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On maximal immediate extensions of valued division algebras

Erwin Schörner

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Abstract. We show an extension theorem for strictly contracting bilinear mappings into a spherically complete valued vector space and we apply this result to prove that every maximal valued division algebra having the same characteristic as its residue division algebra is spherically complete.

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In his classical paper [5], Kaplansky proved that a valued field (K, v, Γ_0) is maximal if and only if any pseudoconvergent sequence has a pseudolimit in K; moreover, under the "Hypothesis A", (K, v, Γ_0) is isomorphic to a Hahn field of formal power series with a factor system. The equivalence of maximality and pseudocompleteness can also be shown for valued abelian groups (see [3], [9] and [11]) and certain classes of valued modules (see [6]). It is still an open question for valued skewfields; in this context, Brungs and Törner gave an example of a maximal right chain ring which is not pseudocomplete (see [2]).

The purpose of this paper is to state the positive result for a valued division algebra in the sense of Zelinsky [16] having the same characteristic as its residue division algebra. This is a generalization of [15, Satz 5] where we give criteria for the embeddability of a valued division algebra into an appropriate Hahn division algebra. Here, we also rely on these Hahn division algebras of formal power series constructed and studied in [12] and [13], but we have to modify their multiplication applying an extension theorem for strictly contracting mappings into a spherically complete valued vector space.

Omitting the algebraic structure of the objects mentioned above, we obtain an ultrametric space (with a totally ordered value set). The theory of ultrametric spaces (even with partially ordered value set) was developed by Prieß-Crampe and Ribenboim in their papers [8], [9] and [10]; for the convenience of the reader we will recall the main results of this theory we are going to make use of in the sequel.

Let X be a set, and let $(\Gamma, \leq)$ be a (totally) ordered set and $\Gamma_0 = \Gamma \cup \{0\}$ with $0 < \gamma$ for all $\gamma \in \Gamma$. A mapping $d : X \times X \to \Gamma_0$ is called an *ultrametric distance*, if the following conditions are satisfied for all x, y and $z \in X$:

- $d(x, y) = 0 \Leftrightarrow x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq \mathrm{Max}\{d(x, y), d(y, z)\}$.

In this situation, (X, d, Γ_0) is called an *ultrametric space*. For $x, y, z \in X$ with $d(x, y) \neq d(y, z)$ we even have $d(x, z) = \mathrm{Max}\{d(x, y), d(y, z)\}$.

An equivalence relation σ on X is called *d-compatible*, if for all $x, x', y, y' \in X$ with $x\sigma y$ and $d(x', y') \leq d(x, y)$ we also have $x'\sigma y'$. The set $\equiv(X)$ of all d-compatible equivalence relations on X is a complete totally ordered set with respect to $\subseteq$. The most important examples are $\equiv_\gamma$ and $\equiv_\gamma^-$ for a $\gamma \in \Gamma$ with $x \equiv_\gamma y \Leftrightarrow d(x, y) \leq \gamma$ and $x \equiv_\gamma^- y \Leftrightarrow d(x, y) < \gamma$ for all $x, y \in X$, respectively. The equivalence classes of $\equiv_\gamma$ and $\equiv_\gamma^-$ are precisely the *balls* $X^\gamma(x) = \{y \in X \mid d(x, y) \leq \gamma\}$ and $X_\gamma(x) = \{y \in X \mid d(x, y) < \gamma\}$ with *centre x* and *radius* γ, respectively. Any set of pairwise non-disjoint balls is a chain with respect to $\subseteq$.

By [7], the following completeness properties are equivalent:

- (X, d, Γ_0) is *spherically complete:* any chain of balls $X^\gamma(x)$ with $x \in X$ and $\gamma \in \Gamma$ has a non-empty intersection.
- (X, d, Γ_0) is *pseudocomplete:* any pseudoconvergent sequence has a pseudolimit in X.
- (X, d, Γ_0) satisfies the ultrametric *Banach's Fixed Point Theorem:* any strictly contracting mapping $f : X \to X$, i.e., $d(f(x), f(y)) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$, has a fixed point in X.

Analyzing the proof of [7, Satz 1], we realize that in a spherically complete ultrametric space any chain of balls has a non-empty intersection.

A subset U of X endowed with the restriction $d = d|_{U \times U}$ of d to U is again an ultrametric space. The extension $(U, d, \Gamma_0) \prec (X, d, \Gamma_0)$ is said to be *immediate*, if $d(U \times U) = d(X \times X)$ holds and if for all $u \in U$ and $x \in X$ with $u \neq x$ there exists $u' \in U$ with $d(u', x) < d(u, x)$. An ultrametric space without any proper immediate extension is called *maximal*. By [10, Theorem 7.9] and [11, Theorem 2.3], an ultrametric space is spherically complete if and only if it is maximal.

The most important examples of spherically complete ultrametric spaces are the Hahn spaces of formal power series; in this paper, we only consider a special case. Let Γ_0 be a totally ordered set as above, and let M be a set with at least two elements and $0 \in M$. The Hahn space $\mathbf{H} = (\mathbf{H}, d_{\mathbf{H}}, \Gamma_0)$ consists of all mappings $\mathbf{f} : \Gamma \to M$ with dually well-ordered support $\mathrm{supp}(\mathbf{f}) = \{\gamma \in \Gamma \mid \mathbf{f}(\gamma) \neq 0\}$, i.e., $\mathrm{supp}(\mathbf{f})$ is well-ordered with respect to the opposite order, and carries the ultrametric distance

$$d_{\mathbf{H}}(\mathbf{f}, \mathbf{g}) = \begin{cases} \mathrm{Max}\{\gamma \in \Gamma \mid \mathbf{f}(\gamma) \neq \mathbf{g}(\gamma)\}, & \text{if } \mathbf{f} \neq \mathbf{g} \\ 0, & \text{if } \mathbf{f} = \mathbf{g} \end{cases}.$$

Usually, the formal power series $\mathbf{f} \in \mathbf{H}$ is symbolized by $\sum_{\gamma \in \Gamma} \mathbf{f}(\gamma) t^\gamma$ using the indeterminate t; thus, mt^γ represents the element of $\mathbf{H}$ with

$$(mt^{\gamma})(\gamma') = \begin{cases} m, & \text{if } \gamma' = \gamma \\ 0, & \text{if } \gamma' \neq \gamma \end{cases}$$

for $m \in M$ and $\gamma \in \Gamma$.

We use the same definition of a valued group and a valued field as [9]; for the convenience of the reader, we recall the notion of a valued division algebra. We consider a division algebra $(N, +, \cdot)$, i.e.,

- $(N, +)$ is an abelian group with neutral element 0,
- $(N^*, \cdot)$ with $N^* = N \backslash \{0\}$ is a loop with neutral element 1,
- $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for all $a, b, c \in N$.

Hence, a division algebra with an associative multiplication is a skewfield. Let Γ be endowed with a multiplication $\cdot$ such that $(\Gamma, \cdot, \varepsilon, \leq)$ becomes a totally ordered loop with neutral element ε; we extend the multiplication to $\Gamma_0 \times \Gamma_0 \to \Gamma_0$ by putting $\gamma \cdot 0 = 0$ and $0 \cdot \gamma = 0$ for all $\gamma \in \Gamma_0$. A mapping $v : N \to \Gamma_0$ is called a *valuation*, if the following conditions are satisfied for all x and $y \in N$:

- $v(x) = 0 \Leftrightarrow x = 0$.
- $v(x \cdot y) = v(x) \cdot v(y)$.
- $v(x + y) \leq \text{Max}\{v(x), v(y)\}$.

In this situation, (N, v, Γ_0) is called a *valued division algebra*. This can be regarded as a special case of the concept of a uniformly valued ternary field developed by Kalhoff in [4]; we should mention that Γ carries the dual order $\leq_d$ in [16], i.e., for all $\gamma, \gamma' \in \Gamma$ we have $\gamma \leq_d \gamma'$ if and only if $\gamma' \leq \gamma$ holds.

For all these algebraic structures, the valuation v induces an ultrametric distance d_v by $d_v(x, y) = v(x - y)$ for all x and y.

In the following, we transfer the general construction of a Hahn ternary field of formal power series given in [12] to our special situation. Therefore, we consider a division algebra K and a totally ordered loop $(\Gamma, \cdot, \varepsilon, \leq)$. For all $\alpha, \beta \in \Gamma$ let $\mu_{\alpha,\beta} : K \times K \to K$ be a biadditive mapping satisfying the following conditions:

- For all $m, b \in K$ with $m \neq 0$ there is a unique $x \in K$ with $\mu_{\alpha,\beta}(m, x) = b$.
- For all $x, b \in K$ with $x \neq 0$ there is $m \in K$ with $\mu_{\alpha,\beta}(m, x) = b$.
- $\mu_{\varepsilon,\varepsilon}(m, x) = m \cdot x$, $\mu_{\alpha,\varepsilon}(m, 1) = m$ and $\mu_{\varepsilon,\beta}(1, x) = x$ hold for all $m, x \in K$.

In this case, the family $(\mu_{\alpha,\beta})_{\alpha,\beta \in \Gamma}$ is called a *factor system* with respect to K and Γ. We define addition and multiplication on the set $\mathbf{H}$ by putting

$$(\mathbf{f} + \mathbf{g})(\gamma) = \mathbf{f}(\gamma) + \mathbf{g}(\gamma) \quad \text{and} \quad (\mathbf{f} \cdot \mathbf{g})(\gamma) = \sum_{\alpha \cdot \beta = \gamma} \mu_{\alpha,\beta}(\mathbf{f}(\alpha), \mathbf{g}(\beta))$$

for all $\mathbf{f}, \mathbf{g} \in \mathbf{H}$ and $\gamma \in \Gamma$. By [12, Satz 6] and [13, Satz 3 and Satz 4], $(\mathbf{H}, +, \cdot)$ is a division algebra with $\mathbf{0} = 0t^{\varepsilon}$ and $\mathbf{1} = 1t^{\varepsilon}$; moreover, $(\mathbf{H}, v_{\mathbf{H}}, \Gamma_0)$ is a valued division algebra with $v_{\mathbf{H}}(\mathbf{f}) = d_{\mathbf{H}}(\mathbf{f}, \mathbf{0})$ for all $\mathbf{f} \in \mathbf{H}$.

First, we want to state and prove the extension theorem for bilinear mappings already mentioned above. Therefore, we need a more general concept of a valued vector space than that given in [17]. We consider a valued field (K, v, Γ_0) and a vector space V over K endowed with a group valuation $\|\cdot\| : V \to \Gamma_0^V$. Let $\cdot : \Gamma \times \Gamma^V \to \Gamma^V$ be an operation of the group Γ on the set Γ^V such that

$$\gamma' < \gamma'' \Rightarrow \gamma' \cdot \gamma_V < \gamma'' \cdot \gamma_V \quad \text{and} \quad \gamma'_V < \gamma''_V \Rightarrow \gamma \cdot \gamma'_V < \gamma \cdot \gamma''_V$$

hold for all $\gamma, \gamma', \gamma'' \in \Gamma$ and $\gamma_V, \gamma'_V, \gamma''_V \in \Gamma^V$; we extend this operation to $\Gamma_0 \times \Gamma_0^V \to \Gamma_0^V$ by putting $\gamma \cdot 0 = 0$ for all $\gamma \in \Gamma_0$ and $0 \cdot \gamma_V = 0$ for all $\gamma_V \in \Gamma_0^V$. In this situation, we call $(V, \|\cdot\|)$ a *valued vector space over* (K, v), if

$$\|\lambda x\| = v(\lambda) \cdot \|x\|$$

is satisfied for all $\lambda \in K$ and $x \in V$. We wish to remark that further assumptions are necessary to ensure that V endowed with the topology $\mathfrak{T}_{\|\cdot\|}$ induced by the valuation $\|\cdot\|$ is a topological vector space over the topological field $(K, \mathfrak{T}_v)$, where $\mathfrak{T}_v$ is given by the valuation v. For example, it is sufficient to ask that $\Gamma \cdot \gamma_V$ is a coinitial subset of Γ^V for all $\gamma_V \in \Gamma^V$.

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be valued vector spaces over the valued field (K, v, Γ_0). A mapping $\circ : \Gamma^X \times \Gamma^Y \to \Gamma^Z$ is called *value-multiplication*, if the properties

$$\gamma'_X < \gamma''_X \Rightarrow \gamma'_X \circ \gamma_Y < \gamma''_X \circ \gamma_Y \quad \text{and} \quad \gamma'_Y < \gamma''_Y \Rightarrow \gamma_X \circ \gamma'_Y < \gamma_X \circ \gamma''_Y$$

and

$$\gamma \cdot (\gamma_X \circ \gamma_Y) = (\gamma \cdot \gamma_X) \circ \gamma_Y = \gamma_X \circ (\gamma \cdot \gamma_Y)$$

hold for all $\gamma \in \Gamma$, $\gamma_X, \gamma'_X, \gamma''_X \in \Gamma^X$ and $\gamma_Y, \gamma'_Y, \gamma''_Y \in \Gamma^Y$; again, we extend this value-multiplication to $\Gamma_0^X \times \Gamma_0^Y \to \Gamma_0^Z$ by defining $\gamma_X \circ 0 = 0$ for all $\gamma_X \in \Gamma_0^X$ and $0 \circ \gamma_Y = 0$ for all $\gamma_Y \in \Gamma_0^Y$. Then, a bilinear mapping $f : X \times Y \to Z$ is called *strictly contracting with respect to the value-multiplication* $\circ$, if

$$\|f(x, y)\|_Z < \|x\|_X \circ \|y\|_Y$$

is satisfied for all $0 \neq x \in X$ and $0 \neq y \in Y$.

Theorem 1. *Let $(X, \|\cdot\|_X, \Gamma_0^X)$, $(Y, \|\cdot\|_Y, \Gamma_0^Y)$ and $(Z, \|\cdot\|_Z, \Gamma_0^Z)$ be valued vector spaces over the valued field (K, v, Γ_0) and $\circ : \Gamma^X \times \Gamma^Y \to \Gamma^Z$ a value-multiplication. Let U be a linear subspace of X and $f : U \times Y \to Z$ a bilinear mapping which is strictly contracting with respect to $\circ$; let V be a linear subspace of Y with $f(x, y) = 0$ for all $x \in U$ and $y \in V$. If $(Z, d_{\|\cdot\|_Z}, \Gamma_0^Z)$ is spherically complete, then f extends to a bilinear mapping $F : X \times Y \to Z$ which is strictly contracting with respect to $\circ$ and satisfies $F(x, y) = 0$ for all $x \in X$ and $y \in V$.*

Proof. Let $\mathfrak{M}$ be the set of all pairs (U', f') where U' is a linear subspace of X containing U and $f' : U' \times Y \to Z$ is a bilinear extension of f which is strictly contracting with respect to $\circ$ and satisfies $f'(x, y) = 0$ for all $x \in U'$ and $y \in V$. Since $(U, f) \in \mathfrak{M}$, the set $\mathfrak{M}$ is non-empty. Moreover, $\mathfrak{M}$ is inductively ordered by

$$(U', f') \leq (U'', f'') \Leftrightarrow U' \subseteq U'' \quad \text{and} \quad f''|_{U' \times Y} = f'$$

and therefore contains a maximal element (U_0, f_0) by Zorn's lemma. In the following, we show that the assumption $U_0 \subsetneq X$ yields a contradiction.

Let $s \in X \setminus U_0$ and $U_1 = U_0 \oplus Ks \subseteq X$. For all $u \in U_0$ we define $\pi_u = \|s - u\|_X \in \Gamma^X$.

Let $\mathfrak{N}$ be the set of all pairs (V', g') where V' is a linear subspace of Y containing V and $g' : V' \to Z$ is a linear mapping with

$$g'(y) \in Z_{\pi_u \circ \|y\|_Y}(f_0(u, y)) \quad \text{for all } u \in U_0 \quad \text{and} \quad 0 \neq y \in V'$$

and $g'(y) = 0$ for all $y \in V$. By $(V, 0) \in \mathfrak{N}$, the set $\mathfrak{N}$ is non-empty. Moreover, $\mathfrak{N}$ is inductively ordered by

$$(V', g') \leq (V'', g'') \Leftrightarrow V' \subseteq V'' \quad \text{and} \quad g''|_{V'} = g'$$

and therefore contains a maximal element (V_0, g_0) by Zorn's lemma. We now show that the assumption $V_0 \subsetneq Y$ is absurd.

Let $t \in Y \setminus V_0$ and $V_1 = V_0 \oplus Kt \subseteq Y$. For $u \in U_0$ and $y \in V_0$ we define the ball

$$B_{u,y} = Z_{\pi_u \circ \|y + t\|_Y}(f_0(u, y + t) - g_0(y))$$

and we show that the intersection of any two of these balls is non-empty. To this end, let $u, u' \in U_0$ with $\pi_u \leq \pi_{u'}$ and $y, y' \in V_0$. In the case $u \neq u'$ and $y = y'$ we have

$$\begin{aligned} &\|(f_0(u, y + t) - g_0(y)) - (f_0(u', y + t) - g_0(y))\|_Z \\ &\quad = \|f_0(u - u', y + t)\|_Z < \|u - u'\|_X \circ \|y + t\|_Y \leq \pi_{u'} \circ \|y + t\|_Y \end{aligned}$$

and $B_{u,y} \subseteq B_{u',y}$. In the case $u = u'$ and $y \neq y'$ we have

$$\begin{aligned} &\|(f_0(u, y + t) - g_0(y)) - (f_0(u, y' + t) - g_0(y'))\|_Z \\ &\quad = \|f_0(u, y - y') - g_0(y - y')\|_Z < \pi_u \circ \|y - y'\|_Y \\ &\quad \leq \operatorname{Max}\{\pi_u \circ \|y + t\|_Y, \pi_u \circ \|y' + t\|_Y\} \end{aligned}$$

and $B_{u,y'} \subseteq B_{u,y}$ or $B_{u,y} \subseteq B_{u,y'}$. Finally, in the case $u \neq u'$ and $y \neq y'$ we obtain

$$\begin{aligned}
&\|(f_0(u,y+t)-g_0(y))-(f_0(u',y'+t)-g_0(y'))\|_Z\\
&\quad=\|(f_0(u,y-y')-g_0(y-y'))+f_0(u-u',y'+t)\|_Z\\
&\quad\le \mathrm{Max}\{\|(f_0(u,y-y')-g_0(y-y'))\|_Z,\|f_0(u-u',y'+t)\|_Z\}\\
&\quad< \mathrm{Max}\{\pi_u\circ\|y-y'\|_Y,\|u-u'\|_X\circ\|y'+t\|_Y\}\\
&\quad\le \mathrm{Max}\{\pi_u\circ\|y+t\|_Y,\pi_{u'}\circ\|y'+t\|_Y\}
\end{aligned}$$

and $B_{u',y'}\subseteq B_{u,y}$ or $B_{u,y}\subseteq B_{u',y'}$. Since $(Z,\|\cdot\|_Z,\Gamma_0^Z)$ is spherically complete, there exists

$$g_1(t)\in\bigcap_{u\in U_0}\bigcap_{y\in V_0}B_{u,y}.$$

Therefore, $g_1:V_1\to Z$ is a linear mapping with $g_1|_{V_0}=g_0$ and satisfying

$$\begin{aligned}
&\|f_0(u,y+\lambda t)-g_1(y+\lambda t)\|_Z\\
&\quad=v(\lambda)\cdot\|(f_0(u,\lambda^{-1}y+t)-g_0(\lambda^{-1}y))-g_1(t)\|_Z\\
&\quad<v(\lambda)\cdot(\pi_u\circ\|\lambda^{-1}y+t\|_Y)=\pi_u\circ(v(\lambda)\cdot\|\lambda^{-1}y+t\|_Y)=\pi_u\circ\|y+\lambda t\|_Y
\end{aligned}$$

for all $u\in U_0$, $y\in V_0$ and $0\neq\lambda\in K$. So we have obtained $(V_1,g_1)\in\mathfrak{N}$ with $(V_0,g_0)<(V_1,g_1)$, which is a contradiction.

Consequently, $V_0=Y$ holds and $g_0:Y\to Z$ is a linear mapping with

$$g_0(y)\in Z_{\pi_u\circ\|y\|_Y}(f_0(u,y))\quad\text{for all } u\in U_0\quad\text{and}\quad 0\neq y\in Y$$

and $g_0(y)=0$ for all $y\in V$. We define

$$f_1(x+\lambda s,y):=f_0(x,y)+\lambda g_0(y)$$

for all $x\in U_0$, $\lambda\in K$ and $y\in Y$; hence, $f_1:U_1\times Y\to Z$ is a bilinear mapping with $f_1|_{U_0}=f_0$ and

$$f_1(x+\lambda s,y)=f_0(x,y)+\lambda g_0(y)=0$$

for all $x\in U_0$, $\lambda\in K$ and $y\in V$. Furthermore, for all $x\in U_0$, $0\neq\lambda\in K$ and $0\neq y\in Y$ we have

$$\begin{aligned}
\|f_1(x+\lambda s,y)\|_Z&=v(\lambda)\cdot\|f_0(-\lambda^{-1}x,y)-g_0(y)\|_Z<v(\lambda)\cdot(\pi_{-\lambda^{-1}x}\circ\|y\|_Y)\\
&=(v(\lambda)\cdot\|-\lambda^{-1}x-s\|_X)\circ\|y\|_Y=\|x+\lambda s\|_X\circ\|y\|_Y.
\end{aligned}$$

Thus, we have $(U_1, f_1) \in \mathfrak{M}$ with $(U_0, f_0) < (U_1, f_1)$ contradicting the maximality of (U_0, f_0).

Consequently, $U_0 = X$ and we can define $F = f_0$. □

Due to the necessity of modifying the multiplication of a Hahn division algebra of formal power series in an appropriate way, we need the following result. For an additive structure $(G, +)$, we call a mapping $f : G \times G \to G$ biadditive, if the equations

$$f(g, h' + h'') = f(g, h') + f(g, h'') \quad \text{and} \quad f(g' + g'', h) = f(g', h) + f(g'', h)$$

hold for all $g, g', g'', h, h', h'' \in G$.

Theorem 2. *Let $(L, +, \cdot)$ be a division algebra endowed with a spherically complete valuation $v : L \to \Gamma_0$, and let $\varphi : L \times L \to L$ be a biadditive mapping satisfying*

- *$\varphi(m, 1) = 0$ and $\varphi(1, x) = 0$ for all $m, x \in L$,*
- *$v(\varphi(m, x)) < v(m) \cdot v(x)$ for all $m, x \in L \backslash \{0\}$.*

Then $(L, +, *)$ with $m * x = m \cdot x + \varphi(m, x)$ for all $m, x \in L$ is again a division algebra endowed with the (spherically complete) valuation v.

Proof. For all $m, n, x, u \in L$, we have

$$(m + n) * x = (m + n) \cdot x + \varphi(m + n, x) = m \cdot x + n \cdot x + \varphi(m, x) + \varphi(n, x)$$

$$= (m \cdot x + \varphi(m, x)) + (n \cdot x + \varphi(n, x)) = m * x + n * x$$

and

$$1 * x = 1 \cdot x + \varphi(1, x) = x$$

and in an analogous way also

$$m * (x + u) = m * x + m * u \quad \text{and} \quad m * 1 = m;$$

thus, it follows $0 * x = 0$ and $m * 0 = 0$.

For all $m, x \in L \backslash \{0\}$ we have $v(\varphi(m, x)) < v(m \cdot x)$ and therefore

$$v(m * x) = v(m \cdot x + \varphi(m, x)) = v(m \cdot x) = v(m) \cdot v(x).$$

Finally, let $m, b \in L$ with $m \neq 0$. Since $(L, +, \cdot)$ is a division algebra, there exists $f(x) \in L$ such that

$$m \cdot f(x) + \varphi(m, x) = b.$$

For all $x, y \in L$ with $x \neq y$ we have

$$m \cdot f(x) - m \cdot f(y) = (b - \varphi(m, x)) - (b - \varphi(m, y)) = \varphi(m, y - x)$$

and therefore

$$v(m) \cdot v(f(x) - f(y)) = v(\varphi(m, y - x)) < v(m) \cdot v(x - y);$$

hence, $f : L \to L$ is a strictly contracting mapping of the spherically complete ultrametric space (L, d_v, Γ_0). By the ultrametric Banach's Fixed Point Theorem [7, Satz 2], there exists exactly one $x_0 \in L$ such that $f(x_0) = x_0$, and we obtain

$$m * x_0 = m \cdot x_0 + \varphi(m, x_0) = b.$$

Similarly one proves that for all $x, b \in L$ with $x \neq 0$ there exists a unique $m_0 \in L$ with $m_0 * x = b$. □

In the sequel, we consider a valuation $v : N \to \Gamma_0$ of the division algebra $(N, +, \cdot)$ with value loop $v(N^*) = \Gamma$, and we assume that N has the same characteristic as its residue division algebra $N_v = A_v/M_v$ with $A_v = N^\varepsilon(0)$ and $M_v = N_\varepsilon(0)$, i.e., N and N_v have the same prime field P.

The division algebras N and N_v as well as the subgroups A_v and M_v of N can be regarded as P-linear spaces, and the canonical mapping

$$v : A_v \to N_v, \quad x \mapsto x + M_v$$

is a P-epimorphism. So there exists a P-linear subspace K of A_v containing P such that $v|_K : K \to N_v$ is a P-isomorphism. Therefore, K is a system of representatives of the equivalence relation $\equiv_\varepsilon^-$ in A_v, i.e., for all $x \in A_v$ there is a unique $k \in K$ with $v(x - k) < \varepsilon$.

Let $(\mathbf{H}, d_{\mathbf{H}}, \Gamma_0)$ be the Hahn space of formal power series $\mathbf{f} : \Gamma \to K$ with dually well-ordered support $\operatorname{supp}(\mathbf{f}) = \{\gamma \in \Gamma \mid \mathbf{f}(\gamma) \neq 0\}$. For all $\gamma \in \Gamma$ we choose elements $u^\gamma \in N$ with $v(u^\gamma) = \gamma$ and $u^\varepsilon = 1$.

For all d_v-compatible equivalence relations $\sigma \in \equiv(N)$, the equivalence class

$$V_\sigma = [0]_\sigma = \{x \in N \mid 0\sigma x\}$$

of 0 with respect to σ is a P-linear subspace of N, and we have $V_\sigma \subseteq V_\tau$ for all $\sigma, \tau \in \equiv(N)$ with $\sigma \subseteq \tau$.

Let U be the P-linear subspace of N generated by $\{k \cdot u^\gamma \mid k \in K \text{ and } \gamma \in \Gamma\}$; for all $\sigma \in \equiv(N)$, the P-linear subspace U_σ of N generated by $\{k \cdot u^\gamma \mid k \in K$ and $\gamma \in \Gamma$ with $\sigma \subsetneq \equiv_\gamma\}$ is a P-linear complement of $V_\sigma \cap U$ in U, and $U_\tau \subseteq U_\sigma$ holds for all $\sigma, \tau \in \equiv(N)$ with $\sigma \subseteq \tau$.

Then, according to Banaschewskis proof of [1, Lemma 4], there exists a family $(\zeta(V_\sigma))_{\sigma \in \equiv(N)}$ of P-linear subspaces of N with

$$N = V_\sigma \oplus \zeta(V_\sigma) \quad \text{for all } \sigma \in \equiv(N),$$

$$K \cdot u^\gamma \subseteq \zeta(V_\sigma) \quad \text{for all } \gamma \in \Gamma \quad \text{and} \quad \sigma \in \equiv(N) \quad \text{with } \sigma \subseteq \equiv_\mu^-$$

and

$$\zeta(V_\tau) \subseteq \zeta(V_\sigma) \quad \text{for all } \sigma, \tau \in \equiv(N) \quad \text{with } \sigma \subseteq \tau.$$

In particular, for all $x \in N$ there exist unique elements $x_\sigma \in V_\sigma$ and $x_\sigma^\zeta \in \zeta(V_\sigma)$ with $x = x_\sigma + x_\sigma^\zeta$.

To define a distance-preserving mapping $\theta : (N, d_v, \Gamma_0) \to (\mathbf{H}, d_{\mathbf{H}}, \Gamma_0)$, let $x \in N$ and $\gamma \in \Gamma$. Since K is a system of representatives of $\equiv_\varepsilon^-$ in A_v, we have $V_{\equiv_\gamma} = V_{\equiv_\gamma^-} \oplus K \cdot u^\gamma$ and therefore

$$N = V_{\equiv_\gamma^-} \oplus K \cdot u^\gamma \oplus \zeta(V_{\equiv_\gamma});$$

thus there is a unique representation

$$x = x_{\equiv_\gamma^-} + \widehat{x}_\gamma \cdot u^\gamma + x_{\equiv_\gamma}^\zeta$$

with $x_{\equiv_\gamma^-} \in V_{\equiv_\gamma^-}$, $\widehat{x}_\gamma \in K$ and $x_{\equiv_\gamma}^\zeta \in \zeta(V_{\equiv_\gamma})$. By putting $\theta(x)(\gamma) = \widehat{x}_\gamma$ we define a mapping $\theta(x) : \Gamma \to K$ with dually well-ordered support. Indeed, suppose there exists a strictly increasing sequence $(\gamma_n)_{n \in \mathbb{N}}$ in the support of $\theta(x)$. Then $\sigma = \bigcup_{n \in \mathbb{N}} \equiv_{\gamma_n}$ is a d_v-compatible equivalence relation, and we obtain

$$x - x_\sigma = x_\sigma^\zeta \in \zeta(V_\sigma) \subseteq \zeta(V_{\equiv_{\gamma_n}}),$$

hence $x_{\equiv_{\gamma_n}} = (x_\sigma)_{\equiv_{\gamma_n}}$ for all $n \in \mathbb{N}$. Since $x_\sigma \in V_\sigma$, there is $n_0 \in \mathbb{N}$ with $x_\sigma \in V_{\equiv_{\gamma_{n_0}}^-}$, which yields

$$x_{\equiv_{\gamma_{n_0}}} = (x_\sigma)_{\equiv_{\gamma_{n_0}}} = x_\sigma \in V_{\equiv_{\gamma_{n_0}}^-}$$

and therefore $\theta(x)(\gamma_{n_0}) = 0$, a contradiction to $\gamma_{n_0} \in \operatorname{supp}(\theta(x))$.

Consequently, the mapping $\theta : N \to \mathbf{H}$ is well-defined, and we observe that

$$d_v(x, y) = d_{\mathbf{H}}(\theta(x), \theta(y)) \quad \text{for all } x, y \in N.$$

In particular, this implies that θ is injective. Since we have

$$\theta(k \cdot u^\gamma) = kt^\gamma \quad \text{for all } k \in K \quad \text{and} \quad \gamma \in \Gamma,$$

$(\theta(N), d_{\mathbf{H}}, \Gamma_0) \prec (\mathbf{H}, d_{\mathbf{H}}, \Gamma_0)$ is an immediate extension of ultrametric spaces.

Next, we define addition and multiplication on $\mathbf{H}$, such that $(\mathbf{H}, v_{\mathbf{H}}, \Gamma_0)$ becomes a valued division algebra. Hereby, we rely on the construction of a Hahn division algebra presented above.

First, we have to endow K with a multiplication $\circ$ such that $(K, +, \circ)$ becomes a division algebra. Since $K \subseteq A_v = K \oplus M_v$ holds, for all $m, x \in K$ we obtain unique elements $m \circ x \in K$ and $r \in M_v$ such that

$$m \cdot x = m \circ x + r$$

is satisfied. Thus, $(K, +, \circ)$ is a not necessarily associative ring with unit 1.

Let $m, b \in K$ with $m \neq 0$. Since N is a division algebra, there exists $h \in N$ with

$$m \cdot h = b,$$

and since $h \in A_v$ there are $x \in K$ and $s \in M_v$ with $h = x + s$. Then, by definition of $\circ$, there is $r \in M_v$ with $m \cdot x = m \circ x + r$, which yields

$$K \ni b - m \circ x = m \cdot h - m \cdot x + r = m \cdot s + r \in M_v$$

and therefore

$$m \circ x = b.$$

For all $y \in K$ with $m \circ y = b$ it follows $m \circ (x - y) = 0$, hence $x = y$. In a similar way we obtain that for all $x, b \in K$ with $x \neq 0$ there is a unique $m \in K$ such that $m \circ x = b$ holds. Thus, $(K, +, \circ)$ is a division algebra.

For all $\alpha, \beta \in \Gamma$ and $m, x \in K$ there exist unique elements $\mu_{\alpha,\beta}(m, x) \in K$ and $r \in V_{\equiv_{\alpha\beta}^{-}}$ such that

$$(m \cdot u^{\alpha}) \cdot (x \cdot u^{\beta}) = \mu_{\alpha,\beta}(m, x) \cdot u^{\alpha\beta} + r$$

holds. With the same arguments as above one proves that the family $(\mu_{\alpha,\beta})_{\alpha,\beta \in \Gamma}$ of mappings $\mu_{\alpha,\beta} : K \times K \to K$ is a factor system with respect to K and Γ. We now endow $\mathbf{H}$ with the corresponding division algebra structure and with the spherically complete valuation $v_{\mathbf{H}}$.

By construction, θ is P-linear, and $v_{\mathbf{H}}(\theta(x)) = v(x)$ holds for all $x \in N$. Moreover, $(\theta(N), v_{\mathbf{H}}, \Gamma_0)$ and $(\mathbf{H}, v_{\mathbf{H}}, \Gamma_0)$ can be regarded as valued vector spaces over the (trivially valued) field P. The mapping

$$\varphi : \theta(N) \times \theta(N) \ni (\theta(m), \theta(x)) \mapsto \theta(m \cdot x) - \theta(m) \cdot \theta(x) \in \mathbf{H}$$

is P-bilinear, and by $\theta(1) = \mathbf{1}$ we have $\varphi(\theta(m), \mathbf{1}) = \mathbf{0}$ and $\varphi(\mathbf{1}, \theta(x)) = \mathbf{0}$ for all $m, x \in N$.

Furthermore, for all $m, x \in N$ with $v(m) = \alpha$ and $v(x) = \beta$ we have unique representations

$$m = m_{\alpha} \cdot u^{\alpha} + m', \quad x = x_{\beta} \cdot u^{\beta} + x' \quad \text{and} \quad m \cdot x = y_{\alpha\beta} \cdot u^{\alpha\beta} + y'$$

with $m_\alpha, x_\beta, y_{\alpha\beta} \in K$ and $m' \in V_{\equiv_\alpha^-}$, $x' \in V_{\equiv_\beta^-}$, $y' \in V_{\equiv_{\alpha\beta}^-}$. Then

$$m \cdot x = (m_\alpha \cdot u^\alpha + m') \cdot (x_\beta \cdot u^\beta + x')$$
$$\in (m_\alpha \cdot u^\alpha) \cdot (x_\beta \cdot u^\beta) + V_{\equiv_{\alpha\beta}^-} = \mu_{\alpha,\beta}(m_\alpha, x_\beta) \cdot u^{\alpha\beta} + V_{\equiv_{\alpha\beta}^-}$$

yields

$$y_{\alpha\beta} = \mu_{\alpha,\beta}(m_\alpha, x_\beta).$$

By

$$\varphi(\theta(m), \theta(x)) = \theta(m \cdot x) - \theta(m) \cdot \theta(x)$$
$$= \mu_{\alpha,\beta}(m_\alpha, x_\beta) t^{\alpha\beta} + \theta(y') - (m_\alpha t^\alpha + \theta(m')) \cdot (x_\beta t^\beta + \theta(x'))$$

we obtain

$$v_{\mathbf{H}}(\varphi(\theta(m), \theta(x))) < v_{\mathbf{H}}(\theta(m)) \cdot v_{\mathbf{H}}(\theta(x));$$

hence, φ is strictly contracting.

By Theorem 1, successively applied to both arguments of φ, there exists a P-bilinear and therefore biadditive extension Φ of φ to $\mathbf{H} \times \mathbf{H} \to \mathbf{H}$ which is strictly contracting and satisfies $\Phi(\mathbf{m}, \mathbf{1}) = \mathbf{0}$ and $\Phi(\mathbf{1}, \mathbf{x}) = \mathbf{0}$ for all $\mathbf{m}, \mathbf{x} \in \mathbf{H}$.

By Theorem 2, $(\mathbf{H}, +, *)$ with $\mathbf{m} * \mathbf{x} = \mathbf{m} \cdot \mathbf{x} + \Phi(\mathbf{m}, \mathbf{x})$ is a division algebra with the spherically complete valuation $v_{\mathbf{H}}$. For all $m, x \in N$ we have

$$\theta(m \cdot x) = \theta(m) \cdot \theta(x) + \varphi(\theta(m), \theta(x)) = \theta(m) * \theta(x),$$

thus θ is a value-preserving monomorphism of division algebras from $(N, +, \cdot)$ to $(\mathbf{H}, +, *)$. Hence, $(\theta(N), v_{\mathbf{H}}, \Gamma_0) \prec (\mathbf{H}, v_{\mathbf{H}}, \Gamma_0)$ is an immediate extension of valued division algebras.

With these considerations we have shown the following

Theorem 3. *Let (N, v, Γ_0) be a valued division algebra having the same characteristic as its residue division algebra. Then the following assertions hold:*

1. (N, v, Γ_0) *is maximal, i.e., without any proper immediate extension of valued division algebras, if and only if* (N, d_v, Γ_0) *is spherically complete.*
2. (N, v, Γ_0) *possesses a maximal immediate extension, and every maximal immediate extension of* (N, v, Γ_0) *is spherically complete.*

This result generalizes [15, Satz 5], which characterizes the valued division algebras admitting an embedding into an appropriate Hahn division algebra of formal power series. Finally, [14] gives an example of a division algebra of characteristic 0 with a maximal discrete valuation, i.e., $\Gamma \cong \mathbb{Z}$, which cannot be regarded as a Hahn division algebra.

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Erwin Schörner, Mathematisches Institut der Universität München, Theresienstraße 39, D-80333 München
schoerner@rz.mathematik.uni-muenchen.de