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TOTO-Modules

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1 Introduction

Given a ring R (with unity), we denote by Mod-R the category of right R-modules. In what follows the term "module" will mean "right module". Let $M \in$ Mod-R. We denote by $E_R(M)$ the injective hull of M and by 1_M the identity endomorphism of M. When the context is clear, we shall write $E(M)$ for $E_R(M)$. Given a submodule $N_R \subseteq M_R$, we denote by $i_N : N_R \to M_R$ the canonical embedding of modules. We shall write $N \subseteq^* M$ whenever N is an essential submodule of M, and $N \subseteq^{\oplus} M$ whenever N is a direct summand of the module M. If $M_R = K_R \oplus L_R$, we denote by $\pi_K : M_R \to K_R$ the canonical projection of modules.

Let $M = M_R$ and $N = N_R$. Recall that $f : M_R \to N_R$ is called *partially invertible* (briefly f is pi) whenever there exists $g : N_R \to M_R$ such that $0 \neq fg = (fg)^2$ (see [6]). It is known that the following to conditions are equivalent:

$$\begin{aligned} &\text{There exists } f \in \mathrm{Hom}_R(N, M) \text{ such that } 0 \neq fg = (fg)^2; \\ &\text{There exists } h \in \mathrm{Hom}_R(N, M) \text{ such that } 0 \neq gh = (gh)^2. \end{aligned} \tag{1}$$

(see [6] or [2]). Further, let $n > 0$, $M_1, M_2, \ldots, M_{n+1} \in$ Mod-R and $g_i \in \mathrm{Hom}_R(M_i, M_{i+1})$, $i = 1, 2, \ldots, n$.

$$\text{If } g_1 g_2 \ldots g_n \text{ is pi. Then each } g_i \text{ is pi} \tag{2}$$

according to [6, 1.3] (see also [2]).

Let $M, N \in$ Mod-R. Then the total $\mathrm{Tot}(M, N)$ of M and N is defined as follows:

$$\mathrm{Tot}(M, N) = \{g \in \mathrm{Hom}_R(M, N) \mid g \text{ is not pi}\}$$

(see [6]).

Theorem 1.1 *Let M be a right R-module. Then the following conditions are equivalent:*

(i) $\text{Tot}(M, N) = 0$ *for any* $N \in$ Mod-R.

(ii) $\text{Tot}(M, C) = 0$ *for some cogenerator* C *of* Mod-R.

(iii) M *is a direct sum of injective simple submodules.*

A module M_R, satisfying the equivalent conditions of Theorem 1.1, is called *a left TOTO-module.* Next, a family $\{K_i \mid i \in I\}$ of submodules of M_R is called independent whenever $\sum_{i\in I} K_i = \oplus_{i\in I} K_i$. Further, M is said to be *semiprime* if for any $0 \neq m \in M$ there exists $f \in \text{Hom}(M_R, R_R)$ such that $mf(m) \neq 0$. Since every endomorphism of the module R_R is a left multiplication by an element of R, we see that the module R_R is semiprime if and only if the ring R is semiprime. We set $\text{Tot}(R) = \text{Tot}(R_R, R_R)$. In view of (1) we have that $\text{Tot}(R) = 0$ if and only if every nonzero left (right) ideal of R contains a nonzero idempotent 9see also [6]). Following [6], a ring R is said to be a TOTO-ring if $\text{Tot}(R) = 0$.

Theorem 1.2 *Let* $0 \neq M \in$ Mod-R. *Then the following conditions are equivalent:*

(i) $\text{Tot}(L, M) = 0$ *for any* $L \in$ Mod-R.

(ii) $\text{Tot}(P, M) = 0$ *for some generator* P *of* Mod-R

(iii) Every nonzero submodule of M *contains a nonzero cyclic projective submodule which is a direct summand of* M.

(iv) Every nonzero submodule of M *contains a nonzero projective submodule which is a direct summand of* M.

(v) M *is a semiprime module with independent family* $\{P_i \mid i \in I\}$ *of projective submodules such that* $\oplus_{i\in I} P_i \subseteq^* M$ *and each* $\text{End}(P_i)$ *is a TOTO-ring.*

A module M_R is called *a right TOTO-module* whenever it satisfies the equivalent conditions of Theorem 1.2. Recall that a module $M \in$ Mod-R is called *torsionless* (in sense of Bass) if for any $0 \neq x \in M$ there exists $f \in M^* = \text{Hom}(M_R, R_R)$ such that $fx \neq 0$.

The goal of the present paper is to study left and right TOTO-modules and TOTO-rings. Besides Theorems 1.1 and 1.2 the following theorem is the main result of the present paper.

Theorem 1.3 *Let* R *be a ring. Then the following conditions are equivalent:*

(i) R *is a TOTO-ring.*

(ii) R *is a semiprime ring and there exists a family* $E = \{e_i \mid i \in I\}$ *of idempotents such that* $\sum_{i\in I} e_i R = \oplus_{i\in I} e_i R \subseteq^* R$ *and each* $e_i R e_i$ *is a TOTO-ring.*

(iii) Every projective right R-module is a right TOTO-module.

(iv) Every torsionless right R-module is a right TOTO-module.

(v) $\mathrm{End}(P)$ *is a TOTO-ring for any projective right R-module P.*

(vi) $\mathrm{End}(M)$ *is a TOTO-ring for any torsionless right R-module M.*

(vii) R has a faithful right TOTO-module.

Further, if R is a TOTO-ring and $0 \neq e = e^2 \in R$, then eRe is a TOTO-ring.

The following result, proved in [9], will be frequently used in the sequel.

Lemma 1.4 *For $g \in \mathrm{Hom}_R(M, N)$ the following conditions are equivalent:*

(a) g is pi.

(b) There exist nonzero submodules $A \subseteq^{\oplus} M$, $B \subseteq^{\oplus} N$ such that the map $A \ni a \mapsto g(a) \in B$ is an isomorphism.

The total was defined in 1982 by F. Kasch and then studied in several papers by F. Kasch and W. Schneider (see [6, 7, 8, 9, 10]). Relationships of the total with Jacobson radical, singular ideal and cosingular ideal in Mod-R have been studied recently in [2]. In the context of the radical theory it was studied in [3] and its applications to the structure of rings were given in [1].

2 Left TOTO-modules

Given a nonempty set I and a module C_R, we denote by C^I the direct product of $|I|$-copies of C_R.

Lemma 2.1 *Let $M, C \in$ Mod-R, let $A_R \subseteq C_R$, let $B \subseteq^{\oplus} M$ and let I be a nonempty set. Suppose that* $\mathrm{Tot}(M, C) = 0$. *Then:*

(i) $\mathrm{Tot}(M, A) = 0$.

(ii) $\mathrm{Tot}(M, C^I) = 0$.

(iii) $\mathrm{Tot}(M/B, C) = 0$.

Proof. (i) Let $\pi : C_R \to A_R$ be the canonical projection. Given a nonzero map $f : M_R \to A_R$, $i_A f$ is pi because $\mathrm{Tot}(M, C) = 0$, and so f is pi by (2). Therefore $\mathrm{Tot}(M, A) = 0$.

(ii) Let $f : M_R \to C^I_R$ be a nonzero map. Then there exists a canonical projection $\pi : C^I \to C$ such that $\pi f \neq 0$. Since $\mathrm{Tot}(M, C) = 0$, we conclude that πf is pi and whence f is pi by (2).

(iii) Let $\pi : M \to M/B$ be the canonical projection. Given a nonzero map $f : \{M/B\}_R \to C_R$, $f\pi$ is pi because $\mathrm{Tot}(M, C) = 0$. Therefore f is pi by (2) forcing $\mathrm{Tot}(M/B, C) = 0$.

Proof of Theorem 1.1. (i)$\Longrightarrow$(ii) is obvious. Assume that C is a cogenerator and $\mathrm{Tot}(M, C) = 0$. First we claim that M is completely reducible. It is well-known that a module is completely reducible if and only if it has no proper essential submodules (see [12, 20.2]). Therefore it is enough to show that M has no proper essential submodules. Assume to the contrary that N is a proper essential submodule of M. Set $L = M/N$. By assumption there exists a nonzero map $f : L_R \to C_R$. Let $\phi : M \to C$ be the composition of the canonical projection $M \to L$ with $f : L \to C$. By assumption $\mathrm{Tot}(M, C) = 0$ and so ϕ is pi. Therefore by Lemma 1.4 there exist nonzero submodules $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} C$ such the map $A \ni a \mapsto \phi(a) \in B$ is an isomorphism. In particular $A \cap N \subseteq A \cap \ker(\phi) = 0$, a contradiction. Thus M is completely reducible and so $M = \oplus_{i \in I} M_i$ where each M_i is a simple submodule of M. Take any $j \in I$. Since C is a cogenerator, there exists a set I and a monomorphism $f : E(M_j) \to C^I$. Set $A = f(E(M_j))$. By Lemma 2.1(ii), $\mathrm{Tot}(M, C^I) = 0$. Therefore Lemma 2.1(i) implies that $\mathrm{Tot}(M, A) = 0$ and so $\mathrm{Tot}(M, E(M_j)) = 0$. Let $\psi : M \to E(M)$ be the composition of the canonical projection $\pi : M \to M_j$ with $i_{M_j} : M_j \to E(M_j)$. Making use of $\mathrm{Tot}(M, E(M_j)) = 0$, we conclude that ψ is partially invertible and so there exist nonzero submodules $A' \subseteq^{\oplus} M$ and $B' \subseteq^{\oplus} E(M_j)$ such that the map $A' \ni a \mapsto \psi(a) \in B'$ is an isomorphism. Clearly $A' \cap \ker(\psi) = 0$ and $\ker(\psi) = \ker(\pi)$. Since M_j is simple, π induces an isomorphism $A' \cong M_j$. Next, as B' is a direct summand of the injective module $E(M_j)$ and the injective hull of a simple module is indecomposable, $B' = E(M_j)$. Since $B' \cong A' \cong M_j$, we conclude that each M_j is an injective simple module and so (iii) is satisfied.

(iii)$\Longrightarrow$(i). Assume now that $M = \oplus_{i \in I} M_i$ where each M_i is an injective simple module. Let N be a right R-module. If $\mathrm{Hom}_R(M, N) = 0$, then $\mathrm{Tot}(M, N) = 0$ as well. Suppose that $\phi : M_R \to N_R$ is a nonzero homomorphism. Then there exists $j \in I$ such that $\phi M_j \neq 0$ and so $M_j \cong \phi M_j$. In particular ϕM_j is injective and whence there exists a submodule L of N with $N = \phi M_j \oplus L$. Since $M = M_j \oplus (\oplus_{i \neq j} M_i)$ and $M_j \ni a \mapsto \phi a \in \phi M_j$ is an isomorphism, Lemma 1.4 implies that ϕ is pi. Therefore $\mathrm{Tot}(M, N) = 0$.

We denote by $\mathcal{T}_\ell$ the subclass of all left TOTO-modules of Mod-R. The following two result follow immediately from Theorem 1.1.

Corollary 2.2 *The class $\mathcal{T}_\ell$ is closed under taking of arbitrary direct sums, submodules and homomorphic images.*

Given $M, N \in$ Mod-R, we set

$$\mathrm{Im}(M, N) = \sum_{f \in \mathrm{Hom}(M,N)} f(M).$$

Corollary 2.3 *Let $M, N \in$ Mod-R. Suppose that* $\mathrm{Im}(M, N) = N$ *and M is a left TOTO-module. Then N is a left TOTO-modules as well.*

Proof. By Theorem 1.1, M is a completely reducible module and every simple submodule of M is injective. Since $\mathrm{Im}(M, N) = N$, we conclude that N is completely reducible and every simple submodule of N is isomorphic to a submodule of M. Therefore every simple submodule of N is injective and so Theorem 1.1 yields that N is a left TOTO-module.

It is easy to see (and well-known) that the maximal condition and the minimal condition for direct summands of a module M are equivalent. If M satisfies one of them, we shall say that M has fccds, the finite chain condition for direct summands.

Corollary 2.4 *Let M be a right R-module. Then the following conditions are equivalent:*

(i) M is a direct sum of a finite number of injective simple modules.

(ii) M is a finitely generated left TOTO-module.

(iii) M has fccds and is a left TOTO-module.

(iv) M has a finite Goldie dimension and is a left TOTO-module.

(v) M is an Artinian (Noetherian) left TOTO-module.

Moreover, R_R is a left TOTO-module if and only if R is a semisimple Artinian ring.

Given a subclass $\mathcal{M}$ of modules of Mod-R, we set

$$r_R(\mathcal{M}) = \{a \in R \mid Ma = 0 \text{ for all } M \in \mathcal{M}\}.$$

Clearly $r_R(\mathcal{M})$ is an ideal of R. We continue our study of properties of the class $\mathcal{T}_\ell$.

Remark 2.5 *The class $\mathcal{T}_\ell$ is closed under essential extensions if and only if every left TOTO-module is injective.*

Proof. Suppose that the class $\mathcal{T}_\ell$ is closed under essential extensions. Take any $0 \neq M \in \mathcal{T}_\ell$. Then $E(M) \in \mathcal{T}_\ell$ and so $E(M)$ is completely reducible by Theorem 1.1. Therefore M is a direct summand of $E(M)$ forcing $M = E(M)$. The converese statement is obvious.

Proposition 2.6 *Let $I = r_R(\mathcal{T}_\ell)$. Then the class $\mathcal{T}_\ell$ is closed under direct products if and only if R/I is a semisimple Artinian ring. Moreover, if $\overline{R}$ is semisimple Artinian, then $\mathcal{T}_\ell =$ Mod-$\overline{R}$.*

Proof. Set $\overline{R} = R/I$. Clearly every $M \in \mathcal{T}_\ell$ is naturally an $\overline{R}$-module. Suppose that the class $\mathcal{T}_\ell$ is closed under direct products. Then it contains a module M such that the right R-module $\overline{R}$ is embeddable into M. By Corollary 2.2, $\overline{R}_R$ is a left TOTO-module. Since $\overline{R}_R$ is a cyclic R-module, it is a completely reducible Artinian right R-module. Therefore $\overline{R}$ is a semisimple Artinian ring.

Conversely, assume that $\overline{R}$ is a semisimple Artinian ring. It is enough to show that $\mathcal{T}_\ell = \text{Mod-}\overline{R}$. Clearly $\mathcal{T}_\ell \subseteq \text{Mod-}\overline{R}$. Consider $\overline{R}$ as a right R-module. Obviously $\overline{R} = \oplus_{i=1}^n M_i$ where each M_i is a simple right R-module (and also a simple right $\overline{R}$-module). Since every module in Mod-$\overline{R}$ is a direct sum of modules each of which is isomorphic to some M_i, in view of Theorem 1.1 it is enough to show that each M_i is an injective R-module. To this end choose any $1 \leq j \leq n$ and let write $M_j = \overline{x}R$, where $x \in R$ and $\overline{x} = x + I \in \overline{R}$. Since $I = r_R(\mathcal{T}_\ell)$ and $x \notin I$, there exists $M \in \mathcal{T}_\ell$ with $mx \neq 0$ for some $m \in M$. Clearly $m\overline{x} = mx$ by the definition of the $\overline{R}$-module structure on M and so $M_j = \overline{x}R \cong mxR$. It follows from Corollary 2.2 that $mxR \in \mathcal{T}_\ell$ and so $M_j \in \mathcal{T}_\ell$. Therefore M_j is injective by Theorem 1.1. The proof is complete.

3 Right TOTO-modules

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Given a nonempty set I and a module P_R, we denote by $P^{(I)}$ the direct sum of $|I|$-copies of P_R. The proof of the following result is similar to that of Lemma 2.1 and is omitted.

Lemma 3.1 *Let* $M, P \in \text{Mod-}R$, *let* $A_R \subseteq P_R$, *let* $B_R \subseteq M_R$ *and let* I *be a nonempty set. Suppose that* $\text{Tot}(P, M) = 0$. *Then:*

(i) $\text{Tot}(P/A, M) = 0$.

(ii) $\text{Tot}(P^{(I)}, M) = 0$.

(iii) $\text{Tot}(P, B) = 0$.

Proof of Theorem 1.2. (i)$\Longrightarrow$(ii) is obvious. Assume that $\text{Tot}(P, M) = 0$ for some generator P. Clearly there exists a set I such that the module R_R is a homomorphic image of $P^{(I)}$. It now follows from Lemma 3.1 that $\text{Tot}(R, M) = 0$. Let N be a nonzero submodule of M. Pick $0 \neq x \in N$ and let a map $f : R_R \to M$ be given by the rule $f(r) = xr$, $r \in R$. since $\text{Tot}(R, M) = 0$, f is pi and so there exist nonzero submodules $A \subseteq^\oplus R$ and $B \subseteq^\oplus M$ such that $A \ni a \mapsto f(a) \in B$ is an isomorphism. Since $A \subseteq^\oplus R$, it is a cyclic projective module. Therefore B_R is also a cyclic projective module. Clearly $B = f(A) \subseteq xR \subseteq N$ and so (iii) is satisfied.

(iii)$\Longrightarrow$(iv) is obvious. We show that (iv)$\Longrightarrow$(i). Let $N \in \text{Mod-}R$ and let $0 \neq f \in \text{Hom}(N_R, M_R)$. Then fN is a nonzero submodule of M and so it contains a projective submodule B of M such that $M = B \oplus D$ for some submodule D of M. Let $\pi : M \to B$ be the canonical projection. Then $\pi f : N \to B$ is an epimorphism and so there exists $h : B_R \to N_R$ such that $(\pi f)h = 1_P$. Therefore f is pi by (2) and whence $\mathrm{Tot}(N, M) = 0$.

(i)$\Longrightarrow$(v). It follows from Zorn's lemma that M contains a maximal independent family $\{P_i \mid i \in I\}$ of projective submodules. Suppose that $P = \oplus_{i \in I} P_i$ is not an essential submodule of M and let L be a nonzero submodule of M with $L \cap P = 0$. By (iv), L contains a nonzero projective submodule, say Q. Clearly the family $\{Q\} \cup \{P_i \mid i \in I\}$ is independent, a contradiction. Therefore $P \subseteq^* M$. Next, let $i \in I$. Given $N \in$ Mod-R, $\mathrm{Tot}(N, M) = 0$ and so $\mathrm{Tot}(N, P_i) = 0$. In particular, $\mathrm{Tot}(P_i, P_i) = 0$ and whence every nonzero element of the ring $\mathrm{End}(P_i)$ is pi. We see that $\mathrm{End}(P_i)$ is a TOTO-ring. Further, let $0 \neq x \in M$. By (iii) the submodule xR contains a a nonzero cyclic projective module C which is a direct summand of M. It is well-known that $C_R \cong eR_R$ for some idempotent $e \in R$. Let $f : C_R \to eR_R$ be an isomorphism and let $\pi : M_R \to C_R$ be the canonical projection. Clearly $\pi(xR) = C$ and so $eR = f\pi(xR) = \{f\pi(x)\}R$ forcing $f\pi(x)r = e$ for some $r \in R$. Denote by L_r the map $L_r : R_R \to R_R$ given by $L_r(a) = ra$, $a \in R$. Set $g = L_r f \pi$ and note that

$$g : M_R \to R_R \quad \text{and} \quad g(x) = rf\pi(x).$$

We now have

$$f\pi(xg(x))r = f\pi(x)g(x)r = f\pi(x)rf\pi(x)r = e^2 = e \neq 0$$

and so $xg(x) \neq 0$. Therefore M_R is semiprime.

(v)$\Longrightarrow$(iv). Let $P = \oplus_{i \in I} P_i$. We claim that $\mathrm{End}(P)$ is a TOTO-ring. Indeed, since M_R is semiprime, it follows directly from the definition that so is every its submodule. In particular, P_R is semiprime. Let $f : P_R \to P_R$ be a nonzero map. Choose $j \in I$ with $f(P_j) \neq 0$. Pick $x \in P_j$ with $f(x) \neq 0$. Since P is semiprime, there exists $g : P_R \to R_R$ such that $f(x)gf(x) \neq 0$. Clearly $f(xgf(x)) = f(x)gf(x) \neq 0$ and so $xgf(x) \neq 0$. Let $h : R_R \to P_R$ be given by $h(r) = xr$. Then:

$$hgf : P_R \to P_R, \quad hgf(P) \subseteq xR \subseteq P_j \quad \text{and} \quad hgf(x) = xgf(x) \neq 0.$$

We see that $hgfi_{P_j}(x) = hgf(x) \neq 0$ and $hgfi_{P_j} : P_j \to P_j$. Since $\mathrm{End}(P_j)$ is a TOTO-ring, $hgfi_{P_j}$ is pi and whence f is pi by (2). Therefore $\mathrm{End}(P)$ is a TOTO-ring.

Now let $0 \neq N_R \subseteq M_R$. Since $P = \oplus_{i \in I} P_i \subseteq^* M$, $N \cap P \neq 0$. Pick $0 \neq y \in N \cap P$. Since M_R is semiprime, there exists $\phi : M_R \to R_R$ with $y\phi(y) \neq 0$. Denote by ψ the composition of ϕ with $R_R \to M_R$, $r \mapsto yr$. Clearly

$$\psi : M_R \to M_R, \quad \psi(M) \subseteq yR \subseteq P \quad \text{and} \quad \psi(y) = y\phi(y) \neq 0.$$

Therefore $\psi i_P(y) = \psi(y) \neq 0$ and so ψi_P is pi because $\mathrm{End}(P)$ is a TOTO-ring. By (2), ψ is pi. Now Lemma 1.4 implies that there exist nonzero submodules

$A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} M$ such that $A \ni a \mapsto \psi(a) \in B$ is an isomorphism. Being a direct sum of projective modules P_i, $i \in I$, P itself is projectve. Next, $B = \psi(A) \subseteq yR \subseteq N \cap P$. Since $B \subseteq^{\oplus} M$, the modular law implies that $B \subseteq^{\oplus} P$. Therefore B is projective. We see that $B \subseteq N$ and $B \subseteq^{\oplus} M$. Thus (iv) is fulfilled and the proof is thereby complete.

We denote by $\mathcal{T}_r$ the subclass of all right TOTO-modules of Mod-R. The following result follows immediately from both Lemma 3.1 and Theorem 1.2.

Corollary 3.2 *The class $\mathcal{T}_r$ is closed under taking of arbitrary direct sums and submodules.*

Given $M, N \in$ Mod-R, we set

$$\mathrm{Ke}(M, N) = \cap_{f \in \mathrm{Hom}(M,N)} \mathrm{Ke}(f).$$

Corollary 3.3 *Let $M, N \in$ Mod-R. Suppose that N is a right TOTO-module and $\mathrm{Ke}(M, N) = 0$. Then M is a right TOTO-module as well.*

Proof. Since $\mathrm{Ke}(M, N) = 0$, M is isomorphic to a submodule of the direct product of some set of copies of N. The result now follows from Corollary 3.2.

Let R be a ring. Then $\mathrm{Tot}(R) = 0$ if and only if every nonzero right (left) ideal of R contains a nonzero right (respectively, left) ideal of R generated by an idempotent. Therefore Theorem 1.2 implies

Corollary 3.4 *The following conditions are equivalent:*

(i) $\mathrm{Tot}(R) = 0$.

(ii) R_R *is a right TOTO-module.*

(iii) ${}_RR$ *is a right TOTO-module.*

It is easy to see that every TOTO-ring is a semiprime ring.

Proposition 3.5 *Let $M \in$ Mod-R be a right TOTO-module. Then M is torsionless.*

Proof. Indeed, let $0 \neq x \in M$. Then by Lemma 1.2 xR contains a projective submodule P which is a direct summand of M. Let $\pi : M_R \to P_R$ be a canonical projection. Since $P \subseteq xR$, $\pi x \neq 0$. Every projective module is torsionless and so there exists $f : P_R \to R_R$ such that $f\pi x \neq 0$. Clearly $f\pi \in \mathrm{Hom}(M_R, R_R)$. Therefore M is torsionless.

Proof of Theorem 1.3. (i)$\Longrightarrow$(ii) is obvious (take $E = \{1\}$). Suppose that (ii) is satisfied. Clearly each e_iR is projective and $\mathrm{End}(e_iR) = e_iRe_i$ is a TOTO-ring. Clearly the family $\{e_iR \mid i \in I\}$ of submodules of R_R is independent. Since

R is semiprime, the module R_R is also semiprime and so Theorem 1.2(v) implies that R_R is a right TOTO-module. By Corollary 3.4, R is a TOTO-ring.

(i)$\Longrightarrow$(iv). Let M be a right torsionless module and let $f : N_R \to M_R$ be a nonzero module map. Then there exists $g : M_R \to R_R$ such that $gf \neq 0$. Clearly $gf \in \mathrm{Hom}(N_R, R_R)$. Since R is a TOTO-ring, R_R is a TOTO-module by Corollary 3.4 and so gf is pi. We see that f is pi by (2) and whence M is a TOTO-module.

(iv)$\Longrightarrow$(iii) because every projective module is torsionless.

(iii)$\Longrightarrow$(i) because R_R is a projective module and so R is a TOTO-ring by Corollary 3.4.

(i)$\Longrightarrow$(vi) Let M_R be torsionless. Then M is a right TOTO-module by (iv). In particular, $\mathrm{Tot}(M, M) = 0$ and so $\mathrm{End}(M_R)$ is a TOTO-ring.

(vi)$\Longrightarrow$(v) is obvious.

(v)$\Longrightarrow$(i) is obvious because R_R is a projective module and $R = \mathrm{End}(R_R)$.

(i)$\Longrightarrow$(vi) Clearly R_R is a faithful right TOTO-module by Corollary 3.4.

(vi)$\Longrightarrow$(i) Let W_R be a faithful right TOTO-module. Given $0 \neq r \in R$, by assumption there exists $w \in W$ with $wr \neq 0$. Define maps $f : R_R \to R_R$ and $g : R_R \to W_R$ by $f(x) = rx$ and $g(x) = wx$, $x \in R$. Then $gf : R_R \to W_R$. Since $gf(1) = wr \neq 0$, also $gf \neq 0$. As $\mathrm{Tot}(R, W) = 0$, gf is pi. According to (2), f is pi which means that there exists $s \in R$ with $0 \neq rs = (rs)^2$. Therefore $\mathrm{Tot}(R) = 0$ and R is a TOTO-ring.

Remark 3.6 *Let R be a TOTO-ring. Then R is left and right nonsingular ring.*

Proof. Let $0 \neq x \in R$. Since $\mathrm{Tot}(R) = 0$, there exists $y \in R$ such that $e = yx$ is a nonzero idempotent of R. Clearly $r_R(x) \subseteq r_R(yx) = (1-e)R$ and so $r_R(x) \cap eR = 0$. Therefore $r_R(x)$ is not an essential right ideal for any $0 \neq x \in R$ and whence R is right nonsingular. Analogously, R is left nonsingular.

Let W_R be a right TOTO-module. Then every simple submodule of W is projective by Theorem 1.2(iv) and so

$$\text{the socle } \mathrm{Soc}(W) \text{ of } W_R \text{ is projective.} \tag{3}$$

Theorem 3.7 *Let $M \in$ Mod-R. Then the following conditions are equivalent:*

(i) M is a right TOTO-module and every its cyclic submodule has fccds.

(ii) M is a projective completely reducible module.

In particular, if R is a TOTO-ring having fccds, then it is a semisimple Artinian ring.

Proof. (i)$\Longrightarrow$(ii) In view of (3) it is enough to show that $M = \mathrm{Soc}(M)$. To this end, pick any $0 \neq x \in M$ and set $L = xR$. Assume that $x \notin \mathrm{Soc}(M)$.

Then $x \notin \mathrm{Soc}(L)$. Let K be a submodule of L maximal with respect to the properties $x \notin K$ and $\mathrm{Soc}(L) \subseteq K$. If $K = 0$, then L is a simple module and so $x \in \mathrm{Soc}(M)$, a contradiction. Therefore $K \neq 0$. By assumption L has fccds and whence K contains a submodule N maximal with respect to the property $N \subseteq^{\oplus} L$. Suppose that $N = K$. Then $K \subseteq^{\oplus} L$ and so $K \oplus K' = L$ for some submodule K' of L. Since K is a maximal submodule of L, K' is simple, forcing $K' \subseteq \mathrm{Soc}(L) \subseteq K$, a contradiction. Therefore $N \subset K$. Choose $N'_R \subseteq L_R$ with $N \oplus N' = L$. By the modular law, $K = N \oplus (K \cap N')$. Clearly $K \cap N' \neq 0$.

Further, since M is a right TOTO-module, it follows from Corollary 3.2 that L is so. By Theorem 1.2(iv), there exists a nonzero submodule T of $K \cap N'$ with $T \subseteq^{\oplus} L$. It now follows from the modular law that $T \subseteq^{\oplus} N'$ and so $N \oplus T \subseteq^{\oplus} L$. Taking into account that $N \subset N \oplus T \subseteq K$, we get a contradiction with the choice of N. Therefore $M = \mathrm{soc}(M)$.

(ii)$\Longrightarrow$(i). Since every submodule of M is its direct summand, we conclude that each submodule of M is projective and so M is a right TOTO-module by Theorem 1.2(iv). The last statement is obvious.

The following result follows immediately from Theorem 3.7.

Corollary 3.8 *Let M be a right R-module. Then the following conditions are equivalent:*

(i) M is a direct sum of a finite number of projective simple modules.

(ii) M has fccds and is a right TOTO-module.

(iii) M has a finite Goldie dimension and is a right TOTO-module.

(iv) M is an Artinian (Noetherian) right TOTO-module.

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