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Regularity in Hom

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1. Introduction

Let R be a ring with $1 \in R$ and denote by M and N R-rightmodules. If $S := \mathrm{End}_R(N)$, $T := \mathrm{End}_R(M)$, then $\mathrm{Hom}_R(M,N)$ is a S-T-bimodule. Denote by U a S-T-submodule of $\mathrm{Hom}_R(M,N)$. Examples for U besides 0 are $\Delta(M,N)$, $\nabla(M,N)$ and RAD(M,N) (definitions later).

In the study of regularity properties of a ring, it is a technical tool, to consider two-sided ideals A of R and derive properties of R from properties of A and the factorring R/A. The similar procedure as in the ring case, that means to work with $\mathrm{Hom}_R(M,N)/U$, is not useful , since this is not any more a "Hom".But we would like, to work still with the good properties of homomorphisms, these are the kernel, the image and the product. Therefore we introduce the following definition.

1.1. Definition

$f \in \mathrm{Hom}_R(M,N)$ is called U-regular $:\Leftrightarrow$ there exist $g \in \mathrm{Hom}_R(N,M)$ and $u \in U$ such that

(1) $\qquad f = fgf + u$.

A subset of $\mathrm{Hom}_R(M,N)$ is called U-regular, if all of its elements are U-regular.

If $U = 0$, then we have the normal regularity. We intend to show, that U-regularity is a valuable notion for the study of regularity in Hom.

2. Largest U-regular submodule of Hom

It is well-known, that in a ring R, there exists a largest regular two-sided ideal A and R/A has no nonzero regular two-sided ideal. We intend to show, that this result is also true in our general situation and can even be extended to the category R-mod of all unitary R-rightmodules.

For $f \in Hom_R(M,N)$ we denote by $\langle f \rangle$ the S-T-submodule of $Hom_R(M,N)$ generated by f. Then we define

(2) $$Reg(U) := \{ f \in Hom_R(M,N) \mid \langle f \rangle \text{ is U-regular} \} .$$

First, we have some trivial remarks about Reg(U).

1.Remark: $U \subseteq Reg(U)$; since if $u \in U$, then $u = u0u + u$ with the zeromapping $0 \in Hom_R(N,M)$. Hence u is U-regular and since $\langle u \rangle \subseteq U$ also $\langle u \rangle$ is U-regular.

2.Remark: If U_1, U_2 are S-T-submodules of $Hom_R(M,N)$, then $U_1 \subseteq U_2$ implies $Reg(U_1) \subseteq Reg(U_2)$.

3.Remark: If f is U-regular with (1) and if $v \in U$, then also f+v is U-regular, since by (1) we have

$$f+v = (f+v)g(f+v) + u_1$$

with

$$u_1 = u+v-fgv-vgf-vgv \quad \in U.$$

This implies also, that if $\langle f \rangle$ is U-regular also $\langle f+v \rangle$ is U-regular.

Now we state our first theorem.

<u>2.1.Theorem</u>

Reg(U) is the largest U-regular S-T-submodule of $Hom_R(M,N)$ and

(3) $$Reg(Reg(U)) = Reg(U) \quad .$$

<u>Proof</u>. We give the proof in five steps.

1.Step. If $f \in Reg(U)$, $s \in S$, then $sf \in \langle f \rangle$ and hence $\langle sf \rangle \subseteq \langle f \rangle$. Therefore also $sf \in Reg(U)$. Similar also $ft \in Reg(U)$ for $t \in T$.

2.Step. We show now: If $f_1, f_2 \in Reg(U)$, then $f := f_1 + f_2 \in Reg(U)$. By assuption there exists $g_1 \in Hom_R(N,M)$, $u_1 \in U$ such that $f_1 = f_1 g_1 f_1 + u_1$ and this implies

(4) $$f - fg_1 f = f_1 - f_1 g_1 f_1 + f_2 - f_1 g_1 f_2 - f_2 g_1 f = u_1 + f_3$$

with

$$f_3 = f_2 - f_1 g_1 f_2 - f_2 g_1 f \in \langle f_2 \rangle$$

and than follows by (4)

(5) $$f_3 = f - fg_1 f - u_1 = f(1_T - g_1 f) - u_1 = (1_S - fg_1) f - u_1 \; .$$

Since $f_3 \in \langle f_2 \rangle$ it is U-regular, hence we have

(6) $$f_3 = f_3 g_3 f_3 + u_3 \quad , \quad u_3 \in U \; .$$

Then (4),(5) and (6) together imply

$$f = fg_1f+f_3+u_1 = fg_1f+f_3g_3f_3+u_1+u_3$$
$$= f(g_1+(1_T-g_1f)g_3(1_S-fg_1))f + u$$

and a computation shows $u \in U$. That means, that $f = f_1+f_2$ is U-regular.

3.Step. We prove now, that $\langle f_1+f_2\rangle$ is U-regular. For this, we consider an arbitrary element of $\langle f_1+f_2\rangle$:

$$\sum_{i=1}^{n} s_i(f_1+f_2)t_i = \sum_{i=1}^{n} s_if_1t_i + \sum_{i=1}^{n} s_if_2t_i \ , s_i \in S, t_i \in T.$$

Since the first sum on the right is in $\langle f_1\rangle$ and the second in $\langle f_2\rangle$, these sums are elements in Reg(U). Then by the 2.step the sum of these elements is U-regular. Therefore $f_1+f_2 \in$ Reg(U). Together we have proved, that Reg(U) is a U-regular S-T-submodule of $Hom_R(M,N)$.

4.Step. If V is also an U-regular S-T-submodule of $Hom_R(M,N)$, then for $h \in V$ also $\langle h\rangle \subseteq V$. But then by definition of Reg(U) $h \in$ Reg(U), hence $V \subseteq$ Reg(U).

5.Step. Still we have (3) to prove. Since Reg(U) is a S-T-submodule of $Hom_R(M,N)$ Reg(Reg(U)) is defined and since $U \subseteq$ Reg(U) it follows Reg(U) $\subseteq$ Reg(Reg(U)). Now we show, that every Reg(U)-regular element is also U-regular. Let f be Reg(U)-regular; then we have

(7) $\qquad f = fgf + w \quad , \quad w \in \text{Reg}(U)$

and for w exists an equation

(8) $\qquad w = whw + u \quad , \quad u \in U$.

By (7) we get

(9) $\qquad w = f(1_T-gf) = (1_S-fg)f$

and (7),(8) and (9) together imply

$$f = fgf + whw + u = f(g+(1_T-gf)h(1_S-fg))f + u ,$$

hence f is U-regular. Now, if $f \in$ Reg(Reg(U)), then $\langle f\rangle$ is Reg(U)-regular, hence also U-regular, hence $f \in$ Reg(U). □

We intend to give examples for Reg(U) and discuss Reg(U) in a special case. But first, we extend theorem 2.1. to the category R-mod.

3. The largest W-regular ideal in R-mod.

An ideal W in R-mod is defined by two conditions:

(Id 1): For arbitrary modules M,N of R-mod is given a subgroup W(M,N) of the additive group of $Hom_R(M,N)$.

(Id 2): For arbitrary modules M,N,X,Y of R-mod and arbitrary

$$f \in W(M,N) \;,\quad h \in Hom_R(X,M) \;,\quad k \in Hom_R(N,Y)$$

is $kfh \in W(X,Y)$

By this definition the ideal W is given by its "components" W(M,N) and therefore $W \cap Hom_R(M,N) = W(M,N)$. Examples for ideals besides the 0-ideal are $\triangle$, ∇ and RAD, for which we now give the definitions:

$$\triangle(M,N) := \{ f \in Hom_R(M,N) \mid \ker(f) \text{ is large in } M \} \;,$$

$$\nabla(M,N) := \{ f \in Hom_R(M,N) \mid \mathrm{ima}(f) \text{ is small in } N \}$$

$$RAD(M,N) := \text{Radical of } Hom_R(M,N) \quad .$$

Now, we come back to the general situation. For $f \in W$ we denote by $\langle\langle f \rangle\rangle$ the ideal in R-mod generated by f. We call $f \in Hom_R(M,N)$ W-regular, if there exist $g \in Hom_R(N,M)$ and $w \in W(M,N)$ such that $f = fgf + w$. Now we define

$$REG(W)(M,N) := \{ f \in Hom_R(M,N) \mid \langle\langle f \rangle\rangle \text{ is W-regular} \} \,.$$

Then $\langle f \rangle \subseteq \langle\langle f \rangle\rangle$ and therefore

$$REG(W)(M,N) \subseteq Reg(W(M,N)) \;.$$

Realise the difference in the writing ! Now we have the analoge theorem to 2.1.

3.1. Theorem.

If W is an ideal in R-mod, then REG(W) is the largest W-regular ideal in R-mod and

(10) $\mathrm{REG}(\mathrm{REG}(W)) = \mathrm{REG}(W)$.

Proof: We use the proof of 2.1. with some obvious modifications. In the steps 2.,3. and 5. only $\langle f \rangle$ has to be substituted by $\langle\langle f \rangle\rangle$. In the 1.step, we have to consider $s \in \mathrm{Hom}_R(N,Y)$, $t \in \mathrm{Hom}_R(X,M)$ for arbitrary modules X,Y. Similar in the 3,step the s_i resp. the t_i have to be in $\mathrm{Hom}_R(N,Y)$ resp. in $\mathrm{Hom}_R(X,M)$ and the sums have to be added in the sense of the 2.step in REG(W)(X,Y) .□

Connected with this result, there are many questions. What are the rings R such that

(11) REG(W) = R-mod

if $W = 0$, Δ, ∇, RAD ? It is obvious, that for a semi-simple ring R (11) is satisfied for W = 0 and hence for any ideal. Are there other rings, for which (11) is satisfied for any proper ideal of R-mod ? How about rings R such that REG(W) = W for one of the examples ? Does there exist for an arbitrary ring R a smallest (or minimal) ideal W such that (11) is true ?

Obviously, it is also possible, to study W-regularity in more general categories than R-mod .

4. Examples and special cases

First we give some results for Reg(U) by using continuity and discretness properties. We need the following conditions (compare |5|).

(C1;M) : Every submodule of M is large in a direct summand of M.

($C2_o$;M): If a submodule of M is isomorphic to M, then it is a direct summand of M.

(C2;M,N): If a submodule of N is isomorphic to a direct summand of M, then it is a direct summand of N.

If (C1;M) and (C2;M):= (C2;M,M) are satisfied, then M is called continuous.

4.1.Teorem

(i) If $(C2_o;M)$ is satisfied, then for every module N

(12) $\Delta(M,N) \subseteq RAD(M,N)$.

(ii) If (C1;M) and (C2;M,N) are satisfied, then

(13) $RAD(M,N) \subseteq \Delta(M,N)$

and

(14) $Reg(\Delta(M,N)) = Hom_R(M,N)$.

Proof:

(i): We consider $f \in \Delta(M,N)$ and an arbitrary $g \in Hom_R(N,M)$. For $x \in ker(f)$ follows $(1_T-gf)(x) = x$, hence

(15) $ker(f) \cap ker(1_T-gf) = 0$ and $ker(f) \subseteq ima(1_T-gf)$.

Since $ker(f) \subseteq {}^*M$, (15) implies

(16) $ker(1_T-gf) = 0$, $ima(1_T-gf) \subseteq {}^*M$.

Since 1_T-gf is a monomorphism, $ima(1_T-gf)$ is isomorphic to M and then by $(C2_o;M)$ it is a direct summand of M. Then (16) implies $ima(1_T-gf) = M$. Together, we see, that 1_T-gf is an automorphism, which means $f \in RAD(M,N)$.

(ii): Assume now $f \in Hom_R(M,N)$, $f \notin \Delta(M,N)$. Then there exists $0 \neq L \subseteq M$ such that $ker(f) \cap L = 0$. By (C1;M) there exists $D \subseteq^{\oplus} M$ with $L \subseteq^* D$; then also $ker(f) \cap D = 0$. If $\iota : D \longrightarrow M$ is the inclusion, then $f\iota : D \rightarrow M$ is a monomorphism. Then by (C2;M,N) $f\iota(D) = f(D)$ is a direct summand of N, hence $N = f(D) \oplus B$. Now, we define $g \in Hom_R(N,M)$ by

$$g(f(x)) := x \quad \text{for} \quad x \in D$$
$$g(b) := 0 \quad \text{for} \quad b \in B \ .$$

Then follows for $x \in D$

$$(1_T-gf)(x) = x - x = 0 \ ,$$

hence $0 \neq D \subseteq \ker(1_T - gf)$ and therefore $f \notin RAD(M,N)$, hence (13) is true. For the proof of (14) we assume $f \in Hom_R(M,N)$ and denote by D a complement of ker(f) in M such that

(17) $\quad \ker(f) \cap D = 0 \quad , \quad \ker(f) + D \subseteq^* M$.

By (C1;M) exists $D_1 \subseteq^{\oplus} M$, $D \subseteq {}^* D_1$. By (17) follows, that also $\ker(f) \cap D_1 = 0$ and this implies $D_1 = D$, since D was maximal with this property. Therefore $D \subseteq^{\oplus} M$. Then as before $N = f(D) \oplus B$ and $D \subseteq \ker(1_T - gf)$ (with the same g as before). Since $f - fgf = f(1_T - gf) = (1_S - fg)f$ we get by using (17)

$$\ker(f) + D \subseteq \ker(f-fgf) \subseteq {}^* M.$$

Then with $u := f - fgf \in \triangle(M,N)$ we have $f = fgf + u$. □

4.2. Corollary

If (C1;M),($C2_o$;M) and (C2;M,N) are satisfied, then

$$\triangle(M,N) = Rad(M,N).$$

A module M is called quasi-continuous, if (C1;M) and the following (C3;M) are satisfied.

(C3;M) : If A and B are direct summands of M with $A \cap B = 0$, then A+B is a direct summand of M.

A homomorphism $f \in Hom_R(M,N)$ is called partially invertible = pi if it is a factor of a nonzero regular element or, equivalent, there exists $g \in Hom_R(N,M)$ with $gf = (gf)^2 \neq 0$. (there are more equivalent conditions for pi). Then we need the total from M to N:

$$TOT(M,N) := \{ f \in Hom_R(M,N) \mid f \text{ is not pi} \} .$$

(For the properties of these notions see |1|...|4|)

4.3 Theorem.

Assume (C1;M),($C2_o$;M),(C2;M,N) and (C3;M) or (C1;M),($C2_o$;M),(C2;M,N) and (C1;N),(C3;N) then

$$\triangle(M,N) = RAD(M,N) = TOT(M,N) .$$

Before we give the proof, we would like to mention some special cases, in which the assumptions look less complicated. If M is injective, then (C1;M),($C2_o$;M),(C2;M,N) and (C3;M) are all satisfied (for arbitrary N !). If M=N , then the conditions above reduce to (C1;M) and (C2;M) (since (C3;M) follows from (C2;M)).

Proof of 4.3.

By 4.2. we have only $\Delta(M,N) = TOT(M,N)$ to show. Since always $\Delta(M,N) \subseteq TOT(M,N)$ holds , only the opposite inclusion is to prove. This we prove by contradiction. Assume $f \in TOT(M,N)$, $f \notin \Delta(M,N)$; then by 4.1. f is $\Delta(M,N)$-regular, hence

(18) $$f = fgf + u \quad , \quad u \in \Delta(M,N) .$$

Since $f \notin \Delta(M,N)$ also

(19) $$gf = (gf)^2 + gu \notin \Delta(M,N) \ , \ fg = (fg)^2 + ug \notin \Delta(M,N) .$$

Here, we used the fact, that Δ is an ideal in R-mod. Now we have also to use the fact, that TOT is a semi-ideal in R-mod. Since $f \in TOT(M,N)$, then also $gf \in TOT(M,M)$ (= TOT(T)) , $gf \in TOT(N,N)$ (= TOT(S)). Now we consider the images $\overline{gf}$ resp. $\overline{fg}$ in $T/\Delta(T)$ resp. $S/\Delta(S)$ (with $\Delta(T) := \Delta(M,M)$, $\Delta(S) := \Delta(N,N)$). Then by (19)

$$\overline{0} \neq \overline{gf} = (\overline{gf})^2 \quad , \quad \overline{0} \neq \overline{fg} = (\overline{fg})^2 .$$

Now, we need an assumption to be able to lift idempotents from $T/\Delta(T)$ to T or from $S/\Delta(S)$ to S. This is the case, if M or N is quasi-continuous ((C1) and (C3)). We consider the first case. Assume $e = e^2 \in T$, such that $\overline{e} = \overline{gf}$ ($\neq \overline{0}$). Then $e = gf + h$, $h \in \Delta(T)$ and by 4.2 also $h \in RAD(T)$. But then follows (|2|,3.3) $e = gf + h \in TOT(T)$, which is impossible for an idempotent $e \neq 0$. Contradiction ! The proof is similar in the second case. □

If we compare 4.1. (including 4.2) with 4.3., we have the following interesting situation. If $f \in Hom_R(M,N)$, $f \notin RAD(M,N)$, then by 4.1. and 4.2.

$$f = fgf + u \quad , \quad u \in RAD(M,N)$$

and by 4.3. there exists $h \in Hom_R(N,M)$ such that

$$hf =: e = e^2 \neq 0 \ .$$

Is there a connection between g and h ? In general: Is there a connection between RAD(M,N)-regular elements,which are not in RAD(M,N), and pi-elements ?

If we dualise the assumptions in 4.1.,4.2. and 4.3., then the dual results are true (For the dual conditions, called discretness conditions, see |5|).

We consider now the special case M = R. Then $Hom_R(R,N)$ is a S-R-bimodule and

$$\beta : Hom_R(M,N) \ni f \longmapsto f(1) \in N$$

is a S-R-isomorphism. If $U \subseteq_S Hom_R(M,N)_R$, then $B := \beta(U)$ is a S-R-submodule of N. Further $Hom_R(N,R) = N^*$ is the dual module of N, which is a R-S-bimodule. This situation with B = 0 was studied by J.Zelmanowitz (|6|).

By applying β on (1), it follows

$$f(1) = f(g(f(1))) + u(1) = f(1)g(f(1)) + u(1).$$

If we write $x := f(1)$, $b := u(1)$ and $gx := g(x)$, then we have the following regularity condition:

$$(21) \qquad x = xgx + b \quad , \quad x \in N \ , \ b \in B \ .$$

Now for $x \in N$ $\langle x \rangle$ is the S-R-submodule of N generated by x. Then

$$Reg(B) := \{ x \in N \mid \langle x \rangle \text{ is B-regular} \}$$

and by theorem 2.1. we know, that Reg(B) is the largest B-regular S-R-submodule of N and

$$Reg(Reg(B)) = Reg(B) \ .$$

Also, we will specialize 4.1.,4.2.and 4.3. on this case. In the assumptions (C1;M) ... we have now M = R, considered as R-rightmodule. We denote for $x \in N$

$$Ann(x) := \{ r \in R \mid xr = 0 \} .$$

Then

$$\beta\Delta(R,N) = \{ x \in N \mid Ann(x) \subseteq^* R_R \}$$

and

$$\beta RAD(R,N) = \{ x \in N \mid \exists\, g \in N \; [1-gx \text{ is a unite in } R] \} .$$

If $r \in R$, $g \in N^*$, then also $rg \in N^*$. Therefore if all $1-rgx$, $r \in R$ are units, then $gx \in Rad(R)$. Since for Rad(N) and all $g \in N^*$ we have $g(Rad(N)) \subseteq Rad(R)$, it follows

(22) $$Rad(N) \subseteq \beta(RAD(R,N)) .$$

Question: Under which conditions for R and N holds the equality in (22) ? If N is projective, then the equality is satisfied (as to see by using a dual basis). What about other conditions ?

Now, we consider the image $\beta(f)$ of a pi-homomorphism $f \in Hom_R(R,N)$. If f is pi, then there exists $g \in N^*$ such that $e := gf$ is an idempotent $\neq 0$. Therefore

$$\begin{aligned} gf(r) &= g(f(1)r) = g(f(1))r \\ &= (gfgf)(r) = gf(f(g(1))r) = g(f(1))g(f(1))r . \end{aligned}$$

By applying β, we get the following condition for $x \in N$ to be a pi-element:

$$x \text{ is pi} \iff \text{there exists } g \in N^* \text{ such that } g(x) \text{ is an idempotent } \neq 0 \text{ in } R.$$

Then

$$Tot(N) := \beta(TOT(R,N)) = \{ x \in N \mid x \text{ is not pi} \} .$$

Now, every one can translate the results 4.1.,4.2. and 4.3. and the dual results in this situation.

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