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Tournaments with Gaps

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Abstract
A standard tournament contract specifies only tournament prizes. If agents' performance is measured on a cardinal scale, the principal can complement the tournament contract by a gap which defines the minimum distance by which the best performing agent must beat the second best to receive the winner prize. We analyze a tournament with two risk averse agents. Under unlimited liability, the principal strictly benefits from a gap by partially insuring the agents and thereby reducing labor costs. If the agents are protected by limited liability, the principal sticks to the standard tournament.

Key words: limited liability; moral hazard; risk aversion; tournament; unlimited liability.
JEL classification: C72; D86.

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1 Introduction

Tournaments are frequently used by private corporations, e.g. in the form of job-promotion tournaments or to decide on relative performance pay. The tournament organizer – the principal – is interested in the optimal design of the tournament, that is, in the prize structure that offers the best compromise between implemented efforts and corresponding labor costs.

Nalebuff and Stiglitz (1983) introduced the idea of complementing a tournament by a gap as a minimum distance by which the best performing agent must beat the second best to become the tournament winner. Such a gap is always feasible if the principal measures performance on a cardinal scale. Nalebuff and Stiglitz consider a competitive market in which expected profits are driven to zero. They base their analysis on the observation that the 'introduction of "gaps" can lower the probability that any prize will be paid while maintaining the same level of marginal incentives' (p. 31). This argument, however, is applicable only if performance measures are verifiable. We follow Prendergast and Topel (1996), among many others, and address the case where the evaluation of agents may involve an element of subjectivity so that performance measures are unverifiable. Such environment typically holds for labor relationships.[1] We analyze under which conditions the introduction of a gap leads to a strict improvement of the standard tournament that solely specifies prizes.

In our paper, we combine contract theory with the theory of contests. We consider a moral-hazard situation in which a risk neutral principal designs the optimal tournament contract for two risk averse agents with either unlimited or limited liability. The contract has three elements – a winner prize, a loser prize and a gap. As emphasized by Malcomson (1984, 1986), Rosen (1988, p. 85) and Milgrom and Roberts (1992, p. 369), restricting attention to the class of tournament contracts in a situation with unverifiable performance measure is justified by the fact that individual incentive schemes like piece

[1] "Objective measures of employee performance are rarely available" (Prendergast and Topel 1996, p. 958).

rates or bonuses do not work: if the performance measure is unverifiable, a rational principal will ex post always claim poor performance of the agents to retain the incentive pay and, hence, to save labor costs. This opportunistic behavior can be anticipated by the agents who consequently choose zero efforts.

Tournament contracts, however, work without verifiable performance because the principal can credible commit to pay out a certain collective amount of money as tournament prizes and this outpayment is verifiable by a third party. Since the principal must distribute the tournament pay among the agents, there is no reason for the principal to misrepresent the agents' performance any longer, which thus restores agents' incentives.

Our results show that under unlimited liability the introduction of a positive gap leads to a better solution of the fundamental trade-off between incentives and insurance, which is inherent in any moral-hazard problem with risk averse agents. This trade-off already exists in the basic model with one agent: The principal should use pay-for-performance and share the income risk with the agent for incentive reasons, but the efficient allocation of risk would require perfect insurance of the risk averse agent by the risk neutral principal. Since such perfect insurance would erase any incentives, the optimal compensation must lead to a compromise between incentives and insurance.

This fundamental logic also applies to tournament contracts. Using a gap yields a partial insurance of the agents when combining it with an optimal prize payment rule for the case that neither contestant has won by the gap. Any random distribution rule (e.g., tossing a coin) cannot be optimal since the agents are risk averse. Giving each agent the average of winner and loser prizes, however, is optimal under risk aversion and unlimited liability. If agents are risk averse and, hence, have a concave utility function, an agent's utility from receiving the average pay is larger than the expected utility from receiving the winner and the loser prizes each with probability one half in a symmetric tournament equilibrium.

Such partial insurance of agents is benefical for the principal under unlimited liability. As is known from the basic one-agent moral-hazard model, an agent's participation constraint is always binding given the optimal contract and unlimited liability. The same rationale holds for two agents and the optimal tournament contract which makes the agents just indifferent between their reservation value and participating in the tournament. An increase of the agents' expected utility via partial insurance directly benefits the principal because he can save money by lowering the loser prize without violating the agents' participation constraint. The principal's optimization problem is complicated by the fact that the use of a positive gap is not free of cost. We can show that, for given tournament prizes, incentives are maximized by a zero gap. Thus, introduction of a gap is detrimental from a pure incentive perspective. However, our results point out that the principal can optimally adapt his flexible tournament prizes so that partial insurance by the gap leads to a first-order gain for the principal that dominates the second-order incentive loss.

If agents are protected by limited liability and earn positive rents, the principal will not be interested in partially insuring the agents against income risks any longer as the agents' participation constraints are not binding in the optimum. Since incentives are maximized by a zero gap, the principal prefers to keep to the standard tournament without gap.

As our paper, Eden (2007) analyzes a tournament model that is based on the seminal paper by Lazear and Rosen (1981). She shows that supplementing a standard tournament by a gap will be optimal if the tournament prizes are exogenously given and if prizes need not to be paid out in any case in order to satisfy Malcomson's (1984) self-commitment property. However, if tournament prizes must always sum up to the same constant, the standard tournament contract without gap will be optimal. This result corresponds to our finding under limited liability. Kono and Yagi (2008, p. 124) argue that, in a related model, introducing a positive gap may increase agents' incentives. In our model, introducing a gap decreases incentives, but we show that the loss is outweighed by the gain due to the insurance effect if agents are risk averse and have unlimited liability. Imhof and Kräkel (2013a) analyze how a gap can be used to balance competition under biased performance evaluation, whereas Imhof and Kräkel (2013b) show how a gap can be used to reduce agents' rents. However, both papers assume agents to be risk neutral so that insurance of agents cannot be an issue.

2 The Model

We consider a situation where a principal must hire two agents in order to run a business.[2] The principal is risk neutral whereas the two agents are assumed to be risk averse. In particular, let agent i's $(i = 1, 2)$ utility from earning income I_i and exerting effort e_i be given by

$$U(I_i, e_i) = u(I_i) - c(e_i) \tag{1}$$

with $u(I_i)$ being monotonically increasing and strictly concave with $u(0) = 0$, and c satisfying $c(0) = c'(0) = 0$ and $c'(e_i), c''(e_i) > 0$ for $e_i > 0$. Hence, we have $U(0, 0) = 0$. Let each agent's reservation utility be $\bar{U} = 0$.

The principal wants to implement a certain effort level at lowest possible cost. For each agent i $(i = 1, 2)$, she observes the unverifiable performance signal $x_i(e_i) = h(e_i) + \theta_i$ with $h(0) = 0$ and $h'(e_i) > 0$, $h''(e_i) \leq 0$. The variables θ_1 and θ_2 denote agents' luck being i.i.d. with density f and cdf F. We assume that $\int_{-\infty}^{\infty} f^2(\theta)\, d\theta < \infty$ to guarantee that $\theta_1 - \theta_2$ has a continuous density g with corresponding cdf G. The principal can neither observe e_i (or $h(e_i)$) nor θ_i so that we have a typical moral-hazard problem.

To induce incentives, the principal uses a tournament that specifies a winner prize w_H, a loser prize $w_L < w_H$ and a gap $\gamma \geq 0$ by which the better performing agent must outperform his opponent to get the winner prize. In other words, agent i will only receive w_H if $x_i(e_i) > x_j(e_j) + \gamma$. In that case, agent j obtains the loser prize w_L. In case of a tie, i.e., $|x_1(e_1) - x_2(e_2)| \leq \gamma$,

[2] Most of the assumptions follow Lazear and Rosen (1981).

each agent will get $(w_H + w_L)/2$.[3] We consider two scenarios: if the agents are not protected by limited liability, there will be no further restriction on the choice of w_H and w_L; if agents are protected by limited liability, we assume that $w_H, w_L \geq 0$ must hold.

3 Solution to the Game

First, we solve the tournament game between the two agents. Then we answer the question how the principal should design w_H, w_L and γ to implement a certain effort level at lowest cost.

Agent 1 maximizes

$$EU_1(e_1) = u(w_H) \cdot [1 - G(h(e_2) - h(e_1) + \gamma)] + u(w_L) \cdot G(h(e_2) - h(e_1) - \gamma) \\ + u\left(\frac{w_H + w_L}{2}\right) \cdot [G(h(e_2) - h(e_1) + \gamma) - G(h(e_2) - h(e_1) - \gamma)] - c(e_1)$$

We assume that an equilibrium in pure strategies exists and is characterized by the first-order conditions.[4] For agent 1 we obtain

$$u(w_H) \cdot g(h(e_2) - h(e_1) + \gamma) - u(w_L) \cdot g(h(e_2) - h(e_1) - \gamma) \\ + u\left(\frac{w_H + w_L}{2}\right) \cdot [g(h(e_2) - h(e_1) - \gamma) - g(h(e_2) - h(e_1) + \gamma)] = \frac{c'(e_1)}{h'(e_1)}.$$

Following Nalebuff and Stiglitz (1983) by restricting the analysis to symmetric Nash equilibria with $e_1 = e_2 = e$ yields

$$[u(w_H) - u(w_L)] \cdot g(\gamma) = \frac{c'(e)}{h'(e)} \tag{2}$$

with $g(\gamma) = g(-\gamma)$ due to symmetry of the convolution.

[3] See also Nalebuff and Stiglitz (1983), pp. 30-32, on this equal-sharing rule.

[4] The problem that the existence of pure-strategy equilibria cannot be guaranteed in general is well-known; see, e.g., Lazear and Rosen (1981), p. 845, Nalebuff and Stiglitz (1983), p. 29. See Schöttner (2008) and Gürtler (2011) for sufficient conditions that guarantee existence.

At the first stage of the game, the principal chooses w_H, w_L and γ to induce a certain effort level at lowest implementation cost $w_H + w_L$, given the incentive constraint (2), the participation constraint $EU_i(e_i) \geq 0$, and – in case of limited liability – the additional limited-liability constraint $w_H, w_L \geq 0$. We will show that under mild conditions on g and u, any feasible tournament with zero gap can be strictly improved by choosing a tournament with $\gamma > 0$, provided agents have unlimited liability. By a feasible tournament we mean a tournament for which the participation constraint is satisfied. The following result covers all strictly concave utility functions u if the convolution g satisfies a smoothness condition. If g is not smooth, we assume agents exhibit hyperbolic absolute risk aversion (HARA) (e.g., Pratt and Zeckhauser 1987), so that, for some α, β, utility function $u : (-\alpha\beta, \infty) \to \mathbb{R}$ satisfies

$$-\frac{u''(I)}{u'(I)} = \frac{1}{\alpha + \frac{I}{\beta}} \qquad \text{for all } I > -\alpha\beta. \tag{3}$$

Proposition 1 *(a) Let agents have unlimited liability. If $g'(0)$ exists, then, given any feasible tournament with $\gamma = 0$, the principal can implement the same effort at a strictly lower cost by choosing a tournament with $\gamma > 0$. The same conclusion holds if g has merely a right derivative at 0 with $g'_+(0) \geq -g^2(0)$, provided u is a HARA utility function with $\alpha \in (0, \infty)$ and $\beta \in [1, \infty]$.*

(b) If agents are protected by limited liability, the principal optimally chooses $\gamma = 0$.

Given unlimited liability, the introduction of a gap leads to a strict improvement of the standard tournament. In particular, if the convolution g is differentiable at zero (e.g., if density f is normal[5]), the optimal gap γ is always positive. Intuitively, the higher the gap the better the agents are insured against income risk: in the symmetric equilibrium under a zero gap, each agent gets the high utility $u(w_H)$ with probability 1/2 and the low utility $u(w_L)$ with the same probability. If the principal imposes a positive

[5] More generally, $g'(0)$ exists whenever f has a bounded derivative.

gap, part of the probability mass is shifted to the event that both agents receive the intermediate utility $u\left(\left(w_H + w_L\right)/2\right)$, which is higher than expected utility $\left[u\left(w_H\right) + u\left(w_L\right)\right]/2$ for a zero gap, due to the concavity of the utility function. In contrast to a standard tournament with zero gap, a tournament with positive gap partially insures the agents against income risk. Such a partial insurance is beneficial for the principal if agents are risk averse, because under unlimited liability, the insurance reduces the risk premium the principal has to pay due to the binding participation constraint.

If g is not differentiable at zero, switching from a zero gap to a positive one is accompanied by a more pronounced incentive loss. However, if insurance advantages dominate incentive disadvantages (i.e., if $g'_+(0) \geq -g^2(0)$), the principal will nevertheless prefer a strictly positive gap,[6] provided u is a HARA utility function as specified above.

If agents are protected by limited liability, the result on the optimal gap completely differs.[7] Given that the agents earn positive rents, the principal no longer cares about the participation constraint and the risk premium as the latter one only reduces the agents' rents. Instead, the principal chooses the gap that leads to highest possible incentives to minimize implementation costs for a given effort level. Since each convolution g has its global maximum at zero, the best a principal can do is choosing a zero gap and, hence, sticking to the standard tournament, which only specifies tournament prizes.

The proof of the proposition shows the technical intuition of our main result on unlimited liability. Introducing a positive gap partially insures the agents but lowers overall incentives because g is maximized at zero. To implement the same effort level as before, the principal can either increase the winner prize or reduce the loser prize. Since the principal wants to implement a certain effort level at lowest cost, he strictly prefers to reduce the loser prize (see equation (5)). This measure is feasible since the agents' utility gain from partial insurance guarantees that the participation constraint still holds.

[6] For instance, if luck θ_i is uniformly distributed on some interval or if $\theta_i = |\nu_i|$, where ν_i is normally distributed with mean 0, then $g'(0)$ does not exist, but $g'_+(0) = -g^2(0)$.

[7] See similarly Eden (2007).

Appendix

Proof of Proposition 1:

The proof uses the following auxiliary result:[8]

Lemma 1 *Let $\alpha \in (0, \infty)$ and $\beta \in [1, \infty]$. Let $u : (-\alpha\beta, \infty) \to R$ be a HARA utility function satisfying (3). Then for all $x, y > -\alpha\beta$ with $x \neq y$,*

$$-\frac{1}{2}[u(x)-u(y)][u'(x)-u'(y))] < \left[2u\left(\frac{x+y}{2}\right) - u(x) - u(y)\right][u'(x)+u'(y)].$$

By inserting the symmetry condition $e_1 = e_2 = e$ in the agents' objective functions, each agent's participation constraint reads as

$$Q(w_H, w_L, \gamma) \geq c(e), \tag{4}$$

where

$$Q(w_H, w_L, \gamma) := [u(w_H) + u(w_L)] \cdot G(-\gamma) + u\left(\frac{w_H + w_L}{2}\right) \cdot [1 - 2G(-\gamma)].$$

To prove (a) suppose agents have unlimited liability. Fix any feasible tournament contract with $(w_H, w_L, \gamma) = (w_H^0, w_L^0, 0)$. Let e_0 denote the agents' common effort level for this tournament. Thus, (2) and (4) are satisfied for $w_H = w_H^0$, $w_L = w_L^0$, $\gamma = 0$ and $e = e_0$.

Consider first the case where g is differentiable at 0. Define a function $\zeta(\gamma)$ for $\gamma \geq 0$ sufficiently small by

$$[u(w_H^0) - u(\zeta(\gamma))]g(\gamma) = \frac{c'(e_0)}{h'(e_0)}. \tag{5}$$

This yields a class of contracts $(w_H^0, \zeta(\gamma), \gamma)$ that implement the same effort e_0, provided the participation constraint is satisfied. If $\gamma > 0$, then $g(\gamma) < g(0)$ and $\zeta(\gamma) < \zeta(0) = w_L^0$. That is, if a contract from the class with a strictly positive gap satisfies the participation constraint, the corresponding cost of implementing effort e_0 is strictly lower than under the

[8] For the proof of Lemma 1 see the online appendix.

given zero-gap contract. It remains to show that the constraint is satisfied for some $\gamma > 0$. Let $q(\gamma) := Q(w_H^0, \zeta(\gamma), \gamma)$. Computing in (5) the right derivative with respect to γ at $\gamma = 0$ and using that $g'(0) = 0$, we obtain $-u'_-(w_L^0)\zeta'_+(0)g(0) = 0$. The concavity of u ensures that u has a left derivative u'_-. Since $g(0) > 0$, $-u'_-(w_L^0)\zeta'_+(0) = 0$, and it follows that

$$q'_+(0) = \frac{1}{2}u'_-(w_L^0)\zeta'_+(0) - [u(w_H^0) + u(w_L^0)]g(0) + 2u\left(\frac{w_H^0 + w_L^0}{2}\right)g(0) > 0.$$

Thus, the participation constraint is indeed satisfied for some positive γ.

Suppose next that g has a right derivative at 0 with $g'_+(0) \geq -g^2(0)$ and that u is a HARA utility function satisfying (3) for some $\alpha \in (0, \infty)$ and $\beta \in [1, \infty]$. Suppose also that $(w_H^0, w_L^0, 0)$ is a zero-gap contract for which the participation constraint (4) binds, the claim being trivial otherwise. Set

$$R(w_H, w_L, \gamma, s) := [u(w_H) - u(w_L)][g(0) + s\gamma]$$

for $w_H > w_L > -\alpha\beta$ and $\gamma, s \in \mathbb{R}$. By the implicit function theorem, there exist continuously differentiable functions $\zeta(\gamma, s), \eta(\gamma, s)$, defined in a neighborhood V of $(0, g'_+(0))$, such that for all $(\gamma, s) \in V$,

$$Q(\zeta(\gamma, s), \eta(\gamma, s), \gamma) = c(e_0), \qquad R(\zeta(\gamma, s), \eta(\gamma, s), \gamma, s) = \frac{c'(e_0)}{h'(e_0)},$$
$$\zeta(0, s) = w_H^0, \qquad \eta(0, s) = w_L^0.$$

The use of the theorem is justified because[9]

$$\det\begin{pmatrix} Q_1(w_H^0, w_L^0, 0) & Q_2(w_H^0, w_L^0, 0) \\ R_1(w_H^0, w_L^0, 0, g'_+(0)) & R_2(w_H^0, w_L^0, 0, g'_+(0)) \end{pmatrix}$$
$$= -u'(w_H^0)u'(w_L^0)g(0) \neq 0.$$

Thus, if $(\gamma, s) \in V$, $\gamma \geq 0$ and s is so chosen that $g(0) + s\gamma = g(\gamma)$, then the

[9] In the following, subscripts of Q, R, ζ, and η denote partial derivatives.

contract $(\zeta(\gamma,s),\eta(\gamma,s),\gamma)$ implements effort e_0. Moreover,

$$
\begin{aligned}
&\zeta_1(0,g'_+(0))+\eta_1(0,g'_+(0))\\
&\quad=\frac{g'_+(0)[u(w_H^0)-u(w_L^0)][u'(w_H^0)-u'(w_L^0)]}{2g(0)u'(w_H^0)u'(w_L^0)}\\
&\qquad-\frac{g(0)\left[2u\left(\frac{1}{2}(w_H^0+w_L^0)\right)-u(w_H^0)-u(w_L^0)\right][u'(w_H^0)+u'(w_L^0)]}{u'(w_H^0)u'(w_L^0)}\\
&\quad<0,
\end{aligned}
$$

where the inequality follows from the assumption that $g'_+(0)\geq -g^2(0)$ and Lemma 1 above. It follows that $\zeta_1+\eta_1$ is positive on some neighborhood of $(0,g'_+(0))$. Consequently, for $\gamma>0$ sufficiently small and $s=[g(\gamma)-g(0)]/\gamma$,

$$\zeta(\gamma,s)+\eta(\gamma,s)<\zeta(0,s)+\eta(0,s)=w_H^0+w_L^0.$$

That is, for these γ and s, the contract $(\zeta(\gamma,s),\eta(\gamma,s),\gamma)$ implements effort e_0 at a strictly lower cost than the given zero-gap contract.

To prove (b) suppose agents have limited liability. Since the agents have zero reservation utilities and $U(0,0)=0$, they will accept any contract with non-negative payments as they can guarantee themselves non-negative expected utilities by choosing zero effort. Hence, the principal does not have to care about the participation constraint when solving for the optimal tournament contract. The optimal loser prize is zero since any positive w_L would decrease incentives and increase implementation costs. The optimal gap therefore maximizes the left-hand side of (2), yielding $\gamma=0$, since g has its global maximum at zero.[10] □

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Online Appendix

Proof of Lemma 1:

The claimed inequality is equivalent to $\Psi(x,y) > 0$, where

$$\Psi(x,y) := 4u\left(\frac{x+y}{2}\right)[u'(x)+u'(y)] - u(x)u'(x) - u(y)u'(y) \\ - 3u(x)u'(y) - 3u'(x)u(y).$$

Suppose first $\beta \in (1,\infty)$. Then $u(x) = A + B(\alpha + x/\beta)^{1-\beta}$ for some constants $A \in \mathbb{R}$, $B \in (-\infty, 0)$. Given $x, y > -\alpha\beta$ with $x \neq y$, set $\overline{x} := \alpha + x/\beta$, $\overline{y} := \alpha + y/\beta$. Suppose without loss of generality that $\overline{x} < \overline{y}$. Define for all $t \in \mathbb{R}$,

$$\phi(t) := \overline{y}^{1-2t} + 3(\overline{x}+\overline{y})(\overline{xy})^{-t} + \overline{x}^{1-2t} - \frac{4}{\overline{x}}\left(\frac{\overline{x}+\overline{y}}{2}\overline{x}\right)^{1-t} - \frac{4}{\overline{y}}\left(\frac{\overline{x}+\overline{y}}{2}\overline{y}\right)^{1-t}.$$

Then $\Psi(x,y) = B^2(\beta-1)\phi(\beta)/\beta$. To show that $\phi(\beta) > 0$ note first that ϕ can have at most 4 zeros counting multiplicities (see, e.g., Pólya, G., Szegö, G., 1976. Problems and theorems in analysis II, fourth ed. Springer, Berlin, pp. 46-47). Moreover,

$$\phi(-1) = \phi(0) = \phi(1) = 0, \qquad \lim_{t\to\infty} \overline{x}^{2t}\phi(t) = \overline{x} > 0$$

and

$$\phi'(1) = \int_{\overline{x}}^{\overline{y}} \frac{1}{t^2\overline{y}} \int_t^{\overline{y}} \frac{(\overline{y}-s)^2}{s(s+\overline{y})}\, ds\, dt > 0.$$

Thus, if $\phi(\beta) \leq 0$, ϕ would have at least 2 zeros in $(1,\infty)$, so that ϕ would have 5 zeros altogether, which is impossible. Hence $\phi(\beta) > 0$.

Suppose next $\beta = 1$. Then $u(x) = A + B\log(\alpha + x)$ for some constants $A \in \mathbb{R}$, $B \in (0,\infty)$. One may verify that

$$\Psi(y,y) = 0, \qquad \left.\frac{\partial}{\partial x}\Psi(x,y)\right|_{x=y} = 0,$$

$$\frac{\partial^2}{\partial x^2}[(x+\alpha)\Psi(x,y)] = \frac{B^2(x-y)^2}{(2\alpha+x+y)(\alpha+x)^2(\alpha+y)}.$$

Thus, for any fixed y, $(x+\alpha)\Psi(x,y)$ is a strictly convex function of x, which attains its minimum 0 at $x=y$. Hence, $\Psi(x,y)>0$ if $x\neq y$.

Suppose finally $\beta=\infty$. Then $u(x)=A+Be^{-x/\alpha}$ for some constants $A\in\mathbb{R}$, $B\in(-\infty,0)$. For any fixed y, $\Psi(x,y)$ is an exponential polynomial in x of degree 4 and so has at most 4 zeros counting multiplicities. We have

$$\Psi(y,y)=0, \qquad \left.\frac{\partial^k}{\partial x^k}\Psi(x,y)\right|_{x=y}=0 \quad \text{for } k=1,2,3,$$
$$\left.\frac{\partial^4}{\partial x^4}\Psi(x,y)\right|_{x=y}=\frac{3B^2e^{-2y/\alpha}}{2\alpha^5}>0.$$

It follows that $\Psi(x,y)>0$ if $x\neq y$. □

Global Maximum of g (see Imhof, L., Kräkel, M., 2013. Optimal Bonus Pools. Mimeo.)*:*

Recall that θ_1 and θ_2 are i.i.d. with density f. In view of the assumption that $\theta_1-\theta_2$ has a continuous density g, $g(\gamma)=\int_{-\infty}^{\infty}f(\theta)f(\theta-\gamma)\,d\theta$ for every $\gamma\in\mathbb{R}$, see, e.g., Mood, A.M., Graybill, F.A., Boes, D.C., 1974. Introduction to the theory of statistics, third ed. McGraw-Hill, Auckland, pp. 185-186, for the convolution formula. Applying the Cauchy-Schwarz inequality leads to[11]

$$g(\gamma)=\int_{-\infty}^{\infty}f(\theta)\,f(\theta-\gamma)\,d\theta\leq\sqrt{\int_{-\infty}^{\infty}[f(\theta)]^2\,d\theta}\sqrt{\int_{-\infty}^{\infty}[f(\theta-\gamma)]^2\,d\theta}$$
$$=\int_{-\infty}^{\infty}[f(\theta)]^2\,d\theta=g(0) \qquad \text{for all } \gamma,$$

so that $\gamma=0$ is a maximum of g. To prove uniqueness suppose that $g(\gamma)=g(0)$ for some $\gamma>0$. Then there must hold equality in the Cauchy-Schwarz inequality, which implies that there is a constant $C>0$ such that $f(\theta)=Cf(\theta-\gamma)$ for almost all θ. As $\int f(\theta)\,d\theta=1=\int f(\theta-\gamma)\,d\theta$, $C=1$. It follows that for every $a\in\mathbb{R}$, $\int_{a-\gamma}^{a}f=\int_{a}^{a+\gamma}f$, which implies that $\int_{-\infty}^{\infty}f\in\{0,\infty\}$. This is impossible, and it follows that $\gamma=0$ is the unique maximum of g. □

[11] The step from line 1 to line 2 uses the fact that $\int_{\underline{x}}^{\bar{x}}y\,(x-\alpha)\,dx=\int_{\underline{x}-\alpha}^{\bar{x}-\alpha}y\,(x)\,dx$.