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Exact and Fast Numerical Algorithms for the Stochastic Wave Equation *

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Abstract

On the basis of integral representations we propose fast numerical methods to solve the Cauchy problem for the stochastic wave equation without boundaries and with the Dirichlet boundary conditions. The algorithms are exact in a probabilistic sense.

1 Statement of the problem and integral representations

Consider the stochastic wave equation

$$\frac{\partial^2 X}{\partial t^2}(t,x) - a^2 \frac{\partial^2 X}{\partial x^2}(t,x) = g(t,x) + f(t,x)\, dW(t,x) \tag{1}$$

with random initial conditions, where W is a Gaussian white noise on the plane. We give shortly the precise statement of the problem. The aim of

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the paper is to construct numerical algorithms in order to solve the Cauchy problem for equation (1) without boundaries and with the Dirichlet boundary conditions, i.e., to simulate realizations of the random field X.

First, we present the necessary formalism for the stochastic wave equation and integral representations for the solutions that underlay our numerical methods. The details can be found in [5], [1] and [2].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, W be an isonormal process indexed by $L^2(\mathbb{R}_+\times\mathbb{R}, \mathcal{B}(\mathbb{R}_+\times\mathbb{R}), \lambda)$ and $\{\mathcal{F}_t;\, t \in \mathbb{R}_+\}$ be a nullset augmented filtration which satisfies

- $W(1_A)$ is $\mathcal{F}_t$-measurable whenever $A \in \mathcal{B}([0,t] \times \mathbb{R})$ with $\lambda(A) < \infty$;
- $W(1_A)$ is independent of $\mathcal{F}_t$ whenever $A \in \mathcal{B}((t,\infty) \times \mathbb{R})$ with $\lambda(A) < \infty$.

Moreover, let $F = \{F(x);\, x \in \mathbb{R}\}$ and $H = \{H(x);\, x \in \mathbb{R}\}$ be almost surely continuous and $\mathcal{F}$-measurable processes such that $\int_{\mathbb{R}} \mathbb{E}\{|F(x)|^2\}\varphi(x)\, dx$ and $\int_{\mathbb{R}} \mathbb{E}\{|H(x)|^2\}\varphi(x)\, dx$ are finite for any smooth φ with compact support. Finally, assume f and g to be real valued functions on $\mathbb{R}_+ \times \mathbb{R}$ such that $f\varphi$ is square integrable and $g\varphi$ is integrable for any smooth φ with compact support, respectively.

Similar to the deterministic case, the solution of equation (1) with initial conditions (F, H) on $\mathbb{R}_+ \times \mathbb{R}$ can be represented in terms of Green's function G_a (hereafter $a > 0$),

$$G_a(t,x,s,y) = \begin{cases} \frac{1}{2a} & \text{if } 0 \le s \le t \text{ and } |x-y| \le a(t-s) \\ 0 & \text{otherwise.} \end{cases}$$

The solution is given by

$$\begin{aligned} X(t,x) = u_0(t,x) \quad &+ \int_{\mathbb{R}_+}\int_{\mathbb{R}} G_a(t,x,s,y) g(s,y)\, dy\, ds \\ &+ \int_{\mathbb{R}_+}\int_{\mathbb{R}} G_a(t,x,s,y) f(s,y)\, dW(s,y), \end{aligned} \tag{2}$$

where $u_0(t,x) = (F(x-at) + F(x+at))/2 + \int_{x-at}^{x+at} H(y)/2a\, dy$.

Remark 1.1 We require the integrability conditions on F and H to have first and second moments for the random field X. An example for a homogeneous wave equation with random initial conditions is studied in [4].

Remark 1.2 The stochastic integral in (2) is defined by

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} h(s,y)\, dW(s,y) := W(h)$$

for any $h \in L^2(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}), \lambda)$ and we use the integral notation for convenience.

Remark 1.3 It holds $X(0,\cdot) = F$ almost surely, and for $f = 0$ we have $\partial X/\partial t(0,\cdot) = H$. However, for $f \neq 0$ the last equation has only formal meaning, because the random field X is not differentiable.

For the Cauchy problem on $\mathbb{R}_+ \times [0,1]$ with Dirichlet boundary conditions Green's function Σ_{G_a} has the following form

$$\begin{aligned}\Sigma_{G_a}(t,x,s,y) \; &= \; \sum_k \Big[G_a(t, x+2k; s,y)1_{[0,1]}(y) \\ &\qquad -G_a(t,-x+2k;s,y)1_{[0,1]}(y)\Big],\end{aligned}$$

and instead of u_0 we consider

$$\begin{aligned}u_{0,D}(t,x) = \frac{1}{2}\sum_k \quad & \Big[F(x+2k-at)1_{(0,1)}(x+2k-at) \\ + \quad & F(x+2k+at)1_{(0,1)}(x+2k+at) \\ - \quad & F(-x+2k-at)1_{(0,1)}(-x+2k-at) \\ - \quad & F(-x+2k+at)1_{(0,1)}(-x+2k+at)\Big] \\ +\frac{1}{2a}\sum_k \quad & \Big[\int_{(x+2k-at)\vee 0}^{(x+2k+at)\wedge 1} H(y)\,dy - \int_{(-x+2k-at)\vee 0}^{(-x+2k+at)\wedge 1} H(y)\,dy\Big].\end{aligned}$$

In this case the solution of (1) with initial conditions (F,H) is

$$\begin{aligned}X(t,x) = u_{0,D}(t,x) \; &+ \; \int_{\mathbb{R}_+}\int_0^1 \Sigma_{G_a}(t,x,s,y) g(s,y)\,dy\,ds \qquad (3) \\ &+ \; \int_{\mathbb{R}_+}\int_0^1 \Sigma_{G_a}(t,x,s,y) f(s,y)\,dW(s,y).\end{aligned}$$

In both cases the solutions are Gaussian random fields and the integral representations give the basis to construct "exact" numerical solutions.

2 Numerical algorithms

A realization of the solution will be simulated on a grid

$$\mathcal{G} = \{(t_j, x_k) = (jh, akh)\}$$

with $j \in \{0, 1/2, 1, 3/2, \ldots\}$, $k \in \{\ldots, -1, -1/2, 0, 1/2, 1, \ldots\}$, where h is a time step, ah is a spatial step and for a grid point the indices j, k are simultaneously either integers or fractional numbers.

In addition to the values $X_k^j = X(jh, akh)$ on the grid $\mathcal{G}$ the algorithm operates with the values μ_k^j on an "adjacent" grid $\mathcal{G}^* = \{(t_j, x_k) = (jh, akh)\}$ with the same sets of indices, but one of the indices is always integer while the other is fractional. Moreover, μ_k^j is given by $\mu_k^j = \int_{x_k-ah/2}^{x_k+ah/2} H_j(y)\,dy$, where $H_j(y)$ is the second part of the initial conditions for the Cauchy problem at time t_j (see Remark 1.3).

2.1 Cauchy problem without boundaries

The algorithm uses the Markov property of X with respect to time, i.e. the values for the next time level depend only on the values obtained at the previous time level. Therefore the complexity of the algorithm is linear.

Initialisation. Set $j = 0$. The values X_k^0 are taken from the initial condition, i.e. $X_k^0 = F(akh) = X(0, akh) = X(t_0, x_k)$. Moreover, set $\mu_k^0 = \int_{x_k-ah/2}^{x_k+ah/2} H(y)\,dy$.

Step 1. Equation (3) gives an explicit rule to calculate $X^{j+1/2}$:

$$X_{k+1/2}^{j+1/2} = \left(\frac{1}{2}\left(X_k^j + X_{k+1}^j\right) + \frac{1}{2a}\mu_{k+1/2}^j\right) + \xi_{k+1/2}^{j+1/2}, \tag{4}$$

where

$$\begin{aligned}\xi_{k+1/2}^{j+1/2} &= \int_{t_j}^{t_{j+1/2}} \int_{x_k}^{x_{k+1}} G_a(t_{j+1/2}, x_{k+1/2}, s, y) g(s,y)\,dy\,ds \\ &+ \int_{t_j}^{t_{j+1/2}} \int_{x_k}^{x_{k+1}} G_a(t_{j+1/2}, x_{k+1/2}, s, y) f(s,y)\,dW(s,y)\end{aligned}$$

is a Gaussian random variable with mean

$$\int_{t_j}^{t_{j+1/2}} \int_{x_k}^{x_{k+1}} G_a(t_{j+1/2}, x_{k+1/2}, s, y) g(s,y)\,dy\,ds$$

and variance

$$\int_{t_j}^{t_{j+1/2}} \int_{x_k}^{x_{k+1}} G_a^2(t_{j+1/2}, x_{k+1/2}, s, y) f^2(s,y)\,dy\,ds.$$

Step 2. The next step is to calculate $\mu_{k+1}^{j+1/2}$. To this end, one observes that X_{k+1}^{j+1} can be expressed in two different ways,

$$X_{k+1}^{j+1} = \left(\frac{1}{2}\left(X_{k+1/2}^{j+1/2} + X_{k+3/2}^{j+1/2}\right) + \frac{1}{2a}\mu_{k+1}^{j+1/2}\right) + \xi_{k+1}^{j+1} \quad (5)$$

and

$$\begin{aligned} X_{k+1}^{j+1} &= \left(\frac{1}{2}\left(X_k^j + X_{k+2}^j\right) + \frac{1}{2a}\left(\mu_{k+1/2}^j + \mu_{k+3/2}^j\right)\right) \quad (6) \\ &+ \left(\xi_{k+1/2}^{j+1/2} + \xi_{k+3/2}^{j+1/2} + \delta_{k+1}^j + \xi_{k+1}^{j+1}\right), \end{aligned}$$

where

$$\begin{aligned} \delta_{k+1}^j &= \frac{1}{2a}\int_{t_j}^{t_{j+1/2}} \int_{x_{k+1}-a(s-t_j)}^{x_{k+1}+a(s-t_j)} g(s,y)\,dy\,ds \quad (7) \\ &+ \frac{1}{2a}\int_{t_j}^{t_{j+1/2}} \int_{x_{k+1}-a(s-t_j)}^{x_{k+1}+a(s-t_j)} f(s,y)\,dW(s,y) \end{aligned}$$

is a Gaussian random variable with mean given by the first double integral in (7) and variance

$$\left(\frac{1}{2a}\right)^2 \int_{t_j}^{t_{j+1/2}} \int_{x_{k+1}-a(s-t_j)}^{x_{k+1}+a(s-t_j)} f^2(s,y)\,dy\,ds.$$

A combination of (5) and (6) yields

$$\begin{aligned} \mu_{k+1}^{j+1/2} &= 2a\left[\frac{1}{2}\left(X_k^j + X_{k+2}^j\right) + \frac{1}{2a}\left(\mu_{k+1/2}^j + \mu_{k+3/2}^j\right)\right. \quad (8) \\ &+ \left.\left(\xi_{k+1/2}^{j+1/2} + \xi_{k+3/2}^{j+1/2}\right) + \delta_{k+1}^j - \frac{1}{2}\left(X_{k+1/2}^{j+1/2} + X_{k+3/2}^{j+1/2}\right)\right]. \end{aligned}$$

Thus, at step two the values $\mu_{k+1}^{j+1/2}$ are simulated according to expression (8). Note, that all the components of the expression, except δ_{k+1}^j, are defined at the previous steps of the algorithm, and the random variable ξ_{k+1}^{j+1} appears in (5) and (6) but cancels out in (8).

Cycling. Set $j = j + 1/2$ and go back to step 1.

Proposition 2.1 *The sequence* $(X_{j,k})$ *on the grid* $\mathcal{G}$ *fulfils* $X_{j,k} = X(t_j, x_k)$ *with probability one.*

Proof. On the grid $\mathcal{G}$ Green's function G_α can be decomposed into triangles with corners given by the grid points. Thus induction on the time level t_j gives the assertion. □

2.2 Cauchy problem with boundary conditions

This algorithm is a modification of the previous one. As ah is a step size in space, here we assume that $N = 1/ah$ is an integer.

Initialisation. Set $j = 0$ and determine X_k^0 for $k \in \{0, \dots, N\}$ and $\mu_{k+1/2}^0$ for $k \in \{0, \dots, N-1\}$ by the initial conditions F and H, respectively.

Step 1. The values $X_{k+1/2}^{j+1/2}$ for $k \in \{0, \dots, N-1\}$ are calculated according to equation (4).

Step 2. The values $\mu_{k+1}^{j+1/2}$ for $k \in \{0, \dots, N-2\}$ are calculated according to equation (8).

Step 3. The values X_k^{j+1} for $k \in \{1, \dots, N-1\}$ are calculated according to equation (4) while X_0^{j+1} and X_N^{j+1} are defined by the boundary conditions, i.e., $X_0^{j+1} = X_N^{j+1} = 0$.

Step 4. Now, the values μ_{k+1}^{j+1} have to be calculated. For $k \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{N-5}{2}\}$ equation (8) is used. To find $\mu_{1/2}^{j+1}$ and $\mu_{N-1/2}^{j+1}$ the boundary conditions should be applied. The value $X_{1/2}^{j+3/2}$ can be expressed in two different ways (note that $X_0^{j+1} = 0$),

$$\begin{aligned}
X_{1/2}^{j+3/2} &= \left(\frac{1}{2}\left(X_0^{j+1} + X_1^{j+1}\right) + \frac{1}{2a}\mu_{1/2}^{j+1}\right) + \xi_{1/2}^{j+3/2} \\
&= \left(\frac{1}{2}X_1^{j+1} + \frac{1}{2a}\mu_{1/2}^{j+1}\right) + \xi_{1/2}^{j+3/2}, \\
X_{1/2}^{j+3/2} &= \left(\frac{1}{2}\left(-X_{1/2}^{j+1/2} + X_{3/2}^{j+1/2}\right) + \frac{1}{2a}\mu_1^{j+1/2}\right) \\
&+ \left(\xi_1^{j+1} + \delta_{1/2}^{j+1/2} + \xi_{1/2}^{j+3/2}\right).
\end{aligned}$$

From these two representations we obtain the formula for simulation,

$$\begin{aligned}
\mu_{1/2}^{j+1} &= 2a\left[\frac{1}{2}\left(-X_{1/2}^{j+1/2} + X_{3/2}^{j+1/2}\right) + \frac{1}{2a}\left(\mu_1^{j+1/2}\right)\right. \\
&\left. + \left(\xi_1^{j+1}\right) + \delta_{1/2}^{j+1/2} - \frac{1}{2}\left(X_1^{j+1}\right)\right].
\end{aligned}$$

Similarly, for $\mu_{N-1/2}^{j+1}$ we have

$$\begin{aligned}
\mu_{N-1/2}^{j+1} &= 2a\left[\frac{1}{2}\left(X_{N-3/2}^{j+1/2} - X_{N-1/2}^{j+1/2}\right) + \frac{1}{2a}\left(\mu_{N-1}^{j+1/2}\right)\right. \\
&\left. + \left(\xi_{N-1}^{j+1}\right) + \delta_{N-1/2}^{j+1/2} - \frac{1}{2}\left(X_{N-1}^{j+1}\right)\right].
\end{aligned}$$

Cycling. Set $j = j + 1$ and go back to Step 1.

The sequence $(X_{j,k})$ fulfils $X_{j,k} = X(t_j, x_k)$ with probability one as before.

2.3 Results of simulation

Results of numerical simulations are presented in Figs. 1 and 2. In addition, in Fig. 3 we present an example of the sample size dependence of the empirical correlation between two points of the solution for the Cauchy problem without boundaries.

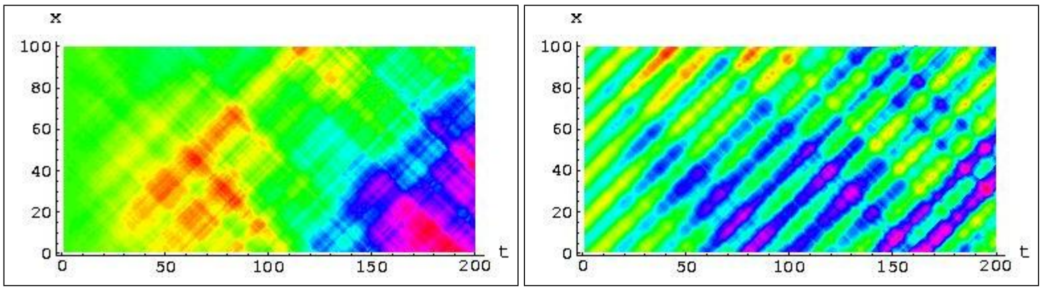

Figure 1: Realization of the solution of the Cauchy problem without boundaries; zero initial conditions and harmonic initial conditions.

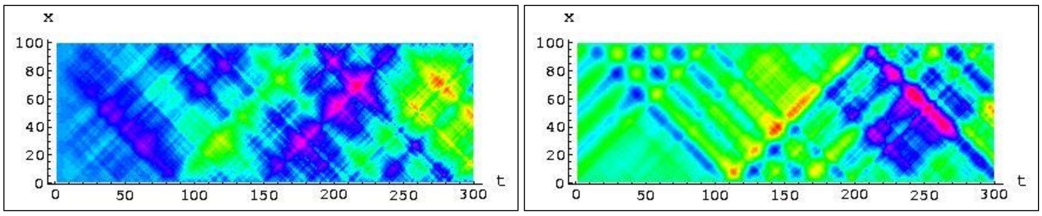

Figure 2: Realization of the solution of the Cauchy problem with Dirichlet boundary conditions; zero initial conditions and harmonic initial conditions.

3 Conclusion

The two proposed methods are "exact" in the sense that the finite-dimensional distributions of the field X on the grid $\mathcal{G}$ coincide with the finite-dimensional distributions of the numerical solution. Moreover, the complexity of the algorithms is linear and they are therefore attractive for numerical simulations.

Remark 3.1 "Exact" algorithms to solve the stochastic Klein-Gordon equation

$$\partial^2 X/\partial t^2(t,x) - \partial^2 X/\partial x^2(t,x) = \alpha X(t,x) + dW(t,x)$$

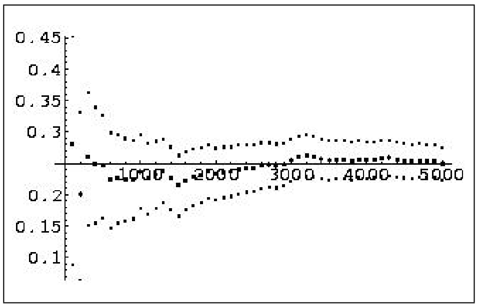

Figure 3: Empirical correlations with confidence intervals of level 0.95 via the sample size; exact correlation is equal to 0.25.

were discussed in [3]. But the algorithms proposed in that paper don't have the Markov property and, practically, result in a general simulation of Gaussian random vectors with correlated components. Because of this fact, it can turn out that the algorithms are not feasible for large grids.

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