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TOTO-Modules

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1 Introduction

Given a ring $R$ (with unity), we denote by $\text{Mod-}R$ the category of right $R$-modules. In what follows the term "module" will mean "right module". Let $M \in \text{Mod-}R$. We denote by $E_R(M)$ the injective hull of $M$ and by $1_M$ the identity endomorphism of $M$. When the context is clear, we shall write $E(M)$ for $E_R(M)$. Given a submodule $N_R \subseteq M_R$, we denote by $i_N : N_R \rightarrow M_R$ the canonical embedding of modules. We shall write $N \subseteq^* M$ whenever $N$ is an essential submodule of $M$, and $N \subseteq^\oplus M$ whenever $N$ is a direct summand of the module $M$. If $M_R = K_R \oplus L_R$, we denote by $\pi_K : M_R \rightarrow K_R$ the canonical projection of modules.

Let $M = M_R$ and $N = N_R$. Recall that $f : M_R \rightarrow N_R$ is called partially invertible (briefly $f$ is pi) whenever there exists $g : N_R \rightarrow M_R$ such that $0 \neq fg = (fg)^2$ (see [6]). It is known that the following to conditions are equivalent:

\begin{align*}
\text{There exists } f \in \text{Hom}_R(N, M) \text{ such that } 0 \neq fg = (fg)^2; \\
\text{There exists } h \in \text{Hom}_R(N, M) \text{ such that } 0 \neq gh = (gh)^2.
\end{align*}

(1)

(see [6] or [2]). Further, let $n > 0$, $M_1, M_2, \ldots, M_{n+1} \in \text{Mod-}R$ and $g_i \in \text{Hom}_R(M_i, M_{i+1})$, $i = 1, 2, \ldots, n$.

If $g_1g_2\ldots g_n$ is pi. Then each $g_i$ is pi

(2)

according to [6, 1.3] (see also [2]).

Let $M, N \in \text{Mod-}R$. Then the total $\text{Tot}(M, N)$ of $M$ and $N$ is defined as follows:

\[ \text{Tot}(M, N) = \{ g \in \text{Hom}_R(M, N) \mid g \text{ is not pi} \} \]

(see [6]).

**Theorem 1.1** Let $M$ be a right $R$-module. Then the following conditions are equivalent:


(i) \( \text{Tot}(M, N) = 0 \) for any \( N \in \text{Mod}-R \).

(ii) \( \text{Tot}(M, C) = 0 \) for some cogenerator \( C \) of \( \text{Mod}-R \).

(iii) \( M \) is a direct sum of injective simple submodules.

A module \( M_R \), satisfying the equivalent conditions of Theorem 1.1, is called a left TOTO-module. Next, a family \( \{K_i \mid i \in I\} \) of submodules of \( M_R \) is called independent whenever \( \sum_{i \in I} K_i = \bigoplus_{i \in I} K_i \). Further, \( M \) is said to be semiprime if for any \( 0 \neq m \in M \) there exists \( f \in \text{Hom}(M_R, R_R) \) such that \( mf(m) \neq 0 \).

Theorem 1.2 Let \( 0 \neq M \in \text{Mod}-R \). Then the following conditions are equivalent:

(i) \( \text{Tot}(L, M) = 0 \) for any \( L \in \text{Mod}-R \).

(ii) \( \text{Tot}(P, M) = 0 \) for some generator \( P \) of \( \text{Mod}-R \)

(iii) Every nonzero submodule of \( M \) contains a nonzero cyclic projective submodule which is a direct summand of \( M \).

(iv) Every nonzero submodule of \( M \) contains a nonzero projective submodule which is a direct summand of \( M \).

(v) \( M \) is a semiprime module with independent family \( \{P_i \mid i \in I\} \) of projective submodules such that \( \bigoplus_{i \in I} P_i \cong M \) and each \( \text{End}(P_i) \) is a TOTO-ring.

A module \( M_R \) is called a right TOTO-module whenever it satisfies the equivalent conditions of Theorem 1.2. Recall that a module \( M \in \text{Mod}-R \) is called torsionless (in sense of Bass) if for any \( 0 \neq x \in M \) there exists \( f \in M^* = \text{Hom}(M_R, R_R) \) such that \( fx \neq 0 \).

The goal of the present paper is to study left and right TOTO-modules and TOTO-rings. Besides Theorems 1.1 and 1.2 the following theorem is the main result of the present paper.

Theorem 1.3 Let \( R \) be a ring. Then the following conditions are equivalent:

(i) \( R \) is a TOTO-ring.

(ii) \( R \) is a semiprime ring and there exists a family \( E = \{e_i \mid i \in I\} \) of idempotents such that \( \sum_{i \in I} e_i R = \bigoplus_{i \in I} e_i R \cong R \) and each \( e_i Re_i \) is a TOTO-ring.
(iii) Every projective right $R$-module is a right TOTO-module.

(iv) Every torsionless right $R$-module is a right TOTO-module.

(v) $\text{End}(P)$ is a TOTO-ring for any projective right $R$-module $P$.

(vi) $\text{End}(M)$ is a TOTO-ring for any torsionless right $R$-module $M$.

(vii) $R$ has a faithful right TOTO-module.

Further, if $R$ is a TOTO-ring and $0 \neq e = e^2 \in R$, then $eRe$ is a TOTO-ring.

The following result, proved in [9], will be frequently used in the sequel.

**Lemma 1.4** For $g \in \text{Hom}_R(M, N)$ the following conditions are equivalent:

(a) $g$ is pi.

(b) There exist nonzero submodules $A \subseteq M$, $B \subseteq N$ such that the map $A \ni a \mapsto g(a) \in B$ is an isomorphism.

The total was defined in 1982 by F. Kasch and then studied in several papers by F. Kasch and W. Schneider (see [6, 7, 8, 9, 10]). Relationships of the total with Jacobson radical, singular ideal and cosingular ideal in Mod-$R$ have been studied recently in [2]. In the context of the radical theory it was studied in [3] and its applications to the structure of rings were given in [1].

## 2 Left TOTO-modules

Given a nonempty set $I$ and a module $C_R$, we denote by $C^I$ the direct product of $|I|$-copies of $C_R$.

**Lemma 2.1** Let $M, C \in \text{Mod}-R$, let $A_R \subseteq C_R$, let $B \subseteq M$ and let $I$ be a nonempty set. Suppose that $\text{Tot}(M, C) = 0$. Then:

(i) $\text{Tot}(M, A) = 0$.

(ii) $\text{Tot}(M, C^I) = 0$.

(iii) $\text{Tot}(M/B, C) = 0$.

**Proof.** (i) Let $\pi : C_R \rightarrow A_R$ be the canonical projection. Given a nonzero map $f : M_R \rightarrow A_R$, $i_Af$ is pi because $\text{Tot}(M, C) = 0$, and so $f$ is pi by (2). Therefore $\text{Tot}(M, A) = 0$.

(ii) Let $f : M_R \rightarrow C^I_R$ be a nonzero map. Then there exists a canonical projection $\pi : C^I \rightarrow C$ such that $\pi f \neq 0$. Since $\text{Tot}(M, C) = 0$, we conclude that $\pi f$ is pi and whence $f$ is pi by (2).
(iii) Let \( \pi : M \to M/B \) be the canonical projection. Given a nonzero map \( f : \{M/B\}_R \to C_R, \) \( f\pi \) is pi because \( \text{Tot}(M,C) = 0. \) Therefore \( f \) is pi by (2) forcing \( \text{Tot}(M/B,C) = 0. \)

**Proof of Theorem 1.1.** (i) \( \implies \) (ii) is obvious. Assume that \( C \) is a cogenerator and \( \text{Tot}(M,C) = 0. \) First we claim that \( M \) is completely reducible. It is well-known that a module is completely reducible if and only if it has no proper essential submodules (see [12, 20.2]). Therefore it is enough to show that \( M \) has no proper essential submodules. Assume to the contrary that \( N \) is a proper essential submodule of \( M. \) Set \( L = M/N. \) By assumption there exists a nonzero map \( f : L_R \to C_R. \) Let \( \gamma : M \to C \) be the composition of the canonical projection \( M \to L \) with \( f : L \to C. \) By assumption \( \text{Tot}(M,C) = 0 \) and so \( \gamma \) is pi. Therefore by Lemma 1.4 there exist nonzero submodules \( A \subseteq M \) and \( B \subseteq C \) such that the map \( A \ni a \mapsto \phi(a) \in B \) is an isomorphism. In particular \( A \cap N \subseteq A \cap \ker(\phi) = 0, \) a contradiction. Thus \( M \) is completely reducible and so \( M = \oplus_{i \in I} M_i \) where each \( M_i \) is a simple submodule of \( M. \) Take any \( j \in I. \) Since \( C \) is a cogenerator, there exists a set \( I \) and a monomorphism \( f : E(M_j) \to C'_j. \) Set \( A = f(E(M_j)). \) By Lemma 2.1(ii), \( \text{Tot}(M,C'_j) = 0. \) Therefore Lemma 2.1(i) implies that \( \text{Tot}(M,A) = 0 \) and so \( \text{Tot}(M,E(M_j)) = 0. \) Let \( \delta : M \to E(M) \) be the composition of the canonical projection \( \pi : M \to M_j \) with \( i_{M_j} : M_j \to E(M_j). \) Making use of \( \text{Tot}(M,E(M_j)) = 0, \) we conclude that \( \delta \) is partially invertible and so there exist nonzero submodules \( A' \subseteq M \) and \( B' \subseteq E(M_j) \) such that the map \( A' \ni a \mapsto \psi(a) \in B' \) is an isomorphism. Clearly \( A' \cap \ker(\psi) = 0 \) and \( \ker(\psi) = \ker(\pi). \) Since \( M_j \) is simple, \( \pi \) induces an isomorphism \( A' \cong M_j. \) Next, as \( B' \) is a direct summand of the injective module \( E(M_j) \) and the injective hull of a simple module is indecomposable, \( B' = E(M_j). \) Since \( B' \cong A' \cong M_j, \) we conclude that each \( M_j \) is an injective simple module and so (iii) is satisfied.

(iii) \( \implies \) (i). Assume now that \( M = \oplus_{i \in I} M_i \) where each \( M_i \) is an injective simple module. Let \( N \) be a right \( R \)-module. If \( \text{Hom}_R(M,N) = 0, \) then \( \text{Tot}(M,N) = 0 \) as well. Suppose that \( \phi : M_R \to N_R \) is a nonzero homomorphism. Then there exists \( j \in I \) such that \( \phi M_j \neq 0 \) and so \( M_j \cong \phi M_j. \) In particular \( \phi M_j \) is injective and whence there exists a submodule \( L \) of \( N \) with \( N = \phi M_j \oplus L. \) Since \( M = M_j \oplus (\oplus_{i \neq j} M_i) \) and \( M_j \ni a \mapsto \phi a \in \phi M_j \) is an isomorphism, Lemma 1.4 implies that \( \phi \) is pi. Therefore \( \text{Tot}(M,N) = 0. \)

We denote by \( T \) the subclass of all left TOTO-modules of Mod-\( R. \) The following two result follow immediately from Theorem 1.1.

**Corollary 2.2** The class \( T \) is closed under taking of arbitrary direct sums, submodules and homomorphic images.

Given \( M, N \in \text{Mod-}R, \) we set
\[
\text{Im}(M,N) = \sum_{f \in \text{Hom}(M,N)} f(M).
\]
Corollary 2.3 Let $M, N \in \text{Mod-}R$. Suppose that $\text{Im}(M, N) = N$ and $M$ is a left TOTO-module. Then $N$ is a left TOTO-modules as well.

**Proof.** By Theorem 1.1, $M$ is a completely reducible module and every simple submodule of $M$ is injective. Since $\text{Im}(M, N) = N$, we conclude that $N$ is completely reducible and every simple submodule of $N$ is isomorphic to a submodule of $M$. Therefore every simple submodule of $N$ is injective and so Theorem 1.1 yields that $N$ is a left TOTO-module.

It is easy to see (and well-known) that the maximal condition and the minimal condition for direct summands of a module $M$ are equivalent. If $M$ satisfies one of them, we shall say that $M$ has fccds, the finite chain condition for direct summands.

Corollary 2.4 Let $M$ be a right $R$-module. Then the following conditions are equivalent:

(i) $M$ is a direct sum of a finite number of injective simple modules.

(ii) $M$ is a finitely generated left TOTO-module.

(iii) $M$ has fccds and is a left TOTO-module.

(iv) $M$ has a finite Goldie dimension and is a left TOTO-module.

(v) $M$ is an Artinian (Noetherian) left TOTO-module.

Moreover, $R_R$ is a left TOTO-module if and only if $R$ is a semisimple Artinian ring.

Given a subclass $\mathcal{M}$ of modules of $\text{Mod-}R$, we set

$$r_R(\mathcal{M}) = \{a \in R \mid Ma = 0 \text{ for all } M \in \mathcal{M}\}.$$

Clearly $r_R(\mathcal{M})$ is an ideal of $R$. We continue our study of properties of the class $\mathcal{T}_f$.

Remark 2.5 The class $\mathcal{T}_f$ is closed under essential extensions if and only if every left TOTO-module is injective.

**Proof.** Suppose that the class $\mathcal{T}_f$ is closed under essential extensions. Take any $0 \neq M \in \mathcal{T}_f$. Then $E(M) \in \mathcal{T}_f$ and so $E(M)$ is completely reducible by Theorem 1.1. Therefore $M$ is a direct summand of $E(M)$ forcing $M = E(M)$. The converse statement is obvious.

Proposition 2.6 Let $I = r_R(\mathcal{T}_f)$. Then the class $\mathcal{T}_f$ is closed under direct products if and only if $R/I$ is a semisimple Artinian ring. Moreover, if $\overline{R}$ is semisimple Artinian, then $\mathcal{T}_f = \text{Mod-}\overline{R}$.
Proof. Set $\overline{R} = R/I$. Clearly every $M \in \mathcal{T}_I$ is naturally an $\overline{R}$-module. Suppose that the class $\mathcal{T}_I$ is closed under direct products. Then it contains a module $M$ such that the right $R$-module $\overline{R}$ is embeddable into $M$. By Corollary 2.2, $\overline{R}$ is a left TOTO-module. Since $\overline{R}$ is a cyclic $R$-module, it is a completely reducible Artinian right $R$-module. Therefore $\overline{R}$ is a semisimple Artinian ring.

Conversely, assume that $\overline{R}$ is a semisimple Artinian ring. It is enough to show that $\mathcal{T}_I = \text{Mod-} \overline{R}$. Clearly $\mathcal{T}_I \subseteq \text{Mod-} \overline{R}$. Consider $\overline{R}$ as a right $R$-module. Obviously $\overline{R} = \bigoplus_{i=1}^{n} M_i$ where each $M_i$ is a simple right $R$-module (and also a simple right $\overline{R}$-module). Since every module in $\text{Mod-} \overline{R}$ is a direct sum of modules each of which is isomorphic to some $M_i$, in view of Theorem 1.1 it is enough to show that each $M_i$ is an injective $R$-module. To this end choose any $1 \leq j \leq n$ and let write $M_j = \overline{x} R$, where $x \in R$ and $\overline{x} = x + I \in \overline{R}$. Since $I = \tau_R(\mathcal{T}_I)$ and $\overline{x} \not\in I$, there exists $M \in \mathcal{T}_I$ with $mx \neq 0$ for some $m \in M$. Clearly $mx \overline{x} = mx$ by the definition of the $\overline{R}$-module structure on $M$ and so $M_j = \overline{x} R \cong mx R$. It follows from Corollary 2.2 that $mx R \in \mathcal{T}_I$ and so $M_j \in \mathcal{T}_I$. Therefore $M_j$ is injective by Theorem 1.1. The proof is complete.

3 Right TOTO-modules

Given a nonempty set $I$ and a module $P_R$, we denote by $P^{(I)}$ the direct sum of $|I|$-copies of $P_R$. The proof of the following result is similar to that of Lemma 2.1 and is omitted.

Lemma 3.1 Let $M, P \in \text{Mod-} R$. let $A_R \subseteq P_R$, let $B_R \subseteq M_R$ and let $I$ be a nonempty set. Suppose that $\text{Tot}(P, M) = 0$. Then:

(i) $\text{Tot}(P/A, M) = 0$.

(ii) $\text{Tot}(P^{(I)}, M) = 0$.

(iii) $\text{Tot}(P, B) = 0$.

Proof of Theorem 1.2. (i)$\Rightarrow$(ii) is obvious. Assume that $\text{Tot}(P, M) = 0$ for some generator $P$. Clearly there exists a set $I$ such that the module $R_R$ is a homomorphic image of $P^{(I)}$. It now follows from Lemma 3.1 that $\text{Tot}(R, M) = 0$. Let $N$ be a nonzero submodule of $M$. Pick $0 \neq x \in N$ and let a map $f : R_R \rightarrow M$ be given by the rule $f(r) = xr$, $r \in R$. since $\text{Tot}(R, M) = 0$, $f$ is pi and so there exist nonzero submodules $A \subseteq R$ and $B \subseteq M$ such that $A \ni a \mapsto f(a) \in B$ is an isomorphism. Since $A \subseteq R$, it is a cyclic projective module. Therefore $B_R$ is also a cyclic projective module. Clearly $B = f(A) \subseteq xR \subseteq N$ and so (iii) is satisfied.

(iii)$\Rightarrow$(iv) is obvious. We show that (iv)$\Rightarrow$(i). Let $N \in \text{Mod-} R$ and let $0 \neq f \in \text{Hom}(N_R, M_R)$. Then $fN$ is a nonzero submodule of $M$ and so it contains a projective submodule $B$ of $M$ such that $M = B \oplus D$ for some
submodule $D$ of $M$. Let $\pi : M \to B$ be the canonical projection. Then $\pi f : N \to B$ is an epimorphism and so there exists $h : B_R \to N_R$ such that $(\pi f)h = 1_R$. Therefore $f$ is $\pi$ by (2) and whence $\text{Tot}(N, M) = 0$.

(i)$\implies$(v). It follows from Zorn’s lemma that $M$ contains a maximal independent family $\{P_i \mid i \in I\}$ of projective submodules. Suppose that $P = \bigoplus_{i \in I} P_i$ is not an essential submodule of $M$ and let $L$ be a nonzero submodule of $M$ with $L \cap P = 0$. By (iv), $L$ contains a nonzero projective submodule, say $Q$. Clearly the family $\{Q\} \cup \{P_i \mid i \in I\}$ is independent, a contradiction. Therefore $P \subseteq M$. Next, let $i \in I$. Given $N \in \text{Mod-}R$, $\text{Tot}(N, M) = 0$ and so $\text{Tot}(N, P_i) = 0$. In particular, $\text{Tot}(P_i, P_i) = 0$ and whence every nonzero element of the ring $\text{End}(P_i)$ is $\pi$. We see that $\text{End}(P_i)$ is a TOTO-ring. Further, let $0 \neq x \in M$. By (iii) the submodule $xR$ contains a a nonzero cyclic projective module $C$ which is a direct summand of $M$. It is well-known that $C_R \cong eR_R$ for some idempotent $e \in R$. Let $f : C_R \to eR_R$ be an isomorphism and let $\pi : M_R \to C_R$ be the canonical projection. Clearly $\pi(xR) = C$ and so $eR = f\pi(xR) = \{f\pi(x)\}R$ forcing $f\pi(x)r = e$ for some $r \in R$. Denote by $L_r$ the map $L_r : R_R \to R_R$ given by $L_r(a) = ra, a \in R$. Set $g = L_rf\pi$ and note that

$$g : M_R \to R_R \quad \text{and} \quad g(x) = rf\pi(x).$$

We now have

$$f\pi(xg(x))r = f\pi(x)g(x)r = f\pi(x)rf\pi(x)r = e^2 = e \neq 0$$

and so $xg(x) \neq 0$. Therefore $M_R$ is semiprime.

(v)$\implies$(iv). Let $P = \bigoplus_{i \in I} P_i$. We claim that $\text{End}(P)$ is a TOTO-ring. Indeed, since $M_R$ is semiprime, it follows directly from the definition that so is every its submodule. In particular, $P_R$ is semiprime. Let $f : P_R \to P_R$ be a nonzero map. Choose $j \in I$ with $f(P_j) \neq 0$. Pick $x \in P_j$ with $f(x) \neq 0$. Since $P$ is semiprime, there exists $g : P_R \to R_R$ such that $f(x)g(x) \neq 0$. Clearly $f(xg(x)) = f(x)g(x) \neq 0$ and so $xg(x) \neq 0$. Let $h : R_R \to P_R$ be given by $h(r) = xa, a \in R$. Set $g = L_rf\pi$ and note that $h : M_R \to R_R$ and $g(x) = rf\pi(x)$.

We see that $hgf : P_R \to P_R, \quad hgf(P) \subseteq xR \subseteq P_j$ and $hgf(x) = xg(x) \neq 0$.

We now have

$$hgf : P_R \to P_R, \quad hgf(P) \subseteq xR \subseteq P_j \quad \text{and} \quad hgf(x) = xg(x) \neq 0.$$
A \subseteq M and B \subseteq M such that A \ni a \mapsto \psi(a) \in B is an isomorphism. Being a direct sum of projective modules $P_i$, $i \in I$, $P$ itself is projective. Next, $B = \psi(A) \subseteq yR \subseteq N \cap P$. Since $B \subseteq M$, the modular law implies that $B \subseteq P$. Therefore $B$ is projective. We see that $B \subseteq N$ and $B \subseteq M$. Thus (iv) is fulfilled and the proof is thereby complete.

We denote by $T_r$ the subclass of all right TOTO-modules of Mod-$.R$. The following result follows immediately from both Lemma 3.1 and Theorem 1.2.

**Corollary 3.2** The class $T_r$ is closed under taking of arbitrary direct sums and submodules.

Given $M, N \in \text{Mod-}R$, we set

$$\text{Ke}(M, N) = \cap_{f \in \text{Hom}(M, N)} \text{Ke}(f).$$

**Corollary 3.3** Let $M, N \in \text{Mod-}R$. Suppose that $N$ is a right TOTO-module and $\text{Ke}(M, N) = 0$. Then $M$ is a right TOTO-module as well.

**Proof.** Since $\text{Ke}(M, N) = 0$, $M$ is isomorphic to a submodule of the direct product of some set of copies of $N$. The result now follows from Corollary 3.2.

Let $R$ be a ring. Then Tot($R$) = 0 if and only if every nonzero right (left) ideal of $R$ contains a nonzero right (respectively, left) ideal of $R$ generated by an idempotent. Therefore Theorem 1.2 implies

**Corollary 3.4** The following conditions are equivalent:

(i) Tot($R$) = 0.

(ii) $R_R$ is a right TOTO-module.

(iii) $R_R$ is a right TOTO-module.

It is easy to see that every TOTO-ring is a semiprime ring.

**Proposition 3.5** Let $M \in \text{Mod-}R$ be a right TOTO-module. Then $M$ is torsionless.

**Proof.** Indeed, let $0 \neq x \in M$. Then by Lemma 1.2 $xR$ contains a projective submodule $P$ which is a direct summand of $M$. Let $\pi : M_R \to P_R$ be a canonical projection. Since $P \subseteq xR$, $\pi x \neq 0$. Every projective module is torsionless and so there exists $f : P_R \to R_R$ such that $f \pi x \neq 0$. Clearly $f \pi \in \text{Hom}(M_R, R_R)$. Therefore $M$ is torsionless.

**Proof of Theorem 1.3.** (i)$\Rightarrow$(ii) is obvious (take $E = \{1\}$). Suppose that (ii) is satisfied. Clearly each $e_iR$ is projective and $\text{End}(e_iR) = e_iR e_i$ is a TOTO-ring. Clearly the family $\{e_iR \mid i \in I\}$ of submodules of $R_R$ is independent. Since
$R$ is semiprime, the module $R_R$ is also semiprime and so Theorem 1.2(v) implies that $R_R$ is a right TOTO-module. By Corollary 3.4, $R$ is a TOTO-ring.

(i)$\implies$(iv). Let $M$ be a right torsionless module and let $f : N_R \to M_R$ be a nonzero module map. Then there exists $g : M_R \to R_R$ such that $gf \neq 0$. Clearly $gf \in \text{Hom}(N_R, R_R)$. Since $R$ is a TOTO-ring, $R_R$ is a TOTO-module by Corollary 3.4 and so $gf$ is pi. We see that $f$ is pi by (2) and whence $M$ is a TOTO-module.

(iv)$\implies$(iii) because every projective module is torsionless.

(iii)$\implies$(i) because $R_R$ is a projective module and so $R$ is a TOTO-ring by Corollary 3.4.

(i)$\implies$(vi) Let $M_R$ be torsionless. Then $M$ is a right TOTO-module by (iv). In particular, $\text{Tot}(M, M) = 0$ and so $\text{End}(M_R)$ is a TOTO-ring.

(v)$\implies$(vi) is obvious.

(vi)$\implies$(i) Let $W_R$ be a faithful right TOTO-module. Given $0 \neq r \in R$, by assumption there exists $w \in W$ with $wr \neq 0$. Define maps $f : R_R \to R_R$ and $g : R_R \to W_R$ by $f(x) = rx$ and $g(x) = wx$, $x \in R$. Then $gf : R_R \to W_R$. Since $gf(1) = wr \neq 0$, also $gf \neq 0$. As $\text{Tct}(R, W) = 0$, $gf$ is pi. According to (2), $f$ is pi which means that there exists $s \in R$ with $0 \neq rs = (rs)^2$. Therefore $\text{Tot}(R) = 0$ and $R$ is a TOTO-ring.

**Remark 3.6** Let $R$ be a TOTO-ring. Then $R$ is left and right nonsingular ring.

**Proof.** Let $0 \neq x \in R$. Since $\text{Tot}(R) = 0$, there exists $y \in R$ such that $e = yx$ is a nonzero idempotent of $R$. Clearly $r_R(x) \subseteq r_R(yx) = (1 - e)R$ and so $r_R(x) \cap eR = 0$. Therefore $r_R(x)$ is not an essential right ideal for any $0 \neq x \in R$ and whence $R$ is right nonsingular. Analogously, $R$ is left nonsingular.

Let $W_R$ be a right TOTO-module. Then every simple submodule of $W$ is projective by Theorem 1.2(iv) and so

$$\text{the socle } \text{Soc}(W) \text{ of } W_R \text{ is projective.}$$

(3)

**Theorem 3.7** Let $M \in \text{Mod-}R$. Then the following conditions are equivalent:

(i) $M$ is a right TOTO-module and every its cyclic submodule has fccds.

(ii) $M$ is a projective completely reducible module.

In particular, if $R$ is a TOTO-ring having fccds, then it is a semisimple Artinian ring.

**Proof.** (i)$\implies$(ii) In view of (3) it is enough to show that $M = \text{Soc}(M)$. To this end, pick any $0 \neq x \in M$ and set $L = xR$. Assume that $x \not\in \text{Soc}(M)$. 

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Then $x \not\in \text{Soc}(L)$. Let $K$ be a submodule of $L$ maximal with respect to the properties $x \not\in K$ and $\text{Soc}(L) \subseteq K$. If $K = 0$, then $L$ is a simple module and so $x \in \text{Soc}(M)$, a contradiction. Therefore $K \neq 0$. By assumption $L$ has fccds and whence $K$ contains a submodule $N$ maximal with respect to the property $N \subseteq L$. Suppose that $N = K$. Then $K \subseteq L$ and so $K \oplus K' = L$ for some submodule $K'$ of $L$. Since $K$ is a maximal submodule of $L$, $K'$ is simple, forcing $K' \subseteq \text{Soc}(L) \subseteq K$, a contradiction. Therefore $N \subset K$. Choose $N' \subseteq L$ with $N \oplus N' = L$. By the modular law, $K = N \oplus (K \cap N')$. Clearly $K \cap N' \neq 0$.

Further, since $M$ is a right TOTO-module, it follows from Corollary 3.2 that $L$ is so. By Theorem 1.2(iv), there exists a nonzero submodule $T$ of $K \cap N'$ with $T \subseteq L$. It now follows from the modular law that $T \subseteq N'$ and so $N \oplus T \subseteq L$. Taking into account that $N \subset N \oplus T \subseteq K$, we get a contradiction with the choice of $N$. Therefore $M = \text{soc}(M)$.

(ii)\(\implies\)(i). Since every submodule of $M$ is its direct summand, we conclude that each submodule of $M$ is projective and so $M$ is a right TOTO-module by Theorem 1.2(iv). The last statement is obvious.

The following result follows immediately from Theorem 3.7.

**Corollary 3.8** Let $M$ be a right $R$-module. Then the following conditions are equivalent:

(i) $M$ is a direct sum of a finite number of projective simple modules.

(ii) $M$ has fccds and is a right TOTO-module.

(iii) $M$ has a finite Goldie dimension and is a right TOTO-module.

(iv) $M$ is an Artinian (Noetherian) right TOTO-module.

**References**


