = (y_n, \dots, y_{n-q+1})'$  is geometric ergodic. In particular, this process has a unique stationary distribution and satisfies the strong mixing condition with geometric rate of convergence. The stationary distribution is defined by the following vector*

$$Y = \zeta_1 + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1} \zeta_k, \quad (2.1)$$

where the  $A_n$  are given by (1.5) and the  $q$ -dimensional vector  $\zeta_n = (\xi_n, 0, \dots, 0)'$ .

**Proof.** We invoke Theorem 3 of Feigin and Tweedie [9]. The process (1.1) is an AR( $q$ ) - process with random coefficients. Furthermore, the vector process  $Y_n$  satisfies assumptions 1-3 of [9], and hence it is geometric ergodic. More precisely,  $Y_n$  converges to the vector  $Y$  in (2.1) exponentially fast.  $\square$

**Remark 2.2.** (a) From equation (2.1) we obtain  $Y \stackrel{d}{=} A_1 Y_1 + \zeta_1$ , where

$$Y_1 = \zeta_2 + \sum_{k=3}^{\infty} A_2 \cdots A_{k-1} \zeta_k \stackrel{d}{=} Y,$$

which implies

$$Y \stackrel{d}{=} A_1 Y + \zeta_1. \quad (2.2)$$

(b) Using (3.3) of Feigin and Tweedie [9] together with basic identities of the Kronecker product, condition  $\mathbf{D}_0$  guarantees that

$$\mathbf{E}|A_n \cdots A_1|^2 \leq c\rho^n$$

for some positive constants  $c > 0$  and  $0 < \rho < 1$ . From this it follows that the series in (2.1) converges a.s. and the second moment of  $Y$  is finite; see Theorem 4 of [9].  $\square$

We require the following additional conditions for the distributions of the coefficient vectors  $(\eta(n))_{n \in \mathbb{N}}$  and the noise variables  $(\xi_n)_{n \in \mathbb{N}}$  in model (1.1).

$\mathbf{D}_1$ ) Define

$$D = \text{diag}(\sigma_1^2, \dots, \sigma_q^2) = \begin{pmatrix} \sigma_1^2 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & \sigma_q^2 \end{pmatrix}.$$

All random variables  $\eta_i(n)$  are symmetric, all moments are finite and, for every  $x \in \mathbb{R}^q$  satisfying  $x'Dx > 0$ , the random variable  $x'\alpha(n)$  has a bounded positive density  $\phi(z, x)$ , which is continuous in  $(z, x) \in \mathbb{R} \times \mathbb{R}^q$ .

$\mathbf{D}_2$ )  $\mathbf{E}(\alpha_1(n) - a_1)^{2m} \in (1, \infty)$  for some  $m \in \mathbb{N}$ .

$\mathbf{D}_3$ )  $\mathbf{E}|\xi_1|^m < \infty$  for all  $m \in \mathbb{N}$ .

$\mathbf{D}_4$ ) For every real sequence  $(c_k)_{k \in \mathbb{N}}$  with  $0 < \sum_{k=1}^{\infty} |c_k| < \infty$ , the random variable

$$\tau = \sum_{k=1}^{+\infty} c_k \xi_k$$

has a symmetric density, which is non-increasing on  $\mathbb{R}_+$ .

Condition  $\mathbf{D}_4$  looks rather awkward and complicated to verify. Therefore, we give a simple sufficient condition, which is satisfied by many distributions. The proof is given in Appendix A1.

**Proposition 2.3.** *If the random variable  $\xi_n$  has bounded, symmetric density  $f$ , continuously differentiable on  $(0, \infty)$  with bounded derivative  $f' \leq 0$ , then condition  $\mathbf{D}_4$  holds.*

The following is our main result.

**Theorem 2.4.** *Assume that conditions  $\mathbf{D}_0 - \mathbf{D}_4$  are satisfied for the model (1.1) with  $a_q^2 + \sigma_q^2 > 0$ . Then for all  $x \in S = \{z \in \mathbb{R}^q : |z| = 1\}$  the distribution of the vector (2.1) satisfies*

$$\lim_{t \rightarrow \infty} t^\lambda \mathbf{P}(x'Y > t) = h(x).$$

*The function  $h(\cdot)$  is strictly positive and continuous on  $S$  and the parameter  $\lambda$  is given as the unique positive solution of*

$$\kappa(\lambda) = 1, \tag{2.3}$$

where

$$\kappa(\lambda) = \lim_{n \rightarrow \infty} (\mathbf{E}|A_n \cdots A_1|^\lambda)^{1/n}$$

and the solution of (2.3) satisfies  $\lambda > 2$ .

The following model describes an important special case.

**Definition 2.5.** *If in equation (1.1) all coefficients and the noise are Gaussian random variables; i.e.  $\eta_i(n) \sim \mathcal{N}(0, 1)$  and  $\xi_n \sim \mathcal{N}(0, 1)$ , we call the model (1.1) a Gaussian linear random coefficient model.*

The proof of the following result is given in Appendix A2.

**Proposition 2.6.** *The Gaussian model (1.1) satisfies conditions  $\mathbf{D}_1 - \mathbf{D}_4$  with  $\sigma_1 > 0$ . Under the condition  $\mathbf{D}_0$  the conditional correlation matrix for  $Y$  is given by*

$$R = \mathbf{E}(YY'|A_i, i \geq 1) = B + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1} B A_{k-1}' \cdots A_1', \tag{2.4}$$

where

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}. \tag{2.5}$$

Moreover,  $R$  is positive definite a.s., i.e. the vector  $Y$  is conditionally non-degenerate Gaussian with finite second moment.

We show that the Gaussian model is in distribution equivalent to an autoregressive model with uncorrelated Gaussian errors, which we specify as autoregressive process with ARCH errors, an often used class of models for financial time series.

**Lemma 2.7.** Define for the same  $q \in \mathbb{N}$ ,  $a_i \in \mathbb{R}$ ,  $\sigma_i > 0$  as in model (1.1),

$$x_n = a_1 x_{n-1} + \dots + a_q x_{n-q} + \sqrt{1 + \sigma_1^2 x_{n-1}^2 + \dots + \sigma_q^2 x_{n-q}^2} \varepsilon_n, \quad n \in \mathbb{N}, \quad (2.6)$$

with the same initial values  $(x_0, \dots, x_{-q+1}) = (y_0, \dots, y_{-q+1})$  as for the process (1.1). Furthermore, let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be iid  $\mathcal{N}(0, 1)$  random variables with initial values  $(x_0, \dots, x_{-q+1})$  independent of the sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$ . Then the stochastic processes  $(x_n)_{n \geq 0}$  and the Gaussian linear random coefficient model (1.1) have the same distribution.

**Proof.** We can rewrite model (1.1) in the form

$$y_n = a_1 y_{n-1} + \dots + a_q y_{n-q} + \sqrt{1 + \sigma_1^2 y_{n-1}^2 + \dots + \sigma_q^2 y_{n-q}^2} \tilde{\varepsilon}_n, \quad n \in \mathbb{N}, \quad (2.7)$$

where

$$\tilde{\varepsilon}_n = \frac{\xi_n + \sigma_1 y_{n-1} \eta_1(n) + \dots + \sigma_q y_{n-q} \eta_q(n)}{\sqrt{1 + \sigma_1^2 y_{n-1}^2 + \dots + \sigma_q^2 y_{n-q}^2}}, \quad n \in \mathbb{N},$$

are iid  $\mathcal{N}(0, 1)$ . This can be seen by calculating characteristic functions.  $\square$

**Example 2.8.** (a) Consider the autoregressive process (2.7) of order 2 with  $\sigma_2 = 0$ ; i.e.

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \sqrt{1 + \sigma_1^2 x_{n-1}^2} \varepsilon_n, \quad n \in \mathbb{N}. \quad (2.8)$$

In this case the corresponding random matrices (2.2) have the following form

$$A_n = \begin{pmatrix} \alpha_1(n) & a_2 \\ 1 & 0 \end{pmatrix}, \quad n \in \mathbb{N}, \quad (2.9)$$

where  $\alpha_1(n) = a_1 + \sigma_1 \eta_1(n)$  and

$$\mathbf{E}A_1 \otimes A_1 = \begin{pmatrix} a_1^2 + \sigma_1^2 & a_1 a_2 & a_1 a_2 & a_2^2 \\ a_1 & 0 & a_2 & 0 \\ a_1 & a_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The stationary distribution for the process (2.8) is given by the two-dimensional random vector (2.1) with matrices (2.9) and with vector  $\zeta_n = (\xi_n, 0)'$ . Theorem 2.4 applies if  $\sigma_1^2 > 0$  and  $a_2 \neq 0$ .

Notice that for  $\sigma_1 = 0$  Theorem 2.4 cannot be applied. In this case the vector  $Y$  is Gaussian.

(b) The model in (a) for  $a_2 = 0$  was investigated in Borkovec and Klüppelberg [1] by



different methods. Stationarity of the model was shown for  $a_1^2 + \sigma_1^2 < 1$ . Under quite general conditions on the noise variables,

$$\kappa(\lambda) = E|a_1 + \sigma_1\varepsilon|^\lambda,$$

the equation  $\kappa(\cdot) = 1$  has a unique positive solution  $\lambda$  and the tail of the stationary random variable  $x_\infty$  satisfies

$$\lim_{t \rightarrow \infty} t^\lambda P(x_\infty > t) = c.$$

Moreover, this also covers infinite variance cases, i.e.  $\lambda$  can be any positive value.  $\square$

### 3 Products of random matrices

**Lemma 3.1.** *Under the conditions of Theorem 2.4, for every  $\lambda > 0$ , there exists the limit*

$$\kappa(\lambda) = \lim_{n \rightarrow \infty} (\mathbf{E}|A_n \cdots A_1|^\lambda)^{1/n}. \quad (3.1)$$

**Proof.** Let again  $S$  denote the sphere in  $\mathbb{R}^q$  and denote by  $\mathbf{C}(S)$  the set of continuous functions on  $S$ . Define for  $\lambda > 0$ ,

$$Q_\lambda : \mathbf{C}(S) \rightarrow \mathbf{C}(S) \quad \text{by} \quad Q_\lambda(f)(x) = \mathbf{E}|A_1 x|^\lambda f(\overline{A_1 x}) \quad (3.2)$$

for  $x \in S$  and  $f \in \mathbf{C}(S)$ , where  $\bar{v} = v/|v|$  for  $v \neq 0$ .

Denote by  $\mathcal{P}(S)$  the set of probability measures on  $S$  and define for every probability measure  $\sigma$ ,

$$T_\sigma : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$$

by

$$T_\sigma(f) = \frac{\int_S Q_\lambda(f)(x) \sigma(dx)}{\int_S Q_\lambda(e)(x) \sigma(dx)}, \quad (3.3)$$

where  $e(x) \equiv 1$ ,  $f \in \mathbf{C}(S)$ .

This function is continuous and by the Schauder-Tykhonov theorem (see Dunford and Schwartz [6]) there exists a fixpoint  $\nu \in \mathcal{P}(S)$  such that

$$T_\nu(f) = \nu(f)$$

for all measurable bounded functions  $f$  on  $S$ , that is

$$\int_S Q_\lambda(f)(x) \nu(dx) = \kappa(\lambda) \int_S f(x) \nu(dx),$$

where

$$\kappa(\lambda) = \int_S Q_\lambda(e)(x)\nu(dx).$$

Notice that

$$\int_S Q_\lambda^{(n)}(f)(x)\nu(dx) = \kappa^n(\lambda) \int_S f(x)\nu(dx) \quad (3.4)$$

for all  $n \in \mathbb{N}$ . Here  $Q^{(n)}$  is the  $n$ th power of the operator  $Q$ . From (3.2) it follows that

$$Q_\lambda^{(n)}(f)(x) = \mathbf{E}|A_n \cdots A_1 x|^\lambda \overline{f(A_n \cdots A_1 x)}. \quad (3.5)$$

Therefore, by (3.4) we have

$$\kappa^n(\lambda) = \int_S Q_\lambda^{(n)}(e)(x)\nu(dx) = \int_S \mathbf{E}|A_n \cdots A_1 x|^\lambda \nu(dx).$$

This implies that

$$\kappa^n(\lambda) \leq \mathbf{E}|A_n \cdots A_1|^\lambda.$$

On the other hand we have

$$\kappa^n(\lambda) = \mathbf{E}|A_n \cdots A_1|^\lambda \int_S |B_n x|^\lambda \nu(dx), \quad (3.6)$$

where  $B_n = A_n \cdots A_1 / |A_n \cdots A_1|$ . We show that

$$c_* = \inf_{|B|=1} \int_S |Bx|^\lambda \nu(dx) > 0. \quad (3.7)$$

Indeed, if  $c_* = 0$  there exists  $B$  with  $|B| = 1$  such that

$$\int_S |Bx|^\lambda \nu(dx) = 0,$$

which means that

$$\nu\{x \in S : Bx \neq 0\} = 0.$$

Set

$$\mathcal{N} = \{x \in S : Bx = 0\} \quad \text{and} \quad g(x) = \chi_{\mathcal{N}^c},$$

where  $\mathcal{N}^c = S \setminus \mathcal{N}$  and for any set  $A$ ,  $\chi_A$  denotes the indicator function of  $A$ . Notice that there exists a vector  $b \neq 0$  such that

$$\mathcal{N} \subset \{x \in \mathbb{R}^q : Bx = 0\} = \{x \in \mathbb{R}^q : b'x = 0\}.$$

Further, by (3.4) we obtain that

$$\int_S Q_\lambda^{(n)}(g)(x)\nu(dx) = \kappa^n(\lambda) \int_S g(x)\nu(dx) = 0,$$

that is for all  $n \in \mathbb{N}$ ,

$$\int_S \mathbf{E}|A_n \cdots A_1 x|^\lambda g(\overline{A_n \cdots A_1 x}) \nu(dx) = 0.$$

This means that  $\overline{A_n \cdots A_1 x} \in \mathcal{N}$  for some  $x \in \mathcal{N}$  implies that

$$b' A_n \cdots A_1 x = 0.$$

But for  $\text{var}(\alpha_1(n)) = \sigma_1^2 > 0$  and by condition  $\mathbf{D}_1$  and taking into account the special form of the matrix  $A_n$  (1.5) this is only possible for  $b = 0$ . But this contradicts  $|B| = 1$ . Thus we obtained (3.7).

Therefore we have that

$$\mathbf{E}|A_n \cdots A_1|^\lambda \geq \kappa^n(\lambda) = \mathbf{E}|A_n \cdots A_1|^\lambda \int_S |B_n x|^\lambda \nu(dx) \geq c_* \mathbf{E}|A_n \cdots A_1|^\lambda,$$

and from this inequality Lemma 3.1 follows.  $\square$

**Lemma 3.2.** *Assume that conditions  $\mathbf{D}_0 - \mathbf{D}_2$  are satisfied. Then equation (2.3) has a unique positive solution.*

**Proof.** Denote

$$\Pi(n) = A_n \cdots A_1 = (\Pi_{ij}(n))_{i,j=1,\dots,q}.$$

Then

$$\begin{aligned} \Pi_{11}(n) &= \alpha_1(n)\Pi_{11}(n-1) + \dots + \alpha_q(n)\Pi_{q1}(n-1) \\ &= (\alpha_1(n) - a_1)\Pi_{11}(n-1) + \mu_n, \end{aligned}$$

where

$$\mu_n = a_1\Pi_{11}(n-1) + \alpha_2(n)\Pi_{21}(n-1) + \dots + \alpha_q(n)\Pi_{q1}(n-1),$$

independent of  $\eta_1(n)$ . By the binomial formula and condition  $\mathbf{D}_1$  (which implies that all odd moments of  $\eta$  are equal to zero) we have for arbitrary  $m \in \mathbb{N}$  with  $C_{2m}^j = \binom{2m}{j}$ ,

$$\begin{aligned} \mathbf{E}(\Pi_{11}(n))^{2m} &= \sum_{j=0}^{2m} C_{2m}^j \sigma_1^j \mathbf{E}[\eta_1^j(n)] \mathbf{E}[(\Pi_{11}(n-1))^j \mu_n^{2m-j}] \\ &= \sum_{j=0}^m C_{2m}^{2j} \mathbf{E}[(\alpha_1(n) - a_1)^{2j}] \mathbf{E}[(\Pi_{11}(n-1))^{2j} \mu_n^{2(m-j)}] \\ &\geq s(m) \mathbf{E}(\Pi_{11}(n-1))^{2m}, \end{aligned}$$

where by  $\mathbf{D}_2$

$$s(m) = E(\alpha_1(n) - a_1)^{2m} > 1$$

for some  $m > 1$ . Thus  $\mathbf{E}(\Pi_{11}(n))^{2m} \geq s(m)^n$ , i.e.

$$\mathbf{E}|\Pi(n)|^{2m} \geq \mathbf{E}(\Pi_{11}(n))^{2m} \geq s(m)^n$$

which implies that  $\kappa(2m) > 1$  for some  $m$  sufficiently large.

From the definition we know that  $\kappa(0) = 1$ . Further,  $\log \kappa(\lambda)$  is convex for all  $\lambda > 0$  and hence continuous. To see the convexity, set

$$\rho_n(\lambda) = \frac{1}{n} \log \mathbf{E}|\Pi(n)|^\lambda, \quad \lambda > 0,$$

and recall that  $\log \kappa(\lambda) = \lim_{n \rightarrow \infty} \rho_n(\lambda)$ . Then for  $\alpha \in (0, 1)$  and  $\beta = 1 - \alpha$  we obtain for  $p = \alpha^{-1}, q = \beta^{-1}$  by Hölder's inequality,

$$\rho_n(\alpha\lambda + \beta\mu) \leq \alpha\rho_n(\lambda) + \beta\rho_n(\mu).$$

These properties are sufficient to guarantee a unique positive root of equation (2.3).  $\square$

**Remark 3.3.** By Remark 2.2(b) condition  $\mathbf{D}_0$  ensures that  $\kappa(\mu) < 1$  for all  $\mu \leq 2$ .  $\square$

We shall obtain the tail behaviour of model (1.1) by applying a renewal theorem to the renewal function based on the Markov process  $(x_n)_{n \geq 0}$  as defined in (1.9), where the matrices  $(A_n)_{n \in \mathbb{N}}$  are given by (1.5). Then  $(x_n)_{n \geq 0}$  is a homogeneous Markov chain with compact state space  $S$  and transition probabilities

$$p(x, \Gamma) = \mathbf{P}_x(x_1 \in \Gamma),$$

for any measurable set  $\Gamma$  on  $S$ .

**Lemma 3.4.** *Under the conditions of Theorem 2.4 the process  $(x_n)_{n \geq 0}$  with values in  $S$  is geometric ergodic with invariant measure  $\pi(\cdot)$  which is equivalent to Lebesgue measure on  $S$ .*

**Proof.** First notice that by Lemma A.7 there exists some  $p_* > 0$  such that

$$\inf_{x \in S} \mathbf{P}_x(x_{2q} \in B) > p_* \Lambda(B), \quad (3.8)$$

where  $\Lambda$  denotes Lebesgue measure on  $S$ . This implies in particular that every measurable set in  $S$  is small.

For  $y \in \mathbb{R}^q$  let  $\langle y \rangle_i$  denote its  $i$ th coordinate. Define  $v : \mathbb{R}^q \rightarrow [1, \infty)$  by

$$v(y) = 1 + |\langle y_1 \rangle|.$$

Then

$$\mathbf{E}_x v(x_1) = 1 + \mathbf{E}\Psi(|\xi|),$$

where  $\xi = x'\alpha(1)$ ,  $\alpha(1) = (\alpha_1(1), \dots, \alpha_q(1))'$ ,  $x = (\langle x \rangle_1, \dots, \langle x \rangle_q)'$  and

$$\Psi(\theta) = \frac{\theta}{\sqrt{\theta^2 + \langle x \rangle_1^2 + \dots + \langle x \rangle_{q-1}^2}}.$$

Notice that  $d\Psi(\theta)/d\theta > 0$  and  $d^2\Psi(\theta)/d\theta^2 < 0$  for  $\theta > 0$  and  $\mathbf{E}|\xi| \leq \mathbf{E}|\alpha(1)| =: \mu$ . By Jensen's inequality we have that

$$\mathbf{E}_x v(x_1) \leq 1 + \Psi(\mathbf{E}|\xi|) \leq 1 + \Psi(\mu) = L(x)v(x),$$

where

$$L(x) = \frac{1}{1 + |\langle x \rangle_1|} \left( 1 + \frac{\mu}{\sqrt{\mu^2 + \langle x \rangle_1^2 + \dots + \langle x \rangle_{q-1}^2}} \right)$$

and

$$\lim_{|\langle x \rangle_1| \rightarrow 1} L(x) = \frac{1}{2} \left( 1 + \frac{\mu}{\sqrt{\mu^2 + 1}} \right) < 1.$$

Thus, for a some  $\varepsilon > 0$  there exists  $0 < r < 1$  such that

$$\sup_{|\langle x \rangle_1| > r} L(x) < 1 - \varepsilon,$$

and we have (A.2) for the set

$$\Gamma = \{x \in S : |\langle x \rangle_1| \leq r\}.$$

By Lemma A.6 the process  $(x_n)_{n \geq 0}$  is geometric ergodic with stationary distribution  $\pi(\cdot)$  on  $S$ , i.e. for all  $x \in S$

$$\|\mathbf{P}^{(n)}(x, \cdot) - \pi(\cdot)\| \leq c\rho^n \tag{3.9}$$

for some  $c > 0$ ,  $0 < \rho < 1$  and  $\|\cdot\|$  denotes total variation of measures on  $S$ .

We show now that  $\pi$  is equivalent to Lebesgue measure.

If  $\Lambda(B) = 0$  then  $\pi(B) = \lim_{n \rightarrow \infty} \mathbf{P}_x(x_n \in B) = 0$ .

If  $\pi(B) = \lim_{n \rightarrow \infty} \mathbf{P}_x(x_n \in B) = 0$  and if  $\Lambda(B) > 0$ , then taking inequality (3.8) into account, we obtain the contradiction

$$\pi(B) = \lim_{n \rightarrow \infty} \mathbf{P}_x(x_{n+2q} \in B) = \lim_{n \rightarrow \infty} \int_S \mathbf{P}_y(x_{2q} \in B) \mathbf{P}^{(n)}(x, dy) \geq p_* \Lambda(B) > 0.$$

Hence  $\pi(\cdot)$  and  $\Lambda(\cdot)$  are equivalent on  $S$ .  $\square$

The proof of the following Lemma is a simplification of Step 2 of Theorem 3 of Kesten [14] adapted to model (1.1); see also Le Page [19], Step 2 of Proposition 1.2.

**Lemma 3.5.** *Under the conditions of Theorem 2.4, for every  $\lambda > 0$ , there exists a continuous function  $h(\cdot) > 0$  such that for  $Q_\lambda$  as defined in (3.2),*

$$Q_\lambda(h)(x) = \kappa(\lambda)h(x), \quad x \in S. \quad (3.10)$$

*If another function  $g$  satisfies this equation, then  $g(\cdot) = \rho h(\cdot)$  for some constant  $\rho > 0$ .*

**Proof.** Recall the notation of the proof of Lemma 3.1, in particular (3.5) and (3.6). Set for  $\lambda > 0$

$$s_n(x) = \frac{Q_\lambda^{(n)}(e)(x)}{\kappa^n(\lambda)} = \frac{\mathbf{E}|A_n \cdots A_1 x|}{\kappa^n(\lambda)}, \quad x \in S.$$

Using (3.7) we have

$$\sup_{x \in S} s_n(x) \leq 1/c_*$$

and for all  $x, y \in S$ ,

$$|s_n(x) - s_n(y)| \leq L|x - y|^\theta, \quad \theta = \min(\lambda, 1),$$

where  $L$  is some positive constant.

By the principle of Arzela-Ascoli there exists a sequence  $(n_k)_{k \in \mathbb{N}}$  with  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$  such that

$$h_k(x) = \frac{1}{n_k} \sum_{j=1}^{n_k} s_j(x) \rightarrow h(x)$$

uniformly on  $S$  and

$$\begin{aligned} Q_\lambda(h)(x) &= \lim_{k \rightarrow \infty} Q_\lambda(h_k)(x) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=1}^{n_k} Q_\lambda(s_j)(x) \\ &= \lim_{k \rightarrow \infty} \frac{\kappa(\lambda)}{n_k} \sum_{j=1}^{n_k} s_{j+1}(x) = \kappa(\lambda)h(x). \end{aligned}$$

Further, if  $h(x) = 0$  for some  $x \in S$  then  $Q_\lambda^{(n)}(h)(x) = 0$  for all  $n \in \mathbb{N}$ , i.e.

$$\mathbf{E}|A_n \cdots A_1 x|^\lambda h(x_n) = 0,$$

where  $x_n = \overline{A_n \cdots A_1 x}$ , which means that  $h(x_n) = 0$  a.s. From Lemma 3.4 we conclude

$$\mathbf{E}_x h(x_n) = 0 \Rightarrow \int_S h(z)\pi(dz) = 0 \Rightarrow \lim_{k \rightarrow \infty} \int_S h_k(z)\pi(dz) = 0.$$

But on the other hand

$$\begin{aligned}
\int_S h_k(z)\pi(\mathrm{d}z) &= \frac{1}{n_k} \sum_{j=1}^{n_k} \int_S s_j(z)\pi(\mathrm{d}z) \\
&= \frac{1}{n_k} \sum_{j=1}^{n_k} \frac{1}{\kappa^j(\lambda)} \int_S Q_\lambda^{(j)}(e)(z)\pi(\mathrm{d}z) \\
&= \frac{1}{n_k} \sum_{j=1}^{n_k} \frac{1}{\kappa^j(\lambda)} \mathbf{E}|A_j \cdots A_1|^\lambda \int_S \frac{|A_j \cdots A_1 z|^\lambda}{|A_j \cdots A_1|^\lambda} \pi(\mathrm{d}z) \\
&\geq c_1 \frac{1}{n_k} \sum_{j=1}^{n_k} \frac{\mathbf{E}|A_j \cdots A_1|^\lambda}{\kappa^j(\lambda)} \\
&\geq c_1/c_*,
\end{aligned}$$

where

$$c_1 = \inf_{|B|=1} \int_S |Bz|^\lambda \pi(\mathrm{d}z).$$

We show that  $c_1 > 0$ . Assume that  $c_1 = 0$ . Then there exists a matrix  $B$  with  $|B| = 1$  such that  $\pi(\mathcal{N}^c \cap S) = 0$  for  $\mathcal{N} = \{x \in \mathbb{R}^q : Bx = 0\}$ . But this is impossible, because  $\mathcal{N}$  is a subspace of  $\mathbb{R}^q$  and  $\pi$  is equivalent to Lebesgue measure on  $S$ . Hence  $h(x) > 0$  for all  $x \in S$ .

Now assume that there exists some function  $g \neq h$  satisfying equation (3.10). By Lemma A.7 every function, satisfying equation (3.10) is continuous, since for all  $n \in \mathbb{N}$

$$g(x) = \kappa^n(\lambda) Q_\lambda^{(n)}(g)(x) = \kappa^n(\lambda) \mathbf{E}_x e^{\lambda v_n} g(x_n), \quad x \in S,$$

where  $v_n = \log |A_n \cdots A_1 x|$  and  $x_n = \overline{A_n \cdots A_1 x}$ . Set

$$\rho = \sup_{x \in S} \frac{g(x)}{h(x)} = \frac{g(x_0)}{h(x_0)} \quad \text{and} \quad l(x) = \rho h(x) - g(x).$$

Notice that  $l(x) \geq 0$  and  $l(x_0) = 0$ . Set

$$L(z) = \frac{l(z)}{h(z)} = \frac{Q_\lambda(l)(z)}{\kappa(\lambda)h(z)} = \cdots = \frac{Q_\lambda^{(n)}(l)(z)}{\kappa^n(\lambda)h(z)} = \frac{Q_\lambda^{(n)}(hL)(z)}{\kappa^n(\lambda)h(z)}.$$

We write

$$\sup_{x \in S} L(x) = L(y_0) = \frac{Q_\lambda^{(n)}(hL)(y_0)}{\kappa^n(\lambda)h(y_0)},$$

equivalently,

$$\mathbf{E}|A_n \cdots A_1 y_0|^\lambda h(x_n) L(x_n) = L(y_0) h(y_0) \kappa^n(\lambda)$$

or

$$\mathbf{E}|A_n \cdots A_1 y_0|^\lambda h(x_n) (L(y_0) - L(x_n)) = 0.$$

Thus  $L(x_n) = L(y_0)$  for all  $n \in \mathbb{N}$ , where  $x_n = \overline{A_n \cdots A_1 y_0}$ . By Lemma 3.4 we get

$$\int_S L(z) \pi(dz) = L(y_0).$$

Therefore for all  $z \in S$

$$L(y_0) = L(z) = L(x_0) = \frac{l(x_0)}{h(x_0)} = 0.$$

Thus  $l(z) = 0$  for all  $z \in S$ .  $\square$

## 4 Renewal theorem for the associated Markov chain

The next result is based on the renewal theorem in Klüppelberg and Pergamenchtchikov [15] for the stationary Markov chain  $(x_n)_{n \geq 0}$  and the process  $(v_n)_{n \geq 0}$  as defined in (1.9) and (1.10), respectively. Denote by  $u_n = \log |A_n x_{n-1}|$ ,  $n \in \mathbb{N}$ , the increments of  $(v_n)_{n \geq 0}$ .

The renewal theorem in [15] gives the asymptotic behaviour of the renewal function

$$G(x, t) = \mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k)$$

under the following conditions:

**C<sub>0</sub>**) *The function  $g : S \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and bounded and satisfies*

$$\sum_{l=-\infty}^{\infty} \sup_{x \in S} \sup_{l \leq t \leq l+1} |g(x, t)| < \infty. \quad (4.1)$$

**C<sub>1</sub>**) *For the processes  $(x_n)_{n \geq 0}$  and  $(u_n)_{n \geq 1}$  define the  $\sigma$ -algebras*

$$\mathcal{F}_0 = \sigma\{x_0\}, \quad \mathcal{F}_n = \sigma\{x_0, x_1, u_1, \dots, x_n, u_n\}, \quad n \in \mathbb{N},$$

where some initial value  $x_0$  is independent of  $(A_n)_{n \in \mathbb{N}}$ .

For every bounded measurable function  $f : \mathbb{R} \times \prod_{i=0}^{\infty} (S \times \mathbb{R}) \rightarrow \mathbb{R}$  and for every  $\mathcal{F}_n$ -measurable random variable  $\eta$

$$\begin{aligned} & \mathbf{E}(f(\eta, x_{n+1}, u_{n+1}, \dots, x_{n+l}, u_{n+l}, \dots) | \mathcal{F}_n) \\ &= \mathbf{E}_{x_n} f(\eta, x_{n+1}, u_{n+1}, \dots, x_{n+l}, u_{n+l}, \dots) \\ &= \Phi(x_n, \eta), \end{aligned} \quad (4.2)$$

i.e.  $\Phi(x, a) = \mathbf{E}_x f(a, x_1, u_1, \dots, x_l, u_l, \dots)$  for all  $x \in S$  and  $a \in \mathbb{R}$ . We assume further that

$$\sup_{x \in S} \mathbf{E}_x |u_1| < \infty. \quad (4.3)$$



**C<sub>2</sub>)** *There exists a probability measure  $\pi(\cdot)$  on  $S$ , which is equivalent to Lebesgue measure such that*

$$\|\mathbf{P}_x^{(n)}(\cdot) - \pi(\cdot)\| \rightarrow 0, \quad n \rightarrow \infty, \quad (4.4)$$

for all  $x \in S$ , where  $\|\cdot\|$  denotes total variation of measures on  $S$ .

Moreover, there exists a constant  $\beta > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{v_n}{n} = \beta \quad \mathbf{P}_x - a.s.$$

for all  $x \in S$ .

**C<sub>3</sub>)** *There exists a number  $m \in \mathbb{N}$  such that for all  $\nu \in \mathbb{R}$  and for all  $\delta > 0$  there exist  $y_{\nu, \delta} \in S$  and  $\varepsilon_0 = \varepsilon_0(\nu, \delta) > 0$  such that  $\forall 0 < \varepsilon < \varepsilon_0$*

$$\inf_{x \in B_{\varepsilon, \delta, \nu}} \mathbf{P}_x(|x_m - y_{\nu, \delta}| < \varepsilon, |v_m - \nu| < \delta) = \rho > 0, \quad (4.5)$$

where  $B_{\varepsilon, \delta, \nu} = \{x \in S : |x - y_{\nu, \delta}| < \delta\}$ .

**C<sub>4</sub>)** *Let  $\Phi : S \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a bounded measurable function. Then there exists some  $l \in \mathbb{N}$  such that the function  $\Phi_1(x, t) = \mathbf{E}_x \Phi(x_l, v_l, t)$  satisfies the following property:*

$$\sup_{|x-y| < \varepsilon} \sup_{t \in \mathbb{R}} |\Phi_1(x, t) - \Phi_1(y, t)| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

**Theorem 4.1.** (Klüppelberg and Pergamenchtchikov [15])

*Assume that conditions **C<sub>0</sub>** – **C<sub>4</sub>** are satisfied. Then*

$$\lim_{t \rightarrow \infty} G(x, t) = \lim_{t \rightarrow \infty} \mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k) = \frac{1}{\beta} \int_S \pi(dx) \int_{-\infty}^{\infty} g(x, t) dt. \quad (4.6)$$

□

We apply this renewal theorem to

$$G(x, t) = \frac{1}{e^t} \int_0^{e^t} u^\lambda \mathbf{P}(x'Y > u) du, \quad x \in S, t \in \mathbb{R}, \quad (4.7)$$

where the vector  $Y$  is given by (2.1) with matrix (1.5) and  $\lambda$  is the positive solution of equation (2.3).

Define

$$\tilde{G}(x, t) = \frac{G(x, t)}{h(x)},$$

where  $h(\cdot) > 0$  satisfies equation (3.10) with positive solution  $\lambda$  for which  $\kappa(\lambda) = 1$ . Further, notice that by Remark 2.2

$$Y \stackrel{d}{=} A_1 Y_1 + \zeta_1,$$

where  $Y_1 = \zeta_2 + \sum_{k=3}^{\infty} A_2 \cdots A_{k-1} \zeta_k \stackrel{d}{=} Y$  independent of  $A_1$  and  $\zeta_1$ . Therefore,

$$\begin{aligned} \tilde{G}(x, t) &= \frac{1}{h(x)e^t} \int_0^{e^t} u^\lambda \mathbf{P}(x' A_1 Y_1 + x' \zeta_1 > u) du \\ &= \phi(x, t) + \psi(x, t), \end{aligned} \quad (4.8)$$

where

$$\phi(x, t) = \frac{1}{h(x)e^t} \int_0^{e^t} u^\lambda \mathbf{P}(x' A_1 Y_1 > u) du, \quad (4.9)$$

$$\psi(x, t) = \frac{1}{h(x)e^t} \int_0^{e^t} u^\lambda \psi_0(x, u) du, \quad (4.10)$$

$$\psi_0(x, u) = \mathbf{P}(x' A_1 Y_1 + x' \zeta_1 > u) - \mathbf{P}(x' A_1 Y_1 > u). \quad (4.11)$$

Denote by  $\tilde{\mathbf{E}}_x$  the expectation with respect to the probability measure  $\tilde{\mathbf{P}}_x$ , which is defined by

$$\tilde{\mathbf{E}}_x F(x_1, u_1, \dots, x_n, u_n) = \frac{1}{h(x)} \mathbf{E} |A_n \cdots A_1 x|^\lambda h(x_n) F(x_1, u_1, \dots, x_n, u_n), \quad (4.12)$$

for each measurable function  $F$ .

**Proposition 4.2.** *Under the conditions of Theorem 2.4, for every  $x \in S$ ,*

$$\tilde{G}(x, t) = \sum_{n=0}^{\infty} \tilde{\mathbf{E}}_x \psi(x_n, t - v_n). \quad (4.13)$$

**Proof.** Consider first  $\phi(x, t)$  as defined in (4.9). Mapping  $u \rightarrow u/|A_1 x|$  and using  $x'_1 = x' A_1 / |A_1 x|$ , we obtain

$$\phi(x, t) = \mathbf{E} \frac{|A_1 x|^\lambda}{h(x) e^{t - \ln |A_1 x|}} \int_0^{e^{t/|A_1 x|}} u^\lambda \mathbf{P}(x'_1 Y > u) du = \tilde{\mathbf{E}}_x \tilde{G}(x_1, t - \ln |A_1 x|).$$

Define the linear operator  $\Theta$  on the continuous functions  $\mathbf{C}(S)$  by

$$\Theta(f)(x, t) = \tilde{\mathbf{E}}_x f(x_1, t - v_1), \quad (4.14)$$

where we have used that  $v_1 = u_1 = \ln |A_1 x|$ . Notice that the  $n$ th power of this operator is given by

$$\Theta^{(n)}(f)(x, t) = \tilde{\mathbf{E}}_x f(x_n, t - v_n). \quad (4.15)$$

Identity (4.8) translates into

$$\tilde{G}(x, t) = \Theta(\tilde{G})(x, t) + \psi(x, t),$$

and we obtain for all  $n \in \mathbb{N}$  from (4.14) iteratively,

$$\tilde{G}(x, t) = \Theta^{(n)}(\tilde{G})(x, t) + \psi(x, t) + \Theta(\psi)(x, t) + \dots + \Theta^{(n-1)}(\psi)(x, t).$$

Furthermore

$$\begin{aligned} \Theta^{(n)}(\tilde{G})(x, t) &= \tilde{\mathbf{E}}_x \tilde{G}(x_n, t - v_n) \\ &= \frac{1}{h(x)e^t} \int_0^{e^t} u^\lambda \mathbf{P}((A_n \cdots A_1 x)' Y > u) du \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

because by condition  $\mathbf{D}_0$ ,

$$\lim_{n \rightarrow 0} \mathbf{E} |A_n \cdots A_1| = 0.$$

This implies (4.13).  $\square$

Next we check conditions  $\mathbf{C}_1 - \mathbf{C}_4$ .

**Proposition 4.3.** *Let  $(x_n)_{n \geq 0}$ ,  $(v_n)_{n \geq 0}$  and  $(u_n)_{n \in \mathbb{N}}$  be defined as in (1.9) and (1.10), respectively, where  $(A_n)_{n \in \mathbb{N}}$  is defined in (1.5). Then conditions  $\mathbf{C}_1 - \mathbf{C}_4$  hold with respect to the measure  $\tilde{\mathbf{P}}_x$  generated by the finite dimensional distributions (4.12).*

**Proof.** First notice that by Lemma A.8 Lemma 3.4 holds with respect to the distribution (4.12). Lemma A.8 ensures immediately condition  $\mathbf{C}_4$ .

Next we check  $\mathbf{C}_1$ . For  $n, l \in \mathbb{N}$  we have

$$x_{n+l} = \frac{A_{n+l} \cdots A_{n+1} x_n}{|A_{n+l} \cdots A_{n+1} x_n|} = h_l(x_n, A_{n+l}, \dots, A_{n+1})$$

and

$$u_{n+l} = \log |A_{n+l} x_{n+l-1}| = \log |A_{n+l} \frac{A_{n+l-1} \cdots A_{n+1} x_n}{|A_{n+l-1} \cdots A_{n+1} x_n|}| = g_l(x_n, A_{n+l}, \dots, A_{n+1})$$

and for any function  $f : \mathbb{R} \times \prod_{i=0}^{\infty} (S \times \mathbb{R}) \rightarrow \mathbb{R}$  we can calculate

$$\begin{aligned} &f(\eta_n, x_{n+1}, u_{n+1}, \dots, x_{n+l}, u_{n+l}, \dots) \\ &= f(\eta_n, h_1(x_n, A_{n+1}), g_1(x_n, A_{n+1}), \dots, \\ &\quad h_l(x_n, A_{n+l}, \dots, A_{n+1}), g_l(x_n, A_{n+l}, \dots, A_{n+1}), \dots) \\ &= f_1(\eta_n, x_n, A_{n+1}, \dots, A_{n+l}, \dots). \end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathbf{E}(f(\eta_n, x_{n+1}, u_{n+1}, \dots, x_{n+l}, u_{n+l}, \dots) | \mathcal{F}_n) \\
&= \mathbf{E}(f_1(\eta_n, x_n, A_{n+1}, \dots, A_{n+l}, \dots) | \mathcal{F}_n) \\
&= \Phi(\eta_n, x_n),
\end{aligned}$$

where (notice that  $(\eta_n, x_n)$  are independent of  $(A_{n+1}, \dots, A_{n+l}, \dots)$ )

$$\begin{aligned}
\Phi(a, x) &= \mathbf{E}f_1(a, x, A_{n+1}, \dots, A_{n+l}, \dots) \\
&= \mathbf{E}f_1(a, x, A_1, \dots, A_l, \dots) \\
&= \mathbf{E}f(a, h_1(x, A_1), g_1(x, A_1), \dots, h_l(x, A_l, \dots, A_1), g_l(x, A_l, \dots, A_1), \dots) \\
&= \mathbf{E}_x f(a, x_1, u_1, \dots, x_l, u_l, \dots).
\end{aligned}$$

From this and (4.12) we get for every  $m \in \mathbb{N}$  and every bounded function  $f_m : \mathbb{R} \times \Pi_{i=0}^{m-1}(S \times \mathbb{R}) \rightarrow \mathbb{R}$

$$\tilde{\mathbf{E}}_x(f_m(\eta_n, x_{n+1}, u_{n+1}, \dots, x_{n+m}, u_{n+m}) | \mathcal{F}_n) = \Phi_m(\eta_n, x_n),$$

where  $\Phi_m(a, x) = \tilde{\mathbf{E}}_x(f_m(a, x_1, u_1, \dots, x_m, u_m))$ .

For every bounded function  $f : \mathbb{R} \times \Pi_{i=0}^{\infty}(S \times \mathbb{R}) \rightarrow \mathbb{R}$  we can write  $f = \lim_{m \rightarrow \infty} f_m$  and, letting  $m$  tend to  $\infty$  in the preceding equality, we obtain (4.2) for the measure (4.12). Further, the function  $\tilde{\mathbf{E}}_x \log |A_1 z|$  is continuous on  $S$  and therefore bounded. From this (4.3) follows. Hence condition  $\mathbf{C}_1$  holds.

Next we check condition  $\mathbf{C}_2$ . Notice first that (by the same method as for the proof of Lemma 3.4) the process  $(x_n)_{n \geq 1}$  is geometric ergodic with respect to the family of distributions  $\tilde{P}_x$  as defined in (4.12). Its stationary distribution is by Lemma 3.4 equivalent to Lebesgue measure on  $S$ . Further we have

$$\frac{v_n}{n} = \frac{1}{n} \sum_{k=1}^n f(x_{k-1}) + \frac{1}{n} \sum_{k=1}^n m_k, \quad (4.16)$$

where

$$f(x) = \frac{1}{h(x)} \mathbf{E} |A_1 x|^\lambda \log |A_1 x| h(\overline{A_1 x}),$$

and

$$m_k = \log |A_k x_{k-1}| - \tilde{\mathbf{E}}_x (\log |A_k x_{k-1}| | \mathcal{F}_{k-1}) = \log |A_k x_{k-1}| - f(x_{k-1}).$$

The last term in (4.16) converges to zero by the martingale convergence theorem.

The first term of the right-hand side of (4.16) converges to the expectation of  $f$  with respect to the invariant measure by the ergodic theorem for  $(x_n)_{n \in \mathbb{N}}$  (see e.g. Theorem

17.0.1, p. 411 in Meyn and Tweedie [16], because this process is a positive Harris chain). Consequently,

$$\lim_{n \rightarrow \infty} \frac{v_n}{n} = \beta = \int_S \tilde{\pi}(dz) \frac{1}{h(z)} \mathbf{E}|A_1 z|^\lambda \log |A_1 z| h(\overline{A_1 z}), \quad \tilde{\pi} - \text{a.s.} \quad (4.17)$$

where  $\tilde{\pi}$  is the stationary distribution of  $(x_n)_{n \in \mathbb{N}}$  with respect to  $\tilde{P}_x$ . This implies

$$\int_S \tilde{\mathbf{P}}_x \left( \lim_{n \rightarrow \infty} \frac{v_n}{n} = \beta \right) \tilde{\pi}(dx) = 1.$$

Since the measure  $\tilde{\pi}$  is equivalent to Lebesgue measure we have that

$$\tilde{\mathbf{P}}_x \left( \lim_{n \rightarrow \infty} \frac{v_n}{n} = \beta \right) = 1 \quad (4.18)$$

for almost all  $x \in S$ . By condition  $\mathbf{C}_1$  we conclude

$$\tilde{\mathbf{P}}_x \left( \lim_{n \rightarrow \infty} \frac{v_n}{n} = \beta \right) = \tilde{\mathbf{E}}_x f(x_l, v_l),$$

where

$$f(x, v) = \tilde{\mathbf{P}}_x \left( \lim_{n \rightarrow \infty} \frac{v_n + v}{n} = \beta \right).$$

By condition  $\mathbf{C}_4$  we obtain that the function  $\tilde{\mathbf{P}}_x \left( \lim_{n \rightarrow \infty} \frac{v_n}{n} = \beta \right)$  is continuous on  $S$  and therefore we obtain (4.18) for all  $x \in S$ .

It remains to show that the constant  $\beta$  in (4.17) is positive. For  $n \in \mathbb{N}$  define

$$\Pi(n) = A_n \cdots A_1.$$

By condition  $\mathbf{D}_0$  there exist  $\delta > 0$ ,  $\gamma > 0$  and  $K > 0$  such that

$$\mathbf{E}|\Pi(n)|^\delta \leq K e^{-\gamma n}.$$

Let  $\gamma_1 > 0$  be such that  $d = \gamma - \delta\gamma_1 > 0$ . Then by Markov's inequality,

$$\mathbf{P}(|\Pi(n)x| > e^{-\gamma_1 n}) \leq \mathbf{P}(|\Pi(n)|^\delta > e^{-\delta\gamma_1 n}) \leq K e^{-dn}.$$

Further, we have for every  $0 < \rho < d/\lambda$  and  $x_n = \Pi(n)x/|\Pi(n)x|$ ,

$$\begin{aligned} \tilde{\mathbf{P}}_x(|\Pi(n)x| < e^{\rho n}) &\leq \frac{1}{h(x)} \mathbf{E}|\Pi(n)x|^\lambda h(x_n) \chi_{\{|\Pi(n)x| < e^{\rho n}\}} \\ &\leq \frac{\max_{z \in S} h(z)}{h(x)} \left( e^{-\lambda\gamma_1 n} + \mathbf{E}|\Pi(n)x|^\lambda \chi_{\{e^{-\gamma_1 n} \leq |\Pi(n)x| < e^{\rho n}\}} \right) \\ &\leq c \left( e^{-\lambda\gamma_1 n} + e^{\lambda\rho n} \mathbf{P}(|\Pi(n)x| > e^{-\gamma_1 n}) \right) \\ &\leq c \left( e^{-\lambda\gamma_1 n} + K e^{-(d-\lambda\rho)n} \right). \end{aligned}$$

By the Lemma of Borel-Cantelli we conclude that for all  $x \in S$

$$\lim_{n \rightarrow \infty} \frac{v_n}{n} > \rho > 0 \quad \tilde{\mathbf{P}}_x - \text{a.s.}$$

This verifies condition  $\mathbf{C}_2$ .

Finally, we check condition  $\mathbf{C}_3$ . We shall show that for  $m = 2q \forall \nu \in \mathbb{R}, \forall \delta > 0 \forall y \in S$  and  $\forall \varepsilon > 0$

$$\rho = \inf_{x \in S} \tilde{\mathbf{P}}_x(|x_m - y| < \varepsilon, |v_m - \nu| < \delta) > 0. \quad (4.19)$$

Indeed,

$$\begin{aligned} & \tilde{\mathbf{P}}_x(|x_m - y| < \varepsilon, |v_m - \nu| < \delta) = \\ & \tilde{\mathbf{P}}_x(|\Pi(m)x / |\Pi(m)x| - y| < \varepsilon, |\log |\Pi(m)x| - \nu| < \delta) = \tilde{\mathbf{P}}_x(\Pi(m)x \in \Gamma_{\varepsilon, \delta}), \end{aligned}$$

where

$$\Gamma_{\varepsilon, \delta} = \{z \in \mathbb{R}^q : |z/|z| - y| < \varepsilon, |\log |z| - \nu| < \delta\}.$$

Notice that, for any  $y \in S$ , this set is a non-empty open set in  $S$ , because the vector  $z = e^\nu y \in \Gamma_{\varepsilon, \delta}, \forall \nu \in \mathbb{R}, \forall \delta > 0, \forall y \in S$  and  $\forall \varepsilon > 0$ . This implies that the Lebesgue measure of  $\Gamma_{\varepsilon, \delta}$  is positive. We conclude that

$$\tilde{\mathbf{P}}_x(|x_m - y| < \varepsilon, |v_m - \nu| < \delta) > 0 \quad (4.20)$$

for any  $x \in S$ . Moreover, by Lemma A.8 the vector  $\Pi(m)x$  (for  $m = 2q$ ) has positive density in  $\mathbb{R}^q$ . By condition  $\mathbf{C}_4$  the function (4.20) is continuous on  $S$  and we obtain the property (4.19). This implies  $\mathbf{C}_3$ .  $\square$

**Lemma 4.4.** *Under the conditions of Theorem 2.4, for every  $x \in S$ , there exists*

$$\lim_{t \rightarrow \infty} G(x, t) = h(x) \frac{1}{\beta} \int_S \tilde{\pi}(dz) \frac{1}{h(z)} \int_0^{+\infty} u^{\lambda-1} \psi_0(z, u) du = h(x) \gamma^* > 0, \quad (4.21)$$

where  $h(\cdot) > 0$  satisfies equation (3.10) with positive solution  $\lambda$  for which  $\kappa(\lambda) = 1$ .

**Proof.** By Proposition 4.2 it suffices to find the limit for the sum in (4.13). We apply Theorem 4.1 to this function. Conditions  $\mathbf{C}_1 - \mathbf{C}_4$  hold by Proposition 4.3.

It remains to show that the function  $\psi$  given by (4.10) satisfies condition  $\mathbf{C}_0$ . By Lemma A.9 (see Appendix A4) it follows that  $\psi(x, t) \geq 0$  and therefore

$$\psi(x, t) \leq h_*(\psi_1^*(x, t) + \psi_2^*(x, t)), \quad (4.22)$$

where  $h_* = 1 / \min_{x \in S} h(x)$  and

$$\begin{aligned} \psi_1^*(x, t) &= \frac{1}{e^t} \int_0^{e^t} u^\lambda \mathbf{P}(\tau_1 > u - n(t)) du - \frac{1}{e^t} \int_0^{e^t} u^\lambda \mathbf{P}(\tau_1 > u) du, \\ \psi_2^*(x, t) &= \frac{e^{(\lambda+1)t}}{(\lambda+1)e^t} \mathbf{P}(\tau_2 > n(t)), \end{aligned}$$

$n(t) = e^{\mu t}$  for some  $\mu > 0$ . We show that the functions  $\psi_i^*(x, t)$  satisfy for sufficiently large  $t > 0$  the inequality

$$\psi_i^*(x, t) \leq ce^{-\nu t} \quad (4.23)$$

for positive constants  $c > 0$  and  $\nu > 0$ .

First notice that immediately from the Lemma 3.2 and Remark 3.3 it follows that  $\kappa(\theta) < 1$  for every  $1 < \theta < \lambda$ ; hence by the definition of  $\kappa(\theta)$  in (3.1) for any  $\nu_\theta \in (\kappa(\theta), 1)$  there exists some  $C > 0$  such that

$$\mathbf{E}|A_n \cdots A_1|^\theta \leq C(\nu_\theta)^n, \quad \forall n \geq 1.$$

From this inequality, Remark 2.2(a) and Hölder's inequality (with  $p = \theta$  and  $q = \theta/(\theta-1)$ ) we obtain that

$$\begin{aligned} \mathbf{E}|\tau_1|^\theta &\leq \mathbf{E}|A_1| \mathbf{E}|Y_1|^\theta \\ &\leq 2^{\theta-1} \mathbf{E}|A_1| \left( \mathbf{E}|\xi_1|^\theta + \mathbf{E} \left( \sum_{k=3}^{\infty} |A_2 \cdots A_{k-1}| |\xi_k| \right)^\theta \right) \\ &\leq 2^{\theta-1} \mathbf{E}|A_1| \left( \mathbf{E}|\xi_1|^\theta + \mathbf{E}|\xi_1|^\theta \sum_{k=3}^{\infty} \rho^{-\theta(k-2)} \mathbf{E}|A_2 \cdots A_{k-1}|^\theta \left( \sum_{k=3}^{\infty} \rho^{\theta(k-2)/(\theta-1)} \right)^{\theta-1} \right) \\ &\leq 2^{\theta-1} \mathbf{E}|A_1| \left( \mathbf{E}|\xi_1|^\theta + C \mathbf{E}|\xi_1|^\theta \sum_{k=3}^{\infty} \rho^{-\theta(k-2)} (\nu_\theta)^{k-2} \left( \sum_{k=3}^{\infty} \rho^{\theta(k-2)/(\theta-1)} \right)^{\theta-1} \right). \end{aligned}$$

By choosing in the last inequality  $\rho = (\nu_\theta)^{\frac{1}{2\theta}}$  we obtain that for every  $1 < \theta < \lambda$  there exists some  $m(\theta) > 0$  such that

$$\sup_{x \in S} \mathbf{E}|\tau_1|^\theta = \sup_{x \in S} \mathbf{E}|x' A_1 Y_1|^\theta < m(\theta) < \infty. \quad (4.24)$$

This means that we can find  $\theta < \lambda$  for which the inequality (4.24) is true and such that  $\delta = \lambda - \theta$  can be chosen very small.

Further we have for sufficiently large  $t > 0$

$$\begin{aligned}
\psi_1^*(x, t) &\leq \frac{1}{e^t} \int_0^{e^t - n(t)} (n(t) + u)^\lambda \mathbf{P}(\tau_1 > u) du \\
&\quad - \frac{1}{e^t} \int_0^{e^t} u^\lambda \mathbf{P}(\tau_1 > u) du + \frac{(n(t))^{\lambda+1}}{e^t} \\
&\leq c \frac{(n(t))^{\lambda+1}}{e^t} + \frac{1}{e^t} \int_{n(t)}^{e^t - n(t)} u^\lambda \left( \left(1 + \frac{n(t)}{u}\right)^\lambda - 1 \right) \mathbf{P}(\tau_1 > u) du \\
&\leq c \frac{(n(t))^{\lambda+1}}{e^t} + M^* \frac{n(t)}{e^t} \int_{n(t)}^{e^t - n(t)} u^{\lambda-\theta-1} du \\
&\leq c \frac{(n(t))^{\lambda+1}}{e^t} + M^* \frac{m(\theta)n(t)}{e^t} \int_{n(t)}^{e^t - n(t)} u^{\delta-1} du \\
&\leq c \left( \frac{n(t)e^{\delta t}}{e^t} + \frac{(n(t))^{\lambda+1}}{e^t} \right) \\
&\leq c \left( e^{-(1-\delta-\mu)t} + e^{-(1-\mu(\lambda+1))t} \right),
\end{aligned}$$

where  $M^* = \sup_{0 < x \leq 1} ((1+x)^\lambda - 1)/x$  and  $c > 0$  is some constant.

To obtain (4.23) for the function  $\psi_1^*(x, t)$  choose the parameters  $\delta$  and  $\mu$  such that  $\delta + \mu < 1$  and  $0 < \mu < (1 + \lambda)^{-1}$ .

The function  $\psi_2^*(x, t)$  satisfies inequality (4.23), because by the condition  $\mathbf{D}_3$  for every  $m > 0$ ,

$$\sup_{x \in S} \mathbf{E}|\tau_2|^m = \sup_{x \in S} \mathbf{E} | \langle x \rangle_1 \xi_1 |^m \leq \mathbf{E}|\xi_1|^m < \infty.$$

On the other hand, if  $t \rightarrow -\infty$ , we have immediately from definition (4.9),

$$\psi(x, t) \leq h_* \frac{1}{e^t} \int_0^{e^t} u^\lambda du \leq h_* e^{\lambda t}$$

and hence condition  $\mathbf{C}_0$  holds.

By Theorem 4.1 and Lemma A.9 (see Appendix A4) we conclude

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{G(x, t)}{h(x)} = \lim_{t \rightarrow \infty} \tilde{G}(x, t) &= \frac{1}{\beta} \int_S \tilde{\pi}(dz) \int_{-\infty}^{+\infty} \psi(z, s) ds \\
&= \frac{1}{\beta} \int_S \tilde{\pi}(dz) \frac{1}{h(z)} \int_{-\infty}^{+\infty} \frac{1}{e^s} \int_0^{e^s} u^\lambda \psi_0(z, u) du ds \\
&= \frac{1}{\beta} \int_S \tilde{\pi}(dz) \frac{1}{h(z)} \int_0^{+\infty} u^{\lambda-1} \psi_0(z, u) du \\
&= \gamma^* > 0,
\end{aligned}$$

where  $\beta > 0$  is defined by (4.17) and  $\tilde{\pi}(\cdot)$  is the stationary measure of the Markov process  $(x_n)_{n \geq 0}$  under the distribution  $\tilde{P}$  as defined in (4.12).  $\square$



**Lemma 4.5.** *Under the conditions of Theorem 2.4, for every  $x \in S$ , there exists*

$$\lim_{t \rightarrow \infty} t^\lambda \mathbf{P}(x'Y > t) = l(x) = \gamma^* h(x) > 0. \quad (4.25)$$

**Proof.** We give here the proof as in Le Page [19], adapted to our situation. Set

$$F(x, t) = \int_0^t u^\lambda \mathbf{P}(x'Y > u) du.$$

By Lemma 4.4 we have

$$\lim_{t \rightarrow \infty} \frac{F(x, t)}{t} = l(x) > 0.$$

Define for  $x \in S$  and  $t \in \mathbb{R}$  the measure ( $1 < a < b < \infty$ )

$$\mu_{t,x}[a, b] = \frac{F(x, tb) - F(x, ta)}{F(x, t)} \rightarrow b - a, \quad t \rightarrow \infty.$$

This measure has representation

$$\mu_{t,x}[a, b] = \int_a^b v^\lambda g_{t,x}(v) dv,$$

where

$$g_{t,x}(v) = \frac{t^{\lambda+1} \mathbf{P}(x'Y > tv)}{\int_0^t u^\lambda \mathbf{P}(x'Y > u) du}.$$

For any  $v \geq 1$

$$\int_0^t u^\lambda \mathbf{P}(x'Y > u) du \geq \frac{1}{\lambda + 1} t^{\lambda+1} \mathbf{P}(x'Y > tv),$$

and hence

$$0 \leq g_{t,x} \leq \lambda + 1.$$

Furthermore, the function  $g_{t,x}$  is decreasing, thus for any sequence  $(t_n)_{n \in \mathbb{N}}$  tending to infinity there exists a subsequence  $(t_{n_k})_{k \in \mathbb{N}}$  and some function  $g_x(\cdot)$ , such that  $t_{n_k} \rightarrow \infty$  as  $k \rightarrow \infty$  and

$$\lim_{k \rightarrow \infty} g_{t_{n_k},x}(v) = g_x(v), \quad a \leq v \leq b,$$

for any point of continuity  $v$  of  $g_x(\cdot)$ . From this we conclude for any  $1 < a < b < \infty$

$$b - a = \lim_{t \rightarrow \infty} \int_a^b v^\lambda g_{t,x}(v) dv = \lim_{k \rightarrow \infty} \int_a^b v^\lambda g_{t_{n_k},x}(v) dv = \int_a^b v^\lambda g_x(v) dv.$$

It follows that  $g_x(v) = 1/v^\lambda$ . From this we obtain

$$\lim_{t \rightarrow \infty} t^\lambda \mathbf{P}(x'Y > t) = l(x) = \gamma^* h(x) > 0.$$

□

This concludes the proof of Theorem 2.4.

# Appendix

## A1) A simple sufficient condition for $D_4$

**Proof of Proposition 2.3.** Let  $l = \inf\{k \geq 1 : |c_k| > 0, \}$ . For  $n \geq l$  set  $\tau_n = \sum_{k=l}^n c_k \xi_k$ . If  $|c_k| > 0$ , then  $c_k \xi_k$  has a symmetric density  $p_k(\cdot)$ , continuously differentiable with derivative  $p'_k(\cdot) \leq 0$  on  $(0, \infty)$ . Therefore  $\tau_l$  has a symmetric density, which is non-increasing on  $(0, \infty)$ . We proceed by induction. Suppose that  $\tau_{n-1}$  has a symmetric density, non-increasing on  $(0, \infty)$ . We show that  $\tau_n$  has a density with these properties. Indeed, if  $c_n = 0$  then  $\tau_n = \tau_{n-1}$  and we have the same distribution for  $\tau_n$ . Let us consider now the case  $c_n > 0$ . Taking into account the properties of  $p_n(\cdot)$  and of the density  $\varphi_{\tau_{n-1}}(\cdot)$  of  $\tau_{n-1}$  we can write the density  $\varphi_{\tau_n}(\cdot)$  of  $\tau_n$  in the following form

$$\begin{aligned} \varphi_{\tau_n}(z) &= \int_{-\infty}^{\infty} p_n(z-u)\varphi_{\tau_{n-1}}(u)du \\ &= \int_0^{\infty} p_n(z+u)\varphi_{\tau_{n-1}}(u)du + \int_0^z p_n(z-u)\varphi_{\tau_{n-1}}(u)du \\ &\quad + \int_z^{\infty} p_n(u-z)\varphi_{\tau_{n-1}}(u)du, \quad z > 0. \end{aligned}$$

Therefore the derivative of this function equals

$$\begin{aligned} \varphi'_{\tau_n}(z) &= \int_0^{\infty} p'_n(z+u)\varphi_{\tau_{n-1}}(u)du + \int_0^z p'_n(z-u)\varphi_{\tau_{n-1}}(u)du - \int_z^{\infty} p'_n(u-z)\varphi_{\tau_{n-1}}(u)du \\ &= \int_z^{\infty} p'_n(u) \left( \varphi_{\tau_{n-1}}(u-z) - \varphi_{\tau_{n-1}}(u+z) \right) du \\ &\quad + \int_0^z p'_n(u) \left( \varphi_{\tau_{n-1}}(z-u) - \varphi_{\tau_{n-1}}(u+z) \right) du \leq 0, \quad z > 0, \end{aligned}$$

since  $p'_n(\cdot) \leq 0$  and  $\varphi_{\tau_{n-1}}(\cdot)$  is non-increasing on  $(0, \infty)$ . Therefore we obtained that for all  $n \geq l$  the random variable  $\tau_n$  has a symmetric continuously differentiable density, which is non-increasing on  $(0, \infty)$ . Further, since  $\tau_n \rightarrow \tau$  a.s. as  $n \rightarrow \infty$  and the random variable  $\tau$  has a continuous density  $\varphi_{\tau}$ , we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} g(z)\varphi_{\tau_n}(z)dz = \int_{-\infty}^{+\infty} g(z)\varphi_{\tau}(z)dz$$

for every bounded measurable function  $g$ . Therefore, for  $0 < a < b$  we have for all  $0 < \delta < (b+a)/2$ ,

$$\int_{b-\delta}^{b+\delta} \varphi_{\tau}(z)dz - \int_{a-\delta}^{a+\delta} \varphi_{\tau}(z)dz = \lim_{n \rightarrow \infty} \left( \int_{b-\delta}^{b+\delta} \varphi_{\tau_n}(z)dz - \int_{a-\delta}^{a+\delta} \varphi_{\tau_n}(z)dz \right) \leq 0.$$

Since the density  $\varphi_{\tau}(\cdot)$  is continuous we conclude

$$\varphi_{\tau}(b) - \varphi_{\tau}(a) = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \left( \int_{b-\delta}^{b+\delta} \varphi_{\tau}(z)dz - \int_{a-\delta}^{a+\delta} \varphi_{\tau}(z)dz \right) \leq 0. \quad \square$$

## A2) Gaussian linear random coefficient models

### Proof of Proposition 2.6.

It is evident that conditions  $\mathbf{D}_1$ – $\mathbf{D}_4$  hold for the Gaussian model.

To show that the conditional correlation matrix (2.4) is positive definite a.s. take some  $x \in \mathbb{R}^q$  such that  $x'Rx = 0$ . Then

$$x'Bx + \sum_{k=1}^{\infty} \theta_k' B \theta_k = 0,$$

where  $B$  is defined in (2.5),  $\theta_0 = x$  and  $\theta_k = A_k' \cdots A_1' x$ ,  $k \in \mathbb{N}$ . If we denote  $\langle x \rangle_i$  the  $i$ th coordinate of  $x \in \mathbb{R}^q$ , the equality above means that  $\langle \theta_k \rangle_1 = 0$  for all  $k \geq 0$ . Taking this into account and by the definition of the vector  $\theta_k$  we obtain that  $\langle \theta_k \rangle_1 = \langle x \rangle_{k+1} = 0$  for all  $0 \leq k \leq q-1$ . From this we conclude that  $x'Rx = 0$  implies  $x = 0$ , which means that  $R$  is positive definite a.s.  $\square$

## A3) Criteria for geometric ergodicity.

**Lemma A.6.** (Meyn and Tweedie [16], p. 355)

Suppose that  $\Gamma$  is a small set and that the measurable function  $v : S \rightarrow [1, \infty)$  satisfies

$$\sup_{x \in \Gamma} \int_S v(y) p(x, dy) < \infty \tag{A.1}$$

and, for some  $\varepsilon > 0$ ,

$$\int_S v(y) p(x, dy) < (1 - \varepsilon)v(x), \quad \text{for all } x \in \Gamma^c. \tag{A.2}$$

Then the process  $(x_n)_{n \in \mathbb{N}}$  is geometric ergodic.

Recall the definition of the  $(A_n)_{n \in \mathbb{N}}$  in (1.5) and set  $\Pi(n) = A_n \cdots A_1$ ,  $n \in \mathbb{N}$ .

**Lemma A.7.** Under the conditions of Theorem 2.4, for  $x \in S$ , the distribution of  $\Pi(2q)x$  has a positive density with respect to Lebesgue measure on  $\mathbb{R}^q$ , which is continuous on  $S$ . This means that the function

$$\Phi_1(x, t) = \mathbf{E}_x \Phi(x_{2q}, v_{2q}, t), \quad x \in S, t \geq 0,$$

is uniformly (with respect to  $t$ ) continuous on  $S$  for every measurable bounded function  $\Phi : S \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . Furthermore, the distribution of the vector  $x_{2q}$  has a positive continuous density with respect to Lebesgue measure on  $S$ .

**Proof.** Let  $x = (x_1, \dots, x_q)' \in S$  and define  $\eta_i = \eta_i(x) = \alpha(i)' \Pi(i-1)x$  with vector  $\alpha(i) = (\alpha_1(i), \dots, \alpha_q(i))'$ . Notice that for  $1 \leq i < q$ ,

$$\Pi(i)x = (\eta_i, \dots, \eta_1, x_1, \dots, x_{q-i})',$$

and for  $i \geq q$

$$\Pi(i)x = (\eta_i, \dots, \eta_{i-q+1})'. \quad (\text{A.3})$$

For  $i > q$  this implies  $\eta_i = \sum_{j=1}^q \alpha_j(i) \eta_{i-j}$ . Therefore by  $\mathbf{D}_1$  if

$$d_i = d_i(\eta_{i-1}, \dots, \eta_{i-q}) = x' \Pi(i-1)' D \Pi(i-1)x = \sum_{j=1}^q \sigma_j^2 \eta_{i-j}^2 > 0,$$

then  $\eta_i = \alpha(i)' \Pi(i-1)x$  has a conditional density (under the condition  $\Pi(i-1)x = (\eta_{i-1}, \dots, \eta_{i-q})'$ ) for  $i > q$

$$\varphi_i(\cdot | \eta_{i-1}, \dots, \eta_{i-q}) = \phi(\cdot, \Pi(i-1)x).$$

By  $\mathbf{D}_2$  we have  $\sigma_1^2 > 0$ . Hence, if  $\eta_q = 0$  with positive probability, then, taking condition  $\mathbf{D}_1$  into account,

$$d_q = \sum_{j=1}^q \sigma_j^2 \langle \Pi(q-1)x \rangle_j^2 = \sum_{j=1}^q \sigma_j^2 \eta_{q-j}^2 = 0,$$

which implies

$$\eta_{q-1} = \langle \Pi(q-1)x \rangle_1 = 0 \quad (\text{because } \sigma_1^2 > 0) \quad \Rightarrow \quad \dots \quad \Rightarrow \quad \eta_1 = 0,$$

i.e.  $\Pi(q)x = 0$  by (A.3). But this is impossible because  $\det \Pi(q) \neq 0$  a.s. and  $x \in S$ , which means that  $\eta_q \neq 0$ . Hence  $d_{q+1} > 0$  a.s.

$$\Rightarrow \quad \eta_{q+1} \neq 0 \quad \Rightarrow \quad d_{q+2} > 0 \quad \Rightarrow \quad \dots,$$

i.e.  $d_i > 0$  a.s. for all  $i > q$ .

This implies that the vector  $\Pi(2q)x = (\eta_{2q}, \dots, \eta_{q+1})'$  has a density in  $\mathbb{R}^q$ :

$$p(z_1, \dots, z_q, x) = \mathbf{E} \prod_{i=q+1}^{2q} \varphi_i(z_{i-q} | z_{i-q-1}, \dots, z_1, \eta_q(x), \dots, \eta_{i-q}(x)). \quad (\text{A.4})$$

From this and  $\mathbf{D}_1$  follows that the function  $p(z_1, \dots, z_q, x)$  is a continuous positive function of  $(z_1, \dots, z_q) \in \mathbb{R}^q$  and  $x \in S$ . Since  $v_{2q} = \log |\Pi(2q)x|$  and  $x_{2q} = \Pi(2q)x / |\Pi(2q)x|$ , we obtain the second part of the lemma.

Further, for any bounded measurable function  $f$  on  $S$  we have

$$\begin{aligned}\mathbf{E}_x f(x_{2q}) &= \int_{\mathbb{R}^q} f\left(\frac{z_1}{|z|}, \dots, \frac{z_q}{|z|}\right) p(z_1, \dots, z_q, x) dz_1 \dots dz_q \\ &= \int_S f(y_1, \dots, y_q) p_S(y_1, \dots, y_q, x) \Lambda(dy_1 \dots dy_q),\end{aligned}$$

where  $\Lambda(\cdot)$  is Lebesgue measure on  $S$  and

$$p_S(y_1, \dots, y_q, x) = \int_0^\infty r^{q-1} p(ry_1, \dots, ry_q, x) dr,$$

where the function  $p(y_1, \dots, y_q, x)$  is defined in (A.4). We show now that the density  $p_S(y_1, \dots, y_q, x)$  is continuous for all  $y = (y_1, \dots, y_q)' \in S$  and  $x \in S$ . To this end, let  $y^0 = (y_1^0, \dots, y_q^0)' \in S$  and  $x^0 \in S$ . We need to show that

$$\lim_{y \rightarrow y^0, x \rightarrow x^0} p_S(y_1, \dots, y_q, x) = p_S(y_1^0, \dots, y_q^0, x^0).$$

For this it suffices to show that for some  $0 < \delta < 1$

$$\lim_{N \rightarrow \infty} \sup_{|y - y^0| \leq \delta, x \in S} \int_N^\infty r^{q-1} p(ry_1, \dots, ry_q, x) dr = 0, \quad (\text{A.5})$$

because  $\int_0^N r^{q-1} p(ry_1, \dots, ry_q, x) dr$  is a continuous function for every positive  $N$ .

Indeed, set  $l = \inf\{j \geq 1 : |y_j^0| > 0\}$ , i.e.  $|y_l^0| > 0$  and  $y_i = 0$  for all  $j < l$ . We choose  $0 < \delta < |y_l^0|/2$ . Then by (A.4) we have for some constant  $C > 0$

$$\begin{aligned}\int_N^\infty r^{q-1} p(ry_1, \dots, ry_q, x) dr &\leq C \mathbf{E} \int_N^\infty r^{q-1} \varphi_{q+l}(ry_l | 0, \dots, 0, \eta_q(x), \dots, \eta_l(x)) dr \\ &\leq C \frac{1}{|y_l|^q} \mathbf{E} \int_{|y_l|N}^\infty a^{q-1} \phi(a, \rho(x)) da \\ &= C \frac{2^q}{|y_l^0|^q} \mathbf{E} \left[ (\alpha'_{q+l} \rho)^{q-1} \chi_{\{\alpha'_{q+l} \rho(x) > |y_l|N\}} \right],\end{aligned}$$

where  $\rho(x) = (0, \dots, 0, \eta_q(x), \dots, \eta_l(x))'$ . By condition  $\mathbf{D}_1$  we have for every  $m \in \mathbb{N}$ ,

$$\sup_{x \in S} \mathbf{E} [|\alpha'_{q+l} \rho(x)|^m] < \infty.$$

From this we obtain (A.5) and the last part of this lemma.  $\square$

**Lemma A.8.** *The assertions of Lemma A.7 remain true with respect to the measure  $\tilde{\mathbf{P}}_x$  as defined in (4.12).*

**Proof.** Indeed, from Lemma A.7 and (4.12) follows that for every bounded measurable function  $f$  we have for  $x \in S$

$$\tilde{\mathbf{E}}_x f(\Pi(2q)x) = \frac{1}{h(x)} \mathbf{E} |\Pi(2q)x|^\lambda h(\overline{\Pi(2q)x}) f(\Pi(2q)x) = \int_{\mathbb{R}^q} f(z) \tilde{p}(z, x) dz,$$

with

$$\tilde{p}(z, x) = \frac{1}{h(x)} |z|^\lambda h(\overline{z}) p(z, x),$$

where  $p(z, x)$  is defined by (A.4) for  $z \in \mathbb{R}^q$  and  $x \in S$ . Therefore the vector  $\Pi(2q)x$  has a continuous positive density  $\tilde{p}(z, x)$  with respect to the measure  $\tilde{\mathbf{P}}_x$ . By the same method as in Lemma A.7 we get that the vector  $x_{2q}$  has a positive continuous density  $\tilde{p}_S(z, x)$  on  $S$  with respect to the measure  $\tilde{\mathbf{P}}_x$  given by

$$\tilde{p}_S(y, x) = \int_0^\infty r^{q-1} \tilde{p}(ry, x) dr = \frac{h(y)}{h(x)} \int_0^\infty r^{q+\lambda-1} p(ry, x) dr.$$

From Lemma A.7 we obtain that this function is continuous in  $y \in \mathbb{R}^q$  and  $x \in S$ .  $\square$

#### A4) A property of $\psi_0$ .

**Lemma A.9.** *Condition  $\mathbf{D}_4$  implies that the function  $\psi_0(x, u)$ , which is defined in (4.11), is non-negative and, for all  $x = (\langle x \rangle_1, \dots, \langle x \rangle_q)' \in S$  with  $\langle x \rangle_1 \neq 0$*

$$\Lambda(\{u \geq 0 : \psi_0(x, u) > 0\}) > 0, \quad (\text{A.6})$$

where  $\Lambda(\cdot)$  denotes Lebesgue measure on  $\mathbb{R}_+$ .

**Proof.** Indeed, by definition we have

$$\psi_0(x, u) = \mathbf{P}(\tau_1 + \tau_2 > u) - \mathbf{P}(\tau_1 > u)$$

with  $\tau_1 = x' A_1 Y_1$  and  $\tau_2 = x' \zeta_1 = \langle x \rangle_1 \xi_1$ . If  $\langle x \rangle_1 = 0$ , then  $\tau_2 = 0$ , and therefore  $\psi_0(x, u) = 0$ . We show that  $\psi_0(x, u) \geq 0$  if  $\langle x \rangle_1 \neq 0$ . By conditioning on  $\tau_2$  we get

$$\psi_0(x, u) = \int_0^\infty (\mathbf{P}(u - t < \tau_1 \leq u) - \mathbf{P}(u < \tau_1 \leq u + t)) p_{\tau_2}(t) dt, \quad (\text{A.7})$$

where  $p_{\tau_2}(\cdot)$  is the density of  $\tau_2$ , which is by  $\mathbf{D}_4$  symmetric and non-increasing on  $(0, \infty)$ . Further, again by  $\mathbf{D}_4$  the conditional density  $p_{\tau_1}(\cdot | \mathcal{A})$  of  $\tau_1$ , where  $\mathcal{A} = \sigma\{A_i, i \geq 1\}$  is symmetric and non-increasing on  $\mathbb{R}_+$ . Hence for  $0 \leq t \leq u$  we have

$$\begin{aligned} & \mathbf{P}(u - t \leq \tau_1 \leq u) - \mathbf{P}(u < \tau_1 \leq u + t) \\ &= \mathbf{E}(\mathbf{P}(u - t \leq \tau_1 \leq u | \mathcal{A}) - \mathbf{P}(u < \tau_1 \leq u + t | \mathcal{A})) \\ &= \mathbf{E}\left(\int_{u-t}^u p_{\tau_1}(a | \mathcal{A}) da - \int_u^{u+t} p_{\tau_1}(a | \mathcal{A}) da\right) \\ &= \mathbf{E}\int_{u-t}^u (p_{\tau_1}(a | \mathcal{A}) - p_{\tau_1}(a + t | \mathcal{A})) da \geq 0. \end{aligned}$$

On the other hand, for  $t > u$  we get

$$\begin{aligned}
& \mathbf{P}(u - t \leq \tau_1 \leq u) - \mathbf{P}(u < \tau_1 \leq u + t) \\
&= \mathbf{E} \left( \int_{u-t}^u p_{\tau_1}(a|\mathcal{A}) da - \int_u^{u+t} p_{\tau_1}(a|\mathcal{A}) da \right) \\
&= \mathbf{E} \left( \int_{u-t}^0 p_{\tau_1}(a|\mathcal{A}) da + \int_0^u p_{\tau_1}(a|\mathcal{A}) da - \int_u^{u+t} p_{\tau_1}(a|\mathcal{A}) da \right) \\
&= \mathbf{E} \left( \int_0^{t-u} p_{\tau_1}(a|\mathcal{A}) da + \int_0^u p_{\tau_1}(a|\mathcal{A}) da - \int_u^{2u} p_{\tau_1}(a|\mathcal{A}) da - \int_{2u}^{u+t} p_{\tau_1}(a|\mathcal{A}) da \right) \\
&= \mathbf{E} \left( \int_0^{t-u} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a + 2u|\mathcal{A})) da + \int_0^u (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a + u|\mathcal{A})) da \right) \geq 0,
\end{aligned}$$

since  $p_{\tau_1}(\cdot|\mathcal{A})$  is non-increasing on  $\mathbb{R}_+$ . This proves the first part of this lemma.

We show now the inequality (A.6). Let  $x \in S$  with  $\langle x \rangle_1 \neq 0$ . By condition  $\mathbf{D}_4$  the function  $p_{\tau_2}(\cdot)$  is non-increasing on  $\mathbb{R}_+$ . Therefore there exists  $t_0 > 0$  such that  $[0, t_0] \subseteq \{u \geq 0 : p_{\tau_2}(u) > 0\}$ . If for this  $x \in S$

$$\Lambda(\{u \geq 0 : \psi_0(x, u) > 0\}) = 0,$$

then  $\psi_0(x, u) = 0$  for all  $u \geq 0$ , because by (A.7) the function  $\psi_0(x, u)$  is continuous on  $u \geq 0$ . This implies that for some  $t \in (0, t_0)$  and for all  $u \geq t$

$$\mathbf{E} \int_{u-t}^u (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a + t|\mathcal{A})) da = 0 \quad \Rightarrow \quad \mathbf{E} \int_0^{+\infty} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a + t|\mathcal{A})) da = 0$$

and by the monotonicity of the function  $p_{\tau_1}(\cdot|\mathcal{A})$  we get

$$p_{\tau_1}(a|\mathcal{A}) = p_{\tau_1}(a + t|\mathcal{A}) \text{ for almost all } a \geq 0.$$

Further we set  $\alpha^* = \max\{a \geq 0 : p_{\tau_1}(a|\mathcal{A}) = p_{\tau_1}(0|\mathcal{A})\}$  and for  $a \in (\alpha^* - \delta, \alpha^*)$  with  $\delta = \min(\alpha^*/2, t/2)$  we obtain

$$p_{\tau_1}(a|\mathcal{A}) = p_{\tau_1}(0|\mathcal{A}) = p_{\tau_1}(a + t|\mathcal{A}),$$

but this contradicts the definition of  $\alpha^*$ , because  $a + t > \alpha^*$ . From this we obtain (A.7).  $\square$

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