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Generalized basic probability assignments

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Generalized Basic Probability Assignments

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Abstract

Dempster-Shafer theory allows to construct belief functions from (precise) basic probability assignments. The present paper extends this idea substantially. By considering *sets* of basic probability assignments, an appealing constructive approach to general interval probability (general imprecise probabilities) is achieved, which allows for a very flexible modelling of uncertain knowledge.

1 Introduction and Sketch of the Argument

In order to model complex uncertain knowledge appropriately, generalizations of the notion of probability and its mathematical formalization have attracted considerable attention (see, e.g., de Cooman, Fine, Moral and Seidenfeld (2001), Breese and Koller (2001), Bernard (2002), and the Imprecise Probability Web Page (de Cooman and Walley (2002)). Most popular, in

particular in artificial intelligence, is the Dempster-Shafer theory of belief functions (Shafer (1976), cf., also, e.g., Yager, Fedrizzi and Kacprzyk (1994)). Though belief function are only a special case of general interval probability or imprecise probabilities (e.g., Walley (1991), Weichselberger (2001)), they are particularly attractive, because they can elegantly be *constructed* from information which is not strong enough to divide the whole probability mass among the singletons and only the singletons: the beliefs in events need not be assigned apriori, but can be calculated by accumulating the corresponding basic probability numbers.

The present paper extends this appealing construction principle substantially. Generalized basic probability assignments are introduced which lead to lower and upper probabilities, providing a vivid constructive approach to general interval probability.

The basic idea of this new concept arises naturally from a closer investigation of the mathematical apparatus behind the Dempster-Shafer framework: Formally, Shafer's constructive approach to interval probability on a measurable space $(\Omega, \mathcal{P}(\Omega))$ is equivalent to assigning a classical probability measure on the "higher-level" measurable space $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$, and the basic probability assignment is merely the corresponding probability mass function. This will be generalized here by allowing for a set $\mathcal{S}$ of mass functions, called *generalized basic probability assignments*. Calculating the envelope of all belief functions generated by elements of $\mathcal{S}$, general interval probability is obtained. A variety of rather different situations can be modelled by adapting methods for handling sets of mass functions (like Kofler and Menges (1976) or Weichselberger and Pöhlmann (1990)) as techniques for dealing with generalized basic probability assignments. For instance, partial orderings on the basic probability numbers can be considered.

The paper is organized as follows: It starts with collecting some basic notions needed later, concerning general interval probability and the Dempster-Shafer approach. In Section 3 the concept of generalized basic probability assignments is introduced, motivated and formalized, and Theorem 3.3 proves that it fits nicely into the frame of general interval probability. Section 4 sketches the possible range of modelling by considering some attractive special cases.

2 Interval Probability and Belief Functions

The usual concept of probability as formalized by Kolmogorov's axioms requires a level of precision and — by the axiom of additivity — a degree of internal consistency of the assignments which often cannot be satisfied. To model more complex uncertainty appropriately, different theories of interval probability have emerged, where an interval $[L(A), U(A)]$ is assigned to every event to describe its probability.

With respect to the intended application, the whole consideration is restricted here to the case of a finitely generated algebra $\mathcal{A}$ based on a sample space Ω. Then, without loss of generality, Ω is finite, and $\mathcal{A}$ is the power set $\mathcal{P}(\Omega)$. Every probability measure in the usual sense, i.e., every set function $p(\cdot)$ satisfying Kolmogorov's axioms, is called a *classical probability*. The set of all classical probabilities on the measurable space $(\Omega, \mathcal{A})$ will be denoted by $\mathcal{C}(\Omega, \mathcal{P}(\Omega))$. Then, as in (1) to (3), axioms for interval-valued probabilities $P(\cdot) = [L(\cdot), U(\cdot)]$ can be obtained by looking at the relation between the non-additive set-function $L(\cdot)$ and $U(\cdot)$ and the set of classical probabilities being in accordance with them.

On a finite sample space, the most important concepts of interval probability coincide. They all are concerned with set-functions

$$
\begin{aligned}
P(\cdot) \;:\; \mathcal{P}(\Omega) &\;\rightarrow\; \{[L, U] \,|\, 0 \le L \le U \le 1\} \\
A &\;\mapsto\; P(A) = [L(A), U(A)]
\end{aligned}
$$

with

$$
\mathcal{M} := \{p(\cdot) \in \mathcal{C}(\Omega, \mathcal{P}(\Omega)) \,|\, L(A) \le p(A) \le U(A), \; \forall A \in \mathcal{P}(\Omega)\} \neq \emptyset \quad (1)
$$

and

$$
\min_{p(\cdot) \in \mathcal{M}} p(A) = L(A), \qquad \forall A \in \mathcal{P}(\Omega), \tag{2}
$$

$$
\max_{p(\cdot) \in \mathcal{M}} p(A) = U(A), \qquad \forall A \in \mathcal{P}(\Omega). \tag{3}
$$

Such $P(\cdot)$, and the corresponding set functions $L(\cdot)$ and $U(\cdot)$, are called lower and upper probability (Huber & Strassen (1973)), envelopes (Walley & Fine (1982), Denneberg (1994)), coherent probability (Walley (1991)) and F-probability (Weichselberger (1995, 2000, 2001)). In game theory $\mathcal{M}$ is the

core (Shapley (1971)). Here Weichselberger's terminology is used, calling $\mathcal{M}$ *structure* of the *F-probability* $P(\cdot)$.

For every F-probability, $L(\cdot)$ and $U(\cdot)$ are *conjugate*, i.e., $L(A) = 1-U(A^c)\,, \forall A \in \mathcal{P}(\Omega)\,.$ Therefore, every F-probability is uniquely determined either by $L(\cdot)$ or by $U(\cdot)$ alone. Here $L(\cdot)$ is used throughout, and $\mathcal{F} = (\Omega, \mathcal{P}(\Omega), L(\cdot))$ is called an *F-probability field.* Specifying an F-probability field $(\Omega, \mathcal{P}(\Omega), L(\cdot))$, it is implicitly required that the conjugate set function $U(\cdot) = 1 - L(\cdot^c)$ describes the upper bound of the interval.

A characteristic special case of F-probability is considered in the Dempster-Shafer approach[1]. The main entities are totally-monotone capacities, called belief functions in this context.

Def. 2.1 (Belief function) *Let $(\Omega, \mathcal{P}(\Omega))$ be a measurable space. A real-valued set-function $Bel(\cdot) : \mathcal{P}(\Omega) \to [0,1]$ with $Bel(\emptyset) = 0$ and $Bel(\Omega) = 1$ is called a* belief function *on $(\Omega, \mathcal{P}(\Omega))$, if for all $n \in \mathbb{N}$ and for all $(A_1 \ldots, A_n)$ with $A_i \in \mathcal{P}(\Omega), i = 1\ldots, n$, it satisfies:*

$$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right). \qquad (4)$$

Typically belief functions are constructed via basic probability assignments.

Def. 2.2 (Basic probability assignment) *A function $m : \mathcal{P}(\Omega) \to \mathbb{R}$ is called a* basic probability assignment *on $(\Omega, \mathcal{P}(\Omega))$, if it satisfies*

$$m(\emptyset) = 0\,, \qquad m(A) \geq 0\,, \quad \forall A \in \mathcal{P}(\Omega)\,, \qquad \sum_{A \in \mathcal{P}(\Omega)} m(A) = 1\,. \qquad (5)$$

Then, for $A \in \mathcal{P}(\Omega)$, the quantity $m(A)$ is called basic probability number *of the event A.*

---

[1]There is quite a couple of approaches, which differ in interpretation but use the same mathematical techniques, namely totally monotone capacities. The argument presented below is technically situated by generalizing the mathematical basis. It immediately carries over to each of the concretely preferred interpretations. Therefore, in this article, it seems to be not necessary to distinguish between the several approaches relying on totally monotone capacities. For simplicity Shafer's (1976) vocabulary and the name 'Dempster-Shafer theory' is used throughout the paper.

The basic probability number $m(A)$ is interpreted as the weight one gives to that part of the information which points solely to A and cannot be divided among proper subsets of A. Then the total belief in A can be calculated as the sum of all basic probability numbers committed to the event A or committed to any nonempty subset of A. For convenience this procedure is called *belief accumulation* in the sequel. It leads to a belief function:

Remark 2.3   *(i) Let $m(\cdot)$ be a basic probability assignment on $(\Omega, \mathcal{P}(\Omega))$. The function $Bel_m : \mathcal{P}(\Omega) \to [0, 1]$ with*

$$Bel_m(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \qquad A \in \mathcal{P}(\Omega), \tag{6}$$

is a belief function, and $(\Omega, \mathcal{P}(\Omega), Bel_m(\cdot))$ is an F-probability field.

(ii) $Bel_m(\cdot)$ will be called the belief function generated by $m(\cdot)$.

While every belief function leads to an F-probability, the converse does not hold: examples of *general F-probability*, i.e., F-probability not satisfying (4), can be constructed easily. Moreover, it can be shown that general F-probability even cannot be reasonably approximated by belief functions, and so belief functions are often judged to be too restrictive to serve as the basis of a powerful generalization of classical probability theory.

3   Generalized Basic Probability Assignments

3.1   Belief Functions as Classical Probabilities

Even more trenchant: belief functions do not really go beyond the scope of classical probability theory, as can be seen by stating belief accumulation formally – an argument which will straightforwardly lead to the generalization proposed.

Prop. 3.1 *(Formalization of belief accumulation)*[2]

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[2]This proposition is essentially already due to Choquet (1953)

1. *Let $p_\Gamma(\cdot)$ be a classical probability measure on the measurable space $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$. Then the function $Bel(\cdot) : \mathcal{P}(\Omega) \to [0,1]$ defined by*

$$Bel(A) := p_\Gamma(\mathcal{P}(A)), \qquad \forall A \in \mathcal{P}(\Omega), \qquad (7)$$

   is a belief function on $(\Omega, \mathcal{P}(\Omega))$.

2. *For every belief function $Bel(\cdot)$ on $(\Omega, \mathcal{P}(\Omega))$ there exists a unique classical probability measure $p_\Gamma(\cdot)$ satisfying (7). The corresponding mass function $m(A) := p_\Gamma(\{A\}), A \in \mathcal{P}(\Omega)$, is a basic probability assignment, generating $Bel(\cdot)$.*

This proposition shows how Dempster-Shafer theory constructs interval probability from indivisible pieces of information. Shafer considers the set $\mathcal{P}(\Omega)$ of all random events as a "higher-level" sample space, assigns a mass function on it and identifies the total belief in A with the probability of the set-system containing all nonempty subsets of A including A. So there is a one-to-one correspondence between belief functions and classical probability measures on the measurable space $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$, and a belief function on $(\Omega, \mathcal{P}(\Omega))$ is nothing else but the restriction of a classical probability measure on $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$ to those events which can be written as the power set of subsets of Ω. Finally that means: belief functions are, in essence, classical probabilities, specified by their mass function, namely the basic probability assignment. According to this, one has to keep in mind that using belief functions finally relies on the assumption that the available information can be adequately quantified by a single probability measure on $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$. In many situations, especially in situations of uncertain knowledge, this might be more than one can honestly require. Therefore an appropriate generalization is highly desirable.

3.2 The Basic Idea

In Dempster-Shafer theory modelling uncertain knowledge, on the one hand, and its quantification by a single basic probability assignment, on the other hand, is seen as Siamese twins, but the discussion above shows that this is not necessary. The attractive constructive character of belief functions is due to a change of the sample space from Ω to $\mathcal{P}(\Omega)$, but not at all to the assignment of a single mass function on it. Separating both aspects, the

generalization suggests itself: *Change the sample space, but permit a more flexible modelling by sets of basic probability assignments!* Note that by allowing for sets of basic probability assignments the flexibility of modelling is enriched substantially because these sets can be gained not only by enumeration of their elements but also by construction from weaker quantified information (see Section 4.2f.).

Every basic probability assignment generates a belief function, and so a set of basic probability assignments generates a set of belief functions. A natural way to handle this set is to assign to every event its minimal belief, i.e., to consider the envelope of all these belief functions. It will be shown that this procedure leads to F-probability.

3.3 Formalization of the Concept

This informal motivation can be formalized rigorously:

Def. 3.2 (Generalized basic probability assignment) *Let $(\Omega, \mathcal{P}(\Omega))$ be a finite measurable space, and denote by $\mathcal{Q}(\Omega, \mathcal{P}(\Omega))$ the set of all basic probability assignments on $(\Omega, \mathcal{P}(\Omega))$. Every nonempty, closed subset $\mathcal{S} \subseteq \mathcal{Q}(\Omega, \mathcal{P}(\Omega))$ is called* generalized basic probability assignment *on $(\Omega, \mathcal{P}(\Omega))$.*

The next theorem embeds generalized basic probability assignments into the theory of general interval probability, by showing that generalized belief accumulation leads to F-probability.

Theorem 3.3 (Generalized belief accumulation) *For every generalized basic probability assignment $\mathcal{S}$ the set-function $\mathbb{L}(\cdot) : \mathcal{P}(\Omega) \to [0, 1]$ with*

$$\mathbb{L}(A) := \min_{m(\cdot) \in \mathcal{S}} \sum_{\emptyset \neq B \subseteq A} m(B), \quad \forall A \in \mathcal{P}(\Omega), \tag{8}$$

is well-defined, and $\mathcal{F}(\mathcal{S}) := (\Omega, \mathcal{P}(\Omega), \mathbb{L}(\cdot))$ is an F-probability field.

Proof: i) To see that $\mathbb{L}(\cdot)$ is well-defined, identify every element $m(\cdot)$ of $\mathcal{S}$ with the $|\mathcal{P}(\Omega)|$-dimensional vector containing the components $(m(A))_{A \in \mathcal{P}(\Omega)}$. Then the mapping

$$
\begin{aligned}
l^A \quad &: \quad \mathbb{R}^{|\mathcal{P}(\Omega)|} \supset \mathcal{S} \quad \to \quad [0, 1] \\
m \quad &\mapsto \quad \sum_{\emptyset \neq B \subseteq A} m(B)
\end{aligned}
$$

describing generalized belief accumulation is continuous in $m(\cdot) \in \mathcal{S}$. Therefore it reaches its extreme values on the closed and bounded, and hence compact set $\mathcal{S}$.

ii) To show that (8) leads to F-probability, one has to verify (1), (2) and (3) with $\mathbb{U}(\cdot) := 1 - \mathbb{L}(\cdot^c)$ and with

$$\mathcal{M} := \{p(\cdot) \in \mathcal{C}\left(\Omega, \mathcal{P}(\Omega)\right) \mid \mathbb{L}(A) \le p(A) \le \mathbb{U}(A), \ \forall A \in \mathcal{P}(\Omega)\}\,.$$

Consider, for every $A \in \mathcal{P}(\Omega)$, that element $m_A(\cdot)$ of $\mathcal{S}$ which produces $\mathbb{L}(A)$, and let $\mathcal{M}_{m_A}$ be the structure of the corresponding F-probability field $(\Omega, \mathcal{P}(\Omega), Bel_{m_A}(\cdot))$. By construction, $Bel_{m_A}(A) = \mathbb{L}(A)$, and so, by (2) applied to $Bel_{m_A}(\cdot)$, there exists a classical probability $p_{m_A}(\cdot) \in \mathcal{M}_{m_A}$ with

$$p_{m_A}(A) = \mathbb{L}(A)\,. \tag{9}$$

$p_{m_A}(\cdot)$ is an element of $\mathcal{M}$, because, for every event $D \in \mathcal{P}(\Omega)$,

$$\begin{aligned} \mathbb{L}(D) &= \min_{m(\cdot) \in \mathcal{S}} \sum_{\emptyset \ne B \subseteq D} m(B) \le \sum_{\emptyset \ne B \subseteq D} m_A(B) = Bel_{m_A}(D) \\ &\le p_{m_A}(D) \le 1 - Bel_{m_A}(D^c) \le 1 - \min_{m(\cdot) \in \mathcal{S}} \sum_{\emptyset \ne B \subseteq D^c} m(B) = \mathbb{U}(D)\,. \end{aligned}$$

Therefore, $\mathcal{M}$ is not empty, and (1) is shown. Furthermore, by the definition of $\mathcal{M}$, $p(A) \ge \mathbb{L}(A)$, for all $p(\cdot) \in \mathcal{M}$, and so (9) leads to (2). Relation (3) is obtained by passing over from A to A^c. $\diamond$

The concept of generalized basic probability assignments and generalized belief accumulation serves as a constructive approach to F-probability, consequentially generalizing Dempster-Shafer theory. The correspondence between F-probability and generalized belief accumulation goes even beyond Theorem 3.3: It can be shown that conversely every F-probability field $\mathcal{F}$ can be obtained by generalized belief accumulation.[3] [4]

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[3]Take the vertices $p_1(\cdot) \ldots, p_q(\cdot)$ of $\mathcal{M}$, which is a convex polyhedron, and apply the procedure described in Subsection 4.1 to the set $\mathcal{S} = \{m_1(\cdot), \ldots, m_q(\cdot)\}$ where, for $j = 1, \ldots, q$, $m_j(A) := p_j(A)$, if A is a singleton, and $m_j(A) := 0$, else.

[4]But this correspondence is not one–to–one: Several generalized basic probability assignments can lead to the same F-probability field $\mathcal{F}$. Only a maximal description $\mathcal{S}_{max}$ can be found, in the sense that $\mathcal{F} = \mathcal{F}(\mathcal{S}_{max})$ and $\mathcal{S} \subseteq \mathcal{S}_{max}$ for all $\mathcal{S}$ with $\mathcal{F}(\mathcal{S}) = \mathcal{F}$.

4   Some Special Cases

The very moderate restrictions on generalized basic probability assignments
– the set $\mathcal{S}$ only has to be closed – allow for a rather high flexibility to model
complex uncertainty. At least two different approaches to generalized basic
probability assignments can be distinguished: The first one takes $\mathcal{S}$ as a finite
set. The second, much richer one, is to apply theories developed for sets of
mass functions on Ω as powerful techniques to construct the generalized basic
probability assignment $\mathcal{S}$ and to develop efficient procedures for generalized
belief accumulation.

Note further that, if one was relying on the Dempster-Shafer theory, one
would be forced to assign a basic probability number to every element of
$\mathcal{P}(\Omega)$. In contrast, generalized basic probability assignments allow for par-
tial specification. Therefore, for instance, a 'vague' description of the sample
space Ω is possible by introducing a residual category and leaving the corre-
sponding basic probability number unspecified.

4.1   Aggregation of Several Basic Probability Assign-ments

First assume $\mathcal{S}$ to consist of q different basic probability assignments, the
judgements of q experts, say. While within the Dempster-Shafer approach
this information has to be mixed to produce a single basic probability as-
signment, Theorem 3.3 allows for an appealing alternative. Generalized belief
accumulation aggregates the assignments in a way that reflects potential con-
flict in the different judgements: the more the assignments differ from each
other, the wider, ceteris paribus, the intervals of the resulting F-probabilty
will be.

Note, that also an analogous aggregation of several generalized basic proba-
bility assignments $(\mathcal{S}_i, i = 1, \ldots, q)$ is possible in this framework by consid-
ering $\mathcal{S} = \cup_{i=1}^{q} \mathcal{S}_i$.

4.2 Linear Partial Basic Probability Assignments

Under the name 'linear partial information', in a series of publications Kofler and Menges (see in particular Kofler and Menges (1976) and Kofler (1989)) have considered sets of probabilities described by linear restrictions. Transferring this concept to the space $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$, a rather flexible modelling is permitted.

Def. 4.1 *A generalized basic probability assignment* $\mathcal{S}$ *is called* linear partial basic probability assignment *if there exists a matrix* $\mathbf{Y}$ *and a vector b so that*

$$\mathcal{S} = \{m(\cdot) \in \mathcal{Q}(\Omega, \mathcal{P}(\Omega)) \mid \mathbf{Y} \cdot \vec{m} \geq b\}.$$

Here $\vec{m}$ is taken as the symbol for the $|\mathcal{P}(\Omega)|$-dimensional vector $(m(A))_{A \in \mathcal{P}(\Omega)}$ for every $m(\cdot) \in \mathcal{Q}(\Omega, \mathcal{P}(\Omega))$.

Note that, as in the example below, linear partial basic probability assignments often arise naturally from qualitative statements on the available information. Among the most promising examples are *comparative basic probability assignments* which arise from a partial ordering on the events of the form $m(A) \leq m(B)$ for some events A and B. ("The information supporting properly A is not weighted higher than that supporting properly B").

Assuming linear partial basic probability assignments, the generalized belief accumulation in (8) is a linear optimization problem, which can be solved by standard routines. Connected to this, note that it furthermore suffices to know the vertices of the polyhedron described by $\mathcal{S}$. In some important special cases explicit expressions for the vertices are available.[5]

Also of particular interest is the case where a generalized basic probability assignment is given in form of intervals

$$d\underline{m}(A) \leq m(A) \leq \overline{m}(A), \qquad A \in \mathcal{P}(\Omega).$$

This leads to probability intervals on $(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$, and the whole framework developed in Weichselberger and Pöhlmann (1990) can be utilized here.

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[5]For instance, for comparative basic probability assignments one can transfer the results of Kofler (1989, p. 26) to the situation under consideration.

In particular one obtains, by adapting their Theorem 2.5, p. 25, an explicit expression for generalized belief accumulation:

$$\mathbb{L}(A) = \max\{ \sum_{\emptyset \neq B \subseteq A} \underline{m}(B),\ 1 - \sum_{\emptyset \neq C \subseteq A^c} \overline{m}(C)\}, \quad \forall A \in \mathcal{P}(\Omega). \qquad (10)$$

The computational complexity is of the same order as it is in Dempster-Shafer theory, but the number of situations which can realistically be modelled is incomparably higher.

4.3 A Toy Example for a Linear Partial Basic Probability Assignment

A patient is supposed to suffer from one of three mutually exclusive diseases A, B, C. Let a medical expert summarize his diagnosis in the following way.

1. None of the three diseases can be excluded with certainty.

2. My weight for the symptoms solely pointing on A is as least as high as that on C.

3. My weight on B is between twice and three times as high as that on disease A.

4. At least half of the total weight is given to the information excluding C, but can not be divided among A and B.

This statement can be transferred immediately into linear restrictions:

$i)$ $m(Z) \geq \delta; \quad Z \in \{A, B, C\} \quad (\delta \neq 0) \qquad ii)$ $m(A) \geq m(C)$

$iii)$ $2 \cdot m(A) \leq m(B) \leq 3 \cdot m(A) \qquad\qquad iv)$ $m(A \cup B) \geq 0.5$.

While Dempster-Shafer theory can not cope adequately with this information, a linear partial basic probability assignment is deduced in a straightforward way. For example, with $\delta = 0.1$ one arrives, after having determined the vertices of the polyhedron arising from these inequalities, at

$$\begin{pmatrix} m(A) \\ m(B) \\ m(C) \\ m(A \cup B) \end{pmatrix} \in \mathrm{conv} \left(\begin{pmatrix} 0.1 \\ 0.3 \\ 0.1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0.2 \\ 0.1 \\ 0.6 \end{pmatrix}, \begin{pmatrix} 4/30 \\ 8/30 \\ 0.1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.125 \\ 0.25 \\ 0.125 \\ 0.5 \end{pmatrix} \right)$$

and $m(X) = 0$, $X \subseteq \{A, B, C\}$ else. Generalized belief accumulation leads to the following F-probability:

$$
\begin{aligned}
P(A) &= [0.1, 0.7] & P(A \cup B) &= [0.875, 0.9] \\
P(B) &= [0.2, 0.8] & P(A \cup C) &= [0.2, 0.8] \\
P(C) &= [0.1, 0.125] & P(B \cup C) &= [0.3, 0.9] \,.
\end{aligned}
$$

5 Concluding Remarks

Relying on sets of basic probability assignments, the paper proposed a general method to deal with uncertain knowledge. While single basic probability assignments correspond to belief functions, generalized basic probability assignments lead to the richer class of F-probability. The theory of general interval probability can be used for concrete calculations.

Level	Dempster-Shafer	Proposal
$(\mathcal{P}(\Omega), \mathcal{P}(\mathcal{P}(\Omega)))$	uncertain knowledge quantified by a basic probability assignment: single mass function	uncertain knowledge quantified by a generalized basic probability assignment: set of mass functions
$(\Omega, \mathcal{P}(\Omega))$	belief function special case of interval probability	F-probability general interval probability

Fig. 1: The modelling of uncertain knowledge quantified by basic probability assignments and generalized basic probability assignments.

By the possibility of applying theories for sets of mass functions as techniques for generalized basic probability assignments a broad field of flexible modelling was opened, but only briefly sketched. Further interesting results can be derived by detailed investigations of more complex procedures of these theories.

The method presented here is general. In principle, it can be used to extend all derived concepts of Dempster-Shafer theory. Another promising aspect is the analogous generalization of other approaches to uncertainty (like possibility theory) which formally can be embedded into Dempster-Shafer theory.

The necessary formal framework is given above, the inherent meanings and interpretations in terms of each concept must be carefully developed case by case.

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