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Extensions of three classical theorems to modules with maximum condition for finite matrix subgroups

Wolfgang Zimmermann

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Abstract. In this article analogues of the Hilbert Basis Theorem, the Artin-Rees Lemma and the Krull Intersection Theorem are shown for modules with ascending chain condition for finite matrix subgroups. The generalized Hilbert Basis Theorem yields an interesting construction principle of $\sum$-pure-injective modules over polynomial rings.

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Introduction and preliminaries

In this article we shall prove three theorems on modules with maximum condition for finite matrix subgroups which generalize resp. modify three classical theorems on noetherian modules over a commutative ring [4, 1.9 Theorem 10, 4.6 Theorem 18, 4.7 Theorem 20]. To give an idea of the class of modules with maximum condition for finite matrix subgroups we note that in case the ground ring is commutative and noetherian it comprises the finitely generated and the flat modules, and with a module M all its pure sub- and factormodules, direct sums $M^{(I)}$ and products M^I. Matrix subgroups have initially been introduced in order to characterize pure-injective and $\sum$-pure-injective modules. For instance it has been proved that a module M is $\sum$-pure-injective (i.e. every direct sum $M^{(I)}$ is pure-injective) if and only if it satisfies the minimum condition for finite matrix subgroups ([2], [9, Folgerung 3.4]). Because of this easy-to-use description and other striking properties of $\sum$-pure-injective modules mainly the last-mentioned chain condition has been explored up to now. But we will give reasons to pay attention to the opposite condition as well. First we recall that both conditions have successfully been applied in the representation theory of artinian rings [1, 6, 10]. Further we point out that basically it does not matter which of the two is studied because they are connected by a good duality principle (e.g. see [6], [10, Proposition 3], or [11, Corollary 1.6]).

However, our most striking argument is the hope that the wide knowledge of noetherian rings and modules might be a source of inspiration for the study of modules with maximum condition for finite matrix subgroups. To some extent this opinion is confirmed by the results of this paper. – It should be noted that matrix subgroups and their use in module theory are a current area of research in model theory as well. In our article [11] we have quoted some relevant literature and exposed how matrix subgroups and the above mentioned duality principle are viewed at by the model theorists. As these aspects are irrelevant for the present paper we do not repeat them here.

In the first section we show a generalization of the Hilbert Basis Theorem. It states that if a module M_R has ascending chain condition for finite matrix subgroups then the polynomial module $M \otimes_R R[U]_{R[U]}$ has this property and even every tensor product $M \otimes_R W_W$ where $R[U_1, \ldots, U_s] \to W$ is a ring homomorphism such that W is a finitely generated left module over $R[U_1, \ldots, U_s]$. Applicating the duality principle we can show a similiar construction of $\sum$-pure-injective modules in Section 2: If M_R has descending chain condition for finite matrix subgroups and W is finitely generated as a right module over $R[U_1, \ldots, U_s]$ then $\operatorname{Hom}(W_R, M_R)_W$ also has descending chain condition for finite matrix subgroups.

The main result of Section 3 is a modification of the Artin-Rees Lemma: If M_R has maximum condition for finite matrix subgroups, T is a finite matrix subgroup of M and A a finitely generated ideal of the center of R then we have $MA^{n+1} \cap T = (MA^n \cap T)A$ for all sufficiently large n. A consequence is a generalization of the Krull Intersection Theorem which we deduce in Section 4. Assuming in addition that all $\operatorname{End}(M)$-submodules of $\bigcap_{n \geq 1} MA^n$ are finite matrix subgroups we have

$$\bigcap_{n \geq 1} MA^n = \{x \in M \mid \exists f \in \operatorname{End}(M)A : (1-f)x = 0\} .$$

All our assumptions are satisfied by a noetherian module M over a commutative ring and using an integrality argument our equality can be given the well-known form

$$\bigcap_{n \geq 1} MA^n = \{x \in M \mid \exists a \in A : x(1-a) = 0\} .$$

Since the question when all $\operatorname{End}(M)$-submodules of a module M_R are finite matrix subgroups is also of interest in other contexts we give some aspects of this problem in Remark 4.3. In an Appendix we have compiled some general facts on the behaviour of modules with chain conditions for finite matrix subgroups under ring extensions. To conclude this summary we note that probably there are analogues of Theorems 1.4 and 3.1 for skew-polynomial rings resp. non-central ideals. This shall be reserved for future research.

For the ease of the reader we gather some notations and facts needed in the following; for completeness see [9] and [11].

We begin by recalling that given a ring R and a set I a pair $(X, \underline{x})$ with a left R-module X and an I-tuple $\underline{x} = (x_i)_{i \in I} \in X^I$ is called an I-pointed left R-module. Every such pair defines two functors $T_{X,\underline{x}}$ and $H_{X,\underline{x}}$ as follows. For every module M_R (resp. ${}_RN$) $T_{X,\underline{x}}(M)$ is the kernel of the map $\tau_{\underline{x}} : M^{(I)} \to M \otimes_R X$ given by $\tau_{\underline{x}}(\underline{m}) = \underline{m} \otimes \underline{x} = \sum_{i \in I} m_i \otimes x_i$ for $\underline{m} = (m_i) \in M^{(I)}$, whereas $H_{X,\underline{x}}(N)$ is the image of the map $\varepsilon_{\underline{x}} : \mathrm{Hom}_R(X, N) \to N^I$, $\varepsilon_{\underline{x}}(h) = h(\underline{x}) = (h(x_i))$. The functors $T_{X,\underline{x}}$ resp. $H_{X,\underline{x}}$ which are predominantly used are those associated with a finitely presented module X and a finite tuple $\underline{x}$. In this case it is optional which of the functors $T_{X,\underline{x}}$ or $H_{X,\underline{x}}$ are used because for each n-pointed finitely presented module $({}_RX, \underline{x})$ there is an n-pointed finitely presented module $(Y_R, \underline{y})$ such that $T_{X,\underline{x}} = H_{Y,\underline{y}}$ and $H_{X,\underline{x}} = T_{Y,\underline{y}}$ [11, Lemma 1.1]. It follows that say for a right module M_R the set of subgroups $T_{X,x}(M)$ coincides with the set of all $H_{Y,y}(M)$, the (X, x) resp. (Y, y) running through all 1-pointed finitely presented left resp. right modules. These subgroups are precisely the finite matrix subgroups of M (f. m. subgroups for short) and the object of this work are modules with the ascending resp. descending chain condition for subgroups of this type (sometimes abbreviated as acc(fm) resp. dcc(fm)). We list some characterizations of these modules to which we refer later on.

The following conditions are equivalent for a module M_R [11, Lemma 2.1 and Theorem 2.5]:

(a) M has ascending chain condition for f. m. subgroups.

(b) Given $n \in \mathbb{N}$ the set of subgroups $T_{X,\underline{x}}(M)$ of M^n has ascending chain condition, the $(X, \underline{x})$ running through all (resp. all finitely presented) n-pointed left modules.

(c) For all modules ${}_RX$ and sets I the map

$$\mu : M^I \otimes_R X \to (M \otimes_R X)^I, \quad \mu((m_i) \otimes x) = (m_i \otimes x),$$

is injective.

For the dual case we assume that M_R in addition is considered as a left module over some rings S such that ${}_SM_R$ is a bimodule ($S = \mathbb{Z}$ or $S = \mathrm{End}_R(M)$ do). Then the following are equivalent ([11, Lemma 3.2 and Theorem 3.8] and [9, Folgerung 3.4]):

(a′) M has descending chain condition for f. m. subgroups.

(b′) Given $n \in \mathbb{N}$ the set of subgroups $H_{Y,\underline{y}}(M)$ of M^n has descending chain condition, the $(Y, \underline{y})$ running through all (resp. all finitely presented) n-pointed right modules.

(c′) For all modules Y_R and all injective modules ${}_SV$ the map

$$v : Y \otimes_R \mathrm{Hom}_S(M, V) \to \mathrm{Hom}_S(\mathrm{Hom}_R(Y, M), V),\ v(y \otimes \varphi)(h) = \varphi(h(y))$$

is injective.

(d′) M is $\sum$-pure-injective.

The meaning of the maps μ and v will be maintained throughout. Finally we want to agree upon writing (M, N) or (M_R, N_R) instead of $\mathrm{Hom}_R(M, N)$ say for right R-modules M and N.

1. Generalization of the Hilbert Basis Theorem

We begin with an easy lemma implicitly occuring in [8, p. 98] which immediately yields the classical Hilbert Basis Theorem and together with a result on matrix subgroups the extension we have in view. First we have to introduce some notations. As usual $\mathbb{Z}[U]$ denotes the polynomial ring in the indeterminate U over $\mathbb{Z}$. If M is an abelian group we put $M[U] = M \otimes_{\mathbb{Z}} \mathbb{Z}[U]$; the elements of $M[\mathrm{U}]$ are written as polynomials with coefficients in M, i.e. instead of $\sum_{j=0}^{i} m_j \otimes U^j$ we simply write $\sum_{j=0}^{i} m_j U^j$. Now we assume that M is a left module over some ring S. Then $M[U]$ is a left module over the polynomial ring $S[U] = S \otimes_{\mathbb{Z}} \mathbb{Z}[U]$ and obviously $M[U] = S[U] \otimes_S M$. There is a natural filtration of $M[U]$ defined by the S-submodules $M[U]^{(i)} = \bigoplus_{j=0}^{i} MU^j$, $i \geq 0$; the induced filtration of a subgroup T of $M[U]$ is given by the $T^{(i)} = T \cap M[U]^{(i)}$, $i \geq 0$. Furthermore let $q_i : T^{(i)} \to T^{(i+1)}$ be the inclusion, $p_i : T^{(i)} \to M$ the projection onto the last factor, i.e. $p_i(\sum_{j=0}^{i} m_j U^j) = m_i$ and $\overline{T}^{(i)}$ the image of p_i.

Lemma 1.1. 1) *The diagram of* S*-modules*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^{(i)} & \xrightarrow{q_i} & T^{(i+1)} & \xrightarrow{p_{i+1}} & \overline{T}^{(i+1)} & \longrightarrow & 0 \\
 & & {\scriptstyle \varrho_U}\big\downarrow & & {\scriptstyle \varrho_U}\big\downarrow & & \cap & & \\
0 & \longrightarrow & T^{(i+1)} & \xrightarrow{q_{i+1}} & T^{(i+2)} & \xrightarrow{p_{i+2}} & \overline{T}^{(i+2)} & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows (ϱ_U *means multiplication by* U).

2) *Given* $i \geq 0$ *we have* $\overline{T}^{(i+1)} = \overline{T}^{(i+2)}$ *if and only if* $T^{(i+2)} = T^{(i+1)} + T^{(i+1)}U$.

3) *Supposing that there is some* $\kappa \geq 0$ *with* $\overline{T}^{(i+1)} = \overline{T}^{(i+2)}$ *for all* $i \geq \kappa$ *we have* $T = \sum_{j \geq 0} T^{(\kappa+1)} U^j$.

Proof. 1) and 2) are checked by elementary calculation, 3) is a consequence of 2) and the equality $T = \bigcup_{i \geq 0} T^{(i)}$. □

For completeness we infer the classical Hilbert Basis Theorem for modules.

Corollary 1.2. *In case ${}_SM$ is noetherian the $S[U]$-module $M[U]$ is noetherian as well.*

Proof. Letting T be an $S[U]$-submodule of $M[U]$ there is some $\kappa \geq 0$ such that $\bar{T}^{(i+1)} = \bar{T}^{(i+2)}$ for all $i \geq \kappa$ and Lemma 1.1 yields $T = \sum_{j \geq 0} T^{(\kappa+1)} U^j$. As ${}_SM$ is noetherian the S-submodule $T^{(\kappa+1)}$ of $M[U]^{(\kappa+1)}$ is finitely generated, hence T is finitely generated over $S[U]$. □

During this article we shall sometimes say that a subgroup T of $M[U]$ is finitely determined if there is some $\kappa \geq 0$ with $T = \sum_{j \geq 0} T^{(\kappa)} U^j$.

In preparation of the proof of the main result of this section we need some further preliminaries. We start with a module M_R over a ring R. Letting act $S = \mathrm{End}_R(M)$ on the left side M becomes an S-, R-bimodule and $\tilde{M} = M[U]$ an $S[U]$-, $R[U]$-bimodule. Given a left $R[U]$-module X and an n-tuple $\underline{x} = (x_1, \ldots, x_n) \in X^n$ we want to analyze the subgroup $T_{X,\underline{x}}(\tilde{M})$ of $\tilde{M}^n$. First we note that in fact it is an $\mathrm{End}_{R[U]}(\tilde{M})$-submodule hence an $S[U]$-submodule of $\tilde{M}^n$. It is convenient to identify $\tilde{M}^n$ with $M^n[U]$ in the following. As before we have the filtrations $M^n[U]^{(i)} = \bigoplus_{j=0}^{i} M^n U^j$, $i \geq 0$, and $T_{X,\underline{x}}(\tilde{M})^{(i)} = T_{X,\underline{x}}(\tilde{M}) \cap M^n[U]^{(i)}$, $i \geq 0$, both consisting of S-submodules. Now we consider the S-isomorphism

$$\tau = \tau_{(U^j)} : (M^n)^{(\mathbb{N}_0)} \to M^n[U], \quad (\underline{m}_j)_{j \geq 0} \mapsto \sum_{j \geq 0} \underline{m}_j U^j .$$

In order to identify the preimages of $T_{X,\underline{x}}(\tilde{M})$ and the $T_{X,\underline{x}}(\tilde{M})^{(i)}$, $i \geq 0$, we introduce the elements $\underline{x}^{(i)} = (\underline{x}, U\underline{x}, \ldots, U^i \underline{x}) \in (X^n)^{i+1}$, $i \geq 0$, and $\underline{x}^{\infty} = (\underline{x}, U\underline{x}, U^2 \underline{x}, \ldots) \in (X^n)^{\mathbb{N}_0}$. Furthermore we let $X^{(i)} = X/\langle \underline{x}^{(i)} \rangle$ where $\langle \underline{x}^{(i)} \rangle$ denotes the R-submodule of X generated by the components $U^j x_k$, $0 \leq j \leq i$, $1 \leq k \leq n$, of $\underline{x}^{(i)}$.

Lemma 1.3. 1) *τ induces S-isomorphisms*

$$\tau^{(i)} : T_{X,\underline{x}^{(i)}}(M_R) \to T_{X,\underline{x}}(\tilde{M})^{(i)}, i \geq 0, \quad \textit{and}$$

$$\tau^{\infty} : T_{X,\underline{x}^{\infty}}(M_R) \to T_{X,\underline{x}}(\tilde{M}) .$$

*These maps are natural transformations, i.e. they commute with morphisms of n-pointed left $R[U]$-*modules.

2) *Given $i \geq 0$ the sequence*

$$0 \longrightarrow T_{X,\underline{x}}(\tilde{M})^{(i)} \xrightarrow{q_i} T_{X,\underline{x}}(\tilde{M})^{(i+1)} \xrightarrow{p_{i+1}} T_{X^{(i)}, U^{i+1}\underline{x}}(M) \longrightarrow 0$$

is exact, q_i denoting the inclusion and p_{i+1} the projection onto the coefficient of U^{i+1}.

Proof. To keep things clear we recall the definition of the groups in question. $T_{X,\underline{x}}(\tilde{M})$ is the set of n-tuples $\underline{p} \in \tilde{M}^n$ with $\underline{p} \otimes \underline{x} = 0$ in $\tilde{M} \otimes_{R[U]} X$, $T_{X,\underline{x}^{(i)}}(M)$ the set of $(i+1)$-tuples $(\underline{m}_0, \ldots, \underline{m}_i)$ with components in M^n such that $\sum_{j=0}^{i} \underline{m}_j \otimes U^j \underline{x} = 0$, and $T_{X,\underline{x}^\infty}(M)$ the union of the chain $T_{X,\underline{x}^{(i)}}(M)$, $i \geq 0$.

1) Using the canonical isomorphism $\tilde{M} \otimes_{R[U]} X \simeq M \otimes_R X, mU^j \otimes y \mapsto m \otimes U^j y$, we have the following equivalences for an n-tuple $\underline{p} = \tau(\underline{m}_0, \ldots, \underline{m}_i, 0, 0, \ldots) \in M^n[U]$:

$$\underline{p} \in T_{X,\underline{x}}(\tilde{M})^{(i)} \Leftrightarrow \underline{p} \otimes \underline{x} = 0 \Leftrightarrow \sum_{j=0}^{i} \underline{m}_j \otimes U^j \underline{x} = 0$$

$$\Leftrightarrow (\underline{m}_0, \ldots, \underline{m}_i) \in T_{X,\underline{x}^{(i)}}(M) .$$

This settles our assertion for the $\tau^{(i)}$, $i \geq 0$, and τ^∞, the naturality being obvious.

2) It has been proven in [11, Lemma 1.3] that the sequence

$$0 \longrightarrow T_{X,\underline{x}^{(i)}}(M) \xrightarrow{q_i} T_{X,\underline{x}^{(i+1)}}(M) \xrightarrow{p_{i+1}} T_{X^{(i)},U^{i+1}\underline{x}}(M) \longrightarrow 0$$

is exact, where q_i is the injection $(\underline{m}_0, \ldots, \underline{m}_i) \mapsto (\underline{m}_0, \ldots, \underline{m}_i, 0)$ and p_{i+1} the projection $(\underline{m}_0, \ldots, \underline{m}_{i+1}) \mapsto \underline{m}_{i+1}$. By 1) we may substitute $T_{X,\underline{x}^{(i)}}(M)$ by $T_{X,\underline{x}}(\tilde{M})^{(i)}$ and $T_{X,\underline{x}^{(i+1)}}(M)$ by $T_{X,\underline{x}}(\tilde{M})^{(i+1)}$ whence our assertion is proved. □

Theorem 1.4. *The following conditions are equivalent for a module M_R.*

1) *M has ascending chain condition for finite matrix subgroups.*

2) *$M[U]_{R[U]}$ has ascending chain condition for finite matrix subgroups.*

3) *Given a left $R[U]$-module X, a number $n \geq 1$ and $\underline{x} = (x_1, \ldots, x_n) \in X^n$ the group $T_{X,\underline{x}}(M[U]_{R[U]})$ is finitely determined.*

Proof. We adhere to the notations introduced above.

1) ⇒ 3) It is easily seen that the diagram of S-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X,\underline{x}}(\tilde{M})^{(i)} & \xrightarrow{q_i} & T_{X,\underline{x}}(\tilde{M})^{(i+1)} & \xrightarrow{p_{i+1}} & T_{X^{(i)},U^{i+1}\underline{x}}(M) & \longrightarrow & 0 \\ & & \varrho_U\downarrow & & \varrho_U\downarrow & & \cap & & \\ 0 & \longrightarrow & T_{X,\underline{x}}(\tilde{M})^{(i+1)} & \xrightarrow{q_{i+1}} & T_{X,\underline{x}}(\tilde{M})^{(i+2)} & \xrightarrow{p_{i+2}} & T_{X^{(i+1)},U^{i+2}\underline{x}}(M) & \longrightarrow & 0 \end{array}$$

is commutative; its rows are exact by the preceding lemma. Condition 1) implies that there is some $\kappa \geq 0$ with $T_{X^{(i)},U^{i+1}\underline{x}}(M) = T_{X^{(i+1)},U^{i+2}\underline{x}}(M)$ for $i \geq \kappa$, hence Lemma 1.1 establishes the equation $T_{X,\underline{x}}(\tilde{M}) = \sum_{j \geq 0} T_{X,\underline{x}}(\tilde{M})^{(\kappa+1)} U^j$.

1) $\Rightarrow$ 2) Let (X_i, x_i), $i \geq 1$, be a family of 1-pointed left $R[U]$-modules satisfying $T_{X_i,x_i}(\tilde{M}) \subset T_{X_{i+1},x_{i+1}}(\tilde{M})$ for all $i \geq 1$. Using [11, Remark to Lemma 1.2] we may assume that there are morphisms $(X_i, x_i) \to (X_{i+1}, x_{i+1})$, $i \geq 1$. Letting $(X, x) = \varinjlim_{i \geq 1}(X_i, x_i)$ we have $T_{X,x}(\tilde{M}) = \bigcup_{i \geq 1} T_{X_i,x_i}(\tilde{M})$ [11, Corollary 1.7] and according to 1) $\Rightarrow$ 3) there is some $\kappa \geq 0$ satisfying $T_{X,x}(\tilde{M}) = \sum_{j \geq 0} T_{X,x}(\tilde{M})^{(\kappa)} U^j$. Condition 1) implies that the chain $T_{X_i,x_i^{(\kappa)}}(M)$, $i \geq 1$, is ultimately constant, i.e. there is a number $\lambda \geq 1$ with $T_{X_i,x_i^{(\kappa)}}(M) = T_{X_{i+1},x_{i+1}^{(\kappa)}}(M)$ for $i \geq \lambda$ (recall that $x_i^{(\kappa)} = (x_i, Ux_i, \ldots, U^\kappa x_i)$). As the $\tau^{(i)}$, $i \geq 0$, are natural isomorphisms we also have $T_{X_i,x_i}(\tilde{M})^{(\kappa)} = T_{X_{i+1},x_{i+1}}(\tilde{M})^{(\kappa)}$ for $i \geq \lambda$. Because $T_{X,x}(\tilde{M})^{(\kappa)} = \bigcup_{i \geq 1} T_{X_i,x_i}(\tilde{M})^{(\kappa)}$ we may conclude $T_{X,x}(\tilde{M})^{(\kappa)} = T_{X_\lambda,x_\lambda}(\tilde{M})^{(\kappa)}$ hence $T_{X,x}(\tilde{M}) = \sum_{j \geq 0} T_{X_\lambda,x_\lambda}(\tilde{M})^{(\kappa)} U^j \subset T_{X_\lambda,x_\lambda}(\tilde{M})$, i.e. the chain $T_{X_i,x_i}(\tilde{M})$, $i \geq 1$, is stationary.

2) $\Rightarrow$ 1) If $\tilde{M}_{R[U]}$ has acc(fm) then $\tilde{M}_R$ has acc(fm) (see Appendix) hence M_R has as it is a direct summand of $\tilde{M}_R$.

3) $\Rightarrow$ 1) We start with a chain $T_{X_1,x_1}(M) \subset T_{X_2,x_2}(M) \subset \cdots$ given by a family of 1-pointed left R-modules (X_i, x_i), $i \geq 1$; as noted in the proof of 1) $\Rightarrow$ 2) we may assume that they are connected by morphisms $\alpha_i : (X_i, x_i) \to (X_{i+1}, x_{i+1})$, $i \geq 1$. To apply condition 3) we construct a 1-pointed left $R[U]$-module (X, x) as follows. We let $X = \coprod_{i \geq 1} X_i$ with the canonical injections $q_j : X_j \to X$, $x = q_1(x_1)$, and define an $R[U]$-module structure on X by $U \cdot q_i(y) = q_{i+1}\alpha_i(y)$ for $y \in X_i$. We note that $U^{i-1} \cdot x = q_i(x_i)$ for $i \geq 1$, $T_{X,x}(\tilde{M}) = \bigoplus_{j \geq 0} T_{X_{j+1},x_{j+1}}(M)U^j$ and $T_{X,x}(\tilde{M})^{(i)} = \bigoplus_{j=0}^{i} T_{X_{j+1},x_{j+1}}(M)U^j$ for $i \geq 0$. Our assumption implies that there is $\kappa \geq 0$ with

$$T_{X,x}(\tilde{M}) = \sum_{k \geq 0} T_{X,x}(\tilde{M})^{(\kappa)} U^k = \sum_{k \geq 0} \sum_{0 \leq j \leq \min\{k,\kappa\}} T_{X_{j+1},x_{j+1}}(M) U^k .$$

Letting $k \geq \kappa + 1$ and comparing coefficients of U^{k-1} we get $T_{X_k,x_k}(M) = \sum_{j=0}^{\kappa} T_{X_{j+1},x_{j+1}}(M) = T_{X_{\kappa+1},x_{\kappa+1}}(M)$. This shows that our chain is ultimately constant. □

Next we extend Theorem 1.4 to polynomial modules in several indeterminates. Let $\mathbb{Z}[U_1, \ldots, U_s]$ be the polynomial ring in s commuting indeterminates $U_1, \ldots, U_s$. Again we consider a module M_R with endomorphism ring S as a bimodule ${}_SM_R$ and the group $M[U_1, \ldots, U_s] = M \otimes_{\mathbb{Z}} \mathbb{Z}[U_1, \ldots, U_s]$ as a bimodule over the rings $S[U_1, \ldots, U_s] = S \otimes_{\mathbb{Z}} \mathbb{Z}[U_1, \ldots, U_s]$ and $R[U_1, \ldots, U_s] = R \otimes_{\mathbb{Z}} \mathbb{Z}[U_1, \ldots, U_s]$. Given natural numbers $\kappa_1 \geq 0, \ldots, \kappa_s \geq 0$ we denote the set of polynomials

$$\sum_{0 \leq j_1 \leq \kappa_1, \ldots, 0 \leq j_s \leq \kappa_s} m_{j_1,\ldots,j_s} U_1^{j_1} \cdot \ldots \cdot U_s^{j_s} \quad \text{in} \quad M[U_1, \ldots, U_s] \quad \text{by}$$

$$M[U_1, \ldots, U_s]^{(\kappa_1,\ldots,\kappa_s)}$$

and we let $T^{(\kappa_1,\ldots,\kappa_s)} = T \cap M[U_1, \ldots, U_s]^{(\kappa_1,\ldots,\kappa_s)}$ for every subgroup T of $M[U_1, \ldots, U_s]$.

Corollary 1.5. *Let M_R be a module with ascending chain condition for finite matrix subgroups.*

1) *The $R[U_1, \dots, U_s]$-module $M[U_1, \dots, U_s]$ has ascending chain condition for finite matrix subgroups.*

2) *For every left $R[U_1, \dots, U_s]$-module X and every n-tuple $\underline{x} = (x_1, \dots, x_n) \in X^n$ the S-module $T_{X,\underline{x}}(M[U_1, \dots, U_s])$ is finitely determined, i.e. there are numbers $\kappa_1 \geq 0, \dots, \kappa_s \geq 0$ with*

$$T_{X,\underline{x}}(M[U_1, \dots, U_s])$$

$$= \sum_{j_1 \geq 0, \dots, j_s \geq 0} T_{X,\underline{x}}\,(M[U_1, \dots, U_s])^{(\kappa_1, \dots, \kappa_s)} U_1^{j_1} \cdot \dots \cdot U_s^{j_s}\,.$$

Proof. Both statements are shown by induction the first one being plain. As for 2) we let $R_i = R[U_1, \dots, U_i]$ and $M_i = M[U_1, \dots, U_i]$ for $1 \leq i \leq s$, hence we have $R_s = R_{s-1}[U_s]$ and $M_s = M_{s-1}[U_s]$. By 1) the R_{s-1}-module M_{s-1} has acc(fm) hence Theorem 1.4 yields a number $\kappa_s \geq 0$ with $T_{X,\underline{x}}(M_s) = \sum_{j_s \geq 0} T_{X,\underline{x}}(M_s)^{(\kappa_s)} U_s^{j_s}$; here $T_{X,\underline{x}}(M_s)^{(\kappa_s)}$ is the set of n-tuples of polynomials whose U_s-degrees are at most κ_s. We have seen that the map

$$\tau_s^{(\kappa_s)}: T_{X,(\underline{x}, U_s\underline{x}, \dots, U_s^{\kappa_s}\underline{x})}(M_{s-1}) \;\to\; T_{X,\underline{x}}(M_s)^{(\kappa_s)}\,,$$

$$(\underline{m}_0, \dots, \underline{m}_{\kappa_s}) \;\mapsto\; \sum_{j=0}^{\kappa_s} \underline{m}_j U_s^j\,,$$

is an $S[U_1, \dots, U_{s-1}]$-isomorphism. To simplify matters we let $\underline{y} = (\underline{x}, U_s\underline{x}, \dots, U_s^{\kappa_s}\underline{x})$. Using the induction hypothesis for $T_{X,\underline{y}}(M_{s-1})$ we see that there are numbers $\kappa_1 \geq 0, \dots, \kappa_{s-1} \geq 0$ with

$$T_{X,\underline{y}}(M_{s-1}) = \sum_{j_1 \geq 0, \dots, j_{s-1} \geq 0} T_{X,\underline{y}}(M_{s-1})^{(\kappa_1, \dots, \kappa_{s-1})} U_1^{j_1} \cdot \dots \cdot U_{s-1}^{j_{s-1}}\,.$$

Now an application of $\tau_s^{(\kappa_s)}$ immediately yields

$$T_{X,\underline{x}}(M_s) = \textstyle\sum_{j_1 \geq 0, \dots, j_s \geq 0} T_{X,\underline{x}}(M_s)^{(\kappa_1, \dots, \kappa_s)} U_1^{j_1} \cdot \dots \cdot U_s^{j_s}\,. \quad \square$$

Corollary 1.6. *Let $R[U_1, \dots, U_s] \to W$ be a ring homomorphism such that W is finitely generated as a left $R[U_1, \dots, U_s]$-module. If M_R has ascending chain condition for finite matrix subgroups then the right W-module $M \otimes_R W$ satisfies the same condition.*

Proof. Let $R_s = R[U_1, \dots, U_s]$. By Corollary 1.5 the right R_s-module $M \otimes_R R_s$ has acc(fm) and it follows from Corollary A.3 that $M \otimes_R W \simeq (M \otimes_R R_s) \otimes_{R_s} W$ has acc(fm) as a right W-module. □

Typical examples for rings W as in Corollary 1.6 are epimorphic images of $R[U_1, \dots, U_s]$.

2. A construction of $\sum$-pure-injective modules over polynomial rings

An application of the well-known dualization principle for modules with chain conditions for f.m. subgroups ([10, Proposition 3] or [11, Corollary 1.6]) to Corollary 1.6 yields our next result. Keep in mind that a module is $\sum$-pure-injective if and only if it satisfies the descending chain condition for finite matrix subgroups.

Theorem 2.1. *Let $R[U_1, \dots, U_s] \to W$ be a ring homomorphism such that W is finitely generated as a right module over $R[U_1, \dots, U_s]$. If M_R satisfies the descending chain condition for finite matrix subgroups then the right W-module $(W_R, M_R)_W$ does as well.*

Proof. For any module L we let $L^+ = (L, \mathbb{Q}/\mathbb{Z})$. Our assumption on M_R implies that ${}_RM^+$ has acc(fm) hence the left W-module $W \otimes_R M^+$ satisfies the same condition by Corollary 1.6. Further dualization shows that the right W-module $(W \otimes_R M^+)^+ \simeq (W_R, M_R^{++})$ has dcc(fm). *As* M is $\sum$-pure-injective the evaluation map $c : M_R \to M_R^{++}$ is a split monomorphism, hence the induced W-homomorphism $(1, c) : (W_R, M_R) \to (W_R, M_R^{++})$ splits as well. This shows that $(W_R, M_R)_W$ has dcc(fm). □

We want to connect Theorem 2.1 with results on inverse power series (for instance see [5]) and therefore take a closer look at the special case $W = R[U_1, \dots, U_s]$. We show that the right $R[U_1, \dots, U_s]$-module $(R[U_1, \dots, U_s]_R, M_R)$ is isomorphic to the module of inverse power series $M[[U_1^{-1}, \dots, U_s^{-1}]]$. As an R-module this is the product $M^{\mathbb{N}_0^s}$ whose elements $(m_{j_1,\dots,j_s})_{(j_1,\dots,j_s)\in\mathbb{N}_0^s}$ are written as power series $\sum_{j_1\geq 0,\dots,j_s\geq 0} m_{j_1,\dots,j_s} U_1^{-j_1} \cdot \ldots \cdot U_s^{-j_s}$ with non-positive exponents. This product is a right $R[U_1, \dots, U_s]$-module if we define $U_1^{-j_1} \cdot \ldots \cdot U_s^{-j_s} \cdot U_1^{k_1} \cdot \ldots \cdot U_s^{k_s}$ to be $U_1^{-(j_1-k_1)} \cdot \ldots \cdot U_s^{-(j_s-k_s)}$ if $0 \leq k_1 \leq j_1, \dots, 0 \leq k_s \leq j_s$, and 0 if $k_1, \dots, k_s, j_1, \dots, j_s$ are elements of $\mathbb{N}_0$ such that there is some i with $k_i > j_i$. It is easily checked that the map

$$\begin{aligned}(R[U_1, \dots, U_s]_R, M_R) &\to M[[U_1^{-1}, \dots, U_s^{-1}]]\\ h &\mapsto \sum_{j_1\geq 0,\dots,j_s\geq 0} h(U_1^{j_1} \cdot \ldots \cdot U_s^{j_s}) U_1^{-j_1} \cdot \ldots \cdot U_s^{-j_s}\end{aligned}$$

is an $R[U_1, \dots, U_s]$-isomorphism.

3. A variant of the Artin-Rees Lemma

Theorem 3.1. *Let M_R be a module with maximum condition for finite matrix subgroups, (X, x) a 1-pointed left R-module and $T = T_{X,x}(M)$. Furthermore let A be the ideal of the center of R generated by central elements $a_1, \dots, a_s$ of R. Then there exists some $\kappa \in \mathbb{N}_0$ such that $MA^{n+1} \cap T = (MA^n \cap T)A$ for $n \geq \kappa$.*

This result is shown by an application of Corollary 1.5. By use of T and the tuple $\underline{a} = (a_1, \dots, a_s)$ we shall define a certain finite matrix subgroup $\tilde{T}$ of $M[U_1, \dots, U_s]$ and the desired number κ is obtained from the fact that $\tilde{T}$ is finitely determined. As in Section 1 we will use the abbreviations $R_s = R[U_1, \dots, U_s]$ and $M_s = M[U_1, \dots, U_s]$; if $f(U_1, \dots, U_s) = \sum_{j_1 \geq 0, \dots, j_s \geq 0} m_{j_1, \dots, j_s} U_1^{j_1} \cdot \ldots \cdot U_s^{j_s}$ is a polynomial in M_s we let $f(\underline{a}) = \sum_{j_1 \geq 0, \dots, j_s \geq 0} m_{j_1, \dots, j_s} a_1^{j_1} \cdot \ldots \cdot a_s^{j_s}$. Now let $R[V]$ be the polynomial ring in a further indeterminate V and $X[V] = R[V] \otimes_R X$; again we write $V^k y$ for $V^k \otimes y$. We define a left R_s-module structure on $X[V]$ by fixing $f \cdot V^k y = f(a_1 V, \dots, a_s V) V^k y$ for $f = f(U_1, \dots, U_s) \in R_s$ and $V^k y \in X[V]$. Note that $f \cdot V^k y = f(\underline{a}) V^{j+k} y$ in case f is homogeneous of degree j. The finite matrix subgroup we aim at is $\tilde{T} = T_{X[V],x}(M_s)$. For better insight into its structure we recall that it is the kernel of the map $\tau_x : M_s \to M_s \otimes_{R_s} X[V]$. Using the identification $M_s \otimes_{R_s} X[V] \simeq M \otimes_R X[V]$, $f \otimes V^k y \mapsto f(\underline{a}) \otimes V^{j+k} y$, where f is homogeneous of degree j, the map $\tau_x : M_s \to M \otimes_R X[V]$ can be calculated as follows: Let $f \in M_s$ and $f = f_0 + \cdots + f_d$ be the decomposition into its homogeneous components (i.e. f_j is zero or homogeneous of degree j) then $\tau_x(f) = \sum_{j=0}^{d} f_j(\underline{a}) \otimes V^j x$.

Lemma 3.2. 1) *A homogeneous polynomial $f \in M_s$ is an element of $\tilde{T}$ iff $f(\underline{a}) \in T$.*

2) *The subgroup $\tilde{T}$ is homogeneous, i.e. a polynomial $f \in M_s$ is an element of $\tilde{T}$ if and only if all its homogeneous components are.*

Proof. 1) We have the following equivalences:

$$f \in \tilde{T} \Leftrightarrow \tau_x(f) = 0 \Leftrightarrow f(\underline{a}) \otimes V^j x = 0 \Leftrightarrow f(\underline{a}) \otimes x = 0 \Leftrightarrow f(\underline{a}) \in T .$$

2) Let $f \in M_s$ and $f = f_0 + \cdots + f_d$ be the homogeneous decomposition. We have

$$f \in \tilde{T} \Leftrightarrow \tau_x(f) = \sum_{j=0}^{d} f_j(\underline{a}) \otimes V^j x = 0 \Leftrightarrow \forall 0 \leq j \leq d : f_j(\underline{a}) \in T$$

$$\Leftrightarrow \forall 0 \leq j \leq d : f_j \in \tilde{T} . \quad \square$$

Proof of Theorem 3.1. Let $\tilde{T} = T_{X[V],x}(M_s)$ be the subgroup just defined. As M_R has acc(fm) this group is finitely determined by Corollary 1.5, hence there are numbers $\kappa_1 \geq 0, \dots, \kappa_s \geq 0$ such that $\tilde{T} = \sum_{j_1 \geq 0, \dots, j_s \geq 0} \tilde{T}^{(\kappa_1, \dots, \kappa_s)} \cdot U_1^{j_1} \cdot \ldots \cdot U_s^{j_s}$ where $\tilde{T}^{(\kappa_1, \dots, \kappa_s)}$ is the set of all $f \in \tilde{T}$ whose U_i-degree is at most κ_i for $1 \leq i \leq s$.

We will show that the number $\kappa = \kappa_1 + \ldots + \kappa_s$ meets our assertion. Letting $n \geq \kappa$ and $0 \neq m \in MA^{n+1} \cap T$ there is a homogeneous polynomial $f \in M_s$ of degree $n+1$ such that $m = f(\underline{a}) \in T$. By Lemma 3.2 we may infer $f \in \tilde{T}$ hence there is a family $(f_{j_1,\ldots,j_s})_{j_1,\ldots,j_s \in \mathbb{N}_0^s}$ of polynomials in $\tilde{T}^{(\kappa_1,\ldots,\kappa_s)}$ almost all of which are zero and for which holds $f = \sum_{j_1 \geq 0,\ldots,j_s \geq 0} f_{j_1,\ldots,j_s} U_1^{j_1} \cdot \ldots \cdot U_s^{j_s}$. By Lemma 3.2 we may in addition assume that the non-zero $f_{j_1,\ldots,j_s}$ are homogeneous with $\deg f_{j_1,\ldots,j_s} = n+1-(j_1+\cdots+j_s)$. We single out such a non-zero polynomial $f_{j_1,\ldots,j_s}$. As its U_i-degree is $\leq \kappa_i$, its total degree is $\leq \kappa_1 + \ldots + \kappa_s = \kappa$, hence there is some $1 \leq i \leq s$ with $j_i \geq 1$. It follows that the degree of $f_{j_1,\ldots,j_s} U_1^{j_1} \cdot \ldots \cdot U_i^{j_i - 1} \cdot \ldots \cdot U_s^{j_s}$ is n hence $f_{j_1,\ldots,j_s}(\underline{a}) a_1^{j_1} \cdot \ldots \cdot a_i^{j_i - 1} \cdot \ldots \cdot a_s^{j_s} \in MA^n \cap T$ and $f_{j_1,\ldots,j_s}(\underline{a}) a_1^{j_1} \cdot \ldots \cdot a_s^{j_s} \in (MA^n \cap T)A$. Because this holds for every non-zero summand of f we have $m = f(\underline{a}) \in (MA^n \cap T)A$. □

4. An extension of the Krull Intersection Theorem

Theorem 4.1. *Let M_R have maximum condition for finite matrix subgroups, let A be a finitely generated ideal of the center of R and $D_A(M) = \bigcap_{n \geq 1} MA^n$.*

1) *For every finite matrix subgroup T contained in $D_A(M)$ we have $TA = T$. In particular $D_A(M)A = D_A(M)$ if $D_A(M)$ itself is a finite matrix subgroup.*

2) *Assuming that every S-submodule of $D_A(M)$ is a finite matrix subgroup we have $D_A(M) = \{x \in M \mid \exists f \in SA : (1-f)x = 0\}$ and even $D_A(M) = 0$ if SA is contained in the Jacobson radical of S.*

As before S denotes the endomorphism ring of M. Regarding the additional assumptions in 1) and 2) we shall exhibit in Remark 4.3 a class of modules for which they are valid.

Proof. 1) If T is an f.m. subgroup of M contained in $D_A(M)$ then we have $MA^{n+1} \cap T = (MA^n \cap T)A$ for all sufficiently large n by Theorem 3.1, hence $T = TA$.

2) Observing 1) and the assumption that the Sx are f.m. subgroups for all $x \in D_A(M)$ we have the following equivalences: $x \in D_A(M) \Leftrightarrow Sx = SxA \Leftrightarrow \exists f \in SA : (1-f)x = 0$. If SA is contained in the radical of S then an equation $(1-f)x = 0$ with $f \in SA$ obviously implies $x = 0$. □

In the next Corollary we shall sharpen the characterization 2) of $D_A(M)$ by imposing further conditions on the modules Sx, $x \in D_A(M)$. In case M is a noetherian module over a commutative ring all assumptions of the Corollary are satisfied, in particular every element of SA is integral over A, hence we obtain a well-known theorem by Krull.

Corollary 4.2. *Let M_R, A and $D_A(M)$ be as in the preceding theorem. Furthermore we assume that M has maximum condition for finite matrix subgroups, every S-submodule of $D_A(M)$ is a finite matrix subgroup, and being given $x \in D_A(M)$ and $f \in SA$ with $(1-f)x = 0$ the endomorphism $Sx \to Sx$, $sx \mapsto fsx$, is integral over A. Then $D_A(M) = \{x \in M \mid \exists a \in A : x(1-a) = 0\}$.*

Proof. We only have to show that every element of $D_A(M)$ is annihilated by some $1-a$, $a \in A$. Letting $x \in D_A(M)$ there is some $f \in SA$ with $(1-f)x = 0$. As a result of the additional assumption the map $\varphi : Sx \to Sx$, $sx \mapsto fsx$, satisfies an equation $\varphi^k + \varphi^{k-1}c_{k-1} + \cdots + \varphi c_1 + 1 \cdot c_0 = 0$ with coefficients $c_0, \ldots, c_{k-1} \in A$. As $\varphi(x) = fx = x$ this yields $x(1 + c_{k-1} + \cdots + c_0) = 0$. □

Remark 4.3. We conclude this section with some comments on those modules M_R for which every cyclic S-submodule ($S = \mathrm{End}_R(M)$) is a finite matrix subgroup, a condition required in Theorem 4.1. We do not have a pleasing characterization of these modules which we call cfm-modules for a moment.

1) It is obvious that a cfm-module having acc(fm) is noetherian over its endomorphism ring.

2) We give an example of a module with acc(fm) which is not a cfm-module. Let $R = K[U_i]_{i \in I}/B^2$ where $K[U_i]_{i \in I}$ is the polynomial ring in an infinite family of indeterminates over a field K and B the ideal of $K[U_i]_{i \in I}$ generated by the indeterminates. R is a local ring with radical $J = B/B^2$ hence the injective hull E of R/J is an injective cogenerator with simple socle. It is well-known that the f.m. subgroups of R are the finitely generated ideals together with J and that R has dcc(fm) (for instance see [9, Satz 6.5]). It follows that $(R_R, E_R) \simeq E_R$ has acc(fm). Assuming that E_R is a cfm-module our first remark shows that E is noetherian over its endomorphism ring S. It follows that the module ${}_S(J_R, E_R)$, being an epimorphic image of ${}_SE$, is noetherian as well contradicting the fact that it is isomorphic to $({}_S\mathrm{Soc}\, E)^I$.

3) To give a positive result we exhibit a class $\mathscr{M}$ of cfm-modules which is likely to be very close to the class of all cfm-modules. The members of $\mathscr{M}$ are the modules M_R satisfying M-dcc, a property introduced in [11, p. 18]. The most striking description which also shows that a module M in $\mathscr{M}$ is a cfm-module reads as follows: For all $n \in \mathbb{N}$ and $\underline{x} \in M^n$ there is an n-pointed finitely presented module $(D, \underline{d})$ such that $\mathrm{End}(M) \cdot \underline{x} = H_{D,\underline{d}}(M)$. The class $\mathscr{M}$ contains the modules with dcc(fm) and the pure-projective modules. It is obvious that it is closed under direct sums; moreover it has the following properties:

a) If M' is a finitely generated pure submodule of some $M \in \mathscr{M}$ then M' and M/M' belong to $\mathscr{M}$ as well.
b) Every direct sum $M^{(I)}$ of a module $M \in \mathscr{M}$ belongs to $\mathscr{M}$.
c) Supposing that a module $M \in \mathscr{M}$ also has acc(fm) every product M^I belongs to $\mathscr{M}$.

Proof. a) If M has M-dcc then M has M'-dcc and M/M' has M-dcc, hence M' and M/M' are in $\mathscr{M}$.

b) Letting $M \in \mathscr{M}$, I be a set and ${}_SV$ an injective module ($S = \mathrm{End}_R(M)$), we have the commutative diagram

$$\begin{array}{ccc} M^{(I)} \otimes_R ({}_SM, {}_SV) & \xrightarrow{\ \nu\ } & ({}_SS^I, {}_SV) \\ \uparrow \cong & & \uparrow \lambda \\ (M \otimes_R ({}_SM, {}_SV))^{(I)} & \xrightarrow{\ \nu^{(I)}\ } & V^{(I)} \end{array}$$

where λ is the monomorphism given by $\lambda((v_i))((s_i)) = \sum_{i \in I} s_i v_i$ for $(v_i) \in V^{(I)}$ and $(s_i) \in S^I$. The lower map $\nu^{(I)}$ is injective by assumption hence the upper map ν is as well. This shows that M has $M^{(I)}$-dcc, hence $M^{(I)}$ has $M^{(I)}$-dcc by [11, Lemma 3.2].

c) This time we consider the commutative diagram

$$\begin{array}{ccc} M^I \otimes_R ({}_SM, {}_SV) & \xrightarrow{\ \nu\ } & ({}_S(M^I, M), {}_SV) \\ \mu \downarrow & & \downarrow \xi \\ (M \otimes_R ({}_SM, {}_SV))^I & \xrightarrow{\ \nu^I\ } & V^I \end{array}$$

in which ξ is defined by $\xi(F) = (F(p_i))_{i \in I}$, $p_i : M^I \to M$ denoting the i-th projection. We want to show that M has M^I-dcc hence have to show that the upper map ν is mono. The maps ν^I and μ are mono because M belongs to $\mathscr{M}$ and has acc(fm), hence ν is mono. Now [11, Lemma 3.2] establishes that M^I has M^I-dcc. □

Appendix: Chain conditions for f.m. subgroups under ring extensions

The following general results are added because we have used part of them in the preceding sections and there does not seem to exist a systematic account of these questions. Some of the statements are well-known and only listed for completeness.

We begin with recalling that in case $\varphi : R \to R'$ is a ring homomorphism and $M_{R'}$ a module having one of the chain conditions for f.m. subgroups then M as an R-module has so as well. This may for instance be seen by noting that f.m. subgroups of $M_{R'}$ resp. M_R are describable by R'-resp. R-matrices and that every R-matrix is an R'-matrix via φ.

Lemma A.1. *Let ${}_RB_S$ be a bimodule, $\underline{b} = (b_k)_{k \in K}$ a generating system of ${}_RB$ and (X, x) a 1-pointed left S-module. Then we have $T_{X,x}(M \otimes_R B_S) = \tau_{\underline{b}} T_{B \otimes_S X, (b_k \otimes x)}(M)$ for every module M_R and $H_{B \otimes_S X, (b_k \otimes x)}(N) = \varepsilon_{\underline{b}} H_{X,x}({}_S({}_RB, {}_RN))$ for every module ${}_RN$.*

Proof. The given data give rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & T_{X,x}(M \otimes_R B_S) & \to & M \otimes_R B_S & \xrightarrow{\tau_x} & M \otimes_R B \otimes_S X \\ & & & & {\scriptstyle \tau_{\underline{b}}}\uparrow & & \| \\ 0 & \to & T_{B \otimes_S X,(b_k \otimes x)}(M) & \to & M^{(K)} & \xrightarrow{\tau_{(b_k \otimes x)}} & M \otimes_R B \otimes_S X \end{array}$$

As $\tau_{\underline{b}}$ is surjective the first formula is readily inferred. The second formula is proved in a similar way; it should be noted that $\varepsilon_{\underline{b}} : ({}_RB, {}_RN) \to N^K$ is injective. □

Corollary A.2. 1) *If ${}_RB$ is finitely generated and M_R has acc(fm) then $M \otimes_R B_S$ has acc(fm) as well.*

2) *If ${}_RB$ is finitely presented and M_R has dcc(fm) then $M \otimes_R B_S$ has the same chain condition.*

3) ([9, Satz 6.1]) *If ${}_RB$ is finitely generated and ${}_RN$ has dcc(fm) then ${}_S({}_RB, {}_RN)$ also has dcc(fm).*

4) *If ${}_RB$ is finitely presented then acc(fm) goes over from ${}_RN$ to ${}_S({}_RB, {}_RN)$.*

Proof. We only show 1) and 2) the remaining items being analogously proven by use of the second formula of Lemma A.1.

1) Let ${}_RB$ be generated by a finite system $\underline{b} = (b_1, \ldots, b_n)$. As M has acc(fm) the product M^n has acc for subgroups of the form $T_{B \otimes_S X,(b_k \otimes x)}(M)$, the (X, x) running through the 1-pointed left S-modules. As $\tau_{\underline{b}}$ is a natural epimorphism the first formula of Lemma A.1 shows that $M \otimes_R B_S$ has acc for the subgroups $T_{X,x}(M \otimes_R B_S)$.

2) Now let ${}_RB$ be finitely presented and M_R have dcc(fm). This time we restrict ourselves to finitely presented pointed modules $({}_SX, x)$. Then the $(B \otimes_S X, (b_k \otimes x))$ are finitely presented n-pointed left R-modules, M^n has dcc for the $T_{B \otimes_S X,(b_k \otimes x)}(M)$ and it follows as in the preceding proof that $M \otimes_R B_S$ has dcc for the $T_{X,x}(M \otimes_R B)$. □

Corollary A.3. *Let $R \to R'$ be ring homomorphism.*

1) *If ${}_RR'$ is finitely generated and M_R has acc(fm) then $M \otimes_R R'_{R'}$ has acc(fm).*

2) *If ${}_RR'$ is finitely presented and M_R has dcc(fm) then $M \otimes_R R'_{R'}$ has dcc(fm).*

3) *If ${}_RR'$ is finitely generated and ${}_RN$ has dcc(fm) then ${}_{R'}({}_RR', {}_RN)$ has dcc(fm).*

4) *If ${}_RR'$ is finitely presented and ${}_RN$ has acc(fm) then ${}_{R'}({}_RR', {}_RN)$ has acc(fm).* □

Lemma A.4. *Let* ${}_S M_R$ *be a bimodule and* F_S *a flat module. Then we have* $F \otimes_S T_{X,x}(M) = T_{X,x}(F \otimes_S M)$ *for every* 1*-pointed left* R*-module* (X, x) *and* $F \otimes_S H_{Y,y}(M) = H_{Y,y}(F \otimes_S M)$ *for every* 1*-pointed finitely presented right* R*-module* (Y, y).

Proof. Tensoring the exact sequence $0 \to T_{X,x}(M) \to M \xrightarrow{\tau_x} M \otimes_R X$ with F_S gives the exact sequence $0 \to F \otimes_S T_{X,x}(M) \to F \otimes_S M \xrightarrow{\tau_x} F \otimes_S M \otimes_R X$ hence the first formula. Letting (Y_R, y) be finitely presented there is a finitely presented module $({}_R X, x)$ with $H_{Y,y} = T_{X,x}$ ([11, Lemma 1.1]), hence $F \otimes_S H_{Y,y}(M) = F \otimes_S T_{X,x}(M) = T_{X,x}(F \otimes_S M) = H_{Y,y}(F \otimes_S M)$. □

Our next result is a direct consequence of this lemma; note that the first part already occurs in [11, Corollary 2.3].

Corollary A.5. 1) *If* M *has acc or dcc for f.m. subgroups then* $F \otimes_S M_R$ *has the same property*.

2) *If* F_S *is faithfully flat and* $F \otimes_S M_R$ *has acc or dcc for f.m. subgroups then* M_R *has this chain condition as well.* □

The usefulness of the first part of this corollary will be illustrated by an example in the end of this Appendix. Here an application of the second part. Let R be a commutative ring and $R \to R'$ a ring homomorphism such that R' is a faithfully flat R-module. If $M \otimes_R R'$ has acc or dcc for f.m. subgroups as an R'- or R-module then M_R satisfies the same condition.

For the next lemma we bring to mind that a ring homomorphism is called a ring epimorphism if the multiplication map $R' \otimes_R R' \to R'$ is bijective [7, Chap. XI, § 1].

Lemma A.6. *Let* $R \to R'$ *be a ring epimorphism and* (X, x) *a* 1*-pointed left* R'*-module. Then we have* $T_{X,x}(M_{R'}) = T_{X,x}(M_R)$ *for every module* $M_{R'}$*, and* $H_{X,x}({}_{R'}N) = H_{X,x}({}_R N)$ *for every module* ${}_{R'}N$.

Proof. These formulae follow from the fact that in the present situation the canonical maps $M \otimes_R X \to M \otimes_{R'} X$ and $({}_{R'}X, {}_{R'}N) \to ({}_R X, {}_R N)$ are isomorphisms. □

Corollary A.7. *Let* $R \to R'$ *be a ring epimorphism and* M *a right* R'*-module.* $M_{R'}$ *has acc resp. dcc for f.m. subgroups iff* M_R *has the respective property.* □

As an illustration we apply Lemmata A.4 and A.6 to show a well-known result on central localization of finite matrix subgroups [3, Lemma 6.31]. Let C be the center of R and Σ a multiplicatively closed subset of C; as usual L_Σ denotes the module of quotients of a C-module L with denominators in Σ. Then we have $T_{X,x}(M_\Sigma) = T_{X,x}(M_R)_\Sigma$ for all M_R and all pointed left R_Σ-modules (X, x), and

$H_{X,x}(N_\Sigma) = H_{X,x}(N_R)_\Sigma$ for all $_RN$ and all finitely presented pointed left R_Σ-modules (X, x). [We deduce the second formula: As $R \to R_\Sigma$ is a ring epimorphism Lemma A.6 gives $H_{X,x}(N_\Sigma) = H_{X,x}((N_\Sigma)_R)$. On the other hand, since $N_\Sigma \simeq C_\Sigma \otimes_C N$ and C_Σ is a flat C-module we have $H_{X,x}((N_\Sigma)_R) = C_\Sigma \otimes_C H_{X,x}(N_R) = H_{X,x}(N_R)_\Sigma$ by Lemma A.4]. These formulae obviously imply that the R_Σ-module M_Σ inherits acc(fm) or dcc(fm) from M_R.

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Wolfgang Zimmermann, Mathematisches Institut der Universität, Theresienstr. 39, D-80333 München, Germany