Forum Math. **13** (2001), 51–90

Forum Mathematicum

Feller semigroups, L^p-sub-Markovian semigroups, and applications to pseudo-differential operators with negative definite symbols

Walter Farkas, Niels Jacob, and René L. Schilling

(Communicated by Masatoshi Fukushima)

Abstract. The question of extending L^p-sub-Markovian semigroups to the spaces L^q, $q > p$, and the interpolation of L^p-sub-Markovian semigroups with Feller semigroups is investigated. The structure of generators of L^p-sub-Markovian semigroups is studied. Subordination in the sense of Bochner is used to discuss the construction of refinements of L^p-sub-Markovian semigroups. The rôle played by some function spaces which are domains of definition for L^p-generators is pointed out. The problem of regularising powers of generators as well as some perturbation results are discussed.

1991 Mathematics Subject Classification: 47D07, 31C25, 60J35, 35S99.

0 Introduction

Probably the most important equation connecting the theory of Markov processes with functional analysis is given by

$$(0.1) \qquad p_t(x, A) = T_t\chi_A(x) = E^x(\chi_A(X_t)).$$

Here $(T_t)_{t\geqslant 0}$ is a semigroup of operators on some function space over $\mathbb{R}^n$ (for simplicity), $((X_t)_{t\geqslant 0}, P^x)_{x\in\mathbb{R}^n}$ is a Markov process with state space $\mathbb{R}^n$ and transition function $p_t(x, A)$; χ_A is the characteristic function of the set A. In order to construct a Markov process using the Kolmogorov theorem we have to know the family $p_t(x, A)$ of (sub-)Markovian kernels. One way to construct $p_t(x, A)$ is to start with a given operator semigroup $(T_t)_{t\geqslant 0}$ and to define $p_t(x, A)$ through (0.1). In this case it is natural to use the theory of strongly continuous contraction semigroups on Banach spaces. The direct approach is, of course, a pointwise construction working with continuous functions. This means that we start with a Feller semigroup $(T_t^{(\infty)})_{t\geqslant 0}$, that is a positivity preserving strongly continuous contraction semigroup on the Banach space $(C_\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ of all continuous functions vanishing at infinity. Now we have a nice structure theorem for the generator of $(T_t^{(\infty)})_{t\geqslant 0}$ due to Ph. Courrège.

However, there are two major drawbacks: in order to obtain non-trivial examples of Feller semigroups, one uses the Hille-Yosida-Ray theorem. This means that one has to solve equations in the Banach space $C_\infty(\mathbb{R}^n)$ which can be quite difficult. Moreover, operators with non-smooth coefficients cannot be treated in general.

M. Fukushima proposed to start with a symmetric L^2-sub-Markovian semigroup $(T_t^{(2)})_{t\geqslant 0}$, i.e., a strongly continuous L^2-contraction semigroup satisfying the sub-Markov property

$$0 \leqslant u \leqslant 1 \quad \text{(a.e.) implies } 0 \leqslant T_t^{(2)} u \leqslant 1 \text{ (a.e.)}$$

Using the potential theory of the associated quadratic form, the Dirichlet form, it is possible to construct the transition function up to an exceptional set, i.e., a set of capacity zero. This method has the advantage that $L^2(\mathbb{R}^n)$ is a Hilbert space where it is easier to solve equations and thus to construct semigroups using the Hille-Yosida theorem; moreover, one can treat operators with non-smooth coefficients. A major problem is, of course, the presence of exceptional sets which implies that the constructed process effectively lives on $\mathbb{R}^n$ *less an exceptional set* and that all considerations have to be done modulo this set. This problem can be overcome if we consider L^2-sub-Markovian semigroups $(T_t^{(2)})_{t\geqslant 0}$ with the property that for all bounded and measurable sets A the functions

$$x \mapsto T_t \chi_A(x) \tag{0.2}$$

are continuous. Recall the result of E. M. Stein that symmetric sub-Markovian semigroups are analytic, hence

$$T_t\chi_A \in \bigcap_{k\geqslant 0} D((A^{(2)})^k)$$

holds, where $D((A^{(2)})^k)$ is the domain of the k-th power of the generator $(A^{(2)}, D(A^{(2)}))$ of $(T_t^{(2)})_{t\geqslant 0}$. We may, therefore, establish the continuity of (0.2) for those cases where we can embed the intersection (of some finite number) of domains of powers of $A^{(2)}$ into $C(\mathbb{R}^n)$. Usually, it is quite hard to obtain precise information on $D((A^{(2)})^k)$ for $k \geqslant 2$ and this requires (in general) higher regularity of the coefficients.

With the Sobolev embedding theorem and the theory of (second order) elliptic differential operators in mind, it might be helpful to pass from the L^2-theory to an L^p-setting, $p > 2$, and to consider operators with domains in some L^p-space such that we may embed these domains into $C(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

The purpose of this paper is to discuss these ideas and to give some examples. Sections 1–5 are of theoretical nature whereas sections 6–9 contain illuminating examples.

In Section 1 we discuss the problem of extending a given L^p-sub-Markovian semigroup to the spaces $L^q(\mathbb{R}^n)$, $q > p$. To do this, we establish first some interpolation results for operators which coincide on $L^p(\mathbb{R}^n) \cap B_b(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, see Theorem 1.4 and Remark 1.6. We pay particular attention to the difference between $B_b(\mathbb{R}^n)$ vs. $L^\infty(\mathbb{R}^n)$, i.e. functions vs. classes of functions, which implies some technical changes to otherwise standard proofs. In Theorem 1.8 we show that any L^p-sub-Markovian semigroup extends to an L^q-sub-Markovian semigroup for $p < q < \infty$. Note that we do not assume symmetry.

A further application is Theorem 1.10 where we show that L^q-sub-Markovian semigroups ($p < q < \infty$) interpolate between L^p-sub-Markovian and Feller semigroups if the operators of the original semigroups coincide on $L^p(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$. These results alone seem not to be too surprising, however, combining them with results of W. Hoh [23]–[27] or [30]–[33] gives many concrete examples of L^p-sub-Markovian semigroups.

Section 2 is devoted to the structure of generators of L^p-sub-Markovian semigroups. The form of generators of Feller semigroups is well known. They satisfy the positive maximum principle and once the domain contains $C_0^\infty(\mathbb{R}^n)$, they are already certain differential–integrodifferential operators with negative definite symbol. Generators of L^p-sub-Markovian semigroups are L^p-Dirichlet operators, i.e., they satisfy for $u \in D(A^{(p)})$

$$\int_{\mathbb{R}^n} (A^{(p)}u)((u-1)^+)^{p-1}\,dx \leqslant 0,$$

which was first proved for $p = 2$ and selfadjoint operators by N. Bouleau and F. Hirsch [7], for the general case we refer to A. Eberle [13], V. Liskevich and Yu. Semenov [41], Z.-M. Ma and M. Röckner [43], E. M. Ouhabaz [48]–[49], and [34]. Using the extension result from Section 1 we conclude that any L^p-Dirichlet operator extends to L^q-Dirichlet operators for all $p < q < \infty$. Under suitable regularity assumptions on the respective domains and the mapping behaviour, see Theorem 2.4 for details, we infer that each L^q-generator satisfies the positive maximum principle and has the same structure as a Feller-generator. This result is quite important since it tells us something about the type of the operator one has to start in order to construct an L^p-sub-Markovian semigroup, or if $p = 2$, a Dirichlet form.

Section 3 recalls just some basic facts on subordination in the sense of Bochner which is applied in Section 4 to discuss the Γ-transform $(V_r^{(p)})_{r\geqslant 0}$ of an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ which is needed to handle refinements of that semigroup. This is, of course, closely related to the work of P. Malliavin and M. Fukushima (with coauthors). New, however, is the observation that the Γ-transformed semigroup is a subordinate semigroup. This enables us to determine $V_r^{(p)}$ as $(\mathrm{id} - A^{(p)})^{-r/2}$, $A^{(p)}$ being the generator of $(T_t^{(p)})_{t\geqslant 0}$, and to identify the abstract Bessel potential space $\mathscr{F}_{r,p}$ with $D((\mathrm{id} - A^{(p)})^{r/2})$, see Theorem 4.1 and Corollary 4.2. For $p = 2$ and a selfadjoint operator $A^{(2)}$ this was proved in [18] and [20] using spectral theory. Our proof is based on a functional calculus for generators of semigroups and Bernstein functions, see [53].

In Section 5 we discuss the problem of constructing refinements of L^p-sub-Markovian semigroups. In particular we are interested in L^p-sub-Markovian semigroups $(T_t^{(p)})_{t\geqslant 0}$ with the property $T_t^{(p)}\chi_A \in C_b(\mathbb{R}^n)$ for all $t > 0$ and all Borel sets A with finite Lebesgue measure. We call these semigroups *strong* L^p-sub-Markovian semigroups in analogy to strong Feller semigroups. Whenever $(T_t^{(p)})_{t\geqslant 0}$ is a strong L^p-sub-Markovian semigroup we may use $p_t(x, A) := T_t^{(p)}\chi_A(x)$ to construct an associated Hunt process without any exceptional set. Otherwise we shall try to reduce the exceptional set whenever possible by using capacities associated with $(T_t^{(p)})_{t\geqslant 0}$ and $\mathscr{F}_{r,p}$ for some suitable r. The key observation (which seems to be new in our context) is that a combination of the regularising effects of an analytic semigroup with the concrete characterisation of the domain(s) (of powers) of the generator, and Sobolev-type embeddings will immediately give the strong L^p-sub-Markov property, see Proposition 5.3 and Theorem 5.4. A first example for this idea is provided by semigroups generated by second order elliptic differential operators. Of course, not every (analytic) L^p-sub-Markovian semigroup is a strong L^p-sub-Markovian semigroup. In this case we use the theory of (r, p)-capacities to get refinements, see [17]–[18] or [20] which is briefly recorded for the reader's convenience.

Our approach can be summed up in the following way: Let $(T_t^{(p)})_{t\geqslant 0}$ be a given L^p-sub-Markovian semigroup with L^q-extensions $(T_t^{(q)})_{t\geqslant 0}$, $p < q < \infty$, and assume that $C_0^\infty(\mathbb{R}^n) \subset \bigcap_{q\geqslant p} D(A^{(q)})$. If each operator $A^{(q)}$ maps $C_0^\infty(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then $A^{(p)}$ (and each $A^{(q)}$) restricted to $C_0^\infty(\mathbb{R}^n)$ is a pseudo-differential operator with negative definite symbol, i.e.,

$$(0.3) \qquad A^{(p)}|_{C_0^\infty(\mathbb{R}^n)} u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x, \xi) \hat{u}(\xi)\, d\xi = -q(x, D)u(x),$$

where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous and negative definite in ξ. If, in addition, the semigroup $(T_t^{(p)})_{t\geqslant 0}$ is analytic and if for some $r > 0$ the space $D(-(-A^{(p)})^r)$ is contained in a space of continuous functions, then $(T_t^{(p)})_{t\geqslant 0}$ is a strong L^p-sub-Markovian semigroup.

Thus it would be very natural to start with $-q(x, D)$ as in (0.3) and to prove that it extends under certain conditions on $q(x, \xi)$ to a generator $A^{(p)}$ of an analytic L^p-sub-Markovian semigroup with some nice function space containing $D((-A^{(p)})^r)$ for some r.

In the case $p = 2$ and symmetric operators many concrete examples are known, mainly due to W. Hoh and some earlier work of the second named author. In the general case several problems arise. First of all there are non-analytic sub-Markovian semigroups. More important however is the fact that a general continuous negative definite function $\xi \mapsto \psi(\xi)$ is neither smooth nor homogeneous implying that standard L^p-analysis tools such as the Calderon–Zygmund theory of singular integrals or multiplier theorems of Michlin–Hörmander or Lizorkin type do not apply. For this reason we cannot (yet) offer a rich theory of L^p generators for $p \neq 2$. In Sections 6–9 however we illustrate our approach using concrete and partly rather new examples.

Section 6 examines those function spaces which should be natural domains of L^p-generators. They are constructed for translation invariant operators, i.e., Lévy processes, and we recall some recent results from [15]. Since we do not dispose of Plancherel's theorem, the L^p-analysis for $p \neq 2$ is much harder than the L^2-analysis.

In Section 7 we concentrate on fractional powers of second order elliptic differential operators generating L^p-sub-Markovian (diffusion) semigroups. We need not assume the analyticity of the original diffusion semigroup since by a result of A. Carasso and T. Kato [9] the subordinate semigroup is automatically analytic if the corresponding Bernstein function is a *complete* Bernstein function. Interpolation results for fractional powers of generators lead to a large class of strong L^p-sub-Markovian semigroups. Moreover we get concrete, non-trivial examples of the structure theorem for generators, see Theorem 2.6.

In Sections 6 and 7 we consider the domain of the generator itself. In Section 8 we are concerned with the problem of regularising powers of generators. We restrict ourselves (as in [19] and [32]) to the Hilbert space case, but handle quite general pseudo-differential operators $q(x, D)$, see W. Hoh [23]–[27] or [31], [33].

In this case it is clear that the semigroup is analytic, see E. M. Stein [55], and that in general the space $H^{\psi,2k}(\mathbb{R}^n)$ which may describe the domain $D((-A^{(2)})^k)$ has better embedding properties the larger k is. In fact, for ψ satisfying asymptotically $\psi(\xi) \geqslant c_0|\xi|^{r_0}$, $c_0 > 0$, $0 < r_0 < 1$, we have $H^{\psi,2k}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n)$ if $k > \dfrac{n}{4r_0}$. But in order to prove that $H^{\psi,2k}(\mathbb{R}^n) = D((-A^{(2)})^k)$ if $H^{\psi,2}(\mathbb{R}^n) = D(-A^{(2)})$, it turns out that the coefficients $x \mapsto q(x, \xi)$ must have increasing regularity with increasing k. In particular the very simple examples taken from [28] show quite detailed which type of regularity of the coefficients is needed to reach finally a space $D((-A^{(2)})^{k_0})$ which is embedded into $C_\infty(\mathbb{R}^n)$. In the intermediate steps the processes can only be constructed up to certain exceptional sets which however become smaller and smaller as k approaches k_0.

The final section treats (from the structural point of view) a simple perturbation of the original generator. Moreover, we restrict our concrete considerations to the easy case $-\psi_1(D) + a$, where $\psi_1(\xi) = \psi(\xi) + 1$, $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function, and $a \in L^\infty(\mathbb{R}^n)$, $a \leqslant 0$. We discuss the associated L^p-generator and study the effects of the regularity of a on the strong L^p-sub-Markov property of the semigroup. Our reasoning is not confined to this case but can easily be applied to similar situations.

Despite the lack of a general L^p-theory for the operators (0.3) our examples show the following:

- good applicability of the L^p-theory to operators obtained from given generators by standard constructions such as subordination or perturbation;
- there is a natural limit in the (r, p)-capacity refinements of L^p-sub-Markovian semigroups, namely the strong L^p-sub-Markovian semigroups;

- the determination of domains in terms of concrete function spaces is the key to get concrete refinement results.

Notation. If X is a Banach space we denote its norm by $\|\cdot|X\|$. Since we work always on $\mathbb{R}^n$ we will from now on drop the $\mathbb{R}^n$ in function spaces, e.g., $L^p = L^p(\mathbb{R}^n)$ or $C_\infty = C_\infty(\mathbb{R}^n)$. All other notation should be standard or self-explanatory.

Acknowledgement. The two first-named authors have been supported through the DFG-project Ja522/7-1 *Funktionenräume in der Theorie der stochastischen Prozesse*.

1 On L^p-sub-Markovian semigroups and Feller semigroups

We start defining the central objects of our investigation.

Definition 1.1 A. A linear contraction $S^{(p)} : L^p \to L^p$ is called an *L^p-sub-Markovian operator* if for $u \in L^p$ with $0 \leqslant u \leqslant 1$ a.e. also $0 \leqslant S^{(p)}u \leqslant 1$ a.e.

B. A family $(T_t^{(p)})_{t\geqslant 0}$ of L^p-sub-Markovian operators is called an *L^p-sub-Markovian semigroup* if $(T_t^{(p)})_{t\geqslant 0}$ is a strongly continuous contraction semigroup on L^p, i.e., we have $\|T_t^{(p)}u|L^p\| \leqslant \|u|L^p\|$, $\lim_{t\to 0}\|T_t^{(p)}u - u|L^p\| = 0$, and $T_t^{(p)} \circ T_s^{(p)} = T_{t+s}^{(p)}$ and $T_0^{(p)} = \mathrm{id}$.

C. A bounded linear operator $S^{(p)} : L^p \to L^p$ is called *positivity preserving* if $u \geqslant 0$ a.e. implies $S^{(p)}u \geqslant 0$ a.e.

Our first problem is the following question: given an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ does it extend to other L^q spaces, i.e., is it possible to find an L^q-sub-Markovian semigroup $(T_t^{(q)})_{t\geqslant 0}$ such that for all $t \geqslant 0$

$$T_t^{(q)}|_{L^p\cap L^q} = T_t^{(p)}|_{L^p\cap L^q}?$$

Without proof we mention the well-known

Lemma 1.2. *Let $1 < p < \infty$ and let $S^{(p)} : L^p \to L^p$ be a sub-Markovian operator. Then $S^{(p)}$ is positivity preserving and $|S^{(p)}u| \leqslant S^{(p)}|u|$ a.e., for any $u \in L^p$. For $u \in L^p \cap L^\infty$ we have*

$$\|S^{(p)}u|L^\infty\| \leqslant \|u|L^\infty\|. \tag{1.1}$$

Now let $S^{(p)} : L^p \to L^p$ be a sub-Markovian operator and $u \in L^p \cap L^\infty$. For any $p < q < \infty$ we find

$$\|S^{(p)}u|L^q\| = \left(\int_{\mathbb{R}^n} |S^{(p)}u|^q\,dx\right)^{1/q}$$

$$= \left(\int_{\mathbb{R}^n} |S^{(p)}u|^{q-p}|S^{(p)}u|^p\,dx\right)^{1/q}$$

$$\leqslant \|S^{(p)}u|L^\infty\|^{(q-p)/q} \cdot \|S^{(p)}u|L^p\|^{p/q}$$

$$\leqslant \|u|L^\infty\|^{(q-p)/q} \cdot \|u|L^p\|^{p/q},$$

thus for $u \in L^p \cap L^\infty$, which is a dense set in L^q, $p < q < \infty$, $S^{(p)}u$ is a function in L^q.

However, we want to prove more, namely that $S^{(p)}$ extends to an L^q contraction. This will be done by interpolation, and—for reasons becoming clear later—we interpolate not the operators $S^{(p)} : L^p \to L^p$ and $S^{(p)}|_{L^\infty \cap L^p} : L^\infty \to L^\infty$, but we will use as second operator a linear contraction on B_b (the Banach space of bounded Borel measurable functions on $\mathbb{R}^n$, normed in the usual way).

First recall the *Hadamard three lines theorem*, see [2, page 195] for a proof.

Theorem 1.3. *Let* $\Omega := \{x + iy : 0 < x < 1, y \in \mathbb{R}\}$ *and* $\bar{\Omega}$ *its closure. Further let F be a bounded continuous function on* $\bar{\Omega}$ *which is analytic in* Ω*. Then the function*

$$M_\gamma := \sup\{|F(\gamma + iy)| : y \in \mathbb{R}\}$$

satisfies

$$M_\gamma \leqslant M_0^{1-\gamma} M_1^\gamma, \quad 0 \leqslant \gamma \leqslant 1.$$

Theorem 1.4. *Let* $1 < p < \infty$, *let* $S^{(p)}$ *be a linear contraction on* L^p *and* $\tilde{S}^{(\infty)}$ *a linear contraction on* B_b *such that*

$$S^{(p)}|_{L^p \cap B_b} = \tilde{S}^{(\infty)}|_{L^p \cap B_b}. \tag{1.2}$$

Then there exists for every q, $p < q < \infty$, *a linear contraction* $S^{(q)}$ *on* L^q *such that*

$$S^{(q)}|_{L^p \cap L^q \cap B_b} = S^{(p)}|_{L^p \cap L^q \cap B_b} = \tilde{S}^{(\infty)}|_{L^p \cap L^q \cap B_b}. \tag{1.3}$$

Remark 1.5. The statements (1.2) and (1.3) need some interpretation. A function $u \in B_b$ is uniquely determined on $\mathbb{R}^n$ whereas an element $v \in L^p$ is an equivalence class of functions which may differ on a set of Lebesgue measure zero. By $u \in B_b \cap L^p$ we mean the uniquely determined element in B_b. Let $S^{(p)} : L^p \to L^p$ and $\tilde{S}^{(\infty)} : B_b \to B_b$ be two linear operators and let $u \in B_b \cap L^p$. Then $\tilde{S}^{(\infty)}u \in B_b$ and $S^{(p)}u \in$

L^p. For the latter we have $S^{(p)}u = S^{(p)}v$ a.e. whenever $v = u$ a.e., i.e., although $u \in B_b \cap L^p$ can be considered as a uniquely determined function, $S^{(p)}u$ is still an equivalence class of functions and all $v \in L^p$ such that $u = v$ a.e. are mapped via $S^{(p)}$ into the equivalence class of $S^{(p)}u$. We write

$$S^{(p)}|_{L^p \cap B_b} = \tilde{S}^{(\infty)}|_{L^p \cap B_b}$$

if, and only if, for all $u \in L^p \cap B_b$ we have

$$S^{(p)}u = \tilde{S}^{(\infty)}u \quad \text{a.e.}$$

Since $\tilde{S}^{(\infty)}u$ is uniquely determined, this allows the interpretation $S^{(p)}u \in B_b \cap L^p$, i.e., we may choose $\tilde{S}^{(\infty)}u$ as representative for $S^{(p)}u$.

Proof of Theorem 1.4. Denote by S the operator $S^{(p)}|_{L^p \cap B_b} = \tilde{S}^{(\infty)}|_{L^p \cap B_b}$. Since for $\dfrac{1}{q} + \dfrac{1}{q'} = 1$

$$\begin{aligned} \|S\|_{L^q \to L^q} &= \sup\{\|Su|L^q\| : u \in L^q, \|u|L^q\| = 1\} \\ &= \sup\left\{\left| \int_{\mathbb{R}^n} (Su)(x)v(x)\,dx \right|\right\}, \end{aligned}$$

where the supremum ranges over all $u \in L^q$ and $v \in L^{q'}$ with $\|u|L^q\| = \|v|L^{q'}\| = 1$, it is sufficient to show that

$$\left| \int_{\mathbb{R}^n} (Su)(x)v(x)\,dx \right| \leqslant 1 \tag{1.4}$$

for all simple functions u, v satisfying $\|u|L^q\| = \|v|L^{q'}\| = 1$. Such functions u and v are of the form

$$u = \sum_{j=1}^{J} a_j \chi_{A_j} \quad \text{and} \quad v = \sum_{k=1}^{K} b_k \chi_{B_k},$$

where the sets A_j, $j = 1, \ldots, J$, and B_k, $k = 1, \ldots, K$, are two families of pairwise disjoint Borel sets with finite Lebesgue measure, and the coefficients a_j and b_k satisfy

$$\sum_{j=1}^{J} |a_j|^q \lambda^{(n)}(A_j) = \sum_{k=1}^{K} |b_k|^{q'} \lambda^{(n)}(B_k) = 1.$$

For each $z \in \mathbb{C}$ let $\alpha(z) = \dfrac{1-z}{p}$ and let $\theta \in (0,1)$ be such that $\alpha(\theta) = \dfrac{1}{q}$, i.e., $\dfrac{1-\theta}{p} = \dfrac{1}{q}$.

Further, for $z \in \mathbb{C}$ define

$$u_z := |u|^{\alpha(z)/\alpha(\theta)} e^{i \arg u} \quad v_z := |v|^{(1-\alpha(z))/(1-\alpha(\theta))} e^{i \arg v}$$

and set

$$F(z) := \int_{\mathbb{R}^n} (Su_z)(x) v_z(x)\, dx.$$

Since

$$F(\theta) = \int_{\mathbb{R}^n} (Su_\theta)(x) v_\theta(x)\, dx = \int_{\mathbb{R}^n} (Su)(x) v(x)\, dx$$

the desired estimate (1.4) will follow from Hadamard's three lines theorem if we show that F is analytic in Ω and bounded, continuous in $\bar{\Omega}$ with

$$|F(iy)| \leqslant 1 \quad \text{and} \quad |F(1+iy)| \leqslant 1 \quad \text{for all } y \in \mathbb{R}.$$

First note that

$$F(z) = \sum_{j=1}^{J} \sum_{k=1}^{K} |a_j|^{\alpha(z)/\alpha(\theta)} |b_k|^{(1-\alpha(z))/(1-\alpha(\theta))} \int_{\mathbb{R}^n} (S\chi_{A_j})(x) \chi_{B_k}(x) e^{i(\arg a_j + \arg b_k)}\, dx$$

which shows that $z \mapsto F(z)$ is an entire function. It also reveals that F is bounded in $\bar{\Omega}$ since the real part of $\alpha(z)$ is bounded there. Applying Hölder's inequality and using the fact that S is an L^p-contraction we have

$$\begin{aligned} (1.5) \qquad |F(iy)| &\leqslant \int_{\mathbb{R}^n} |(Su_{iy})(x) v_{iy}(x)|\, dx \\ &\leqslant \|Su_{iy}|L^p\|\, \|v_{iy}|L^{p'}\| \leqslant \|u_{iy}|L^p\| \|v_{iy}|L^{p'}\|. \end{aligned}$$

Moreover, since $\alpha(\theta) = \dfrac{1}{q}$ and $\operatorname{Re}\alpha(iy) = \dfrac{1}{p}$, we have

$$(1.6) \qquad \|u_{iy}|L^p\|^p = \sum_{j=1}^{J} ||a_j|^{\alpha(iy)/\alpha(\theta)}|^p \lambda^{(n)}(A_j) = \sum_{j=1}^{J} |a_j|^q \lambda^{(n)}(A_j) = 1$$

and similarly, since $1 - \alpha(\theta) = \dfrac{1}{q'}$ and $\operatorname{Re}(1 - \alpha(iy)) = \dfrac{1}{p'}$

$$(1.7) \qquad \|v_{iy}|L^{p'}\|^{p'} = \sum_{k=1}^{K} ||b_k|^{(1-\alpha(iy))/(1-\alpha(\theta))}|^{p'} \lambda^{(n)}(B_k) = \sum_{k=1}^{K} |b_k|^{q'} \lambda^{(n)}(B_k) = 1.$$

From (1.5)–(1.7) we conclude that $|F(iy)| \leqslant 1$. In order to estimate $|F(1+iy)|$ observe that

$$(1.8) \qquad |u_{1+iy}(x)| = ||u(x)|^{\alpha(1+iy)/\alpha(\theta)}| = 1$$

since $\alpha(\theta) = \dfrac{1}{q}$ and $\operatorname{Re}\alpha(1+iy) = 0$, and similarly

$$(1.9) \qquad |v_{1+iy}(x)| = ||v(x)|^{(1-\alpha(1+iy))/(1-\alpha(\theta))}| = |v(x)|^{q'}$$

since $1 - \alpha(\theta) = 1 - \dfrac{1}{q} = \dfrac{1}{q'}$ and $\operatorname{Re}(1 - \alpha(1+iy)) = 1$. Using (1.8), (1.9), and the fact that S is also a contraction on B_b we arrive at

$$\begin{aligned} |F(1+iy)| &\leqslant \int_{\mathbb{R}^n} |(Su_{1+iy})(x)v_{1+iy}(x)|\,dx \\ &\leqslant \|Su_{1+iy}|L^\infty\| \int_{\mathbb{R}^n} |v_{1+iy}(x)|\,dx \leqslant \int_{\mathbb{R}^n} |v(x)|^{q'}dx = 1, \end{aligned}$$

and the theorem is proved. □

Remark 1.6. Clearly, Theorem 1.4 is a type of Riesz-Thorin theorem where the space L^∞ of the interpolation couple is substituted by B_b. We have given a proof of this result since we could not find a precise reference in the literature for this situation. We followed the standard proof, see C. Bennett and R. Sharpley [2, Theorem 2.2, page 196].

From Remark 1.6 it is clear that we may apply the result of Theorem 1.4 also to the situation where B_b is substituted by L^∞.

Corollary 1.7. *Let $1 < p < \infty$, let $S^{(p)}$ be a linear contraction on L^p, and let $S^{(\infty)}$ be a linear contraction on L^∞ such that*

$$S^{(p)}|_{L^p\cap L^\infty} = S^{(\infty)}|_{L^p\cap L^\infty}.$$

Then there exists for every q, $p < q < \infty$, a linear contraction $S^{(q)}$ on L^q such that

$$S^{(q)}|_{L^p\cap L^q\cap L^\infty} = S^{(p)}|_{L^p\cap L^q\cap L^\infty} = S^{(\infty)}|_{L^p\cap L^q\cap L^\infty}.$$

Now we may use Corollary 1.7 to answer the extension problem for sub-Markovian semigroups:

Theorem 1.8. *Let $1 < p < \infty$ and let $(T_t^{(p)})_{t\geqslant 0}$ be an L^p-sub-Markovian semigroup on L^p. Then for any $p < q < \infty$ there exists on L^q an L^q-sub-Markovian semigroup*

$(T_t^{(q)})_{t\geqslant 0}$ *such that for any* $t \geqslant 0$

$$T_t^{(q)}|_{L^p\cap L^q} = T_t^{(p)}|_{L^p\cap L^q}.$$

Proof. By Lemma 1.2 and Corollary 1.7 each of the operators $T_t^{(p)}$, $t > 0$, extends to an L^∞-contraction $\tilde{T}_t^{(\infty)}$ and an L^q-contraction $T_t^{(q)}$. Clearly $(T_t^{(q)})_{t\geqslant 0}$ is a semigroup, and each of the operators $T_t^{(q)}$ is sub-Markovian. It remains to prove that $(T_t^{(q)})_{t\geqslant 0}$ is strongly continuous. For $u \in L^p \cap L^\infty$ we have

$$\begin{aligned}\|T_t^{(q)}u - u|L^q\| &\leqslant \|\tilde{T}_t^{(\infty)}u - u|L^\infty\|^{(q-p)/p} \cdot \|T_t^{(p)}u - u|L^p\|^{p/q}\\ &\leqslant 2\|u|L^\infty\|^{(q-p)/p} \cdot \|T_t^{(p)}u - u|L^p\|^{p/q},\end{aligned}$$

which yields $\lim_{t\to 0}\|T_t^{(q)}u - u|L^q\| = 0$. For general $u \in L^q$ the claim follows with a standard approximation argument. □

We return to Theorem 1.4 and we will use it to interpolate between L^p-sub-Markovian semigroups and Feller semigroups.

Definition 1.9. A *Feller semigroup* $(T_t^{(\infty)})_{t\geqslant 0}$ is a strongly continuous, positivity preserving contraction semigroup on the space C_∞ of continuous functions on $\mathbb{R}^n$ vanishing at infinity.

It is well-known that a Feller semigroup gives rise to sub-Markovian kernels $(p_t(x,\cdot))_{t\geqslant 0}$ which may be used to extend $T_t^{(\infty)}$ to an operator $\tilde{T}_t^{(\infty)}$ on B_b by

$$\tilde{T}_t^{(\infty)}u(x) = \int_{\mathbb{R}^n} u(y)p_t(x,dy), \quad u \in B_b. \tag{1.10}$$

It is easy to see that $(\tilde{T}_t^{(\infty)})_{t\geqslant 0}$ is a contraction semigroup on B_b and each of the operators $\tilde{T}_t^{(\infty)}$, $t > 0$, is positivity preserving. However, in general, this semigroup is not strongly continuous. Applying Theorem 1.4 we get

Theorem 1.10. *Let* $(T_t^{(\infty)})_{t\geqslant 0}$ *be a Feller semigroup with extension* $(\tilde{T}_t^{(\infty)})_{t\geqslant 0}$ *on* B_b *and let* $(T_t^{(p)})_{t\geqslant 0}$ *be an* L^p*-sub-Markovian semigroup. If for all* $t \geqslant 0$ *we have* $\tilde{T}_t^{(\infty)}|_{L^p\cap B_b} = T_t^{(p)}|_{L^p\cap B_b}$ *then* $(T_t^{(p)}|_{L^p\cap B_b})_{t\geqslant 0}$ *extends for all* $p < q < \infty$ *to a strongly continuous contraction semigroup* $(T_t^{(q)})_{t\geqslant 0}$ *of sub-Markovian operators on* L^q *satisfying*

$$T_t^{(q)}|_{L^q\cap L^p\cap B_b} = T_t^{(p)}|_{L^q\cap L^p\cap B_b} = \tilde{T}_t^{(\infty)}|_{L^q\cap L^p\cap B_b}.$$

Proof. As in the proof of Theorem 1.8 it remains to prove that $(T_t^{(q)})_{t\geqslant 0}$ is strongly continuous on L^q and this can be done in the same way as in that proof. □

For later purposes let us introduce the notion of strong Feller semigroups and an analogous notion for L^p-sub-Markovian semigroups.

Definition 1.11 A. A Feller semigroup $(T_t^{(\infty)})_{t\geqslant 0}$ is called a *strong Feller semigroup* if for all $t \geqslant 0$ the operators $\tilde{T}_t^{(\infty)}$ defined by (1.10) map B_b into C_b.

B. An L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ is called a *strong L^p-sub-Markovian semigroup* if each of the operators $T_t^{(p)}$ maps L^p into $L^p \cap C$.

Suppose that $(T_t^{(p)})_{t\geqslant 0}$ is a strong L^p-sub-Markovian semigroup. In this case for any bounded Borel set $A \subset \mathbb{R}^n$ we find $T_t^{(p)}\chi_A \in C_b$. This observation will be used later on to avoid exceptional sets when constructing Markov processes starting with L^p-semigroups.

Remark 1.12. For the study of one-parameter semigroups acting simultaneously on different L^p-spaces and for some interpolation results with consequences for the spectrum of generators we refer to the works [58] and [59] of J. Voigt.

2 Generators of L^p-sub-Markovian semigroups and the positive maximum principle

Let $(T_t^{(\infty)})_{t\geqslant 0}$ be a Feller semigroup with generator $(A^{(\infty)}, D(A^{(\infty)}))$ such that $C_0^\infty \subset D(A^{(\infty)}) \subset C_\infty$. It is well-known that $A^{(\infty)}$ satisfies the *positive maximum principle*, i.e.,

$$u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geqslant 0 \quad \text{implies } A^{(\infty)}u(x_0) \leqslant 0.$$

Due to a result of Ph. Courrège [10] we know that on C_0^∞ the operator $A^{(\infty)}$ is a pseudo-differential operator

$$A^{(\infty)}u(x) = -q(x,D)u(x) = -(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}q(x,\xi)\hat{u}(\xi)\,d\xi \tag{2.1}$$

where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a measurable, locally bounded function such that for every $x \in \mathbb{R}^n$ the function $q(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is continuous and *negative definite* (in the sense of I. J. Schoenberg), i.e., $q(x,0) \geqslant 0$ and $\xi \mapsto e^{-tq(x,\xi)}$ is for all $t > 0$ and all $x \in \mathbb{R}^n$ *positive definite* (in the usual sense). Alternatively, $q(x,\cdot)$ satisfies the following *Lévy-Khinchin formula*

$$q(x,\xi) = -c(x) + i\sum_{j=1}^n b_j(x)\xi_j + \sum_{k,l=1}^n a_{kl}(x)\xi_j\xi_k$$
$$+ \int_{y\neq 0}\left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2}\right)\nu(x,dy)$$

where $c \leqslant 0$, $(a_{kl})_{kl} \in \mathbb{R}^{n \times n}$ is a symmetric, positive semidefinite matrix, $b \in \mathbb{R}^n$, and $\nu(x, dy)$ is a kernel satisfying $\int_{y \neq 0} \min\{|y|^2, 1\} \nu(x, dy) < \infty$.

Using the Lévy-Khinchin formula we can derive another representation of $A^{(\infty)}$, namely

$$(2.2) \qquad A^{(\infty)}u(x) = L(x, D)u(x) + S(x, D)u(x)$$

where

$$L(x, D)u(x) = \sum_{k,l=1}^{n} a_{kl}(x) \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x) \frac{\partial u(x)}{\partial x_j} + c(x)u(x)$$

is a second order differential operator with non-negative characteristic form and $c(x) \leqslant 0$, and

$$S(x, D)u(x) = \int_{\mathbb{R}^n \setminus \{0\}} \left(u(x + y) - u(x) + \sum_{j=1}^{n} \frac{y_j}{1 + |y|^2} \frac{\partial u(x)}{\partial x_j} \right) \nu(x, dy).$$

(This representation shows immediately that *any* pseudo-differential operator $-q(x, D)$ with a negative definite symbol $q(x, \xi)$ naturally satisfies the positive maximum principle, independent of the question whether $-q(x, D)$ extends to the generator of a Feller semigroup.)

Thus the structure of the generators of Feller semigroups is (essentially) known. In case of a (symmetric) sub-Markovian semigroup $(T_t^{(2)})_{t \geqslant 0}$ on L^2, N. Bouleau and F. Hirsch [7] showed that its generator $(A^{(2)}, D(A^{(2)}))$ is a Dirichlet operator in the sense that

$$(2.3) \qquad \int_{\mathbb{R}^n} (A^{(2)}u)((u - 1)^+) \, dx \leqslant 0$$

holds for all $u \in D(A^{(2)})$. For non-symmetric sub-Markovian semigroups on L^2 this result is shown in the monograph [43] by Z.-M. Ma and M. Röckner. However, from (2.3) we cannot deduce a structure theorem like Courrège's result. The notion of a Dirichlet operator in the context of L^p-sub-Markovian semigroups were introduced by the second author, see [34, 35], where also related and independent results of A. Eberle [13], V. Liskevich and Yu. Semenov [41], and E. M. Ouhabaz [48, 49] are discussed. Let us call $(A^{(p)}, D(A^{(p)}))$, $D(A^{(p)}) \subset L^p$, an *L^p-Dirichlet operator* if

$$\int_{\mathbb{R}^n} (A^{(p)}u)((u - 1)^+)^{p-1} \, dx \leqslant 0$$

holds for all $u \in D(A^{(p)})$. It is easy to see, compare [34, 35] that for an L^p-Dirichlet operator

$$\text{(2.4)} \qquad \int_{\mathbb{R}^n} (A^{(p)}u)(u^+)^{p-1}\,dx \leqslant 0 \quad \text{and} \quad \int_{\mathbb{R}^n} (A^{(p)}u)(u^-)^{p-1}\,dx \geqslant 0$$

hold for all $u \in D(A^{(p)})$. The following result was proved in [34].

Theorem 2.1. *Let $(A^{(p)}, D(A^{(p)}))$ be an L^p-Dirichlet operator which generates a strongly continuous contraction semigroup $(T_t^{(p)})_{t\geqslant 0}$ on L^p. Then $(T_t^{(p)})_{t\geqslant 0}$ is sub-Markovian.*

Conversely, if $(A^{(p)}, D(A^{(p)}))$ is the generator of a sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ on L^p, then $(A^{(p)}, D(A^{(p)}))$ is an L^p-Dirichlet operator.

In [34], see also [35], it was proved that if an operator $(A^{(\infty)}, D(A^{(\infty)}))$ generates a Feller semigroup and extends to a generator $(A^{(p)}, D(A^{(p)}))$ of a strongly continuous contraction semigroup on L^p, then $A^{(p)}$ is an L^p-Dirichlet operator.

Theorem 2.2. *Let $(A^{(\infty)}, D(A^{(\infty)}))$ be the generator of a Feller semigroup $(T_t^{(\infty)})_{t\geqslant 0}$. Moreover, suppose that $U \subset D(A^{(\infty)})$ is a dense subspace of L^p. If $A^{(\infty)}|_U$ extends to a generator $A^{(p)}$ of a strongly continuous contraction semigroup $(T_t^{(p)})_{t\geqslant 0}$ on L^p such that $V := (\lambda - A^{(p)})^{-1}U$ is an operator core for $A^{(p)}$, then $(A^{(p)}, D(A^{(p)}))$ is an L^p-Dirichlet operator and $(T_t^{(p)})_{t\geqslant 0}$ is sub-Markovian.*

Moreover, in Section 1 we proved the possibility of interpolating Feller semigroups with their extension to L^p-sub-Markovian semigroups to obtain L^q-Dirichlet operators for $p < q < \infty$. It remains, however, to get some structure results for L^p-Dirichlet operators. We will now show a result in this direction: if $(A^{(p)}, D(A^{(p)}))$ is a Dirichlet operator for all $p \geqslant p_0$ such that on $D(A^{(p_0)}) \cap D(A^{(p)})$ we have always $A^{(p_0)}u = A^{(p)}u$, then the operator

$$A^{(p)}|_{\bigcap\limits_{p\geqslant p_0} D(A^{(p)})}$$

satisfies the positive maximum principle. To do this, we need the following lemma which is proved in the same way as $\lim_{p\to\infty} \|v|L^p\| = \|v|L^\infty\|$.

Lemma 2.3. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be two functions such that for some $r \geqslant 1$ we have $f^{r-1}g \in L^1$ and $f \in L^\infty$. Then*

$$\lim_{p\to\infty} \left(\int_{\mathbb{R}^n} g(x) f^{p-1}(x)\,dx \right)^{1/p} = \operatorname{ess\,sup}\{f(x) : x \in \{g > 0\}\}.$$

Now suppose that for all $p \geqslant p_0$ we have a family of L^p-Dirichlet operators

$(A^{(p)}, D(A^{(p)}))$ satisfying the following conditions: there is a vector space $D \neq \emptyset$

$$D \subset \bigcap_{p \geqslant p_0} D(A^{(p)}) \cap C_b \tag{2.5}$$

such that

$$A^{(p)}|_D = A^{(q)}|_D \quad \text{for all } p, q \geqslant p_0; \tag{2.6}$$

there exists a function $\varphi \in C_0^\infty$ such that $\varphi(0) = 1$, $\varphi(x) < 1$ for all $x \neq 0$, $\operatorname{supp} \varphi \subset \overline{B_1(0)}$, and for all $k \in \mathbb{N}$ and $y \in \mathbb{R}^n$

$$\varphi(k(\cdot - y)) \in D; \tag{2.7}$$

$$A := A^{(p)}|_D \quad \text{maps } D \text{ into } C. \tag{2.8}$$

Theorem 2.4. *Suppose that the family* $(A^{(p)}, D(A^{(p)}))$, $p \geqslant p_0$, *of* L^p*-Dirichlet operators satisfies conditions* (2.5)–(2.8) *from above. Then* A *satisfies on* D *the positive maximum principle.*

Proof. Let $u \in D \subset C_b$. From (2.4) we deduce for $p \geqslant p_0$

$$\int_{\mathbb{R}^n} (Au)(u^+)^{p-1}\, dx \leqslant 0. \tag{2.9}$$

Since

$$\int_{\mathbb{R}^n} (Au)(u^+)^{p-1}\, dx = \int_{\{Au \leqslant 0\}} (Au)(u^+)^{p-1}\, dx + \int_{\{Au \geqslant 0\}} (Au)(u^+)^{p-1}\, dx$$

we get using (2.9)

$$0 \leqslant \int_{\{Au \geqslant 0\}} (Au)(u^+)^{p-1}\, dx \leqslant \int_{\{Au \leqslant 0\}} (-Au)(u^+)^{p-1}\, dx$$

which yields for $u \in D$ and $p \geqslant p_0$

$$\left(\int_{\{Au \geqslant 0\}} (Au)(u^+)^{p-1}\, dx \right)^{1/p} \leqslant \left(- \int_{\{Au \leqslant 0\}} (Au)(u^+)^{p-1}\, dx \right)^{1/p}. \tag{2.10}$$

Now we can apply Proposition 2.3 to the left-hand side of (2.10) with $g = \chi_{\{Au \geqslant 0\}} Au$ and $f = u^+$, and the right-hand side of (2.10) with $g = -\chi_{\{Au \leqslant 0\}} Au$ and $f = u^+$. Since $D \subset \bigcap_{p \geqslant p_0} L^p \cap C_b$ we find

$$\text{(2.11)} \quad \operatorname{ess\,sup}\{u^+(x) : x \in \{Au > 0\}\} \leqslant \operatorname{ess\,sup}\{u^+(x) : x \in \{Au < 0\}\}.$$

Suppose first that u has only one isolated absolute positive maximum at x_0, $y_0 = u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geqslant 0$. Since Au is continuous, $x_0 \in \{Au \leqslant 0\}$. Indeed, if $x_0 \notin \{Au \leqslant 0\}$, then $x_0 \in \{Au > 0\}$, and (2.11) would entail that

$$y_0 = \operatorname{ess\,sup}\{u^+(x) : x \in \{Au > 0\}\} \leqslant \operatorname{ess\,sup}\{u^+(x) : x \in \{Au < 0\}\} < y_0,$$

since y_0 is the *unique* isolated supremum. Hence $x_0 \in \{Au \leqslant 0\}$ and $Au(x_0) \leqslant 0$.

Next we suppose that there are several positive maxima. We set

$$\mathscr{M}(u) := \left\{ y : u(y) \geqslant 0 \text{ and } u(y) = \sup_{x \in \mathbb{R}^n} u(x) \right\}.$$

Take some arbitrary $x_0 \in \mathscr{M}(u)$ and choose φ as in (2.7). For $n \in \mathbb{N}$ the function

$$\varphi_n(x) := r_n \varphi(n(x - x_0))$$

belongs to D where r_n is given by the real number

$$r_n^{-1} := n \sup_{|x - x_0| \leqslant 1/n} |(A\varphi(n(\cdot - x_0)))(x)|.$$

Note, that by our assumptions on φ we have $\varphi(0) > \varphi(x)$ for all $x \neq 0$. It follows that $v_n := u + \varphi_n \in D$ and

$$v_n(x_0) = u(x_0) + \varphi_n(x_0) \geqslant u(x) + \varphi_n(x_0) > u(x) + \varphi_n(x) = v_n(x).$$

Thus v_n has a single isolated absolute maximum at the point x_0 with $v_n(x_0) = u(x_0) + r_n$. We may apply the result of the first case to get

$$0 \geqslant (Av_n)(x_0) = (Au)(x_0) + (A\varphi_n)(x_0).$$

However for $(A\varphi_n)(x_0)$ we have

$$\begin{aligned} |(A\varphi_n)(x_0)| &= r_n |(A(\varphi(n(\cdot - x_0))))(x_0)| \\ &= \frac{1}{n} \frac{|(A(\varphi(n(\cdot - x_0))))(x_0)|}{\sup\limits_{|x - x_0| \leqslant 1/n} |(A\varphi(n(\cdot - x_0)))(x)|} \leqslant \frac{1}{n}, \end{aligned}$$

thus $\lim_{n \to \infty} |(A\varphi_n)(x_0)| = 0$, implying finally $0 \geqslant \lim_{n \to \infty} (Av_n)(x_0) = (Au)(x_0)$. Since $x_0 \in \mathscr{M}(u)$ was arbitrarily chosen, the theorem is proved. □

Remark 2.5. After this paper was finished in [54] the third-named author extended the results mentioned so far in this section into several directions. First he gave an extension of the notion of a Dirichlet operator for the case $p = 1$. Further he worked on an arbitrary measure spaces $(X, \mathscr{B}, m)$ when handling L^p-sub-Markovian semigroups and their generators.

Now we may combine the results in this section with those of Section 1. Let $(T_t^{(p)})_{t\geqslant 0}$, $1 < p < \infty$, be an L^p-sub-Markovian semigroup with L^q-extensions $(T_t^{(q)})_{t\geqslant 0}$, $p < q < \infty$. Suppose that there is a vector space D satisfying (2.5)–(2.8) when $(A^{(q)}, D(A^{(q)}))$ denotes the generator of $(T_t^{(q)})_{t\geqslant 0}$, $p \leqslant q < \infty$. Then it follows that the conclusion of Theorem 2.4 applies to $A := A^{(p)}|_D$. In particular it follows that $A^{(p)}|_D$ satisfies the positive maximum principle for $p \leqslant q < \infty$. In many concrete situations we can take $D = C_0^\infty$.

Theorem 2.6. *Let $(T_t^{(p)})_{t\geqslant 0}$ be an L^p-sub-Markovian semigroup and denote its L^q-extensions by $(T_t^{(q)})_{t\geqslant 0}$, $p < q < \infty$. Suppose that each of the generators $(A^{(q)}, D(A^{(q)}))$, maps C_0^∞ into C_b. Then $A^{(q)}|_{C_0^\infty}$ satisfies the positive maximum principle and hence, by the theorem of Ph. Courrège it has the structure* (2.1) *or* (2.2), *respectively.*

Remark 2.7 A. Having Theorem 2.6 in mind, it is clear that for constructing L^p-sub-Markovian semigroups one should start with operators defined on C_0^∞ having the structure (2.1) or (2.2) respectively.

B. With a different technique, see [54, Theorem 2.10], it is enough to assume in Theorem 2.6 that $A^{(q)}$ maps C_0^∞ into C_b for some $q \geqslant p$.

3 Subordination in the sense of Bochner

Subordination is a technique to obtain new semigroups from a given one. On the level of infinitesimal generators, subordination gives rise to a functional calculus. S. Bochner developed these ideas in his 1949 paper [5] and in his monograph [6].

Our references for this section are the monographs of C. Berg, G. Forst [4], and of S. Bochner [6] and the papers of R. S. Phillips [50], F. Hirsch [22], C. Berg, Kh. Boyadzhiev and R. deLaubenfels [3], and R. L. Schilling [52, 53].

Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on L^p (or C_∞) with generator $(A, D(A))$ and let $(\mu_t)_{t\geqslant 0}$ be a convolution semigroup of sub-probability measures supported in $[0, \infty)$. It is well known that these convolution semigroups are in one-to-one correspondence with Bernstein functions. This correspondence is given by

$$\int_{[0,\infty)} e^{-sx} \mu_t(ds) = e^{-tf(x)}, \quad t, x \geqslant 0.$$

In this situation the Bochner integral

$$(3.1)\qquad T_t^f u := \int_0^\infty T_s u \mu_t(ds), \quad u \in L^p \quad (\text{or } C_\infty),$$

is well-defined and gives a strongly continuous semigroup on L^p (or C_∞). Properties like contractivity, Markov or Feller property are passed over from $(T_t)_{t \geqslant 0}$ to $(T_t^f)_{t \geqslant 0}$.

Definition 3.1. Let $(T_t)_{t \geqslant 0}$ be a strongly continuous semigroup on L^p or C_∞ and let $(\mu_t)_{t \geqslant 0}$ be a vaguely continuous semigroup of sub-probability measures on $[0, \infty)$. The semigroup $(T_t^f)_{t \geqslant 0}$ given by (3.1) is called *subordinate semigroup*. Its generator is given by $(A^f, D(A^f))$.

For f to be a Bernstein function it is necessary and sufficient to satisfy the following Lévy-Khinchin-type representation

$$(3.2)\qquad f(x) = a + bx + \int_{(0,\infty)} (1 - e^{-tx}) \mu(dt)$$

with $a, b \geqslant 0$ and a measure μ on $(0, \infty)$ such that $\int_{(0,\infty)} t/(1+t) \mu(dt) < \infty$.

We will be mainly interested in the set of *complete Bernstein functions*, $\mathscr{CBF}$, which consists of those Bernstein functions f satisfying

$$\mu(dt) = m(t)\, dt, \quad m(t) = \int_{(0,\infty)} e^{-rt} \rho(dr)$$

with a measure ρ on $(0, \infty)$ such that $\int_{(0,\infty)} (1+t)^{-1} \frac{\rho(dt)}{t} < \infty$. It is not hard to see that $f \in \mathscr{CBF}$ has the representation

$$(3.3)\qquad f(x) = a + bx + \int_{(0,\infty)} \frac{x}{t+x} \frac{\rho(dt)}{t}, \quad x \geqslant 0.$$

Examples for complete Bernstein functions are the fractional powers, $f_\alpha(x) = x^\alpha$ $(0 \leqslant \alpha \leqslant 1)$ or the logarithm $f(x) = \log(1+x)$.

Using (3.2) and (3.3) one can obtain representation formulae for A^f. This problem was first investigated by R. S. Phillips in [50] for general Bernstein functions. Here we follow F. Hirsch [22], C. Berg, Kh. Boyadzhiev and R. deLaubenfels [3], R. L. Schilling [53], where it was shown (independently) that for $f \in \mathscr{CBF}$

$$A^f u = -au + bAu + \int_{(0,\infty)} A(\lambda\, \mathrm{id} - A)^{-1} u \frac{\rho(d\lambda)}{\lambda}, \quad u \in D(A),$$

holds. This is a straightforward generalisation of Balakrishnan's formula for fractional powers, e.g., Yosida [60, Chapter IX.11], in the sense that

$$A^{(x^\alpha)}|_{D(A)} = -(-A)^\alpha|_{D(A)}, \quad 0 < \alpha \leqslant 1.$$

In fact, as it was shown in [53], we have even

$$(3.4) \qquad D(A^f) = \left\{ u \in L^p(\text{or } C_\infty) : \lim_{k\to\infty} \int_0^\infty (T_t u - u) m_k(t)\, dt \text{ exists strongly} \right\}$$

where $m_k(t) = \int_0^k e^{-rt}\rho(dr)$. (A similar result is due to F. Hirsch [22].)

As in the case of fractional powers, subordination gives rise to a functional calculus that is in agreement with the classical Dunford-Taylor-integral, cf. Dunford-Schwartz [12, VII.9]. The next theorem collects some general results on $\mathscr{CBF}$ as well as material from [52, 53] on the functional calculus.

Theorem 3.2 A. *$\mathscr{CBF}$ is a convex cone that is stable under pointwise limits and composition of functions.*

B. *$A^f = -f(-A)$ with the resolvent of $-f(-A)$ being given by the Dunford-Taylor-integral.*

C. $A^{\alpha f} = \alpha A^f$, $A^{f+g} = \overline{A^f + A^g}$, $A^{f\circ g} = (A^g)^f$, $A^{\alpha + x + f} = -\alpha\,\mathrm{id} + A + A^f$.

D. *If $f \cdot g \in \mathscr{CBF}$ then $A^{fg} = -A^f A^g = -A^g A^f$.*

E. *If $f \in \mathscr{CBF}$ then $g(x) = \dfrac{x}{f(x)} \in \mathscr{CBF}$ and $A = -A^f A^g = -A^g A^f$.*

F. *If $f_n \in \mathscr{CBF}$ for any $n \in \mathbb{N}$ and $f_n \to f$ (pointwise) then $f \in \mathscr{CBF}$ and $A^{f_n} u \to A^f u$ strongly.*

The above equalities have to be understood as equalities between closed operators, their domains being given by (3.4).

Remark 3.3 A. Subordination has a nice stochastic interpretation: if there is a stochastic process related to $(T_t)_{t\geqslant 0}$, then $(T_t^f)_{t\geqslant 0}$ gives rise to a stochastic process and this process is obtained by a random time-change of the original process.

B. It is possible to extend Theorem 3.2 to the algebra generated by $\mathscr{CBF}$. For such f, however A^f is, in general, not any longer a generator of a semigroup but merely a closed operator. This was investigated in [53].

4 The Γ-transform of L^p-sub-Markovian semigroups

As before, $(T_t^{(p)})_{t\geqslant 0}$ denotes an L^p-sub-Markovian semigroup, $1 < p < \infty$. For $u \in L^p$ and $r > 0$ we define the *gamma-transform* of $(T_t^{(p)})_{t\geqslant 0}$ by

$$V_r^{(p)}u := \frac{1}{\Gamma\left(\frac{r}{2}\right)}\int_0^\infty t^{r/2-1}e^{-t}T_t^{(p)}u\,dt.$$

According to our considerations in Section 3, $(V_r^{(p)})_{r\geqslant 0}$ is an L^p-sub-Markovian semigroup obtained from $(T_t^{(p)})_{t\geqslant 0}$ by subordination in the sense of Bochner. The corresponding Bernstein function is given by $f(s) = \frac{1}{2}\log(1+s)$, and the corresponding convolution semigroup $(\eta_t)_{t\geqslant 0}$ is given by

$$\eta_t(ds) = \chi_{[0,\infty)}(s)\frac{1}{\Gamma\left(\frac{t}{2}\right)}s^{t/2-1}e^{-s}\,ds.$$

Thus we have $\|V_r^{(p)}u|L^p\| \leqslant \|u|L^p\|$ and $V_{r_1}^{(p)}V_{r_2}^{(p)} = V_{r_1+r_2}^{(p)}$. Moreover, according to a result of A. Carasso and T. Kato, see [9], the semigroup $(V_r^{(p)})_{r\geqslant 0}$ is always analytic.

Theorem 4.1. *Let $(A^{(p)}, D(A^{(p)}))$ be the generator of the L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$. For all $r > 0$ and all $u \in L^p$ we have*

$$V_r^{(p)}u = (\mathrm{id} - A^{(p)})^{-r/2}u.$$

In particular, each of the operators $V_r^{(p)}$ is injective.

Proof. Denote by f the Bernstein function $f(s) = \frac{1}{2}\log(1+s)$. We already know that $V_r^{(p)} = T_t^{(p),f}$. On $D(A^{(p)})$ the resolvent at $\lambda > 0$ of $(A^{(p)}, D(A^{(p)}))$ satisfies

$$((\lambda+1)\,\mathrm{id} - A^{(p)})^{-1}u = \int_0^\infty e^{-(\lambda+1)t}T_t^{(p)}u\,dt, \quad u \in D(A^{(p)}).$$

From the Dunford-Taylor calculus for unbounded operators we have

$$(\mathrm{id} - A^{(p)})^{-s}u = \frac{1}{2\pi i}\int_\Gamma \xi^{-s}((\xi+1)\mathrm{id} - A^{(p)})^{-1}u\,d\xi, \tag{4.1}$$

for any arc Γ extending from $-\infty$ to $+\infty$ inside the resolvent set of $(\mathrm{id} - A^{(p)})$. By standard techniques for analytic (operator valued) integrals we get from (4.1) the representation

$$(\mathrm{id} - A^{(p)})^{-s}u = \frac{\sin s\pi}{\pi}\int_0^\infty \lambda^{-s}((\lambda+1)\mathrm{id} - A^{(p)})^{-1}u\,d\lambda$$

see T. Kato [39, V.11, Lemma 3.4] for the Hilbert space case or [37, Lemma 6.1] for the Banach space situation. Taking $0 < s < 1$ we obtain

$$(\mathrm{id} - A^{(p)})^{-s}u = \frac{\sin s\pi}{\pi}\int_0^\infty \lambda^{-s}\left(\int\limits_0^\infty e^{-(\lambda+1)t}T_t^{(p)}u\,dt\right)d\lambda$$

$$= \frac{\sin s\pi}{\pi}\int_0^\infty\int_0^\infty \lambda^{-s}e^{-(\lambda+1)t}T_t^{(p)}u\,dt\,d\lambda.$$

Observe that for $0 < s < 1$ the term $\lambda^{-s}e^{-(\lambda+1)t}$ is positive. By Tonelli's theorem we get

$$\int_0^\infty\int_0^\infty \lambda^{-s}e^{-(\lambda+1)t}\,dt\,d\lambda = \int_0^\infty\int_0^\infty \lambda^{-s}e^{-(\lambda+1)t}\,d\lambda\,dt$$

$$= \int_0^\infty\int_0^\infty \left(\frac{\mu}{t}\right)^{-s} e^{-\mu}e^{-t}\frac{d\mu}{t}\,dt$$

$$= \int_0^\infty \mu^{-s}e^{-\mu}\,d\mu\int_0^\infty t^{s-1}e^{-t}\,dt < \infty.$$

Since $\|\lambda^{-s}e^{-(\lambda+1)t}T_t^{(p)}u|L^p\| \leqslant \lambda^{-s}e^{-(\lambda+1)t}\|u|L^p\|$, we find for $0 < s < 1$ that

$$(\mathrm{id} - A^{(p)})^{-s}u = \frac{\sin s\pi}{\pi}\int_0^\infty\int_0^\infty \lambda^{-s}e^{-(\lambda+1)t}d\lambda T_t^{(p)}u\,dt$$

$$= \frac{\sin s\pi}{\pi}\int_0^\infty\int_0^\infty \lambda^{-s}e^{-\lambda t}\,d\lambda e^{-t}T_t^{(p)}u\,dt$$

$$= \frac{\sin s\pi}{\pi}\int_0^\infty\int_0^\infty \mu^{-s}e^{-\mu}\,d\mu t^{s-1}e^{-t}T_t^{(p)}u\,dt$$

$$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t}T_t^{(p)}u\,dt = V_{2s}^{(p)}u.$$

Thus we have proved for $0 < r < 2$ that

(4.2) $\quad (\mathrm{id} - A^{(p)})^{-r/2}u = V_r^{(p)}u, \quad u \in D(A^{(p)}).$

Due to the semigroup property of $V_r^{(p)}$ and the well-known functional calculus, see N. Dunford and J. Schwartz, [12, VIII.9, Theorem 8] we may extend equality (4.2) to all $r > 0$. The strong continuity of $(V_r^{(p)})_{r\geqslant 0}$ finally shows (4.2) also for $r = 0$. □

Since each of the operators $V_r^{(p)}$ is injective we may define for $1 < p < \infty$ the spaces

(4.3) $\quad \mathscr{F}_{r,p} := V_r^{(p)}(L^p) \quad$ and $\quad \|u|\mathscr{F}_{r,p}\| := \|v|L^p\| \quad$ for $u = V_r v$.

Clearly $(\mathscr{F}_{r,p}, \|\cdot|\mathscr{F}_{r,p}\|)$ is a separable Banach space.

Corollary 4.2. *In the situation of Theorem 4.1 we have* $\mathscr{F}_{r,p} = D((\mathrm{id} - A^{(p)})^{r/2})$.

Proof. Clearly, $u \in \mathscr{F}_{r,p}$ if, and only if, $u = V_r^{(p)} v$ for some $v \in L^p$. Since $V_r^{(p)} = (\mathrm{id} - A^{(p)})^{-r/2}$, we have $u \in \mathscr{F}_{r,p}$ if, and only if, $u = (\mathrm{id} - A^{(p)})^{-r/2} v$ which implies that $D((\mathrm{id} - A^{(p)})^{r/2}) = \mathscr{F}_{r,p}$. □

Remark 4.3. In the Hilbert space case, i.e. $p = 2$, and if $(A^{(2)}, D(A^{(2)}))$ is a selfadjoint generator, the results of Theorem 4.1 and its corollary are well known (see our comments in the next section) and are proved by the spectral theorem for selfadjoint operators.

We want to give some representation formulae for the generator $(A^{(p),f}, D(A^{(p),f}))$ of the semigroup $(V_r^{(p)})_{r \geqslant 0}$. The next result is contained in Theorem 4.1 and Example 4.2 of [52].

Corollary 4.4. *For all* $u \in D(A^{(p)})$ *it follows that*

$$A^{(p),f} u = \frac{1}{2} \int_1^\infty \frac{1}{\lambda} A^{(p)} (\lambda\, \mathrm{id} - A^{(p)})^{-1} u \, d\lambda$$

$$= \frac{1}{2} \int_0^\infty \frac{1}{\lambda + 1} A^{(p)} ((\lambda + 1)\, \mathrm{id} - A^{(p)})^{-1} u \, d\lambda.$$

The very definition of the logarithm of an operator in Banach space, cf. V. Nollau [47], proves the following auxiliary result.

Corollary 4.5. *For* $u \in D(A^{(p)})$ *we have* $A^{(p),f} u = -\dfrac{1}{2} \log(\mathrm{id} - A^{(p)} u)$.

We will now examine the domain of the operator $A^{(p),f}$ in greater detail.

Proposition 4.6. *For all* $\alpha > 0$ *we have* $D([A^{(p)}]^\alpha) \subset D(A^{(p),f})$.

Proof. By Corollary 2.10 in [52] it is sufficient to show that for $u \in D([A^{(p)}]^\alpha)$ the integral

$$\int_1^\infty \frac{1}{\lambda} A^{(p)} (\lambda\, \mathrm{id} - A^{(p)}) u \, d\lambda$$

converges strongly. According to V. Nollau [46, Lemma 2] we have for $\lambda > 0$

$$\| [A^{(p)}]^{1-\alpha} (\lambda\, \mathrm{id} - A^{(p)})^{-1} \|_{L^p \to L^p} \leqslant 2 \frac{\sin \pi(1-\alpha)}{\pi(1-\alpha)} \frac{\lambda^{-\alpha}}{\alpha}.$$

If $u \in D([A^{(p)}]^\alpha)$ then

$$\left\| \frac{1}{\lambda} A^{(p)}(\lambda \,\mathrm{id} - A^{(p)})^{-1} u | L^p \right\|$$

$$\leqslant \frac{1}{\lambda} \| [A^{(p)}]^{1-\alpha} (\lambda \,\mathrm{id} - A^{(p)})^{-1} \|_{L^p \to L^p} \| [A^{(p)}]^{\alpha} u | L^p \|,$$

which yields

$$\| A^{(p),f} u | L^p \| \leqslant c_\alpha \int_1^\infty \lambda^{-\alpha-1} \, d\lambda \| [A^{(p)}]^\alpha u | L^p \|$$

and this proves the proposition. □

Corollary 4.7. *We have* $\bigcup_{\alpha>0} D([A^{(p)}]^\alpha) \subset D(A^{(p),f})$ *and*

$$A^{(p),f}|_{\bigcup\limits_{\alpha>0} D([A^{(p)}]^\alpha)} = -\frac{1}{2} \log(\mathrm{id} - A^{(p)})|_{\bigcup\limits_{\alpha>0} D([A^{(p)}]^\alpha)}.$$

5 Refinements for analytic L^p-sub-Markovian semigroups

Given an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t \geqslant 0}$, $1 < p < \infty$, it is of course possible to define for any Borel set $A \subset \mathbb{R}^n$ with finite Lebesgue measure $\lambda^{(n)}(A) < \infty$ the function $p_t(x, A) = T_t^{(p)} \chi_A(x)$. As an element in L^p, the function $x \mapsto p_t(x, A)$ is only *almost everywhere* determined; it is therefore not possible to use the family $p_t(x, A)$, $t \geqslant 0$, $x \in \mathbb{R}^n$, $A \in \mathscr{B}^n$, in order to construct a Markov process. However, if it would be possible to find for each $A \in \mathscr{B}^n$, with $\lambda^{(n)}(A) < \infty$, and all $t > 0$ a *unique* representative $\tilde{p}_t(\cdot, A)$ of $x \mapsto p_t(x, A)$ such that the Chapman-Kolmogorov equations

$$\tilde{p}_{t+s}(x, A) = \int_{\mathbb{R}^n} \tilde{p}_t(y, A) \tilde{p}_s(x, dy)$$

hold, we could construct a Markov process starting in *every* point $x \in \mathbb{R}^n$. Clearly, if $T_t^{(p)}$ maps for all $t > 0$ the space L^p into $C \cap L^p$ (in the sense that $T_t^{(p)} u$ is considered to be a uniquely determined continuous function), i.e., if $(T_t^{(p)})_{t \geqslant 0}$ is strongly L^p-sub-Markovian in the sense of Definition 1.11.B, then we are done.

Another approach is to use capacities and to define the process only up to a set $\mathscr{N}$ of capacity zero in the state space. The drawback of this method is that—unless the set $\mathscr{N}$ is the empty set—the process is only defined on $\mathbb{R} \backslash \mathscr{N}$, i.e., it can only start at points *outside* $\mathscr{N}$. This was the idea of M. Fukushima in [16] where he used Dirichlet forms to construct Hunt processes.

In this section we will first discuss conditions for an L^p-sub-Markovian semigroup to be strongly L^p-sub-Markovian, and then the theory of (r, p)-capacities and its application to analytic L^p-sub-Markovian semigroups.

The concept of (r, p)-capacities was introduced by P. Malliavin in [44], see also [45], and many investigations have been done in the context of sub-Markovian semigroups by M. Fukushima and H. Kaneko, see [17, 18], [20] and [38].

In order to study capacities we need further properties of the spaces $\mathscr{F}_{r,p}$ defined in Section 4, (4.2)–(4.3).

Lemma 5.1 A. *For all* $s, r \geqslant 0$ *we have* $\mathscr{F}_{r+s,\,p} \subset \mathscr{F}_{r,p}$.

B. *For* $k \in \mathbb{N}$ *the space* $\mathscr{F}_{k+2,p}$ *is dense in* $\mathscr{F}_{k,p}$.

Proof. A. Since $\mathscr{F}_{r+s,p} = V_{r+s}^{(p)}(L^p) = V_r^{(p)} V_s^{(p)}(L^p)$, we have $\mathscr{F}_{r+s,p} = V_r^{(p)}(\mathscr{F}_{s,p})$. Using $V_r^{(p)} = (\mathrm{id} - A^{(p)})^{-r/2}$, we find

$$\begin{aligned}\|u|\mathscr{F}_{r,p}\| &= \|(\mathrm{id} - A^{(p)})^{r/2} u|L^p\| \\ &= \|(\mathrm{id} - A^{(p)})^{-s/2}(\mathrm{id} - A^{(p)})^{(r+s)/2} u|L^p\| \\ &\leqslant \|(\mathrm{id} - A^{(p)})^{(r+s)/2} u|L^p\| \\ &= \|u|\mathscr{F}_{r+s,p}\|,\end{aligned}$$

where we used that $V_s^{(p)}$ is a contraction.

B. We know from Corollary 4.2 that the space $\mathscr{F}_{k,p}$ is given by $(\mathrm{id} - A^{(p)})^{-k/2}(L^p)$. Let $u \in \mathscr{F}_{k,p}$ with the representation $u = (\mathrm{id} - A^{(p)})^{-k/2} f$, $f \in L^p$. Since $D(A^{(p)})$ is dense in L^p we find for every $\varepsilon > 0$ some $\omega_\varepsilon \in D(A^{(p)})$ such that $\|f - \omega_\varepsilon|L^p\| < \varepsilon$.

Set $h_\varepsilon := (\mathrm{id} - A^{(p)})\omega_\varepsilon$. Then we have $h_\varepsilon \in L^p$ and $\omega_\varepsilon = (\mathrm{id} - A^{(p)})^{-1} h_\varepsilon$. It follows

$$(\mathrm{id} - A^{(p)})^{-k/2}\omega_\varepsilon = (\mathrm{id} - A^{(p)})^{(-k-2)/2} h_\varepsilon \in \mathscr{F}_{k+2,p}.$$

Furthermore,

$$\begin{aligned}&\|u - (\mathrm{id} - A^{(p)})^{(-k-2)/2} h_\varepsilon|\mathscr{F}_{k,p}\| \\ &\quad = \|(\mathrm{id} - A^{(p)})^{-k/2}(f - (\mathrm{id} - A^{(p)})^{-1}) h_\varepsilon|\mathscr{F}_{k,p}\| \\ &\quad = \|f - (\mathrm{id} - A^{(p)})^{-1} h_\varepsilon|L^p\| \\ &\quad = \|f - \omega_\varepsilon|L^p\| < \varepsilon\end{aligned}$$

which proves the lemma. □

The spaces $\mathscr{F}_{r,p}$ should be considered as abstract Bessel potential spaces associated with the generator $(A^{(p)}, D(A^{(p)}))$ of $(T_t^{(p)})_{t\geqslant 0}$. Clearly one can try to associate the corresponding Riesz potential spaces. This was done in [15, Section 1.5] and we just quote the important estimates

$$\gamma_{r,p}(\|u|L^p\| + \|(-A^{(p)})^r u|L^p\|) \leqslant \|u|\mathscr{F}_{r,p}\| \leqslant \tilde{\gamma}_{r,p}(\|u|L^p\| + \|(-A^{(p)})^r u|L^p\|)$$

which hold for all $r \geqslant 0$ and all $u \in D((-A^{(p)})^{[r]+1}) = D((\mathrm{id} - A^{(p)})^{[r]+1})$.

Definition 5.2. Let $(T_t^{(p)})_{t\geqslant 0}$ be an L^p-sub-Markovian semigroup and $\mathscr{F}_{r,p}$ as above. We call $\mathscr{F}_{r,p}$ *regular* if $\mathscr{F}_{r,p} \cap C$ is dense in $(\mathscr{F}_{r,p}, \|\cdot|\mathscr{F}_{r,p}\|)$.

Proposition 5.3. *Let $k \in \mathbb{N}$ and suppose that the set $C \cap D([A^{(p)}]^k)$ is an operator core for $[A^{(p)}]^k$. Then $\mathscr{F}_{2k,p}$ is regular.*

Proof. First note that $D((\mathrm{id} - A^{(p)})^k) = D([A^{(p)}]^k)$, hence we have

$$C \cap \mathscr{F}_{2k,p} = C \cap D([A^{(p)}]^k).$$

Since $C \cap \mathscr{F}_{2k,p}$ is an operator core for $[A^{(p)}]^k$, we may choose for every $u \in \mathscr{F}_{2k,p}$ a sequence $(u_\nu)_{\nu\in\mathbb{N}}$, $u_\nu \in C \cap \mathscr{F}_{2k,p}$, which converges in the graph norm of $[A^{(p)}]^k$ to u. This implies

$$\|u_\nu - u|\mathscr{F}_{2k,p}\| = \|(V_{2k}^{(p)})^{-1}(u_\nu - u)|L^p\| = \|(\mathrm{id} - A^{(p)})^k(u_\nu - u)|L^p\| \to 0. \qquad \square$$

Thus the regularity problem for $\mathscr{F}_{r,p}$ can be reduced to find a good operator core for $A^{(p)}$ or $[A^{(p)}]^k$. Our next theorem gives a first answer when one can find a good version of $p_t(x, A)$.

Theorem 5.4. *Let $(T_t^{(p)})_{t\geqslant 0}$ be an analytic L^p-sub-Markovian semigroup with generator $(A^{(p)}, D(A^{(p)}))$. If for some $k_0 \in \mathbb{N}$ the space $D([A^{(p)}]^{k_0})$ is contained in $C \cap L^p$, then all the spaces $\mathscr{F}_{r,p}$, $r > 0$, are regular, and $(T_t^{(p)})_{t\geqslant 0}$ is a strong L^p-sub-Markovian semigroup, i.e., maps L^p into $L^p \cap C$.*

Proof. Because of the analyticity of $(T_t^{(p)})_{t\geqslant 0}$ we find $T_t^{(p)} u \in \bigcap_{k\in\mathbb{N}} D([A^{(p)}]^k)$. By assumption, there is some $k_0 \in \mathbb{N}$ such that $D([A^{(p)}]^{k_0}) \subset C \cap L^p$, hence, $\mathscr{F}_{2k_0,p}$ is regular, and for all $t > 0$ we have $T_t^{(p)} u \in C \cap L^p$. From Lemma 5.1.B we deduce further that in this situation $\mathscr{F}_{r,p}$ is regular for all $r > 0$. $\square$

Let us give a first instructive example for an application of Theorem 5.4. It is well known that many second order elliptic differential operators

$$L(x, D)u(x) = \sum_{k,l=1}^{n} a_{kl}(x) \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x) \frac{\partial u(x)}{\partial x_j} + c(x)u(x)$$

are L^p-Dirichlet operators, see [35], and extend to generators $A^{(p)}$ of analytic L^p-contraction semigroups, see [42]. Under mild regularity assumptions on the coefficients one can prove that the domain $D(A^{(p)})$ of the generator is the Sobolev space W_p^2, see again [42]. By the Sobolev embedding theorem,

$$W_p^2 \hookrightarrow C_\infty \quad \text{for } p > \frac{n}{2}.$$

and the analyticity of $(T_t^{(p)})_{t \geqslant 0}$ for $p > \frac{n}{2}$ implies

$$T_t^{(p)} u \in \bigcap_{k \geqslant 0} D((-A^{(p)})^k) \subset W_p^2 \subset C_\infty.$$

This, however, means that $(T_t^{(p)})_{t \geqslant 0}$ is already a strong L^p-sub-Markovian semigroup.

This example suggests a strategy to find strong L^p-sub-Markovian semigroups: Determine the domain of its generator in terms of function spaces and prove good embedding results for these function spaces.

Clearly one cannot expect every L^p-sub-Markovian semigroup to be strongly L^p-sub-Markov. Therefore we aim to find good representatives of $T_t^{(p)} \chi_A(\cdot)$ on a subset $\mathbb{R}^n \backslash N$ where N is negligible in an appropriate sense. This can be achieved by introducing a capacity $\mathrm{cap}_{r,p}$ in each of the spaces $\mathscr{F}_{r,p}$.

Let us recall some results due to M. Fukushima and H. Kaneko. For an open set $G \subset \mathbb{R}^n$ we define the (r, p)-capacity by

$$\mathrm{cap}_{r,p}(G) := \inf\{\|u|\mathscr{F}_{r,p}\|^p : u \in \mathscr{F}_{r,p} \text{ and } u \geqslant 1 \text{ a.e. on } G\}.$$

Defining for an arbitrary set $E \subset \mathbb{R}^n$

$$\mathrm{cap}_{r,p}(E) = \inf\{\mathrm{cap}_{r,p}(G) : E \subset G \text{ and } G \text{ open}\},$$

$\mathrm{cap}_{r,p}$ extends to an outer capacity. The following lemma can be found in [20].

Lemma 5.5. *Let $(T_t^{(p)})_{t \geqslant 0}$ be an L^p-sub-Markovian semigroup.*

A. *For any measurable set $E \subset \mathbb{R}^n$ we have:* $\lambda^{(n)}(E) \leqslant \mathrm{cap}_{r,p}(E)$.

B. *Whenever $E \subseteq F \subset \mathbb{R}^n$, $r \leqslant r'$, or $p \leqslant p'$ then* $\mathrm{cap}_{r,p}(E) \leqslant \mathrm{cap}_{r',p'}(F)$.

C. *For any sequence $(E_j)_{j \in \mathbb{N}}$ of subsets of $\mathbb{R}^n$ we have* $\mathrm{cap}_{r,p}(\bigcup_{j=1}^\infty E_j) \leqslant \sum_{j=1}^\infty \mathrm{cap}_{r,p}(E_j)$.

Using $\mathrm{cap}_{r,p}$ we may introduce the concepts of exceptional sets and quasi-continuous functions.

Definition 5.6. Let $(T_t^{(p)})_{t \geqslant 0}$ be an L^p-sub-Markovian semigroup.

A. A set $N \subset \mathbb{R}^n$ satisfying $\mathrm{cap}_{r,p}(N) = 0$ is called (r,p)-exceptional (w.r.t. $(T_t^{(p)})_{t \geqslant 0}$).

B. A statement is said to hold (r,p)-*quasi-everywhere* (w.r.t. $(T_t^{(p)})_{t \geqslant 0}$) if there exists an (r,p)-exceptional set N such that the statement holds on $\mathbb{R}^n \backslash N$. We will use the abbreviation (r,p)-q.e. for (r,p)-quasi-everywhere.

C. A real valued function u defined (r,p)-quasi-everywhere on $\mathbb{R}^n$ is called (r,p)-*quasi-continuous* (w.r.t. $(T_t^{(p)})_{t \geqslant 0}$) if for any $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $\mathrm{cap}_{r,p}(G) < \varepsilon$ and $u|_{G^c}$ is continuous.

The following theorem is again taken from [20].

Theorem 5.7. *Let $(T_t^{(p)})_{t \geqslant 0}$ be an L^p-sub-Markovian semigroup and assume $\mathscr{F}_{r,p}$ is regular.*

A. *If u is (r,p)-quasi-continuous and $u \geqslant 0$ a.e. on an open set G, then $u \geqslant 0$ (r,p)-q.e. on G.*

B. *Each $u \in \mathscr{F}_{r,p}$ admits an (r,p)-quasi-continuous modification denoted by $\tilde{u}$, and we have*

$$\mathrm{cap}_{r,p}(\{|\tilde{u}| > \varrho\}) \leqslant \frac{1}{\varrho^p} \|u|\mathscr{F}_{r,p}\|^p, \quad \varrho > 0.$$

Further we have, see [20],

Proposition 5.8. *For any $A \subset \mathbb{R}^n$ with finite (r,p)-capacity there exists a unique function $e_A \in \{u \in \mathscr{F}_{r,p} : \tilde{u} \geqslant 1 \ (r,p)\text{-q.e. on } A\}$ minimising the norm $\|\cdot|\mathscr{F}_{r,p}\|$. The function e_A is non-negative and satisfies*

$$\mathrm{cap}_{r,p}(A) = \|e_A|\mathscr{F}_{r,p}\|^p.$$

For the next results of this section one should note that the semigroup $(T_t^{(p)})_{t \geqslant 0}$ has to be *symmetric* and *analytic*. The next proposition is due to H. Kaneko, see [38].

Proposition 5.9. *Let $(T_t^{(p)})_{t \geqslant 0}$ be a symmetric, analytic L^p-sub-Markovian semigroup and suppose that $\mathscr{F}_{r,p}$ is regular. For each $u \in L^p$ we can choose a function $\widetilde{T_t^{(p)}u}$ such that the function $(x,t) \mapsto \widetilde{T_t^{(p)}u}(x)$ has the following properties:*

(i) *For each $t > 0$ the function $x \mapsto \widetilde{T_t^{(p)}u}(x)$ is an (r,p)-quasi-continuous version of $T_t^{(p)}u$. Moreover, for any $\varepsilon > 0$ there exists an open set G independent of t such that $\mathrm{cap}_{r,p}(G) < \varepsilon$ and the functions $x \mapsto \widetilde{T_t^{(p)}u}(x)$ are continuous on $\mathbb{R}^n \backslash G$ for all $t > 0$.*

(ii) *For (r,p)-quasi-every $x \in \mathbb{R}^n$ the function $t \mapsto \widetilde{T_t^{(p)}u}(x)$ is analytic.*

For our purposes it is important to note that Proposition 5.9 allows to select a nice representative for the function $p_t(x,B) = T_t^{(p)}\chi_B(x)$. In particular, suppose that we can find a real number r_0 such that $\mathrm{cap}_{r_0,p}(A) = 0$ implies $A = \emptyset$. Then it follows that we have even a *continuous* representative for $x \mapsto T_t^{(p)}u(x)$, $u \in L^p$, and $(T_t^{(p)})_{t\geqslant 0}$ is strongly L^p-sub-Markovian. This proves

Theorem 5.10. *Let $(T_t^{(p)})_{t\geqslant 0}$ be a symmetric, analytic L^p-sub-Markovian semigroup and suppose that for some $r_0 > 0$ the space $\mathscr{F}_{r_0,p}$ is regular and that for every $A \subset \mathbb{R}^n$ such that $\mathrm{cap}_{r_0,p}(A) = 0$ it follows that $A = \emptyset$. Then $(T_t^{(p)})_{t\geqslant 0}$ is a strong L^p-sub-Markovian semigroup, i.e., each $T_t^{(p)}$ maps L^p into $L^p \cap C$.*

We have already remarked that the regularity problem for $\mathscr{F}_{r,p}$ can be solved by characterising these spaces or the spaces $D((-A^{(p)})^k)$ in terms of function spaces. A criterion for the condition

$$(5.1) \qquad \mathrm{cap}_{r,p}(A) = 0 \quad \text{implies } A = \emptyset.$$

can also be derived when characterising the spaces $\mathscr{F}_{r,p}$ or the spaces $D((-A^{(p)})^k)$ in terms of function spaces. Thus if $\mathscr{F}_{r,p} \hookrightarrow C_\infty$ for some values of r and p then we have

$$\inf\{\mathrm{cap}_{r,p}(E) : \emptyset \neq E \subset \mathbb{R}^n\} > 0,$$

i.e., every statement which holds (r,p)-quasi-everywhere reduces already to a statement which holds everywhere. Therefore we are looking for Sobolev–type embeddings for the spaces $\mathscr{F}_{r,p}$. Conversely, (5.1) already implies the inclusion $\mathscr{F}_{r,p} \subset C_\infty$.

Remark 5.11. We may combine Proposition 5.9 and Corollary 4.2 with H. Kaneko's construction of Hunt processes associated to a symmetric, analytic L^p-sub-Markovian semigroup to find that if $D(A^{(p)})$ is regular, then the process is determined up to a $\mathrm{cap}_{2,p}$-exceptional set. In particular, for $p = 2$, i.e., the Dirichlet form situation, it follows that the process is always determined up to an exceptional set determined by the regular domain of the generator, not only up to an exceptional set determined by the domain of the Dirichlet form.

6 L^p-domains of generators of Lévy processes

In this section we recall some results from [15] and outline possible applications. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function with representation

$$(6.1) \qquad \psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\nu(dx)$$

where ν is a Lévy measure integrating the function $x \mapsto 1 \wedge |x|^2$. For any $R > 0$ we decompose ψ according to $\psi(\xi) = \psi_R(\xi) + \tilde{\psi}_R(\xi)$ where

$$\psi_R(\xi) := \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(x\cdot\xi))\chi_{B(0,R)}(x)\nu(dx) \quad \text{and} \quad \tilde{\psi}_R(\xi) := \psi(\xi)-\psi_R(\xi).$$

Both ψ_R and $\tilde{\psi}_R$ are continuous and negative definite. Moreover, ψ_R is smooth and polynomially bounded, whereas $\tilde{\psi}_R$ is just bounded. We define on $\mathscr{S}$ the norm

$$\|u\|_{\psi,R,p} := \|(\mathrm{id} + \psi_R(D))u|L^p\|, \quad R > 0,\ 1 < p < \infty.$$

The following result can be found in [15].

Theorem 6.1. *For $R > 0$ and $S > 0$ the norms $\|\cdot\|_{\psi,R,p}$ and $\|\cdot\|_{\psi,S,p}$ are equivalent and each of these norms is equivalent to the norm $\|(\mathrm{id} + \psi(D))u|L^p\|$.*

Further, let us define the space

$$H_p^{\psi,2} := \{u \in L^p : \|u|H_p^{\psi,2}\| < \infty\}$$

where

$$\|u|H_p^{\psi,2}\| := \|(\mathrm{id} + \psi(D))u|L^p\|.$$

Then we find that $\mathscr{S}$ is dense in $H_p^{\psi,2}$ and, in addition, the next theorem holds.

Theorem 6.2. *The generator of the L^p-sub-Markovian semigroup associated with the continuous negative definite function* (6.1) *has as its domain the space $H_p^{\psi,2}$.*

Using Theorem 6.2 we may extend $H_p^{\psi,2}$ now to the scale

$$H_p^{\psi,s} := \mathscr{F}_{s,p,\psi}, \quad 0 \leqslant s < \infty,$$

where $\mathscr{F}_{s,p,\psi}$ is the (abstract) Bessel potential space associated with the L^p-generator $-\psi(D)$. Again it is possible to prove that $\mathscr{S}$ is dense in $H_p^{\psi,s}$ and for the purposes of this paper we have the following important embedding results.

Theorem 6.3. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function as in* (6.1), *$1 < p < \infty$, and $s > 0$. Then $H_p^{\psi,s} \hookrightarrow C_\infty$ if, and only if,*

$$\mathscr{F}^{-1}[(1 + \psi(\cdot))^{-s/2}] \in L^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{6.2}$$

Note that (6.2) means that $(1 + \psi(\cdot))^{-s/2}$ must be a Fourier multiplier of type (p, ∞). Here is a sufficient criterion for (6.2) to hold:

Theorem 6.4. *Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function with representation* (6.1) *and such that*

$$(6.3) \qquad 1 + \psi(\xi) \geqslant c_0(1 + |\xi|^2)^{r_0}, \quad \xi \in \mathbb{R}^n,$$

holds for some constant $c_0 > 0$ *and some* $0 < r_0 \leqslant 1$. *For* $0 < \varepsilon < 1$ *and* $\frac{1}{2-\varepsilon} < \theta < 1$ *we have the following continuous embedding*

$$H_p^{\psi, \theta n/r_0} \hookrightarrow C_\infty \quad \text{if } p = p_{\varepsilon,\theta} := \frac{1 + \theta\varepsilon}{1 + (\varepsilon - 1)\theta}.$$

Let us point out by a simple but concrete example that Theorem 6.4 gives indeed non-trivial results. Choose $\varepsilon = \frac{1}{2}$, $\theta = \frac{3}{4}$, and $r_0 = \frac{34}{8}$. It follows that $H_{11/5}^{\psi,2}$ is embedded into C_∞ provided $n = 1, 2$.

The last fact which we need about the spaces $H_p^{\psi,s}$ is that they are compatible under complex interpolation $[\cdot\,,\cdot]_\theta$.

Theorem 6.5. *Let* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous negative definite function as in* (6.1), *let* $s_0, s_1 \geqslant 0$, *let* $1 < p_0,\ p_1 < \infty$, *and* $0 < \theta < 1$. *Set*

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$[H_{p_0}^{\psi,s_0}, H_{p_1}^{\psi,s_1}]_\theta = H_p^{\psi,s}.$$

One should note that the representation (6.1) excludes the continuous negative definite function $\xi \mapsto |\xi|^2$. However, this function leads to the classical Bessel potential spaces H_p^s, see e.g. [56], and these spaces are well understood. Observe that in this case the homogeneity of $\xi \mapsto |\xi|^2$ leads to sharper results. For example, we have the embedding

$$H_p^s \hookrightarrow C_\infty \quad \text{for } s > \frac{n}{p}.$$

Let us point out the difference between the Hilbert space $(p = 2)$ and the Banach space settings. The norm in $H_p^{\psi,s}$ is given by

$$\|(\mathrm{id} + \psi(D))^{s/2}u|L^p\| = \|\mathscr{F}^{-1}[(1 + \psi(\cdot))^{s/2}\hat{u}]|L^p\|.$$

In the Hilbert space case we may apply Plancherel's theorem to get

$$\|u|H_2^{\psi,s}\| = \|(1 + \psi(\cdot))^{s/2}\hat{u}|L^2\|,$$

and in this case estimates for ψ directly imply to norm estimates! This makes the

Hilbert space case much easier to handle. As in our previous papers we will write $H^{\psi,s} = H_2^{\psi,s}$.

Now let us indicate how one can use the results from this section provided a good L^p-theory for the operators $q(x,D)$ does exist. For this, suppose that $-q(x,D)$ extends to a generator of an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ with domain $H_p^{\psi,2}$ where ψ is a fixed continuous negative definite function. If, in addition, ψ satisfies the assumptions of Theorem 6.3, or the more concrete assumptions of Theorem 6.4, and if the semigroup $(T_t^{(p)})_{t\geqslant 0}$ is analytic, then $(T_t^{(p)})_{t\geqslant 0}$ is a strong L^p-sub-Markovian semigroup and we may associate with $-q(x,D)$ a Hunt process *without any exceptional set*.

Remark 6.6. Much work has been done for parabolic pseudo-differential operators with symbols in some *classical* symbol classes, for example for operators of type $\frac{\partial}{\partial t} - q(x,D)$, with $q \in S^m_{\rho\delta}$. Clearly, under the assumption that $q(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function and $q \in S^m_{\rho\delta}$, we can apply the existing parabolic theory of pseudo-differential operators to construct sub-Markovian or Feller semigroups and processes. For this we refer to the monographs of H. Kumano-go [40] and that of G. Grubb [21] as well as to the survey of S. D. Ejdel'man [14]. The simplest example is the operator $(1-\Delta)^t$, $0 < t < 1$, which is an L^p-Dirichlet operator with domain H_p^{2t}; obviously, $\xi \mapsto (1+|\xi|^2)^t$ is a continuous negative definite function.

7 Subordination of second order elliptic differential operators

We start with a result which follows from the estimates for elliptic differential operators given by F. Browder in [8]. A detailed treatment can be found in [36], in particular Theorem 6.1.44.

Theorem 7.1. *Let*

$$L(x,D) = \sum_{k,l=1}^{n} a_{kl}(x)\frac{\partial^2}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x)$$

be a uniformly elliptic operator of second order with coefficients $a_{kl} = a_{lk} \in C_b^2$, $b_j \in C_b^1$ and $c \in C_b$. In addition suppose that $c(x) \leqslant 0$ and

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(b_j(x) - \sum_{k=1}^{n} \frac{\partial}{\partial x_j} a_{kj}(x)\right) \geqslant 0.$$

Then $(L(x,D), W_p^2)$ generates an L^p-sub-Markovian semigroup for $1 < p < \infty$.

For convenience we denote the L^p-generator $(L(x,D), W_p^2)$ by $(A^{(p)}, W_p^2)$ and the corresponding L^p-sub-Markovian semigroup by $(T_t^{(p)})_{t\geqslant 0}$. In particular, the semigroup $(T_t^{(p)})_{t\geqslant 0}$ is a strongly continuous contraction semigroup on L^p.

According to H. Triebel, [57, page 91], the operator $-(A^{(p)} - \lambda\,\mathrm{id})$ is for $\lambda > 0$ *positive* in the sense that

$$\|(\kappa\,\mathrm{id} - (A^{(p)} - \lambda\,\mathrm{id}))^{-1}\| \leqslant \frac{c_\lambda}{1+\kappa}, \quad \kappa > 0,$$

holds. Moreover, $(A^{(p)} - \lambda\,\mathrm{id}, W_p^2)$ is also the generator of an L^p-sub-Markovian semigroup, namely the semigroup $(T_{\lambda,t}^{(p)})_{t\geqslant 0}$ where $T_{\lambda,t}^{(p)} = e^{-\lambda t} T_t^{(p)}$. Consequently, $A_\lambda^{(p)} := A^{(p)} - \lambda\,\mathrm{id}$ has a positivity preserving resolvent and $(e^{-\lambda t} T_t^{(p)})_{t\geqslant 0}$ is of negative type since

$$\|e^{-\lambda t} T_t^{(p)}\|_{L^p \to L^p} \leqslant e^{-\lambda t}, \quad t \geqslant 0.$$

From Example 4.7.3.(c) in H. Amann [1] we deduce that $A_\lambda^{(p)}$, $1 < p < \infty$, has bounded imaginary powers,

$$\|(A_\lambda^{(p)})^{i\kappa}\|_{L^p \to L^p} \leqslant c(1+\kappa^2) e^{\pi|\kappa|/2}, \quad \kappa \in \mathbb{R}.$$

Now we may apply Theorem 1.15.3 from [57] to deduce

Theorem 7.2. *Suppose $L(x,D)$ fulfils the assumptions of Theorem 7.1 and let $\lambda > 0$ be fixed. Each of the operators $(A_\lambda^{(p)}, W_p^2)$ generates an L^p-sub-Markovian semigroup and the same holds for the fractional powers $-(-A_\lambda^{(p)})^\alpha$, $0 < \alpha < 1$. The domains of these operators are determined by complex interpolation leading to*

$$D((-A_\lambda^{(p)})^\alpha) = [L^p, W_p^2]_\alpha = H_p^{2\alpha}.$$

Moreover, since $0 \in \rho(-A_\lambda^{(p)})$ the operator $-(-A_\lambda^{(p)})^{-\beta}$, $0 < \beta < 1$, maps L^p into $H_p^{2\beta}$.

Denote by $(T_{\lambda,t}^{(p),\alpha})_{t\geqslant 0}$ the L^p-sub-Markovian semigroup generated by $(-(-A_\lambda^{(p)})^\alpha, H_p^{2\alpha})$. From the results of A. Carasso and T. Kato [9] it follows that these semigroups are analytic.

Corollary 7.3. *Let $L(x,D)$ and $\lambda > 0$ be as in Theorem 7.2 and $(T_{\lambda,t}^{(p),\alpha})_{t\geqslant 0}$ be the L^p-sub-Markovian semigroup generated by $(-(-A_\lambda^{(p)})^\alpha, H_p^{2\alpha})$. Then for all $u \in L^p$ we have*

$$T_{\lambda,t}^{(p),\alpha} u \in H_p^{2\alpha}.$$

In particular, for $p > \dfrac{n}{2\alpha}$ the semigroup $(T_{\lambda,t}^{(p),\alpha})_{t\geqslant 0}$ is strongly L^p-sub-Markovian.

Clearly, $C_0^\infty \subset H_p^{2\alpha}$ for all $1 < p < \infty$ and therefore

$$-(-A_\lambda^{(p)})^\alpha u \in H_p^{2(1-\alpha)} \quad \text{for any } u \in C_0^\infty.$$

Thus for $p > \dfrac{n}{2(1-\alpha)}$ each of the operators $-(-A_\lambda^{(p)})^\alpha$ maps C_0^∞ into C_∞. Therefore we may apply Theorem 2.6 implying that for $0 < \alpha < 1$ the operator $-(-A_\lambda^{(p)})^\alpha|_{C_0^\infty}$ is indeed a pseudo-differential operator

$$q_{\lambda,\alpha}(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q_{\lambda,\alpha}(x,\xi)\hat{u}(\xi)\, d\xi$$

with negative definite symbol $q_{\lambda,\alpha}(x,\xi)$. Of course for $q(x,\xi) = \psi(\xi)$ we have $q_{\lambda,\alpha}(x,\xi) = (\psi(\xi) + \lambda)^\alpha$. In the general situation one should expect an asymptotic expansion of type

$$q_{\lambda,\alpha}(x,\xi) = (q(x,\xi) + \lambda)^\alpha + \text{lower order remainder.}$$

We close this section with the following

Conjecture 7.4. Let f be a complete Bernstein function and $A_\lambda^{(p)}$ be as in Theorem 7.2. We should expect

$$D(-(-A_\lambda^{(p)})^f) = H_p^{f(|\cdot|^2),2}.$$

8 On regularising effects of powers of generators in the Hilbert space case

We have already seen in the previous sections that good knowledge of the spaces $D((A^{(p)})^k)$ does help to decide whether an analytic L^p-sub-Markovian semigroup is strongly L^p-sub-Markovian. In this section we will discuss how the improvement of the regularity of the coefficients will turn an analytic L^p-sub-Markovian semigroup into a strong L^p-sub-Markovian semigroup. The result depends essentially on the fact that higher regularity of the coefficients allows us to determine the domains of (large integer) powers of the generator in terms of function spaces.

Due to the massive problems in the L^p-analysis of pseudo-differential operators with negative definite symbols we restrict ourselves to the Hilbert space case. The results we discuss here are closely related to earlier considerations of M. Fukushima, H. Kaneko and the second named author in [19], as well as in [32]. Our point of view, however, is now somewhat different.

First let us fix the class of operators. For this let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that $\xi \mapsto q(x,\xi)$ is negative definite. Further assume that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a fixed continuous negative definite function satisfying

$$(8.1) \qquad 1 + \psi(\xi) \geqslant c_0(1 + |\xi|^2)^{r_0}, \quad c_0 > 0,\ 0 < r_0 < 1.$$

Assumption 8.1. *The pseudo-differential operator $q(x, D)$ maps $H^{\psi,2}$ continuously into L^2 and $(-q(x, D), H^{\psi,2})$ is the generator of a symmetric L^2-sub-Markovian semigroup $(T_t^{(2)})_{t\geqslant 0}$.*

Remark 8.2. Due to a result of E. M. Stein [55], $(T_t^{(2)})_{t\geqslant 0}$ is an analytic semigroup.

There are many concrete examples of operators satisfying Assumption 8.1. Essentially, one starts with the symbol $q(x,\xi)$ and imposes suitable analytic conditions. In particular we refer to [30]–[33], [28], as well as to the results of W. Hoh [23], [24], [25]–[27].

The fact that $q(x,D) : H^{\psi,2} \to L^2$ is continuous, i.e., the estimate

$$\|q(x,D)u|L^2\| \leqslant c\|u|H^{\psi,2}\|,$$

requires some regularity and boundedness properties of $x \mapsto q(x,\xi)$. Let us discuss the question under which circumstances we can prove

$$D(q(x,D)^k) = H^{\psi,2k}. \tag{8.2}$$

From the general theory of (selfadjoint) operators we know that

$$\begin{aligned} D(q(x,D)^k) &= \{u \in D(q(x,D)^{k-1}) : q(x,D)^{k-1}u \in D(q(x,D))\} \\ &= \{u \in D(q(x,D)) : q(x,D)u \in D(q(x,D)^{k-1})\}. \end{aligned} \tag{8.3}$$

By induction, we are lead to the problem to assure that $f \in L^2$ and $q(x,D)^k u = f$ implies $u \in H^{\psi,2k}$. If $q_\lambda(x,D)^k$ is invertible, we need, in particular, estimates of the type

$$\|q(x,D)^k u|L^2\| \leqslant c\|u|H^{\psi,2k}\|.$$

for $u \in H^{\psi,2k}$. But (8.3) requires for larger $k \in \mathbb{N}$ more regularity of $x \mapsto q(x,\xi)$. The condition (8.1) implies that $H^{\psi,m} \hookrightarrow C_\infty$ for $m > \dfrac{n}{2r_0}$ and this last fact combined with Theorem 5.4 leads to

Theorem 8.3. *Suppose that $q(x,D)$ satisfies Assumption 8.1 and that* (8.1) *as well as* (8.2) *hold for $2k > \dfrac{n}{2r_0}$. Then the semigroup $(T_t^{(2)})_{t\geqslant 0}$ is strongly L^2-sub-Markovian.*

The simplest and already quite instructive examples to study regularity effects caused by coefficients were constructed in [28]. These operators are of form

$$q(x,D) = \sum_{j=1}^n b_j(x)\psi_j(D_j)$$

where $\psi_j : \mathbb{R}^n \to \mathbb{R}$ are fixed continuous negative definite functions and the coefficients b_j are L^∞-functions independent of the variable x_j. Under the assumption

$b_j(x) \geqslant d_0 > 0$ it follows that $-q(x,D)$ generates a symmetric Dirichlet form with domain $H^{\psi,1}$, $\psi(\xi) = \sum_{j=1}^n \psi_j(\xi_j)$. Thus we may associate with $-q(x,D)$ a Hunt process which is defined up to an exceptional set in $H^{\psi,1}$. If, in addition, the oscillations of the coefficients are controlled by $\frac{d_0}{2n}$, then $H^{\psi,2}$ is the domain of $q(x,D)$, and we may construct the process up to an exceptional set in $H^{\psi,2}$. In order to identify $D(q(x,D)^k)$ with $H^{\psi,2k}$ we need bounds for certain iterated commutators

$$[\psi_{j_0}(D_{j_0}),[\psi_{j_1}(D_{j_1}),[\cdots[\psi_{j_{k-1}}(D_{j_{k-1}}),b_j]\cdots]]]$$

which require regularity of the coefficients b_j! It is thus obvious that stronger regularity assumptions on the coefficients will guarantee the strong L^2-sub-Markovian property provided we have for some $m \in \mathbb{N}$ the embedding $H^{\psi,m} \hookrightarrow C_\infty$; now we can construct a process associated to $-q(x,D)$ without any exceptional set!

Let us finally point out that in an L^p-setting the condition on k in Theorem 8.3 should read $2k > \frac{n}{pr_0}$. This means that there is a trade-off between k (i.e. regularity of the coefficients) and p (i.e. integrability): better integrability requires less regularity. This holds at least for some cases of negative definite functions.

9 Some perturbation results

In this section we will discuss how we may apply the results from Section 5 to simple perturbations of generators. We want to start with an unperturbed generator $A^{(p)}$ of an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ which is analytic. Moreover, we suppose that $D(A^{(p)})$ is explicitly known as a function space. So far we have seen two examples, namely the operators $(-\psi_1(D), H_p^{\psi,2})$ with $\psi_1(D) = \mathrm{id} + \psi(D)$ and a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ (cf. Section 6), and the operator $(-(-L(x,D))^r + \mathrm{id}), H_p^{2r})$, $0 < r < 1$ (cf. Section 7).

We will now only consider the operator $(-\psi_1(D), H_p^{\psi,2})$, all results have an analogue for $(-((-L(x,D))^r + \mathrm{id}), H_p^{2r})$. Actually, they carry over to any $(A_1^{(p)}, H_p^{\psi,2})$, $A_1^{(p)} = A^{(p)} - \mathrm{id}$, generating an L^p-sub-Markovian semigroup. Our main purpose is, again, to show how the regularity of the coefficients determines which type of refinement result one can apply.

Considering $-\psi_1(D)$ or $-((-L(x,D))^r + \mathrm{id})$ rather than $-\psi(D)$ or $-(-L(x,D))^r$ is just a technical point; clearly we have $H_p^{\psi_1,r} = H_p^{\psi,r}$.

From H. Amann [1, Theorem V.1.2.4, page 259] we deduce for any $0 \leqslant s < 1$ the estimate

$$\|\psi_1^s(D)u|L^p\| \leqslant \varepsilon\|\psi_1(D)u|L^p\| + c(\varepsilon)\|u|L^p\|$$

for all $\varepsilon > 0$ and suitable constants $c(\varepsilon)$. In particular, for $a \in L^\infty$ we have

$$\|au|L^\infty\| \leqslant \varepsilon\|\psi_1(D)u|L^p\| + c(\varepsilon)\|u|L^p\|,$$

that is,

$$\|au|L^\infty\| \leqslant \varepsilon\|u|H_p^{\psi,2}\| + c(\varepsilon)\|u|L^p\|. \tag{9.1}$$

If $a \leqslant 0$ it follows that

$$\int_{\mathbb{R}^n} (au)((u-1)^+)^{p-1}\,dx = \int_{u\geqslant 1} (au)(u-1)^{p-1}\,dx \leqslant 0$$

and from (9.1) we get that $(-\psi_1(D) + a, H_p^{\psi,2})$ generates an analytic L^p-sub-Markovian semigroup $(T_{a,t}^{(p)})_{t\geqslant 0}$ for $a \in L^\infty$, $a \leqslant 0$. In general, the operator $u \mapsto au$ does not map C_0^∞ into C_∞ and therefore $-\psi_1(D) + a$ will not map C_0^∞ into C_∞. This shows that, in general, we cannot expect $-\psi_1(D) + a$ to be the generator of a *Feller* semigroup (with C_0^∞ in the domain of its generator).

By the general theory of H. Kaneko, see Section 5 for a brief review, we can associate with $(-\psi_1(D) + a, H_p^{\psi,2})$ a Hunt process up to an $\mathrm{cap}_{H_p^{\psi,2}}$-exceptional set. If

$$H_p^{\psi,2} \hookrightarrow C_\infty,$$

then we do not have any exceptional set at all, but as we know this embedding does not hold for general ψ and p.

Let us examine the problem of determination $D(-(\psi_1(D) + a)^k)$. We will restrict ourselves to $k = 2$. In order to verify

$$D(-(\psi_1(D) + a)^2) = H_p^{\psi,4}$$

we need (to calculate and) to estimate $-(-\psi_1(D) + a)(-\psi_1(D) + a)u$ for $u \in H_p^{\psi,4}$.

Formally,

$$-(-\psi_1(D) + a)(-\psi_1(D) + a)u = -\psi_1^2(D)u + a\psi_1(D)u - a^2u + \psi_1(D)(au).$$

It is only the last term that could cause trouble: for $a \in L^\infty$ and $u \in H_p^{\psi,4}$ the operator $u \mapsto \psi_1(D)(au)$ is not bounded in L^p and we do not have the estimate

$$\|\psi_1(D)(au)|L^p\| \leqslant c\|u|H_p^{\psi,4}\|.$$

(With the classical theory in mind one should look for $\|\psi_1(D)(au)|L^p\| \leqslant \tilde{c}\|u|H_p^{\psi,2}\|$.)

However, if we could bound the commutator $[\psi_1(D), a]u := \psi_1(D)(au) - a\psi_1(D)u$,

$$\|[\psi_1(D), a]u|L^p\| \leqslant \|u|H_p^{\psi,2}\|,$$

(we should expect $\|[\psi_1(D), a]u|L^p\| \leqslant \|u|H_p^{\psi,1}\|$) then we can identify $D(-(\psi_1(D)+a)^2)$ with $H_p^{\psi,4}$. Since the embedding $H_p^{\psi,4} \hookrightarrow C_\infty$ holds often for a larger range of p (depending of ψ and n) than the embedding $H_p^{\psi,2} \hookrightarrow C_\infty$, we see that higher regularity of a will give smaller exceptional sets which might even vanish at all.

References

[1] Amann, H.: Linear and quasi-linear parabolic problems, Vol. I Abstract linear theory. Monographs in Mathematics, vol. 89. Birkhäuser, Basel 1995

[2] Bennett, C. and Sharpley, R.: Interpolation of operators. Pure Appl. Math. vol. 129. Academic Press, Boston 1988

[3] Berg, C., Boyadzhiev, Kh. and deLaubenfels, R.: Generation of generators of holomorphic semigroups. J. Austral. Math. Soc. (Ser. A) **55** (1993), 246–269

[4] Berg, C. and Forst, G.: Potential theory on locally compact abelian groups. Ergebnisse der Mathematik und ihre Grenzgebiete, II. Ser. vol. 87. Springer Verlag, Berlin 1975

[5] Bochner, S.: Diffusion equation and stochastic processes. Proc. Natl. Acad. Sci. **35** (1949), 368–370

[6] Bochner, S.: Harmonic analysis and the theory of probability. California Monographs in Math. Sci. University of California Press, Berkeley CA 1955

[7] Bouleau, N. and Hirsch, F.: Formes de Dirichlet générales et densité des variables aléatoires réelles sur l'espace de Wiener. J. Funct. Anal. **69** (1986), 229–259

[8] Browder, F.: On the spectral theory of elliptic differential operators I. Math. Ann. **142** (1961), 22–130

[9] Carasso, A. S. and Kato, T.: On subordinated holomorphic semigroups. Trans. Amer. Math. Soc. **327** (1991), 867–877

[10] Courrège, Ph.: Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans $\mathbb{C}$ satisfaisant au principe du maximum. Sém. Théorie du Potentiel 1965/66, Exposé 2, 38 pp

[11] Davies, E. B.: One-parameter semigroups. London Mathematical Society Monographs, vol. 15. Academic Press, London 1980

[12] Dunford, N. and Schwartz, J. T.: Linear Operators I. Pure Appl. Math. vol 7. Interscience Publishers, New York 1957

[13] Eberle, A.: Uniqueness and non-uniqueness of singular diffusion operators. Dissertation, Universität Bielefeld, 1998

[14] Ejdel'man, S. D.: Parabolic equations. In: Yu. Egorov and M. Shubin (eds.), Partial differential equations VI. Encyclopaedia of Mathematical Sciences, Springer Verlag, Berlin 1994, 203–316

[15] Farkas, W., Jacob, N. and Schilling, R. L.: Function spaces related to continuous negative definite functions: ψ-Bessel potential spaces. Dissertationes Mathematicae, to appear

[16] Fukushima, M.: Dirichlet spaces and strong Markov processes. Trans. Amer. Math. Soc. **162** (1971), 181–224

[17] Fukushima, M.: (r,p)-capacities and Hunt processes in infinite dimensions. In: A. N. Shiryaev et al. (eds.), Probability theory and mathematical statistics. World Scientific, Singapore 1992, 96–103

[18] Fukushima, M.: Two topics related to Dirichlet forms: quasi-everywhere convergence and additive functionals. In: G. Dell'Antonio and U. Mosco (eds.), Dirichlet forms. Lecture Notes in Mathematics, vol. 1563. Springer Verlag, 1993, 21–53

[19] Fukushima, M., Jacob, N. and Kaneko, H.: On $(r,2)$-capacities for a class of elliptic pseudo-differential operators. Math. Ann. **293** (1992), 343–348

[20] Fukushima, M. and Kaneko, H.: On (r,p)-capacities for general Markovian semigroups. In: S. Albeverio (ed.), Infinite dimensional analysis and stochastic processes. Research Notes in Mathematics, vol. 124. Pitman Publishing Inc. Boston MA 1985, 41–47

[21] Grubb, G.: Functional calculus of pseudo-differential boundary problems. Progress in Mathematics, vol. 65. Birkhäuser, Basel 1986, 2^{nd} ed. 1996

[22] Hirsch, F.: Domaines d'opérateurs représentés comme intégrales de resolvantes. J. Funct. Anal. **23** (1976), 239–264

[23] Hoh, W.: Pseudo differential operators with negative definite symbols and the martingale problem. Stoch. and Stoch. Rep. **55** (1995), 225–252

[24] Hoh, W.: Feller semigroups generated by pseudo-differential operators. In: Zh.-M. Ma, M. Röckner and A. Yan (eds.), Intern. Conf. Dirichlet forms and stochastic processes. Walter de Gruyter, Berlin 1995, 199–206

[25] Hoh, W.: A symbolic calculus for pseudo-differential operators generating Feller semigroups. Osaka J. Math. **35** (1998), 798–820

[26] Hoh, W.: Pseudo-differential operators with negative definite symbols of variable order. Revista Mat. Iberoamericana. to appear

[27] Hoh, W.: Pseudo-differential operators generating Markov processes. Habilitationsschrift, Universität Bielefeld, 1998

[28] Hoh, W. and Jacob, N.: Some Dirichlet forms generated by pseudo-differential operators. Bull. Sci. Math. (2^e sèr.) **116** (1992), 383–398

[29] Jacob, N.: Feller semigroups, Dirichlet forms, and pseudo-differential operators. Forum Math. **4** (1992), 433–446

[30] Jacob, N.: Further pseudo-differential operators generating Feller semigroups and Dirichlet forms. Revista Mat. Iberoamericana **9** (1993), 373–407

[31] Jacob, N.: A class of Feller semigroups generated by pseudo-differential operators. Math. Z. **215** (1994), 215–151

[32] Jacob, N.: An application of $(r,2)$-capacities to pseudo-differential operators. Stoch. and Stoch. Rep. **47** (1994), 193–200

[33] Jacob, N.: Non-local (semi-) Dirichlet forms generated by pseudo-differential operators. In: Zh.-M. Ma, M. Röckner and A. Yan (eds.), Intern. conf. Dirichlet forms and stochastic processes. Walter de Gruyter, Berlin 1995, 223–233

[34] Jacob, N.: Generators of Feller semigroups as generators of L^p-sub-Markovian semigroups. In: G. Lumer and L. Weis (eds.), Proc. 6th. Intnl. conf. on evolution equations and their applications. M. Dekker, New York (to appear)

[35] Jacob, N.: Pseudo-differential operators and Markov processes. Vol. 1: Fourier analysis and semigroups. Monograph in preparation

[36] Jacob, N.: Pseudo-differential operators and Markov processes. Vol. 2: Generators and potential theory. Monograph in preparation

[37] Jacob, N. and Schilling, R. L.: Fractional derivatives, non-symmetric and time—dependent Dirichlet forms, and the drift form. Z. Anal. Anwend. to appear

[38] Kaneko, H.: On (r,p)-capacities for Markov processes. Osaka J. Math. **23** (1986), 325–336

[39] Kato, T.: Perturbation theory for linear operators. Grundlehren der mathematischen Wissenschaften, vol. 132. Springer Verlag, New York 1966

[40] Kumano-go, H.: Pseudo-differential operators. MIT Press, Cambridge MA 1974

[41] Liskevich, V. A. and Semenov, Yu. A.: Some problems on Markov semigroups. In: M. Demuth et al. (eds.), Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras. Mathematical Topics, vol. 11. Akademie Verlag, Berlin 1996, 167–217

[42] Lunardi, A.: Analytic semigroups and optimal regularity in parabolic problems. Progress in nonlinear differential equations and their applications, vol. 16. Birkhäuser, Basel, 1995
[43] Ma, Zh.-M. and Röckner, M.: An introduction to the theory of (non-symmetric) Dirichlet forms. Universitext, Springer Verlag, Berlin 1992
[44] Malliavin, P.: Implicit functions in finite corank on the Wiener space. In: K. Itô (ed.), Proc. of the Taniguchi Symp. Stoch. Anal. North Holland Publishing Company, Amsterdam 1984, 369–389
[45] Malliavin, P.: Stochastic analysis. Grundlehren der mathematischen Wissenschaften, vol. 313. Springer Verlag, Berlin 1997
[46] Nollau, V.: Über Potenzen von linearen Operatoren in Banachschen Räumen. Acta Sci. Math. **28** (1967), 107–121
[47] Nollau, V.: Über den Logarithmus abgeschlossener Operatoren in Banachschen Räumen. Acta Sci. Math. **30** (1969), 161–174
[48] Ouhabaz, E. M.: L^∞-contractivity of semigroups generated by sectorial forms. J. London Math. Soc. (2) **46** (1992), 529–542
[49] Ouhabaz, E. M.: L^p contraction semigroups for vector valued functions. Prépublication de l'Équipe d'Analyse et de Mathématiques Appliquées 10/98, Université de Marne-la-Vallee, 1998
[50] Phillips, R. S.: On the generation of semigroups of linear operators. Pacific J. Math. **2** (1952), 343–369
[51] Rogers, L. C. G. and Williams, D.: Diffusions, Markov processes, and martingales. vol. 1: Foundations (2^{nd} ed.). Wiley series in probability and mathematical statistics Chichester 1994
[52] Schilling, R. L.: On the domain of the generator of a subordinate semigroup. In: L. Kral et al (eds.), Potential Theory—ICTP 94. Proceedings Intnl. Conf. Potential Theory, Kouty (CR), 1994. Walter de Gruyter, Berlin 1996, 446–462
[53] Schilling, R. L.: Subordination in the sense of Bochner and a related functional calculus. J. Austral. Math. Soc. (Ser. A) **64** (1998), 368–396
[54] Schilling, R. L.: Dirichlet operators and the positive maximum principle. Integral Equations and Op. Theory, to appear
[55] Stein, E. M.: Topics in harmonic analysis related to the Littlewood–Paley theory. Annals of Mathematics Studies, vol. 63. Princeton University Press, Princeton NJ 1970
[56] Stein, E. M.: Singular integrals and differentiability of functions. Princeton University Press, Princeton NJ 1970
[57] Triebel, H.: Interpolation theory, function spaces, differential operators. North Holland, Amsterdam 1978
[58] Voigt, J.: Interpolation for (positive) C_0-semigroups on L_p-spaces. Math. Z. **188** (1985), 283–286
[59] Voigt, J.: One-parameter semigroups acting simultaneously on different L_p spaces. Bull. Soc. Royale Sci. Liège **61** (1992), 465–470
[60] Yosida, K.: Functional Analysis. Grundlehren der mathematischen Wissenschaften, vol 123. Springer Verlag, Berlin 1980 (6^{th} edition)

Received June 21, 1999; in final form April 10, 2000

Walter Farkas, Ludwig-Maximilians-Universität München, Mathematisches Institut, Theresienstr. 39, 80333 München, Germany
farkas@rz.mathematik.uni-muenchen.de

current address: Universität München, Mathematischen Institut, Theresienstrasse 39, 80333 München, Germany

Niels Jacob, University of Wales Swansea, Department of Mathematics, Singleton Park, Swansea SA2 8PP, U.K. United Kingdom
N.Jacob@swansea.ac.uk

René L. Schilling, School of Mathematical Sciences, University of Sussex, Falmer, Brighton BN1 9QH, U.K. United Kingdom
r.schilling@sussex.ac.uk