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The Total of Modules and Rings

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Introduction

In these notes we give a complete and detailed presentation of all results connected with the notions " partially invertible" (=pi) and "the total" . We include also some results about regularity .

The questions , notions and most of the results originate from the authors of these notes (see [1] , [2] , [5]) . Some interesting results were also contributed by A. Zöllner (especially II. 3.4 and in IV. 3. the ideal-property of TOT_{k_0} and TOT_{k_3} ; see [7] , [8]) . One of the authors had also very stimulating conversations with H. Kleisli and B. Pareigis . Especially , B. Pareigis gave a nice characterization for pi (I. 2.4. (3)) and collaborated with us to get I. 6.5 and I. 6.6 . For the example at the end of I.7 we owe thanks for a hint to H. Zöschinger . Finally , the interesting theorem I. 4.8. 1 was proved by T. Martin .

We use several well-known results from the literature without mentioning always the sources . Especially , for regularity (in the sense of von Neumann) the paper [6] of J . Zelmanowitz was a foundation . Very stimulating for us were the results of M. Harada about what we call "Harada-modules" (see lecture notes of F. Kasch [3]) . For some well-known results about exchange modules , which we include here for completeness with proofs , we give no references (References are for example in [5]) . The definition of the radical of a category and IV. 1.2 is taken from G.M. Kelly [4] .

For the reader of these notes it will be obvious that our ideas and results can be extended and generalized in several directions . These notes may be a foundation to do that and may stimulate further work in this connection .

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Contents

I. General foundation in a Morita-context

§1. Assumptions and examples

We consider rings S , T with 1-element and unitary bimodules $A = {}_SA_T$, $B = {}_TB_S$.
Let further

$$\sigma : A\times B \to S \quad , \quad \tau : B\times A \to T$$

be mappings , for which we assume first only the properties (M1) and (M2) .

(M1) $\sigma(sa,b) = s\sigma(a,b)$, $\sigma(a,bs) = \sigma(a,b)s$,
$\sigma(at,b) = \sigma(a,tb)$,
$\tau(tb,a) = t\tau(b,a)$, $\tau(b,at) = \tau(b,a)t$,
$\tau(bs,a) = \tau(b,sa)$
for $a\in A$, $b\in B$, $s\in S$, $t\in T$.

(M2) Associative laws :
$\sigma(a,b)a_1 = a\tau(b,a_1)$, $\tau(b,a)b_1 = b\sigma(a,b_1)$
for $a , a_1 \in A$, $b , b_1 \in B$.

If there is no danger of confusion , we write for abbreviation

$$ab := \sigma(a,b) \quad , \quad ba := \tau(b,a) .$$

If we have a meaningful product of elements of A , B , S , T , by (M2) we can avoid using brackets . For further considerations σ and τ have also to be additive .

(M3) Additivity :

$$\sigma(a+a_1,b+b_1) = \sigma(a,b)+\sigma(a,b_1)+\sigma(a_1,b)+\sigma(a_1,b_1)$$

$$\tau(b+b_1,a+a_1) = \tau(b,a)+\tau(b,a_1)+\tau(b_1,a)+\tau(b_1,a_1)$$

If (M1) , (M2) , (M3) are satiesfied, then these conditions define a Morita-context and the mappings σ and τ can be factorized via the tensor products $A\otimes_T B$ resp. $B\otimes_S A$.

The induced homomorphisms we denote by $\hat{\sigma}$ and $\hat{\tau}$:

$$\hat{\sigma} : A \underset{T}{\otimes} B \to S \quad , \quad \hat{\tau} : B \otimes A \to T$$

First we assume only (M1) and (M2) , in which case A and B have only to be sets and S and T multiplicative monoids. Our fundamental notions can be defined under these week conditions and this fact may be of some relevance for semi groups.

To have later the possibility for short quotations , we mention here three examples for a Morita-context.

<u>(E1) Ring case</u>

For a ring R with $1 \in R$ let $A=B=S=T := R$ and $\sigma(r_1,r_2)=\tau(r_1,r_2)=r_1r_2$, $r_1 , r_2 \in R$. Then all conditions are satisfied .

<u>(E2) Hom case</u>

Let R be a ring with identity and let M_R , N_R be unitary R-modules . Denote

$$S := \mathrm{End}(N_R) \quad , \quad T := \mathrm{End}(M_R) \quad ,$$

$${}_SA_T := \mathrm{Hom}_R(M,N) \quad , \quad {}_TB_S := \mathrm{Hom}_R(N,M)$$

and $\sigma(f,g) := fg$, $\tau(g,f) := gf$, $f \in A$, $g \in B$.

Then (M1) , (M2) , (M3) are satisfied .

<u>(E3) Dual module case</u>

Let T be a ring with identity and let A_T be a unitary T-module . Denote

$$S := \mathrm{End}(A_T) \quad , \quad B := A^* = \mathrm{Hom}_T(A,T) \quad .$$

Then ${}_SA_T$, ${}_TB_S$ are bimodules . For $a \in A$, $g \in B$ define

$$\sigma(a,g) := ag : A \ni x \mapsto ag(x) \in A \quad ,$$

hence $ag \in S$. Further define

$$\tau(g,a) := ga = g(a)$$

(g applied on a) , hence $ga \in T$.

Then again (M1) , (M2) , (M3) are satisfied . By a slight change , this situation can also be considered as a special case of (E2) .

To see this , one has to substitute ${}_SA_T$ by the S-T-isomorphic module $Hom_T(T,A)$ with the isomorphism

$$\psi : A \ni a \mapsto (T \ni x \mapsto ax \in A) \in Hom_T(T,A) .$$

By this substitution σ and τ change to the mappings in (E2). In the following it is easy to see that the isomorphism ψ preserves all the notions defined in this paper.

§2. Definitions and multiplicative properties

In the following we have to make use of idempotents of a ring . Here we mention some properties of idempotents . An element d of a ring S is called an idempotent , iff $d^2 = d$. Then also 1-d is an idempotent and we have the decompositions

$$S = dS \oplus (1-d)S \quad , \quad S = Sd \oplus S(1-d)$$

in right resp. left ideals . Contrary , if $S = U \oplus V$ is a decomposition in right ideals and if $1 = u+v$, $u \in U$, $v \in V$, then u and v are idempotents and $v = 1-u$, $uS = U$, $vS = V$. We use these facts without any quotation .

In this section we assume only (M1) , (M2) .

2.1 Lemma

For $a \in A$ the following properties are equivalent :

(i) $\exists\, b \in B$ [ab is an idempotent $\neq 0$ in S]

(ii) $\exists\, b_1 \in B$ [b_1a is an idempotent $\neq 0$ in T]

(iii) $\exists\, c \in B$ [ac is an idempotent $\neq 0$ in S $\wedge$ ca is an idempotent $\neq 0$ in T]

Proof:

(i)$\Rightarrow$(ii),(iii) : $ab = d = d^2 \neq 0 \;\Rightarrow\; abd = d \wedge (bda)(bda) = bd(ab)da = bd^3a = bda$. Also $a(bda)b = d^3 = d \neq 0 \;\Rightarrow\; bda \neq 0$. Hence (i) and (iii) are satisfied with $b_1 = c = bd$. Similar proof for (ii) $\Rightarrow$ (i),(iii) and (iii) $\Rightarrow$ (i),(ii) is obvious .

2.2. Definition

Let be $a \in A$.

1) a is called **partielly invertible** , abbreviation pi ,
   $:\Leftrightarrow$ the conditions of 2.1 are satisfied .
2) The **total** of A = Tot(A) := set of elements of A , which are not pi .
3) a is called **regular** $:\Leftrightarrow \exists\, b \in B \; [aba = a]$

We underline the fact , that these notions are independent of the side and that there is a close relation to regularity .

Remark :

If a is pi and ab is an idempotent , then b is in general not uniquely determined by this property . Also in the definition aba = a of regularity b is in general not uniquely dtermined . But if b is uniquely determined , then $ab = 1 \in S$, $ba = 1 \in T$ since

$$aba = a(b + 1 - ab)a = a(b + 1 - ba)a \quad .$$

If we use these notions for rings, then always in the sense of example (E1) . For example $s \in S$ is pi iff there exists $s' \in S$ such that ss' is an idempotent $\neq 0$ in S. Our notion for regularity coincides in the ring case with the classical notion .

2.3. Corollary

Let be $a \in A$, $b \in B$, $s \in S$, $t \in T$.

(1) If sat is pi, then s,a,t are pi (in S resp. B resp. T).
(2) If ab is pi, then a,b are pi (in B resp. A). If ba is pi, then a,b are pi .
(3) $STot(A)T = Tot(A)$, $TTot(B)S = Tot(B)$,
$Tot(A)B \subset Tot(S)$, $ATot(B) \subset Tot(S)$,
$BTot(A) \subset Tot(T)$, $Tot(B)A \subset Tot(T)$,
$Tot(S)A \subset Tot(A)$, $ATot(T) \subset Tot(A)$,
$BTot(S) \subset Tot(B)$, $Tot(T)B \subset Tot(B)$,
$STot(S)S = Tot(S)$, $TTot(T)T = Tot(T)$.

Proof :

(1) : Since sat is pi, there exists $b \in B$ such that $satb = d = d^2 \neq 0$. Then $d = s(atb)$, hence s is pi. Similar on the other side for t . By $d = (sa)(tb)$ and the proof of 2.1 we see, that $(tbd)(sa) = (tbds)a$ is an idempotent $\neq 0$, hence a is pi .

(2) : ab is pi $\Rightarrow \exists\ s \in S$ [abs and sab are idempotent $\neq 0$] $\Rightarrow$ a,b are pi. Similar for ba .

(3) : If $a \in Tot(A)$, $s \in S$, $t \in T$, then by (1) sat cannot be pi , hence $sat \in Tot(A) \Rightarrow STot(A)T \subset Tot(A)$. Since $1 \in S$, $1 \in T$ also $Tot(A) \subset STot(A)T$. Similar in all other cases .

Obviously implies (3): If in a meaningful product of elements of A,B,S,T at least one factor is in the Tot , then the product is in Tot .

2.4. Corollary

Notation as before .

(1) If $aba = a \neq 0$, then ab and ba are idempotents $\neq 0$. Hence regular elements $\neq 0$ are pi .

(2) If $ab = d = d^2 \neq 0$ resp. $ba = e = e^2 \neq 0$, then da , bd , eb , ae are regular elements $\neq 0$.

(3) a is pi $\Leftrightarrow \exists\ c \in B$ $[cac = c \neq 0]$

(4) If $aba = a \Rightarrow a(bab)a = a$, $(bab)a(bab) = bab$.

Proof:

(1) : $aba = a \neq 0 \Rightarrow abab = ab \neq 0 \Rightarrow$ a is pi . Similar for ba .

(2) : $(da)b(da) = d^3a = da \Rightarrow$ da regular . $dab = d^2 = d \neq 0 \Rightarrow da \neq 0$. Similar in the other cases.

(3)$\Rightarrow$: $ab = d = d^2 \neq 0 \Rightarrow (bd)a(bd) = bd^3 = bd$. $a(bd) = d^2 \neq 0 \Rightarrow bd \neq 0$. For $c := bd$ (3) is satisfied .

(3)$\Leftarrow$: $cac = c \neq 0 \Rightarrow caca = ca \neq 0 \Rightarrow$ a is pi .

(4) : Compute .

By (2) we see that we can produce regular elements by pi elements. (3) shows that the pi elements are exactly those who occure in the definition of regular elements in the "middle".

By (4) we see that in the definition of regular elements the element in the middle can always be taken from the two-sided ideal generated by a and to be a regular element .

2.5. Corollary

If $aba = a$, $d := ab$, $e := ba$, then

$$Sa \ni sa \mapsto sd \in Sd$$

$$aT \ni at \mapsto et \in eT$$

are isomorphisms , hence Sa resp. aT are projective S- resp. T-modules .

Proof:

The given mappings are obviously epimorphisms. If $sd = sab = 0$, then $saba = sa = 0$, hence also injective. Since Sd resp. eT are projective, also Sa resp. aT are projective.

2.6. Corollary

For $a \in A$ we have

(1) a is pi $\Leftrightarrow \exists\, d \in S$, $d = d^2 \neq 0$ $[dS \subset aB \wedge dA \subset aT]$

$\Leftrightarrow \exists\, e \in T$, $e = e^2 \neq 0$ $[Te \subset Ba \wedge Ae \subset Sa]$

(2) a is regular $\Leftrightarrow \exists\, d \in S$, $d = d^2$ $[dS = aB \wedge dA = aT]$

$\Leftrightarrow \exists\, e \in T$, $e = e^2$ $[Te = Ba \wedge Ae = Sa]$

Proof:

(1)$\Rightarrow$: a is pi $\Rightarrow \exists\, b \in B$, $d = d^2 \neq 0$ $[ab = d]$ $\Rightarrow$ $dS = a(bS) \subset aB \wedge dA = a(bA) \subset aT$.

$\Leftarrow$: $dS \subset aB \Rightarrow \exists\, b \in B$ $[ab = d = d^2 \neq 0]$ $\Rightarrow$ a is pi .

Similar for the second "$\Leftrightarrow$" .

(2)$\Rightarrow$: a is regular $\Rightarrow \exists\, b \in B$ $[aba = a]$. For $d := ab$ we have $d = d^2$ and $dS = abS \subset aB \wedge aB = abaB \subset abS = dS \Rightarrow dS = aB$. Similar proof for $dA = aT$.

$\Leftarrow$: $dS = aB \Rightarrow \exists\, b \in B$ $[d = ab]$. $dA = aT \Rightarrow \exists\, a_1 \in A$ $[da_1 = a] \Rightarrow d^2 a_1 = da_1 = da = a$. Then $aba = da = a$.

Similar for the second "$\Leftrightarrow$" .

§3. Additive properties

Now , we assume (M1) , (M2) , (M3) , that is , we have a Morita-context .

Further, we have to use the following homomorphisms (with $a \in A$, $b \in B$) :

$$(-b)a : A \ni x \mapsto (xb)a \in Sa \quad ,$$
$$a(b-) : A \ni x \mapsto a(bx) \in aT \quad .$$

If f is a homomorphism, then we denote by Ke(f) the kernel of f and by Im(f) the image of f .

3.1. Theorem

If $a \in A$, $b \in B$ and if $aba = a$, then

$$A = Sa \oplus Ke((-b)a) = aT \oplus Ke(a(b-))$$

Proof :

Let $\iota : Sa \to A$ be the inclusion and $1_{Sa} : Sa \to Sa$ the identity , then the diagramm

$$\begin{array}{ccc} Sa & \xrightarrow{\ \iota\ } & A \\ & {\scriptstyle 1_{Sa}}\searrow & \downarrow{\scriptstyle (-b)a} \\ & & Sa \end{array}$$

is commutative . Hence $A = Im(\iota) \oplus Ke((-b)a)$. Similar for the second decomposition .

3.2. Corollary

If $a \in A$ is regular , then Sa resp. aT are projective, direct summands of ${}_SA$ resp. A_T .

Later we will consider the question if the converse of this statement is true .

Here we continue first in our general considerations .

3.3. Corollary

Let $a \in A$, $b \in B$ and $ab = d = d^2$, $ba = e = e^2$, then

$$A = Sda \oplus Ke((-b)da) \ , \quad Sa = Sda \oplus S(1-d)a \ ,$$
$$A = aeT \oplus Ke(ae(b-)) \ , \quad aT = aeT \oplus a(1-e)T \ .$$

Proof :

Since $(da)b(da) = d^3a = da$, we have the first decomposition by 3.1 .
$Sa = Sda + S(1-d)a$ is obvious. Assume $sda = s_1(1-d)a$, $s , s_1 \in S$, then multiplication with b from the right implies $sd^2 = sd = s_1(1-d)d = 0$. Hence $Sa = Sda \oplus S(1-d)a$. Similar for the other side .

For later considerations the following characterization of pi resp. regular is useful. For this we need the following notation: A **operates faithfully** on B iff for each $x \in A$, $x \neq 0$ also $xB \neq 0$.

3.4. Theorem

Assume $a \in A$, then

(1) a is pi $\Leftrightarrow$ $\exists\, B_o \subseteq^{\oplus} B_S$, $0 \neq D \subseteq^{\oplus} S_S$
[$B_o \ni y \mapsto ay \in D$ is an isomorphism]

(2) a is regular $\Rightarrow$ $\exists\, B_S = B_o \oplus B_1$, $D \subseteq^{\oplus} S_S$
[$B_o \ni y \mapsto ay \in D$ is an isomorphism $\wedge$ $aB_1 = 0$]

If A operates faithfully on B , also the converse of this implication is true.

Proof :

(1)$\Rightarrow$: Let be $ab = d = d^2 \neq 0$, then 3.3 (for bd) implies $B_o := bdS \subseteq^{\oplus} B$.
If $D := dS$, then $B_o \ni bds \mapsto abds = ds \in D$
obviously is an isomorphism and $D \neq 0$.

(1)$\Leftarrow$: Since $0 \neq D \subseteq^{\oplus} S_S$ there exists $d \in S$, $d = d^2 \neq 0$, $D = dS$. Then there exists $b \in B_o$ such that $ab = d$.

(2)$\Rightarrow$: By 2.4. (4) we can assume $aba = a$, $bab = b$. By 3.1. for B we have

$$B = bS \oplus Ke(b(a-)) \ .$$

Define $B_o := bS$, $d := ab$, $D := dS$, then

$$B_o \ni bs \mapsto abs = ds \in D$$

is an isomorphism . For $y \in Ke(b(a-))$ we have

$$ay = (aba)y = a(b(ay)) = 0 \quad ,$$

hence with $B_1 := Ke(b(a-))$ the proof is complete .

(2)$\Leftarrow$: Since $D \subseteq^{\oplus} S_S$ we have $D = dS$, $d = d^2$. By the isomorphism there exists $b \in B_o$ such that $ab = d$. Since $b \in B_o$, also $bS \subseteq B_o$ and since

$$bS \ni bs \mapsto abs = ds \in dS = D$$

is already an isomorphism , we get $B_o = bS$. Then elements $y \in B$ can be written in the form $y = bs + y_1$, $y_1 \in B_1$.

Then by the assumption $aB_1 = 0$ we have

$$ay = abs + ay_1 = ds = d^2s = (aba)bs = (aba)bs + (aba)y_1 = (aba)y \quad ,$$

hence $(a - aba)y = 0$ for all $y \in B$. Since A operates faithfully on B , this implies $a = aba$.

Now we have to consider Tot(A) = the set of all elements of A, which are not pi . As shown in 2.3. (3) Tot(A) is closed under multiplication with elements of S and of T , that is STot(A)T = Tot(A) . But in general Tot(A) is not closed under addition . For example $Tot(\mathbb{Z}) = \mathbb{Z} \setminus \{-1;1\}$. It is a fundamental question of our considerations, under which conditions Tot(A) is closed under addition . Then Tot(A) is a S-T-submodule of A . In the ring case Tot(S) is then a twosided ideal of S .

3.5. Definition

If Tot(A) is closed under addition, then A is called a **total module** (with respect to the given Morita-context) .

If S is a ring and if Tot(S) is closed under addition, then S is called a **total ring** .

As mentioned before, Tot(A) is in general not additively closed, but there is always an important closure property .

To state this, we need the radical of ${}_SA$ resp. A_T , denoted by $\mathrm{Rad}({}_SA)$ resp. $\mathrm{Rad}(A_T)$. As wellknown , $\mathrm{Rad}({}_SA)$ is the sum of all small (=superfluous) submodules of ${}_SA$.

3.6. Proposition

(1) $\mathrm{Rad}({}_SA) + \mathrm{Tot}(A) = \mathrm{Rad}(A_T) + \mathrm{Tot}(A) = \mathrm{Tot}(A)$

(2) $\mathrm{Rad}({}_SA) + \mathrm{Rad}(A_T) \subset \mathrm{Tot}(A)$

Proof :

(1) : $\mathrm{Rad}({}_SA) + \mathrm{Tot}(A) \subset \mathrm{Tot}(A)$:

Let $u \in \mathrm{Rad}({}_SA)$, $v \in \mathrm{Tot}(A)$ and assume $u + v \notin \mathrm{Tot}(A)$, that is $u + v$ pi . Then there exists $(u + v)b = d = d^2 \neq 0$. Since

$${}_SA \ni x \mapsto xb \in S$$

is a homomorphism , $\mathrm{Rad}({}_SA)b \subset \mathrm{Rad}(S)$. Therefore $ub \in \mathrm{Rad}(S)$ and then Sub is a small submodule of ${}_SS$. This implies

$$S = Sd \oplus S(1-d) = Sub + Svb + S(1-d) = Svb + S(1-d)$$

$\Rightarrow Sd = Svbd \Rightarrow \exists\, s \in S\ [\ d = svbd\ ]$. Since d is pi , by 2.3. v must be pi , in contradiction to $v \in \mathrm{Tot}(A)$. Hence we have $\mathrm{Rad}({}_SA) + \mathrm{Tot}(A) \subset \mathrm{Tot}(A)$. Since $0 \in \mathrm{Rad}({}_SA)$ the inclusion in the opposite direction is also satisfied . Similar proof for $\mathrm{Rad}(A_T)$.

(2) : Since $0 \in \mathrm{Tot}(A)$ (1) implies

$$\mathrm{Rad}({}_SA) \subset \mathrm{Tot}(A) \quad , \quad \mathrm{Rad}(A_T) \subset \mathrm{Tot}(A)$$

and then (1) implies (2) .

3.7. Proposition

If S or T is a total ring , then A and B are total modules .

Proof :

Let S be a total ring and let $u , v \in \mathrm{Tot}(A)$. Assume $u + v$ is pi . Then there exists $b \in B$, $d \in S$ such that

$$(u + v)b = ub + vb = d = d^2 \neq 0 \ .$$

By 2.3. $ub, vb \in \mathrm{Tot}(S)$ and by assumption $ub + vb \in \mathrm{Tot}(S)$, but $d \notin \mathrm{Tot}(S)$ ↯.
Similar for the other cases .

§4. Morita-equivalence

We defined a ring S as a total ring iff Tot(S) is additively closed, that means, Tot(S) is a twosided ideal in S . Further we know by 3.6. that $Rad(S) \subset Tot(S)$. We define now two special types of total rings .

4.1. Definition

1) S is called a **radicaltotal ring** $:\Leftrightarrow$ Rad(S) = Tot(S) .
2) S is called **totalfree** $:\Leftrightarrow$ Tot(S) = 0 .

Obviously a totalfree ring is radicaltotal and - since Rad(S) is a twosided ideal - a radicaltotal ring is total . Now we study total rings and the just defined interesting special cases of total rings .

In this section we intend to prove, that the notions "total" , "radicaltotal" and "totalfree" are preserved under Morita-equivalence . If the rings S and T are Morita-equivalent , we write $S \approx T$. In this case , there exists a progenerator A_T such that $S \cong End(A_T)$. Since our notions are obviously preserved under ringisomorphisms , we assume $S := End(A_T)$ and the case (E3) , where $B = A^* = Hom_T(A,T)$. If A_T is a progenerator , then $Im(\hat{\sigma}) = S$, $Im(\hat{\tau}) = T$. But we have not to use always all the properties , which are given by the assumption $S \approx T$. We state in each case , what we really need .

4.2. Lemma

If $S = End(A_T)$, $Im(\hat{\sigma}) = S$ and Tot(A) additively closed , then S is a total ring .

Proof :

Let s_1 , $s_2 \in Tot(S)$ and assume $s_1 + s_2 \notin Tot(S)$. Then there exists $s \in S$ such that

$$s(s_1 + s_2) = d = d^2 \neq 0 \ .$$

Since ss_1A , $ss_2A \subset \mathrm{Tot}(A)$ by 2.3. , then by assumption also $dA \subset \mathrm{Tot}(A)$. Then by $\mathrm{Im}(\hat{\sigma}) = S$ this implies $dS \subset \mathrm{Tot}(S)$, hence $d \in \mathrm{Tot}(S)$ ↯ .

4.3. Corollary

If $S \approx T$ and T is total , then S is total .

Proof :

Since $S \approx T$ we have $AA^* = S$. Since T is total , by 3.7. A is additively closed . Then we can apply 4.2 .

4.4. Lemma

1) If T is radicaltotal and A_T is projective , then $\mathrm{Rad}(A_T) = \mathrm{Tot}(A)$.
2) If $S = \mathrm{End}(A_T)$ and A_T is finitely generated and projective and if $\mathrm{Rad}(A_T) = \mathrm{Tot}(A)$, then S is radicaltotal .

Proof :

1) : Since $\mathrm{Rad}(A_T) \subset \mathrm{Tot}(A)$, we have only to show : $\mathrm{Tot}(A) \subset \mathrm{Rad}(A_T)$. Let $a \in A$, then since A_T is projective , we can write a with a dual basis : $a = \sum a_i\psi_i(a)$, $a_i \in A$, $\psi_i \in A^*$. For $a \in \mathrm{Tot}(A)$ by 2.3. follows $\psi_i(a) \in \mathrm{Tot}(T) = \mathrm{Rad}(T)$. Since $A\mathrm{Rad}(T) \subset \mathrm{Rad}(A_T)$, then $a_i\psi_i(a) \in \mathrm{Rad}(A_T)$, hence $a \in \mathrm{Rad}(A_T)$.

2) : Again , only $\mathrm{Tot}(S) \subset \mathrm{Rad}(S)$ is to prove . For $s \in \mathrm{Tot}(S)$ follows $sA = \mathrm{Im}(s) \subset \mathrm{Tot}(A) = \mathrm{Rad}(A_T)$. Since A_T is finitely generated , $\mathrm{Rad}(A_T)$ is small in A_T , hence sA is small in A_T . Since A_T is projective , that implies $s \in \mathrm{Rad}(S)$.

4.5. Corollary

If $S \approx T$ and if T is radicaltotal , then S is radicaltotal .

Proof : By 4.4 .

4.6. Lemma

If T is totalfree and A_T is projective , then $\mathrm{Tot}(A) = 0$ and $S = \mathrm{End}(A_T)$ is totalfree .

Proof :

Assume $a \in \mathrm{Tot}(A)$. Then in the dual basis representation $a = \sum a_i\psi_i(a)$ all $\psi_i(a) \in \mathrm{Tot}(T) = 0$, hence $a = 0$. Assume $s \in \mathrm{Tot}(S)$, then for all $a \in A$ $s(a) \in \mathrm{Tot}(A) = 0$, hence $s = 0$.

4.7. Corollary

If $S \approx T$ and T is totalfree , then S is totalfree .

Proof : By 4.6.

Until now , we transfered properties from T to A and S . But also the converse is possible . By 3.7. we know already : If S is a total ring , then A_T is a total module .

4.8. Proposition

1) If A_T is projective and $S = \mathrm{End}(A_T)$ is radicaltotal, then $\mathrm{Rad}(A_T) = \mathrm{Tot}(A)$.
2) If A_T is a generator and $S = \mathrm{End}(A_T)$ is totalfree , then $\mathrm{Tot}(A) = 0$.

Proof :

1) : We have only to prove $\mathrm{Tot}(A) \subset \mathrm{Rad}(A_T)$. Assume $a \in \mathrm{Tot}(A)$ and let $U \subsetneq A_T$ such that

$$A = aT + U \ .$$

If $(u_i \mid i \in I)$ is a family of generators of U , then there exists a dual-basis of A_T of the form $((a , u_i \mid i \in I) , (\psi , \psi_i \mid i \in I))$ where ψ belongs to a and ψ_i to u_i . Since $a \in \mathrm{Tot}(A)$ $a\psi \in \mathrm{Tot}(S) = \mathrm{Rad}(S)$. Since A_T is projective , $\mathrm{Im}(a\psi) = a\psi(A)$ is small in A . Then

$$A = a\psi(A) + \sum_{i \in I} u_i\psi_i(A) = \sum u_i\psi_i(A) \subsetneq U ,$$

hence $U = A$. This implies aT is small in A_T , hence $a \in \mathrm{Rad}(A_T)$.

2) : Assume $a \in \mathrm{Tot}(A)$, then $aA^* \subset \mathrm{Tot}(S) = 0$. Since A_T is a generator this implies $aT = 0$, hence $a = 0$.

§5. Simple properties of total , radicaltotal and totalfree rings

5.1. Proposition

If S is a total ring , then S/Tot(S) is totalfree .

Proof : Let $\overline{s} \in \overline{S} := S/\mathrm{Tot}(S)$, $\overline{s} \neq 0$, then $s \notin \mathrm{Tot}(S)$. Then there exists $r \in S$ such that $sr = d = d^2 \neq 0$. Since $d \notin \mathrm{Tot}(S)$, we have $\overline{s}\,\overline{r} = \overline{d} = \overline{d}^2 \neq 0$, hence $\overline{s} \notin \mathrm{Tot}(\overline{S})$, therefore $\mathrm{Tot}(\overline{S}) = 0$.

5.2. Proposition

Let be $\nu : S \to S/\mathrm{Rad}(S)$ and $s \in S$. Then

1) s is pi $\Rightarrow$ $\nu(s)$ is pi

2) If idempotents can be lifted from S/Rad(S) to S , then :

   s is pi $\Leftrightarrow$ $\nu(s)$ is pi .

Proof :

1) : s is pi $\Rightarrow \exists\ st = d = d^2 \neq 0$, $t \in S \Rightarrow \nu(s)\nu(t) = \nu(d) = \nu(d)^2 \neq 0$, since $d \notin \mathrm{Rad}(S)$ $\Rightarrow$ $\nu(s)$ is pi .

2) : We have only to prove : $\nu(s)$ is pi $\Rightarrow$ s is pi .

$\nu(s)$ is pi $\Rightarrow \exists\ \nu(s)\nu(t) = \nu(e) = \nu(e)^2 \neq 0$. By assumption there exists an idempotent $d \in S$ with $\nu(d) = \nu(e)$. Then we have $\nu(s)\nu(t) = \nu(st) = \nu(e) = \nu(d)$ $\Rightarrow st = d + u$, $u \in \mathrm{Rad}(S) \Rightarrow d = -u + st$. Assume s is not pi $\Rightarrow st \in \mathrm{Tot}(S)$ $\Rightarrow -u + st \in \mathrm{Tot}(S)$ by 3.6 . But $d \notin \mathrm{Tot}(S)$, since d is an idempotent $\neq 0$ ↯.

5.3. Corollary

Assumptions as in 5.2 . Then

1) $\mathrm{Tot}(S/\mathrm{Rad}(S)) \subset \nu(\mathrm{Tot}(S))$

2) If idempotents can be lifted from S/Rad(S) to S , then

   $\mathrm{Tot}(S/\mathrm{Rad}(S)) = \nu(\mathrm{Tot}(S))$

3) If idempotents can be lifted from S/Rad(S) to S and if $\mathrm{Tot}(S/\mathrm{Rad}(S)) = 0$, then S is radicaltotal .

Proof :

1) : By 5.2. 1) we have $\nu(s) \in \mathrm{Tot}(S/\mathrm{Rad}(S)) \Rightarrow s \in \mathrm{Tot}(S)$. Then follows $\nu(s) \in \nu(\mathrm{Tot}(S))$, hence 1) .

2) : By 5.2. 2) we have $s \in \mathrm{Tot}(S) \Leftrightarrow \nu(s) \in \mathrm{Tot}(S/\mathrm{Rad}(S))$. Then follows $\nu(\mathrm{Tot}(S)) \subset \mathrm{Tot}(S/\mathrm{Rad}(S))$. The converse inclusion is 1) .

3) : By assumption and 5.2. 2) we have $\mathrm{Tot}(S/\mathrm{Rad}(S)) = \nu(\mathrm{Tot}(S)) = 0 \Rightarrow \mathrm{Tot}(S) \subset \mathrm{Rad}(S) \Rightarrow \mathrm{Rad}(S) = \mathrm{Tot}(S)$.

5.4. Remarks and examples

1) It is well-known, that idempotents can be lifted from S/Rad(S) to S if Rad(S) is a nilideal .

2) Semi-simple and - more general - regular rings are totalfree . The converse is not true . We give an example for a totalfree ring , which is not regular . Let K be a field and R a subring $\neq 0$ of K which is not a field (for example : $\mathbb{Q}$ and $\mathbb{Z}$) . Then we consider the following subring of $K^{\mathbb{N}}$:

   $$S := \{ (x_i) \in K^{\mathbb{N}} \mid \exists\, m \in \mathbb{N}, r \in R \;\; \forall\, i \geq m \;\; [x_i = r] \} \ .$$

   Since R is not a field , there exists $0 \neq r_o \in R$ with $r_o^{-1} \notin R$. Define $(r_o) = (r_o\, r_o\, r_o \ldots)$, then (r_o) is not a regular element in S :

   Assume $(r_o)(x_i)(r_o) = (r_o)$, then $r_o r r_o = r_o$ for $i \geq m$, hence $r = r_o^{-1} \in R$ ↯.

   Therefore S is not a regular ring , but we show Tot(S) = 0 :

   Assume $0 \neq (x_i) \in S$ and $x_j \neq 0$, then

   $$(x_i)(0 \ldots 0\; x_j^{-1}\; 0 \ldots) = (0 \ldots 0\; 1\; 0 \ldots)$$

   is an idempotent $\neq 0$. Hence every element $\neq 0$ is pi .

3) If S is f-semi-perfect (= semi-regular) , then S/Rad(S) is regular and idempotents can be lifted from S/Rad(S) to S . Hence by 5.3. 3) these rings are radicaltotal . But there exist radicaltotal rings , which are not f-semi-perfect . An example for this is again the ring in 2) since Tot(S) = Rad(S) = 0 and S/Rad(S) = S is not regular .

4) Please remember in this connection for the well-known fact : For rings hold the following implications : artinian ⇒ perfect ⇒ semi-perfect ⇒ f-semi-perfect . Therefore , all these rings are radicaltotal .

Now we would like to consider for an idempotent $e \in S$ the ring eSe which has e as 1-element . It is well-known that

(1) $eRad(S)e = Rad(S) \cap eSe = Rad(eSe)$

holds . The same relation is true for Tot (without the assumption that S is a total ring) .

5.5. Proposition

If $e \in S$ is an idempotent , then

(2) $eTot(S)e = Tot(S) \cap eSe = Tot(eSe)$.

Proof :

We prove first for $s \in S$:

(3) ese is pi in eSe $\Leftrightarrow$ ese is pi in S .

$\Rightarrow$: This is obvious , since an idempotent in eSe is also an idempotent in S .

$\Leftarrow$: Let ese be pi in S and

$$eset = d = d^2 \neq 0$$

then $esetd = d = ed$. This implies $dede = d^2e = de$, hence $de = (ese)(etde)$ is an idempotent in eSe . Further $ded = d^2 = d \neq 0$, hence $de \neq 0$. That means , that (3) is true . For $s \in S$ (3) implies

$$ese \in Tot(eSe) \Leftrightarrow ese \in Tot(S)$$

and this means

$$Tot(eSe) = Tot(S) \cap eSe .$$

For $t = ese \in Tot(S) \cap eSe$ follows $t = ete \in eTot(S)e$. Since $STot(S)S = Tot(S)$ we have conversely : For $s \in Tot(S)$, hence $ese \in eTot(S)e$ follows $ese \in Tot(S) \cap eSe$. Therefore we have also

$$Tot(S) \cap eSe = eTot(S)e .$$

5.6. Corollary

Let $e \in S$ be an idempotent , then

1) Total S $\Rightarrow$ total eSe ,
2) Radicaltotal S $\Rightarrow$ radicaltotal eSe ,
3) Totalfree S $\Rightarrow$ totalfree eSe .

<u>Proof :</u>

1) and 3) follow from (2) in 5.5 .

2) follows from (1) and (2) in 5.5 .

We would like to mention that 5.6. also can be derived from 4.2. , 4.4. and 4.6. (with S in place of T) by using the finitely generated and projective module $A_S := eS$.

In this connection, it is useful to realize the following fact : If $e \in S := \mathrm{End}(A_R)$ is an idempotent $\neq 0,1$ and if $\iota : e(A) \to A$ is the inclusion and $\pi : A \to e(A)$ is the projection belonging to $A = e(A) \oplus (1-e)(A)$, then $e = \iota\pi$ and $1_{e(A)} = \pi\iota$.

What is End(e(A)) ? It is not eSe , since this is a subring of S and for $s \in S$

$$\mathrm{dom}(ese) = \mathrm{codom}(ese) = A$$

and not e(A) . To be precise :

$$\mathrm{End}(e(A)) = \pi S \iota \ .$$

But there exists the ringisomorphism

$$\rho : \mathrm{End}(eS) \ni \pi s \iota \mapsto \iota(\pi s \iota)\pi =. ese \in eSe$$

and for $x \in A$

$$\pi s \iota(e(x)) = ese(e(x)) \ ,$$

where on the left side e(x) is considered as an element in e(A) and on the right side as an element in A .

If we have an idempotent $\neq 0$ in End(e(A)) , then the image under this isomorphism is an idempotent $\neq 0$ in eSe , hence also in S .

Now we do the same , what is done very often in the literature , we write

$$\mathrm{End}(e(A)) = eSe \ .$$

This is not correct , but convenient and cannot imply confusions . By this , we can avoid to deal always with the ringisomorphism ρ . The same holds for

$$\mathrm{End}(eS) = eSe \ .$$

Now we intend to consider some properties of totalfree rings .

The right- resp. left-socle of the ring S we denote by $Soc(S_S)$ resp. $Soc(_SS)$.

5.7. Proposition

If S is totalfree , then $Soc(S_S) = Soc(_SS)$.

Proof :

The endomorphismring of a simple module ($\neq 0$) is a division ring . If $e \in S$ is an idempotent , then $eSe \cong End(eS)$. Therefore , if eS is simple , then eSe is a division ring with the 1-element e . We intend to show that also Se is simple . Consider $se \neq 0$, $s \in S$. Since S is totalfree there exists $t \in S$ such that tse is an idempotent $\neq 0$, that is $tsetse \neq 0$, hence $etse \neq 0$. Then there exists $eae \in eSe$ with $eaeetse = e$, which implies $Sse = Se$, therefore Se is simple . The same is true for the other side . Then follows $Soc(S_S) = Soc(_SS)$.

5.8. Lemma

If e and d are idempotents of S , then

$$eS \subsetneq dS \Leftrightarrow S(1-d) \subsetneq S(1-e)$$

Proof :

$\Rightarrow$: Since $eS \subset dS$ we have $de = e$. Then $(1-d)e = 0$, hence $S(1-d)$ is contained in the left-annihilator $S(1-e)$ of e . Assume $S(1-d) = S(1-e)$, then follows $d = ed$, hence $dS \subset eS$ in contradiction to the assumption .

$\Leftarrow$: Same proof .

5.9. Proposition

If the totalfree ring S satisfies the maximum condition for rightideals (or leftideals) , which are direct summands , then S is semi-simple .

Proof :

We assume the maximum condition for the right side .

1. Part : We show first , that every leftideal $\neq 0$ contains a simple leftideal of the form Se , $e = e^2$.

The proof for this is indirect . Assume A_1 is a leftideal , which does not contain a simple leftideal. Let $a \in A$, $a \neq 0$, then there exists $b \in S$ such that $ba = e_1 = e_1^2 \neq 0$. Since Se_1 is not simple , there exists a proper subideal $A_2 \subsetneqq Se_1$. Let $0 \neq e_2 \in A_2$ be an idempotent , then Se_2 is not simple . By induction , there exists a sequence

$$Se_1 \supsetneqq Se_2 \supsetneqq Se_3 \supsetneqq \dots$$

with idempotents $e_1 , e_2 , e_3 , \dots$ By 5.8. follows

$$(1-e_1)S \subsetneqq (1-e_2)S \subsetneqq (1-e_3)S \subsetneqq \dots$$

in contradiction to our assumption .

If B is a simple leftideal and $b \in B$, $b \neq 0$, there exists $s \in S$ such that $sb = e = e^2 \neq 0$ and $B = Se$.

2. Part : The proof of 5.9. is indirect . Assume S is not semi-simple (that is : not a direct sum of simple leftideals). Then we prove by induction :

For every $n \in \mathbb{N}$ there exists a decomposition

(4) $$S = Se_1 \oplus \dots \oplus Se_n \oplus Sd_{n+1}$$

with orthogonal idempotents $e_1 , \dots , e_n , d_{n+1}$ and simple $Se_1 , \dots , Se_n$. By the 1. part we have a simple leftideal Se_1 . With $d_2 := 1-e_1$ the case $n=1$ is satisfied . In the case n (see (4)) Sd_{n+1} cannot be 0 or simple since then S would be semi-simple . Therefore Sd_{n+1} contains a simple leftideal Se , $e = e^2$. Then

$$Sd_{n+1} = Se \oplus (S(1-e) \cap Sd_{n+1}) .$$

Let $d_{n+1} = e_{n+1} + d_{n+2}$, $e_{n+1} \in Se$, $d_{n+2} \in S(1-e) \cap Sd_{n+1}$,

then $Se = Se_{n+1}$ is simple and $d_{n+2} \neq 0$. Since $e_{n+1} , d_{n+2} \in Sd_{n+1}$ we have

$$e_{n+1}d_{n+1} = e_{n+1} , \quad d_{n+2}d_{n+1} = d_{n+2} .$$

By this follows , that e_{n+1} , d_{n+2} are orthogonal idempotents and further

$$e_{n+1}e_i = 0 , \quad d_{n+2}e_i = 0 , \quad i = 1, \dots, n .$$

Also

$$e_id_{n+1} = 0 = e_ie_{n+1} + e_id_{n+2} , \quad i = 1, \dots, n$$

implies $e_ie_{n+1} = e_id_{n+2} = 0$. With this , induction $n \to n+1$ is complete . Realize also that the $e_1 , \dots , e_n$ did not change by going from n to n+1 . To the sequence $e_1 , e_2 , e_3 , \dots$ of orthogonal idempotents we consider the sequence of rightideals

$$e_1S \subsetneqq (e_1 + e_2)S \subsetneqq (e_1 + e_2 + e_3)S \subsetneqq \dots$$

These are direct summands of S_S , since by the orthogonality $e_1 + \ldots + e_n$ is an idempotent . This is a contradiction to our assumption .

This result includes the well-known fact , that regular , onesided noetherian rings are semi-simple .

As an example , we consider $\mathbb{Z}/n\mathbb{Z}$, $n > 1$. Let be

$$n = p_1^{k_1} \ldots p_m^{k_m} \quad , \quad k_i \geq 1$$

the primnumber decomposition of n . Denote by $\varphi(n)$ the Euler-function , by $\nu(n)$ the number of regular elements and by $\kappa(n)$ the number of pi elements of $\mathbb{Z}/n\mathbb{Z}$. Since $\mathbb{Z}/n\mathbb{Z}$ is artinian , it is a radicaltotal ring . Hence $\kappa(n)$ is also the number of elements not in $\mathrm{Rad}(\mathbb{Z}/n\mathbb{Z})$.

5.10. Proposition

For $a \in \mathbb{Z}$ holds :

(i) $a + n\mathbb{Z}$ is regular $\Leftrightarrow$ $\forall\, i = 1, \ldots, m \quad [\, p_i | a \Rightarrow p_i^{k_i} | a \,]$

(ii) $\nu(n) = \prod_{i=1}^{m} (\varphi(p_i^{k_i}) + 1)$

(iii) $a + n\mathbb{Z}$ is pi $\Leftrightarrow$ $\exists\, i \in \{1, \ldots, n\} \quad [\, p_i \nmid a \,]$

(iv) $\kappa(n) = n(1 - \frac{1}{p_1 p_2 \cdots p_m})$

Proof :

(i): $a + n\mathbb{Z}$ is regular iff there exists $b \in \mathbb{Z}$ such that

$$(a + n\mathbb{Z})(b + n\mathbb{Z})(a + n\mathbb{Z}) = a + n\mathbb{Z} \Leftrightarrow aba \equiv a \pmod n \Leftrightarrow$$
$$a(ba - 1) \equiv 0 \pmod n \ .$$

$\Rightarrow$: If $p_i | a \Rightarrow p_i \nmid (ba - 1)$; since $p_i^{k_i} | a(ba-1)$, we get $p_i^{k_i} | a$.

$\Leftarrow$: For $a \in \mathbb{Z}$, which satisfies the condition in (i) , we define

$$I := \{ i \mid i \in \{1, \ldots, m\} \wedge p_i^{k_i} | a \} \ ,$$
$$I' := \{1, \ldots, m\} \setminus I$$

and

$$r_I := \prod_{i \in I} p_i^{k_i} \quad (\text{with } r_\emptyset = 1) \quad , \quad s_I := \frac{n}{r_I} = r_{I'} \quad , \quad a_o = \frac{a}{r_I} \ .$$

Then by assumption $\gcd(a_o, s_I) = 1$ and also $\gcd(a, s_I) = 1$. Then there exist $b, c \in \mathbb{Z}$ such that

$$ba + cs_I = 1 \quad \Rightarrow \quad aba + acs_I = a \quad .$$

Since $acs_I = r_I a_o c \frac{n}{r_I} = a_o cn \Rightarrow aba \equiv a \pmod{n}$.

(ii): We have to count the integers a with $1 \le a \le n$ and which satisfy the condition in (i), that is, which are of the form

$$a = r_I a_o \quad , \quad \gcd(a_o, s_I) = 1 \quad .$$

For fixed r_I there exist exactly $\varphi(s_I)$ such integers.

Now we consider $I_1, I_2 \subset \{1, \ldots, m\}$, $I_1 \neq I_2$. We show that

$$r_{I_1} a_1 = r_{I_2} a_2 \quad , \quad \gcd(a_1, s_{I_1}) = \gcd(a_2, s_{I_2}) = 1$$

is not possible. Since $I_1 \neq I_2$, we can assume, that there exists $i \in I_1$, $i \notin I_2$. Then $p_i^{k_i} \mid r_{I_1}$, $p_i^{k_i} \nmid r_{I_2}$, hence $p_i^{k_i} \mid a_2$, which contradicts $\gcd(a_2, s_{I_2}) = 1$. Therefore $\varphi(s_{I_1})$ and $\varphi(s_{I_2})$ do not count the same regular element twice.

If I runs through all subsets of $\{1, \ldots, m\}$, then also I' and $s_I = r_{I'}$. Therefore

$$\nu(n) = \sum_{I \subset \{1,\ldots,m\}} \varphi(s_I)$$
$$= \sum_{I \subset \{1,\ldots,m\}} \varphi(r_I) = \prod_{i=1}^{m} (\varphi(p_i^{k_i}) + 1)$$

For the second equation the multiplicative property of the φ-function is used and the fact, that $\varphi(r_\emptyset) = \varphi(1) = 1$.

(iii),(iv): Immediate consequences of the fact that

$$\mathrm{Tot}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Rad}(\mathbb{Z}/n\mathbb{Z})$$

is the ideal generated by $p_1 \ldots p_m + n\mathbb{Z}$.

§6. Partially invertible and regular elements in Hom

Now we consider the Hom case (E2), where

$${}_S A_T = \mathrm{Hom}_R(M,N) \quad , \quad {}_T B_S = \mathrm{Hom}_R(N,M)$$
$$S = \mathrm{End}(N_R) \quad , \quad T = \mathrm{End}(M_R) \quad .$$

<u>6.1. Lemma</u>

If $h \in B$ is regular and

$$hfh = h \quad , \quad f \in A \quad ,$$

then

(5) $$M = h(N) \oplus (1-hf)(M) \quad .$$

<u>Proof :</u>

Denote $e := hf$, then $e = e^2 \in T$. Consider

$$e(M) = hf(M) \subset h(N) = hfh(N) \subset hf(M) = e(M) \quad ,$$

hence $e(M) = h(N)$ and therefore

$$M = e(M) \oplus (1-e)(M) = h(N) \oplus (1-hf)(M) \quad .$$

If $f \in A$ is pi and $fg = d = d^2 \neq 0$, $g \in B$, then $h := gd$ is regular and $hfh = h$. By (5) we have now

(6) $$M = gd(N) \oplus (1-gdf)(M) \quad .$$

<u>6.2. Proposition</u>

Assume $f \in A$, then

1) f is pi $\Leftrightarrow \exists \; M_o \subsetneq^{\oplus} M \;,\; 0 \neq N_o \subsetneq^{\oplus} N$
$[\, M_o \ni x \mapsto f(x) \in N_o$ is an isomorphism $]$

2) f is regular $\Leftrightarrow \exists \; M = M_o \oplus M_1 \;,\; N_o \subsetneq^{\oplus} N$
$[\, M_o \ni x \mapsto f(x) \in N_o$ is an isomorphism $\wedge \; f(M_1) = 0 \,]$

<u>Proof :</u>

1)$\Rightarrow$: Define $M_o := gd(N)$ as in (6) and $N_o := d(N)$. Then for $y \in N$ we have $f(gd(y)) = d^2(y) = d(y) \in N_o$. Further $f(gd(y)) = 0$ implies $gd(y) = 0$. Therefore

$$M_o \ni x_o \mapsto f(x_o) \in N_o$$

is an isomorphism and $N_o \neq 0$, since $d \neq 0$.

1)$\Leftarrow$: Denote with $\iota_{M_o} : M_o \to M$ the inclusion , π_{N_o} the projection $N \to N_o$ belonging to $N = N_o \oplus N_1$ and $\phi := \hat{f}^{-1}$, where $\hat{f}$ is the given isomorphism .

Then for $y = y_o + y_1$, $y_o \in N_o$, $y_1 \in N_1$ follows

$$f \iota_{M_o} \phi \pi_{N_o} (y_o + y_1) = f \iota_{M_o} \phi (y_o) = f(\phi(y_o)) = y_o$$

and this implies , that $f \iota_{M_o} \phi \pi_{N_o}$ is an idempotent $\neq 0$. Hence f is pi .

2)$\Rightarrow$: Assume $fgf = f$. Take the same isomorphism as in the proof of 1) . Then also

$$f(1-gdf)(M) = (f - fgfgf)(M) = (f - f)(M) = 0 \; .$$

Hence with $M_1 := (1 - gdf)(M)$ we have the statement .

2)$\Leftarrow$: We consider the same situation as in the proof of 1) and further $M = M_o \oplus M_1$ with $f(M_1) = 0$. For $x = x_o + x_1$, $x_o \in M_o$, $x_1 \in M_1$ we have by the given isomorphism $f(x_o) \in N_o$ and then

$$f \iota_{M_o} \phi \pi_{N_o} f(x_o + x_1) = f \iota_{M_o} \phi \pi_{N_o} (x_o) = f(x_o) = f(x_o + x_1) \; ,$$

hence $f(\iota_{M_o} \phi \pi_{N_o}) f = f$. Therefore f is regular .

This result is similar to 3.4. , but realize the difference !

If $f \in \mathrm{Tot}(\mathrm{Hom}_R(M,N))$, then by 6.2. f does not induce an isomorphism between any direct summands $\neq 0$ of M and N . Therefore we called f a **total nonisomorphism** . The total of $\mathrm{Hom}_R(M,N)$ is then the set of all total nonisomorphisms . In this way the word "total" came into the game .

Later we have to use 6.2., since it is a good tool to check if a homomorphism f is a total nonisomorphism or not .

Now we consider the question : Under which conditions for M_R , N_R is the converse of 3.2. satisfied ? We show first that it is always true in the dual case (E3) , where A_T is arbitrary and $S = \mathrm{End}(A_T)$, $B = (A_T)^*$.

<u>6.3. Proposition</u>

Assume $a \in A$ such that aT is a projective , direct summand of A_T , then a is regular with respect to $B = A^*$, that is there exists $h \in (A_T)^*$ such that $ah(a) = a$.

Proof :

Since aT is projective the epimorphism

$$T \ni t \mapsto at \in aT$$

splits . That implies that there exists a decomposition $T = eT \oplus (1-e)T$, $e = e^2$, such that

$$eT \ni et \mapsto aet \in aT$$

is an isomorphism and $a(1-e)T = 0$. This implies $a = ae$. The inverse isomorphism we denote by

$$\psi : aT \ni at = aet \mapsto et \in eT .$$

Then $\psi(a) = e$. By assumption we have a decomposition $A = aT \oplus A_1$. Denote by $\iota : eT \to T$ the inclusion , then we define $h \in (A_T)^*$ by

$$h_{|aT} := \iota\psi \quad , \quad h(A_1) := 0 .$$

Then $h(a) = e$ and $ah(a) = ae = a$, what we had to show .

Now we come back to the general case (E2) . For $g \in Hom_R(N,M)$, we consider the right T-homomorphism

$$g^* : Hom_R(M,N)_T \ni f \mapsto gf \in T_T = Hom_R(M,M)_T .$$

Then

$$\Delta : {}_T Hom_R(N,M) \ni g \mapsto g^* \in Hom_T(Hom_R(M,N)_T , T_T)$$

is a left T-homomorphism .

6.4. Remark

Assume $f \in Hom_R(M,N)$ and fT is a projective direct summand and Δ is surjective , then f is regular .

Proof :

By 6.3. there exists $h \in (Hom_R(M,N)_T)^*$ such that $fhf = f$. Since Δ is surjective , there exists $g^* = h$, hence $fg^*f = fgf = f$.

6.5. Proposition

1) If M_R is a generator and N_R is arbitrary , then Δ is an isomorphism .

2) If $M^k = N \oplus U$ with $k \in \mathbb{N}$, then Δ is an epimorphism .

Proof :

1) Δ is surjective : Given $\psi \in \mathrm{Hom}_T(\mathrm{Hom}_R(M,N)_T, T_T)$, then we intend to define $g \in \mathrm{Hom}_R(M,N)$ such that $gf(x) = \psi(f)(x)$ for all $f \in \mathrm{Hom}_R(M,N)$ and $x \in M$. Since M_R is a generator , every $y \in N$ can be written in the form $y = \sum_{i=1}^{m} f_i(x_i)$, $f_i \in \mathrm{Hom}_R(M,N)$, $x_i \in M$. Also $1 \in R$ has a representation $1 = \sum_{j=1}^{n} h_j(m_j)$, $h_j \in \mathrm{Hom}_R(M,N)$, $m_j \in M$.

We define

$$g(y) := \sum_{i=1}^{m} \psi(f_i)(x_i) \quad .$$

Then

$$g(y) = \sum_{i=1}^{m} \psi(f_i)(x_i 1) = \sum_{i=1}^{m}\sum_{j=1}^{n} \psi(f_i)(x_i h_j m_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} \psi(f_i x_i h_j)(m_j)$$

since $x_i h_j \in T$. We continue

$$g(y) = \sum_{j=1}^{n} \psi(y h_j)(m_j)$$

and this equation shows , that $g(y)$ ist independent of the representation $y = \sum_{i=1}^{m} f_i(x_i)$. One can easily verify , that g is an R-isomorphism .

For $f \in \mathrm{Hom}_R(M,N)$ follows by the definition of g

$$gf(x) = \psi(f)(x)$$

hence $gf = g^* f = \psi(f)$.

Δ is injective : If $g \neq 0$ and $g(y) \neq 0$, then if $y = \sum_{i=1}^{m} f_i(x_i)$, then $gf_i \neq 0$ for at least one i . Therefore $g^* \neq 0$.

2) We denote

$\pi_N \ : \ M^k \rightarrow N$ the projection along $M^k = N \oplus U$,

$\iota_N \ : \ N \rightarrow M^k$ the inclusion ,

$\pi_i \ : \ M^k \rightarrow M_i$ the projection on the i-th' comp. ,

$\iota_i \ : \ M_i \rightarrow M^k$ the inclusion ,

$\alpha_i \ : \ M \rightarrow M_i$ the isomorphism .

Then $\alpha_i \alpha_i^{-1} = 1_{M_i}$, $\sum_{i=1}^{k} \iota_i \pi_i = 1_{M^k}$, $\pi_N \iota_N = 1_N$. For ψ as before , we define g by

$$g(y) := \sum_{i=1}^{k} \psi(\pi_N \iota_i \alpha_i)(\alpha_i^{-1} \pi_i \iota_N y) \quad , \quad y \in N \ .$$

Again , it is easy to check , that $g \in Hom_R(N,M)$. For $f \in Hom_R(M,N)$, $x \in M$ follows

$$gf(x) = \sum_{i=1}^{k} \psi(\pi_N \iota_i \alpha_i)(\alpha_i^{-1} \pi_i \iota_N f(x)) = \psi(\sum_{i=1}^{k} \pi_N \iota_i \alpha_i \alpha_i^{-1} \pi_i \iota_N f)(x) = \psi(f)(x)$$

since $\alpha_i^{-1} \pi_i \iota_N f \in T$. Again we have $gf = \psi(f)$.

It is a natural question if similar or "dual" results hold for $S = End(N_R)$ in place of T . For the 2) statement in 6.5. this is true , but not for 1) .

6.6. Proposition

If $N^k = M \oplus U$ with $k \in \mathbb{N}$, then for every $\psi \in Hom_S({}_S Hom_R(M,N) , {}_S S)$ there exists $g \in Hom_R(N,M)$ such that $fg = \psi(f)$ for every $f \in Hom_R(M,N)$.

Proof :

Similar notations as in in the proof of 6.5. 2) . Now we define g by

$$g(y) := \sum_{i=1}^{k} \pi_M \iota_i \alpha_i \psi(\alpha_i^{-1} \pi_i \iota_M)(y)$$

then

$$fg(y) = \sum_{i=1}^{k} f\pi_M \iota_i \alpha_i \psi(\alpha_i^{-1} \pi_i \iota_M)(y) = \psi(\sum_{i=1}^{k} f\pi_M \iota_i \alpha_i \alpha_i^{-1} \pi_i \iota_M)(y) = \psi(f)(y)$$

since $f\pi_M \iota_i \alpha_i \in S$. Therefore $fg = \psi(f)$.

Not always are Δ and the corresponding mapping for S surjective .

Counterexample :

$R = \mathbb{Z}$, $M = \mathbb{Q}$, $N = \mathbb{Q}/\mathbb{Z}$, then $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}$, $Hom_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$, $End(\mathbb{Q}_{\mathbb{Z}}) \cong \mathbb{Q}$, $End((\mathbb{Q}/\mathbb{Z})_{\mathbb{Z}}) \cong \mathbb{Q}$, $Hom_{\mathbb{Q}}(Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}) \cong \mathbb{Q} \neq 0$.

We have the following conjecture :
If for fixed M_R and all N_R Δ is an isomorphism , then M_R is a generator .

§7. The dual case

In the following we assume $B = A^*$ (E 3) . In §4. we proved already several results in this case in connection with Morita-equivalence . Further in 3.2 and 6.3 we proved already under this assumption the following result .

7.1. Proposition

Assume $a \in A_T$. Then a is regular iff aT is a projective direct summand of A_T .

We repeat one part of the proof . Let be $afa = a$, $f \in A^*$ and denote $e := fa$, then $e = e^2 \in T$. The mapping

$$aT \ni at \mapsto fat = et \in eT$$

is then an isomorphism and $ae = afa = a$. Since eT is projective , also aT is projective . Further

$$A = aT \oplus Ke(af-) .$$

We use these properties in the following .

7.2. Proposition

For an arbitrary module A_T one of the following conditions is satisfied :

(i) $A = Tot(A)$

(ii) $A = \bigoplus_{i=1}^{n} a_iT \oplus U$ with $n \geq 1$, $U \subset Tot(A)$ and $a_iT \cong e_iT$, where e_i is an idempotent $\neq 0$ in T and $a_ie_i = a_i$, $i = 1,\dots,n$ (a_iT is projective) .

(iii) A contains a locally direct summand of the form $\bigoplus_{i=1}^{\infty} a_iT$, where the a_iT have the same properties as in (ii) .

Proof :

If $A = Tot(A)$, then (i) is satisfied . Assume now $Tot(A) \neq A$. Then there exist $a \in A$, $f \in A^*$ such that $fa =: e_1$ is an idempotent $\neq 0$ in T . Then $a_1 := ae_1$ is regular by 2.4. and hence we have

$$A = a_1T \oplus U_1 \quad , \quad a_1T \cong e_1T \quad , \quad a_1e_1 = a_1 \quad .$$

If $U_1 \subset Tot(A)$, we have (ii) . If $U_1 \not\subset Tot(A)$, there exists a regular element $a_2 \in U_1$ with the properties as a_1 and

$$A = a_2T \oplus B_2 \quad .$$

Since $a_2T \subsetneq U_1$ this implies

$$U_1 = a_2T \oplus (U_1 \cap B_2) \quad .$$

With the notation $U_2 := U_1 \cap B_2$, we get

$$A = a_1T \oplus a_2T \oplus U_2 \quad .$$

If $U_2 \not\subset Tot(A)$ we continue by induction . Either this construction stops with $U_n = U \subset Tot(A)$, that is (ii) , or continues indefinitely , that is (iii) . If $I \neq \emptyset$ is a finite subset of $\mathbb{N}$ and if $n = \max\{i \mid i \in I\}$, then the decomposition in (ii) shows , that $\bigoplus_{i=1}^{\infty} a_iT$ is a locally direct summand of A .

7.3. Corollary

If $A_T \neq 0$ is a projective , radicaltotal module , then one of the following conditions is satisfied :

1) condition (ii) with $U = 0$

2) condition (iii) .

Proof :

For a projective module $A \neq 0$ always $Rad(A) \neq A$, hence (i) cannot occure . Now , consider (ii) . Since $U \subsetneq^{\oplus} A$, U is also projective and $Rad(U) = U \cap Rad(A)$. By assumption we have

$$U \subsetneq Tot(A) = Rad(A) \quad ,$$

hence $Rad(U) = U$, hence $U = 0$.

In this connection it is good to know by 4.4. , that for a radicaltotal ring T every projective module A_T is radicaltotal .

7.4. Corollary

If $A_T \neq 0$ is directly indecomposable , then exactly one of the following conditions is satisfied :

(i) $Tot(A) = A$,

(ii) $A = aT$, A is projective and there exists an idempotent $e \in T$, $e \neq 0$ such that $aT \cong eT$, $ae = e$, $a \notin Tot(A)$.

This implies that a directly indecomposable module , which is not projective or not cyclic , satisfies $Tot(A) = A$.

Now , we consider the situation $u \in U \subset A_T$. Then u is pi as an element of A iff there exists $f \in A^*$ such that fu (= $f(u)$) is an idempotent $\neq 0$ in T . Then u is also pi as an element of U , since $f|_U \in U^*$. This implies $Tot(U) \subset Tot(A)$, hence

$$Tot(U) \subset U \cap Tot(A) .$$

In general , the converse inclusion is not true . But if $U \subsetneq^{\oplus} A$ or if T_T is injective , then it is satisfied .

If $U \subsetneq^{\oplus} A$, then this follows from the fact , that every $g \in U^*$ can be extended to an element in A^* .

With the injective case we deal in the following proposition .

7.5. Proposition

Let T be a right-injective ring . Assume $u \in U \subsetneq A_T$, then : u is pi in A iff u is pi in U .

Proof :

⇒: Already proved by the foregoing remark .

$\Leftarrow$: Assume that there exists $g \in U^*$ such that $gu = e = e^2 \neq 0$. Then by 2.4. ue is a regular element and $ueT \cong eT$. Since T_T is injective , also eT and ueT are injective . Then the inclusion

$$\iota : ueT \to A$$

splits : $A = ueT \oplus B$. We define $f \in A^*$ by

$$f_{|ueT} := g_{|ueT} \quad , \quad f_{|B} := 0 \quad .$$

Then $fue = gue = e$ and this implies

$$(efu)(efu) = efu$$

and

$$efue = e \neq 0$$

hence efu is an idempotent $\neq 0$.

With $f \in A^*$ also $ef \in A^*$. Therefore u is pi as an element of A .

As an immediate consequence , we have

7.6. Corollary

Assumption as in 7.5 . Then

$$Tot(U) = U \cap Tot(A) \quad .$$

If A is total (that is $Tot(A) \subsetneq A$) , then $Tot(Tot(A)) = Tot(A)$.

It is well-known , that there exist modules A_T with $Rad(A) = A$ (for example $\mathbb{Q}_{\mathbb{Z}}$) . But if A_T is a projective module $\neq 0$, then $Rad(A) \neq A$. Is the same true for $Tot(A)$? We show by an example , that the answer is "no" . We give this example with all details , in spite of the fact that some properties could be taken from more general results in the literature .

1. Remark :

In a commutative ring T without zerodivisors any ideal is directly indecomposable .

Proof :

Assume the ideal $A \neq 0$ has a decomposition

$$A = A_1 \oplus A_2 \quad , \quad A_1 \neq 0 \quad .$$

For $\alpha_1 \in A_1$, $\alpha_1 \neq 0$ and $\alpha_2 \in A_2$ follows

$$\alpha_1\alpha_2 = \alpha_2\alpha_1 \in A_1 \cap A_2 = 0$$

hence $\alpha_2 = 0$, hence $A_2 = 0$.

Now we consider the ring $T = \mathbb{Z}[\sqrt{-5}]$, which is a subring of the field of complex numbers . Then T is a commutative ring without zerodivisors . We apply the norm of the complex numbers on T . If $a + b\sqrt{-5} \in T$, $(a,b \in \mathbb{Z})$, then $N(a+b\sqrt{-5}) = a^2 + 5b^2$. In T we consider the ideal A generated by 3 and $1 + \sqrt{-5}$. The elements of A have then the form

$$\alpha = 3(a_1 + a_2\sqrt{-5}) + (1+\sqrt{-5})(b_1 + b_2\sqrt{-5})$$
$$= (3a_1 + b_1 - 5b_2) + (3a_2 + b_1 + b_2)\sqrt{-5} \quad , \quad a_1, a_2, b_1, b_2 \in \mathbb{Z} \quad .$$

Denote $c_1 := 3a_1 + b_1 - 5b_2$, $c_2 := 3a_2 + b_1 + b_2$,

then $N(\alpha) = c_1^2 + 5c_2^2$.

We intend to show

$$N(\alpha) \geq 5 \quad \text{for} \quad 0 \neq \alpha \in A \quad .$$

If $c_2 \neq 0$, then $N(\alpha) \geq 5$. If $c_2 = 0$, $c_1 \neq 0$

then $c_2 = 3a_2 + b_1 + b_2 = 0 \quad \Rightarrow$

$$b_1 = -3a_2 - b_2 \quad \Rightarrow$$
$$c_1 = 3a_1 - 3a_2 - b_2 - 5b_2 = 3(a_1 - a_2 - 2b_2)$$

hence $N(\alpha) \geq 9$.

2. Remark : A is not cyclic .

Proof :

Assume $A = \alpha_o T$, then there exist β , $\gamma \in T$ such that

$$\alpha_o\beta = 3 \quad , \quad \alpha_o\gamma = 1 + \sqrt{-5} \quad .$$

These imply

$$N(\alpha_o\beta) = N(\alpha_o)N(\beta) = 9 \quad ,$$
$$N(\alpha_o\gamma) = N(\alpha_o)N(\gamma) = 6 \quad .$$

Therefore $N(\alpha_o)$ is a common divisor of 9 and 6 , that is 1 or 3 , in contradiction to $N(\alpha_o) \geq 5$.

Since A is directly indecomposable and not cyclic , 7.4. implies already Tot(A) = A . Since T is a Dedekind-ring , A must be projective . We give a proof in this special case .

3. Remark : A is projective .

Proof :

We show , that A has a dual basis . First we have

$$3\left(1 - \frac{2}{1+\sqrt{-5}}\right) = 2 + \sqrt{-5} \in T ,$$

$$(1+\sqrt{-5})\left(1 - \frac{2}{1+\sqrt{-5}}\right) = -1 + \sqrt{-5} \in T .$$

Denote by f_1 the multiplication of A by $1 - \frac{2}{1+\sqrt{-5}}$ and by f_2 the multiplication of A by -1 , then $f_1 , f_2 \in A^*$ and for $\alpha \in A$ follows

$$(3f_1 + (1+\sqrt{-5})f_2)(\alpha) = (2 + \sqrt{-5} - 1 - \sqrt{-5})\alpha = \alpha .$$

Hence

$$3f_1 + (1+\sqrt{-5})f_2 = 1_A ,$$

that is , we have a dual basis and A is projective .

In III . §3 we give further results in the dual case for direct decompositions .

II. Total properties and exchange properties

§1. Exchange properties

Before we go in the details , we mention one of the main goals of this chapter : A module with a total endomorphismring can be characterized by an exchange property , which is somewhat weeker than the well-known 2-exchange property . This includes that the 2-exchange modules have total endomorphismrings . Especially are the 2-exchange rings total rings . This shows that the class of total rings is a fairly interesting class of rings .

In this § we state some notions and results about exchange properties . To make these selfcontained , we include the proofs for the facts which we need in the following .

1.1 Definition

1) A module A_R has the exchange property (= EP) resp. the n-exchange property (= n-EP) for $n \in \mathbb{N}$

:⇔ for every situation

(1) $$M = A \oplus B = \bigoplus_{i \in I} C_i \quad \text{with I arbitrary resp. } I = \{1, \dots, n\} ,$$

there exists $C_i' \subsetneq C_i$ such that

(2) $$M = A \oplus (\bigoplus_{i \in I} C_i') .$$

2) A module A_R has the D2-exchange property = D2-EP

:⇔ for every $A_o \subsetneq^{\oplus} A$, $A_o \neq 0$ and for every situation

(3) $$M = A_o \oplus B = C \oplus D$$

at least one of the following conditions is satisfied :

(i) there exist $A_o' \subsetneq^{\oplus} A_o$, $A_o' \neq 0$ and $C' \subsetneq C$ such that

(4) $$M = A_o' \oplus C' \oplus D$$

(ii) there exist $A_o'' \subsetneq^{\oplus} A_o$, $A_o'' \neq 0$ and $D' \subsetneq D$ such that

(5) $$M = A_o'' \oplus C \oplus D' .$$

In these definitions it would be possible to write for A resp. A_o a module isomorphic to A resp. A_o . At the first sight , that looks more general , but at the second , it is to realize , that it gives the same notions . By the substitutionprinciple*) all these exchange properties are preserved under isomorphisms . Our notation is easier since we have not always to handle with a superfluous isomorphism . It is also trivial , that direct summands of modules with D2-EP have also this property .

The modular law implies that in the definitions C_i' , C' and D' are not only submodules but direct summands of C_i resp. C resp. D .

1.2. Corollary

For a module A the following conditions are equivalent :

(i) A has the D2-EP

(ii) every nonzero direct summand of A has the D2-EP

(iii) every nonzero direct summand of A contains a nonzero direct summand , which has the D2-EP .

Proof : (i) ⇒ (ii) ⇒ (iii) is trivial .

(iii) ⇒ (i) : Consider $0 \neq A_o \subsetneq^{\oplus} A$ and the situation

$$M = A_o \oplus B = C \oplus D .$$

By assumption there exists $0 \neq A_1 \subsetneq^{\oplus} A_o$, which has the D2-EP . Let $A_o = A_1 \oplus A_2$, then

$$M = A_1 \oplus (A_2 \oplus B) = C \oplus D .$$

Now , there exists $0 \neq A_1' \subsetneq^{\oplus} A_1$, such that

$$M = A_1' \oplus C' \oplus D , \quad C' \subsetneq C ,$$

or there exists $0 \neq A_1'' \subsetneq^{\oplus} A_1$, such that

$$M = A_1'' \oplus C \oplus D' , \quad D' \subsetneq D .$$

This proves (i) .

---

*) Substitutionprinciple : If M_R is a R-module , if $U \subsetneq M_R$ and if $f : U \to A$ is a R-isomorphism , then there exists a R-module N_R with $A \subsetneq N_R$ and a R-isomorphism $F : M_R \to N_R$ with $F_{|U} = f$.

We are here only interested in the 2-EP and D2-EP . But we would like to mention that the 2-EP implies the n-EP for every $n \in \mathbb{N}$, $n>2$. It is still an open question if the 2-EP implies the general EP .

We give now the proofs for two wellknown results about modules with 2-EP .

1.3. Lemma

Assume A has the 2-EP and

(6) $M = U \oplus A \oplus B = U \oplus C \oplus D$.

Then there exist $C' \subsetneq C$, $D' \subsetneq D$ such that

(7) $M = U \oplus A \oplus C' \oplus D'$.

Proof :

Denote by π the projection of M onto $C \oplus D$ along (6) . Then the restriction of π onto $A \oplus B$, that is $\pi|_{A\oplus B}$, is an isomorphism . Therefore

(8) $\pi(A) \oplus \pi(B) = C \oplus D$

and also $\pi(A)$ has the 2-EP . Therefore exist $C' \subsetneq C$, $D' \subsetneq D$ such that

(9) $\pi(A) \oplus \pi(B) = \pi(A) \oplus C' \oplus D'$.

We claim

$$M = U \oplus A \oplus C' \oplus D' .$$

First we have by (8) and (9)

(10) $M = U \oplus \pi(A) \oplus C' \oplus D'$.

If in (6) $a = u + c + d$, $a \in A$, $u \in U$, $c \in C$, $d \in D$, then $\pi(a) = c + d = -u + a$ and $a = u + \pi(a)$. Therefore $U \oplus \pi(A) = U \oplus A$. This and (10) imply (7) , what was to show .

1.4. Lemma

Assume $A = A_1 \oplus A_2$. Then : A has 2-EP $\Leftrightarrow$ A_1 and A_2 have 2-EP .

Proof :

$\Rightarrow$: Assume $M = A_1 \oplus B = C \oplus D$, then consider

$M \amalg A_2 = A_1 \oplus A_3 \oplus B = A_3 \oplus C \oplus D$ with $A_3 \cong A_2$, $A_1 \oplus A_3 \cong A_1 \oplus A_2 = A$. We use now the 2-EP of $A_1 \oplus A_3$ in the decomposition $M \amalg A_2 = (A_3 \oplus C) \oplus D$. There exist $U \subsetneq A_3 \oplus C$, $D' \subsetneq D$ such that

$$M \amalg A_2 = (A_1 \oplus A_3) \oplus U \oplus D' .$$

By $A_3 \oplus U \subset A_3 \oplus C$ follows $A_3 \oplus U = A_3 \oplus (C \cap (A_3 + U))$. For $C' = C \cap (A_3 + U)$ follows $M \amalg A_2 = A_1 \oplus A_3 \oplus C' \oplus D'$.
The projection of $M \amalg A_2$ onto M then delivers

$$M = A_1 \oplus C' \oplus D' .$$

$\Leftarrow$: Assume $M = A \oplus B = C \oplus D$. Since A_1 has the 2-EP , there exist $C' \subsetneq C$, $D' \subsetneq D$ such that

$$M = A_1 \oplus A_2 \oplus B = A_1 \oplus C' \oplus D' .$$

Now since A_2 has 2-EP we apply 1.3 . Then there exist $C'' \subsetneq C'$, $D'' \subsetneq D'$ such that

$$M = A_1 \oplus A_2 \oplus C'' \oplus D'' .$$

Therefore A has 2-EP .

<u>1.5. Proposition</u>

If A has the 2-EP , then A has the D2-EP .

<u>Proof :</u>

If A has 2-EP by 1.4. every direct summand A_o of A has also 2-EP . Without loss of generality we can therefore assume

$$M = A \oplus B = C \oplus D$$

and by assumption there exist $C = C' \oplus C''$, $D = D' \oplus D''$ such that

$$M = A \oplus C' \oplus D' .$$

If $C' = C$ or $D' = D$, then the proof is finished . We assume now $D' \neq D$. From

$$M = A \oplus C' \oplus D' = C \oplus D = C' \oplus C'' \oplus D' \oplus D''$$

follows

$$A \cong C'' \oplus D''$$

and therefore C'' has also 2-EP . We apply this now on the decomposition

$$M = C' \oplus C'' \oplus D = A \oplus C' \oplus D'$$

in the sense of 1.3 . Then there exist $A' \subsetneq A$, $D^* \subsetneq D'$ with

$$M = A' \oplus (C' \oplus C'') \oplus D^* = A' \oplus C \oplus D^* \ .$$

In this equation $A' = 0$ is not possible , then $A' = 0$ would imply $D^* = D' = D$.

§2. Partially invertible endomorphisms and exchange properties

Now we study connections between our notions and exchange properties .

For a module M_R we consider two decompositions

(11) $\quad M = A \oplus B \ , \ A \neq 0 \ ,$

(12) $\quad M = C \oplus D \ .$

We denote

$\iota_A : A \to M$ the inclusion

$\pi_A : M \to A$ the projection belonging to (11)

$e_C : M \to M$ resp. $e_D : M \to M$

the projector on C resp. D belonging to (12) .

Then $\pi_A e_C \iota_A$, $\pi_A e_D \iota_A \in S := \mathrm{End}(A)$ and

$$1_M = e_C + e_D$$

$$1_A = \pi_A 1_M \iota_A = \pi_A e_C \iota_A + \pi_A e_D \iota_A$$

2.1. Proposition

Notations as before . Then

1) $\pi_A e_C \iota_A$ is pi $\Leftrightarrow$ there exist decompositions

(13) $\quad M = A' \oplus C' \oplus D \ , \quad 0 \neq A' \subsetneq A \ , \ C' \subsetneq C \ ,$

(14) $\quad M = e_C(A') \oplus A'' \oplus B \ , \quad A'' \subsetneq A \ .$

2) $\pi_A e_C \iota_A$ is invertible (= automorphism of A) $\Leftrightarrow$ there exist decompositions

(15) $\quad M = A \oplus (B \cap C) \oplus D \ ,$

(16) $\quad M = e_C(A) \oplus B \ .$

3) If $\pi_A e_C \iota_A$ is invertible , the mapping

(17) $\quad A \ni x \to e_C(x) \in e_C(A)$

is an isomorphism .

Proof :

1)⇒: By assumption there exist $g, d \in S$ such that $g\pi_A e_C \iota_A = d = d^2 \neq 0$.
For the idempotent d holds

(18) $$A = d(A) \oplus (1-d)(A) \quad , \quad d(A) \neq 0 \; .$$

Denote by

$\iota : d(A) \to A$ the inclusion and

$\pi : A \to d(A)$ the projection belonging to (18) .

Then $d = \iota\pi$, $1_{d(A)} = \pi\iota$ and

(19) $$\pi g \pi_A e_C \iota_A \iota = \pi d \iota = 1_{d(A)} \; .$$

This gives the commutative diagram

$$\begin{array}{ccc} d(A) & \xrightarrow{\iota_A \iota} & M \\ & \searrow^{1_{d(A)}} & \downarrow \pi g \pi_A e_C \\ & & d(A) \end{array}$$

which implies

(20) $$M = \mathrm{Im}(\iota_A \iota) \oplus \mathrm{Ke}(\pi g \pi_A e_C) = d(A) \oplus \mathrm{Ke}(\pi g \pi_A e_C)$$

Since $\mathrm{Ke}(e_C) = D \subseteq \mathrm{Ke}(\pi g \pi_A e_C)$ and by the modular law we get by (12)

$$\mathrm{Ke}(\pi g \pi_A e_C) = C' \oplus D \quad , \quad C' \subseteq C \; .$$

Denote still $A' := d(A)$, then we have (13) .

By (19) follows similarly

$$M = \mathrm{Im}(e_C \iota_A \iota) \oplus \mathrm{Ke}(\pi g \pi_A) = e_C(d(A)) \oplus A'' \oplus B = e_C(A') \oplus A'' \oplus B \; ,$$

which is (14) .

2)⇒: Now , $\pi_A e_C \iota_A$ is invertible in S , that means , that there exists an automorphism $g \in S$ such that $g\pi_A e_C \iota_A = 1_A$. This implies

$$M = \mathrm{Im}(\iota_A) \oplus \mathrm{Ke}(g\pi_A e_C) = A \oplus (\mathrm{Ke}(g\pi_A) \cap C) \oplus D =$$
$$= A \oplus (B \cap C) \oplus D$$

and

$$M = \mathrm{Im}(e_C \iota_A) \oplus \mathrm{Ke}(g\pi_A) = e_C(A) \oplus B \; .$$

1)⇐: We intend to show , that $\pi_A e_C \iota_A$ induces an isomorphism between the direct summands A' and $\pi_A e_C(A')$. First , by (13) and the modular law follows that A' is a direct summand of A . Since $B = \mathrm{Ke}(\pi_A)$ by (14) follows $A = \pi_A e_C(A') \oplus A''$, hence also $\pi_A e_C(A')$ is a direct summand of A . Since $D = \mathrm{Ke}(e_C)$ and by (13) we see that $e_C \iota_A$ induces a monomorphism from A' to M . Since $B = \mathrm{Ke}(\pi_A)$ and by (14) we see that π_A induces a monomorphism

from $e_C(A')$ to A . Together we have the result , that $\pi_A e_C \iota_A$ induces an isomorphism between A' and $\pi_A e_C(A')$. By I. 6.2. $\pi_A e_C \iota_A$ is pi .
2)$\Leftarrow$: Now , as before , $\pi_A e_C \iota_A$ induces an isomorphism between A and $\pi_A e_C(A)$. By (16) $\pi_A e_C \iota_A(A) = A$, therefore $\pi_A e_C \iota_A$ is now an automorphism .
3) : That the mapping (17) is an isomorphism follows from (15) and $Ke(e_C) = D$.

2.2. Corollary

Let $A \neq 0$. If $S = End(A)$ is a total ring , then A has the D2-exchange property .

Proof :

If $0 \neq A_o \subsetneq^{\oplus} A$, then by I. 5.6. $End(A_o)$ ($\cong eSe$) is also a total ring . Therefore , in the following proof we can assume $A_o = A$ and also (11) and (12). Since $1_A = \pi_A e_C \iota_A + \pi_A e_D \iota_A$ at least one of $\pi_A e_C \iota_A$ or $\pi_A e_D \iota_A$ must be pi . In the first case we have (13) . Similar , in the second case holds

$$M = A_1 \oplus C \oplus D_1 \quad , \quad 0 \neq A_1 \subsetneq A \quad , \quad D_1 \subsetneq D \ .$$

That means , that the D2-EP is satisfied .

2.3. Corollary

Let $A \neq 0$ and assume (11) and (12) . If $S = End(A)$ is a radicaltotal ring , then one of the following conditions is satisfied :

(i) $M = A \oplus (B \cap C) \oplus D = e_C(A) \oplus B$

(ii) $M = A \oplus C \oplus (B \cap D) = e_D(A) \oplus B$

(iii) $M = A' \oplus C' \oplus D = e_C(A') \oplus A'' \oplus B$, $0 \neq A' \subsetneq A$, $A'' \subsetneq A$, $C' \subsetneq C$
$\wedge$ $M = A^* \oplus C \oplus D' = e_D(A^*) \oplus A^{**} \oplus B$, $0 \neq A^* \subsetneq A$, $A^{**} \subsetneq A$, $D' \subsetneq D$.

Proof :

We consider again $1_A = \pi_A e_C \iota_A + \pi_A e_D \iota_A$ and distinguish three cases .
1) case : $\pi_A e_D \iota_A \in Tot(S) = Rad(S)$, then

$$1_A - \pi_A e_D \iota_A = \pi_A e_C \iota_A$$

is an automorphism (since Rad(S) is quasi-regular) . (i) follows then from 2.1. 2) .

2) case : $\pi_A e_C \iota_A \in Tot(S) = Rad(S)$. Similar , this implies (ii) .
3) case : $\pi_A e_C \iota_A$ and $\pi_A e_D \iota_A$ are both pi . Now 2.1. 1) implies (iii) .

Later , we will prove , that the converse of 2.3. is true , if we assume (i) , (ii) , (iii) for certain modules A , B , C , D .

§3. Exchange properties imply total properties

One of the main results of this paragraph is , that the converse of 2.2. is true. Together with 2.2. we have then the result : $S = End(A_R)$ is a total ring iff A_R has D2-exchange property . Since S is the endomorphism ring of itself (by left-multiplication) , this includes the special case : S is a total ring iff S_S has D2-exchange property .

The foundation for the following considerations is a lemma for which we need some notations . Given A_R and let be $S := End(A_R)$. Assume $f \in S$ and write $g := 1_A - f$. Further define

$$M := A \times A = \{ (a_1,a_2) \mid a_1, a_2 \in A \} ,$$
$$A_1 := \{ (a,0) \mid a \in A \} ,$$
$$A_2 := \{ (0,a) \mid a \in A \} ,$$
$$C := \{ (f(a),-g(a)) \mid a \in A \} ,$$
$$D := \{ (a,a) \mid a \in A \} .$$

Then we consider the following mappings :

$$\alpha_1 : A \ni a \mapsto (a,0) \in A_1 ,$$
$$\alpha_2 : A \ni a \mapsto (0,a) \in A_2 ,$$
$$\gamma : A \ni a \mapsto (f(a),-g(a)) \in C ,$$
$$\delta : A \ni a \mapsto (a,a) \in D .$$

It is obvious , that α_1 , α_2 , δ are isomorphisms . But also γ is an isomorphism ; for this we have only to check the injectivity. Assume $(f(a),-g(a)) = (0,0)$, then $f(a) = 0$ and $-g(a) = f(a) - a = -a = 0$.
Further we have $M = A_1 \oplus A_2$. Also $M = C \oplus D$ is true : For $a_1, a_2 \in A$ we have

$$(a_1, a_2) = (f(a_1-a_2), -g(a_1-a_2)) + (a_1 - f(a_1-a_2), a_1 - f(a_1-a_2)) \quad ,$$

hence $M = C + D$; assume $(f(a),-g(a)) = (a_1, a_1) \in C \cap D$, then $f(a) = a_1$, $-g(a) = f(a) - a = a_1$ and this implies $a = 0$, $a_1 = 0$. Together we have

(21) $$M = A_1 \oplus A_2 = C \oplus D \quad , \quad A \cong A_1 \cong A_2 \cong C \cong D \quad .$$

3.1. Lemma

(i) $M = C \oplus A_1' \oplus A_2'$ with $A_1' \subsetneq A_1$, $A_2' \subsetneq A_2$, $A_2' \neq 0$, then f is pi (in $S = End(A_R)$) ,
(ii) $M = C' \oplus A_1' \oplus A_2$ with $A_1' \subsetneq A_1$, $C' \subsetneq C$, $C' \neq 0$, then f is pi ,
(iii) $M = C' \oplus A_1 \oplus A_2'$ with $A_2' \subsetneq A_2$, $C' \subsetneq C$, $C' \neq 0$, then g is pi .

Proof :

For the proof we use I . 6.2 , that is we show , that f induces an isomorphism between nontrivial direct summands of A .
(i) : By the modular law and the assumption follows $A_2 = A_2' \oplus (C \oplus A_1')$; denote $A_2'' = A_2 \cap (C \oplus A_1')$. Then

$$A = \alpha_2^{-1}(A_2') \oplus \alpha_2^{-1}(A_2'') \quad , \quad \alpha_2^{-1}(A_2') \neq 0 \quad .$$

Let π_D be the projection of M onto D belonging to $M = C \oplus D$. Since $Ke(\pi_D) = C$ and by (i) π_D induces an isomorphism of $A_1' \oplus A_2'$ onto D , therefore

$$D = \pi_D(A_1') \oplus \pi_D(A_2') \quad , \quad \pi_D(A_2') \neq 0$$

Then

$$A = \delta^{-1}(D) = \delta^{-1}\pi_D(A_1') \oplus \delta^{-1}\pi_D(A_2') \quad , \quad \delta^{-1}\pi_D(A_2') \neq 0 \quad .$$

We intend to prove , that f induces an isomorphism from $\alpha_2^{-1}(A_2')$ onto $\delta^{-1}\pi_D(A_2')$

via

$$\alpha_2^{-1}(A_2') \xrightarrow{\hat{\alpha}_2} A_2' \xrightarrow{\hat{\pi}_D} \pi_D(A_2') \xrightarrow{\hat{\delta}^{-1}} \delta^{-1}\pi_D(A_2')$$

with the isomorphisms $\hat{\alpha}_2$, $\hat{\pi}_D$, $\hat{\delta}^{-1}$ induced by α_2 , π_D , δ^{-1} . For $x \in \alpha_2^{-1}(A_2')$ we have

(22) $\alpha_2(x) = (0,x) = (f(-x), x + f(-x)) + (f(x), f(x))$

with $(f(-x), x + f(-x)) \in C$, $(f(x), f(x)) \in D$

then follows

$$\pi_D\alpha_2(x) = (f(x), f(x)) \Rightarrow \delta^{-1}\pi_D\alpha_2(x) = f(x) ,$$

hence

$$\alpha_2^{-1}(A_2') \ni x \mapsto f(x) \in \delta^{-1}\pi_D(A_2')$$

is an isomorphism and therefore f is pi .

(ii) : Similar proof as for (i) . Now (ii) implies $C = C' \oplus C''$, $C'' := C \cap (A_1' + A_2)$. Then follows

$$A = \gamma^{-1}(C') \oplus \gamma^{-1}(C'') , \quad \gamma^{-1}(C') \neq 0$$

and

(23) $A_2 = \alpha_2\gamma^{-1}(C') \oplus \alpha_2\gamma^{-1}(C'') , \quad \alpha_2\gamma^{-1}(C') \neq 0$.

By π we denote the projection from M onto $C'' \oplus D$ belonging to $M = C' \oplus C'' \oplus D$; then $Ke(\pi) = C'$. By (ii) π induces an isomorphism between $A_1' \oplus A_2$ and $C'' \oplus D$, hence by (23)

(24) $C'' \oplus D = \pi(A_1') \oplus \pi(A_2) = \pi\alpha_2\gamma^{-1}(C') \oplus L , \quad \pi\alpha_2\gamma^{-1}(C') \neq 0$

with $L := \pi\alpha_2\gamma^{-1}(C'') \oplus \pi(A_1')$.

We claim : $\pi\alpha_2\gamma^{-1}(C') \subsetneq D$. For $x \in \gamma^{-1}(C')$ follows

$$\gamma(x) = (f(x), -x + f(x)) \in C' \Rightarrow -\gamma(x) \in C'$$

This and (22) imply

(25) $\pi\alpha_2(x) = \pi(0,x) = \pi(f(-x), x + f(-x)) + \pi(f(x), f(x))$

$= \pi(-\gamma(x)) + \pi(f(x), f(x)) = (f(x), f(x)) \in D$.

Now , we can apply the modular law on (24) to get

$$D = \pi\alpha_2\gamma^{-1}(C') \oplus (D \cap L) , \quad \pi\alpha_2\gamma^{-1}(C') \neq 0 .$$

Finally , we show that f induces an isomorphism between the nonzero direct summands $\gamma^{-1}(C')$ and $\delta^{-1}\pi\alpha_2\gamma^{-1}(C')$

via

$$\gamma^{-1}(C') \xrightarrow{\hat{\alpha}_2} \alpha_2\gamma^{-1}(C') \xrightarrow{\hat{\pi}} \pi\alpha_2\gamma^{-1}(C') \xrightarrow{\hat{\delta}^{-1}} \delta^{-1}\pi\alpha_2\gamma^{-1}(C')$$

with the isomorphisms $\hat{\alpha}_2$, $\hat{\pi}$, $\hat{\delta}^{-1}$ induced by α_2 , π , δ^{-1} . For $x \in \gamma^{-1}(C')$ we have by (25)

$$\delta^{-1}\pi\alpha_2(x) = \delta^{-1}(f(x),f(x)) = f(x) \quad ,$$

hence

$$\gamma^{-1}(C') \ni x \mapsto f(x) \in \delta^{-1}\pi\alpha_2\gamma^{-1}(C')$$

is an isomorphism and therefore f is pi .

(iii) : The proof is similar to the proof of (ii) , but not symmetric , since in C $(f(x),-g(x))$ is not symmetric with respect to f and $g = 1-f$. Now , we have in place of (23)

$$A_1 = \alpha_1\gamma^{-1}(C') \oplus \alpha_1\gamma^{-1}(C'') \quad , \quad \alpha_1\gamma^{-1}(C') \neq 0 \quad .$$

Then π denotes the projection from M onto $C'' \oplus D$ belonging to $M = C' \oplus C'' \oplus D$. Then π induces an isomorphism between $A_1 \oplus A_2'$ and $C'' \oplus D$. In place of (24) we have now

$$C'' \oplus D = \pi(A_1) \oplus \pi(A_2') = \pi\alpha_1\gamma^{-1}(C') \oplus L \quad , \quad \pi\alpha_1\gamma^{-1}(C') \neq 0$$

$$\text{with } L := \pi\alpha_1\gamma^{-1}(C'') \oplus \pi(A_2') \quad .$$

Again holds $\pi\alpha_1\gamma^{-1}(C') \in D$: For $x \in \gamma^{-1}(C')$ follows $\gamma(x) = (f(x), -x + f(x)) \in C'$; then

(26) $$\pi\alpha_1(x) = \pi(x,0) = \pi(f(x),-x+f(x)) + \pi(x-f(x), x-f(x))$$
$$= \pi(\gamma(x)) + \pi(g(x),g(x)) = (g(x),g(x)) \in D \quad .$$

Therefore

$$D = \pi\alpha_1\gamma^{-1}(C') \oplus (D \cap L) \quad , \quad \pi\alpha_1\gamma^{-1}(C') \neq 0 \quad .$$

Now we consider the induced isomorphisms

$$\gamma^{-1}(C') \xrightarrow{\hat{\alpha}_1} \alpha_1\gamma^{-1}(C') \xrightarrow{\hat{\pi}} \pi\alpha_1\gamma^{-1}(C') \xrightarrow{\hat{\delta}^{-1}} \delta^{-1}\pi\alpha_1\gamma^{-1}(C') \quad .$$

For $x \in \gamma^{-1}(C')$ follows by (26)

$$\delta^{-1}\pi\alpha_1(x) = \delta^{-1}(g(x),g(x)) = g(x) \quad ,$$

hence g is pi .

<u>3.2. Proposition</u>

Given A_R and $S := End(A_R)$.

If A_R has D2-exchange property , then S is a total ring .

<u>Proof :</u>

Indirect proof . Assume $f , g \in Tot(S)$ and $f + g \notin Tot(S)$, hence f + g is pi .
Then there exists $h \in S$ such that $h(f+g) = e = e^2 \neq 0$.
We assume first e = 1 and use the construction $M = A \times A = A_1 \oplus A_2 = C \oplus D$ with hf in place of f and hg in place of g . Since A has D2-EP and $A \cong C$ also C has D2-EP . We apply this on $M = A_1 \oplus A_2 = C \oplus D$ and get either

$$M = C' \oplus A_1' \oplus A_2 \quad \text{or} \quad M = C' \oplus A_1 \oplus A_2' \quad , \quad 0 \neq C' \subsetneq^{\oplus} C \; .$$

Then by 3.1.(ii) resp. 3.1.(iii) follows hf or hg is pi and therefore f or g is pi ↯.
In the general case $h(f+g) = e = e^2 \neq 0$ we denote by $\iota : e(A) \mapsto A$ the inclusion and by $\pi : A \mapsto e(A)$ the projection belonging to $A = e(A) \oplus (1-e)(A)$.
Then $1_{e(A)} = \pi\iota$, $e = \iota\pi$ and by $h(f+g) = e$ follows $\pi h(f+g)\iota = \pi e \iota = 1_{e(A)}$.
With A also e(A) has the D2-EP . Now we are again in the case e = 1 and know $\pi hf\iota$ or $\pi hg\iota$ is pi in End(e(A)) = eSe , hence f or g is pi in S ↯ .

By 3.2. and 2.2. together we have one of our main results , where we use that for a ring S with $1 \in S$ $End(S_S) = S$ holds .

<u>3.3. Theorem</u>

(i) Let be A_R and $S := End(A_R)$, then :
S is a total ring iff A_R has D2-EP .

(ii) Let be S a ring with $1 \in S$, then :
S is a total ring iff S_S has D2-EP .

There is another interesting theorem , which connects exchange properties with total properties .

<u>3.4. Theorem</u>

(i) If A_R is a module with 2-EP , then $S := End(A_R)$ is a radicaltotal ring .

(ii) If S is a ring with $1 \in S$ and if S_S has 2-EP , then S is a radicaltotal ring (Short : Exchange rings are radicaltotal rings) .

<u>Proof :</u>

(i): Since $Rad(S) \subset Tot(S)$ we have only to show : If $f \in Tot(S)$, then $f \in Rad(S)$. Since for $f \in Tot(S)$ also $fS \subset Tot(S)$, we have only to prove , that f is quasi-regular , that is , $1-f$ is an automorphism of A . We use again the construction in 3.1. with f and $g = 1-f$. Since with A also C has the 2-EP , there exist $A_1' \subsetneq A_1$, $A_2' \subsetneq A_2$ such that $M = C \oplus A_1' \oplus A_2'$. Since f is not pi by 3.1 (i) follows $A_2' = 0$, that is $M = C \oplus A_1'$. Now the 2-EP will be used for A_2 and the decompositions

$$M = A_1 \oplus A_2 = C \oplus A_1' \ .$$

Then

$$M = C' \oplus A_1'' \oplus A_2 \quad , \quad C' \subsetneq C \quad , \quad A_1'' \subsetneq A_1' \ .$$

But now by 3.1. (ii) follows $C' = 0$, since otherwise f would be pi . Therefore

$$M = A_1'' \oplus A_2 \ .$$

Since $M = A_1 \oplus A_2$, this implies

$$A_1'' = A_1' = A_1$$

and we have

(27) $$M = C \oplus A_1' = C \oplus A_1 \ .$$

With this decomposition we show , that $g = 1-f$ is an automorphism . $1-f$ is surjective : For $x \in A$, there exist $y, z \in A$ such that

$$(0,-x) = (f(y), f(y)-y) + (z,0) \Rightarrow x = (1-f)(y) \ .$$

$1-f$ is injective : Assume $(1-f)(y) = 0 \Rightarrow (f(y), f(y)-y) = (f(y),0) \in C \cap A_1 = 0$ $\Rightarrow f(y) = y = 0$.

The proof for (i) is complete and (ii) is a special case of (i) .

If there is an implication , there is always the question , if the converse is true . We show later by an example , that the converse of 3.4. is not true . But a converse of 2.3. is satisfied .

3.5. Proposition

If A_R is a R-module and $S := End(A_R)$. Then the following conditions are equivalent :

(I) S is radicaltotal

(II) For every situation

$M = A^* \oplus B = U \oplus V$ with $A^* \cong A$

one of the following conditions is satisfied :

(i) There exists $U' \subsetneq U$ such that

$M = A^* \oplus U' \oplus V$

(ii) there exists $V' \subsetneq V$ such that

$M = A^* \oplus U \oplus V'$

(iii) there exist $0 \neq A' \subsetneq A^*$, $U' \subsetneq U$ such that

$M = A' \oplus U' \oplus V$

and there exist $0 \neq A'' \subsetneq A^*$, $V' \subsetneq V$ such that

$M = A'' \oplus U \oplus V'$.

Proof :

(I) ⇒ (II) : By 2.3. , where $A = A'$, $C = U$, $D = V$.

(II) ⇒ (I) : For $f \in Tot(A)$ we consider (21) and identify $A^* = C$, $B = B$, $U = A_1$, $V = A_2$, then $C \cong A$.

By assumption we have now three cases :

(i) : $M = C \oplus A_1' \oplus A_2$; then by 3.1. (i) f would be pi

(ii) : $M = C \oplus A_1 \oplus A_2'$; then by 3.1. (i) A_2' must be 0 , then otherwise f would be pi . Hence we have now $M = C \oplus A_1$, which is (27) .

(iii) : $M = C' \oplus A_1' \oplus A_2$, $C' \neq 0$ and

$M = C'' \oplus A_1 \oplus A_2'$, $C'' \neq 0$. But by 3.1. (ii) the first equation would imply that f is pi .

There is only (27) left over . We proved already , that (27) implies , that 1 - f is an automorphism . Since with $f \in Tot(S)$ also $fS \subset Tot(S)$, this is a quasi-regular right ideal , hence $f \in Rad(S)$.

§4. The special case : Directly indecomposable modules

First we repeat some well-known facts about directly indecomposable modules and local rings . A ring S is called a local ring iff the set of noninvertible elements in S is closed under addition or , what is the same , is a two-sided ideal of S .

Now , we have the following connection between A_R and $S := \mathrm{End}(A_R)$.

4.1. Remarks

The following conditions are equivalent :

1) A_R is directly indecomposable .
2) S contains only the idempotents 0 and 1 .
3) Tot(S) = set of not invertible elements of S .

Proof :

Only 3) $\Rightarrow$ 2) : If $0 \neq e$ is an idempotent , then e is pi , hence by 3) invertible : $es = 1 \Rightarrow e = e^2s = es = 1$.

4.2. Proposition

For a directly indecomposable module A_R and $S := \mathrm{End}(A_R)$ the following conditions are equivalent :

1) S is a local ring ,
2) S is a total ring ,
3) S is a radicaltotal ring ,
4) A_R has 2-EP ,
5) A_R has D2-EP .

Proof :

(1) $\Leftrightarrow$ (2) : By 4.1.

(3) $\Rightarrow$ (2) : Clear .

(2) $\Rightarrow$ (3) : We have always Rad(S) $\subset$ Tot(S) . Now , for $t \in$ Tot(S) we consider $1 - t$. This cannot be in Tot(S) since then by (2) $1 \in$ Tot(S) $\lightning$. Therefore $1 - t$ is pi , which means now , that $1 - t$ is invertible . Therefore Tot(S) is a quasi-regular ideal , hence Tot(S) $\subset$ Rad(S) .

(2) $\Leftrightarrow$ (5) : By 3.3.

(4) $\Rightarrow$ (5) : By 1.5.

(5) $\Rightarrow$ (4) : Since A_R is directly indecomposable , every nonzero direct summand of A is A itself . Hence by (5) in the situation

$$M = A \oplus B = C \oplus D$$

we have $M = A \oplus C' \oplus D$, $C' \subsetneq C$ or $M = A \oplus C \oplus D'$, $D' \subsetneq D$, which shows , that (4) is satisfied .

If A_R is directly indecomposable , but S is not a local ring , then Rad(S) $\neq$ Tot(S) . For example , $\mathbb{Z}_{\mathbb{Z}}$ is directly indecomposable , $\mathrm{End}(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z}$ (leftmultiplications) and $\mathrm{Rad}(\mathbb{Z}) = 0$, $\mathrm{Tot}(\mathbb{Z}) = \mathbb{Z} \setminus \{-1;1\}$.

§5. An example for a radicaltotal ring , which is not regular and not a 2-EP ring

For the example we need a special case of a result about exchange modules . In order to have these notes self-contained , we give a proof for this special case . First we have to introduce some notations . Let again S be a ring with $1 \in S$ and let $a \in S$. Then consider

(28) $$S^2 = S \times S = S_1 \oplus S_2 = U \oplus V$$

with

$$S_1 := \{(s,0) \mid s \in S\} \quad , \quad S_2 := \{(0,s) \mid s \in S\} \quad ,$$

$$U := \{(as\,,(1-a)s) \mid s \in S\} \quad ,$$

$$V := \{(s,-s) \mid s \in S\} \quad .$$

and isomorphisms

$$\delta_1 : S \ni s \mapsto (s,0) \in S_1$$
$$\delta_2 : S \ni s \mapsto (0,s) \in S_2$$
$$\mu : S \ni s \mapsto (as,(1-a)s) \in U$$
$$\nu : S \ni s \mapsto (s,-s) \in V .$$

The modules S_1 , S_2 correspond to A_1 and A_2 in §3 , but U and V do not correspond to C and D by having the negative sign in the second coefficient . The proofs for (28) and the isomorphisms are as simple as in §3 . Further we need the epimorphism

$$\rho : S^2 \ni (s_1,s_2) \mapsto s_1 + s_2 \in S$$

for which $Ke(\rho) = V$. By ι_{S_1} , ι_{S_2} , ι_U , ι_V we denote the inclusions of S_1 , S_2 , U , V in S^2 . Then follows

(29) $\mu\rho\iota_U = 1_U \quad , \quad \rho\iota_U\mu = 1_S$.

We need also

$$\mu_1 : S \ni s \mapsto (as,0) \in S_1 ,$$
$$\mu_2 : S \ni s \mapsto (0,(1-a)s) \in S_2$$

for which $\mu_1 + \mu_2 = \mu$.

5.1. Lemma

S_S is a 2-EP ring $\Leftrightarrow$

$\forall a \in S \quad \exists d \in S \quad [d = d^2 \ \wedge \ d \in Sa \ \wedge \ 1-d \in S(1-a)]$.

Proof :

$\Rightarrow$: Since S_S is a 2-EP ring and $S_S \cong U_S$, there exist $S_1' \subsetneq S_1$, $S_2' \subsetneq S_2$ such that

(30) $S^2 = U \oplus S_1' \oplus S_2'$.

By the modular law follows $S_1' \subsetneq^{\oplus} S_1$, $S_2' \subsetneq^{\oplus} S_2$. Let be

$$S_1 = S_1' \oplus S_1'' \quad , \quad S_2 = S_2' \oplus S_2''$$

and if $T := S_1'' \oplus S_2''$

then

(31) $S^2 = T \oplus S_1' \oplus S_2'$.

Denote by π the projection of S^2 onto T belonging to (31) , then $Ke(\pi) = S_1' \oplus S_2'$ and by (30) π induces an isomorphism $\tau := \pi|_U$ between U and T .

Further let w_1 , w_2 be the projections of $T = S_1'' \oplus S_2''$ onto S_1'' resp. S_2'' and ι_1 , ι_2 the inclusions of S_1'' resp. S_2'' in T . Then $e_i := \iota_i w_i$, i = 1,2 are idempotents in $End(T_S)$ with

(32) $$e_1 e_2 = e_2 e_1 = 0 \quad , \quad e_1 + e_2 = 1_T \quad .$$

Now we define

(33) $$d_i := \rho \iota_U \tau^{-1} e_i \tau \mu \quad , \quad i = 1,2$$

and compute with (29) and (32)

$$d_i^2 = \rho \iota_U \tau^{-1} e_i \tau (\mu \rho \iota_U) \tau^{-1} e_i \tau \mu = d_i$$

$$d_1 d_2 = d_2 d_1 = 0 \quad , \quad d_1 + d_2 = 1_S \quad .$$

We define $d := d_1(1)$, then

$$d_1(1)^2 = d_1(1 d_1(1)) = d_1(d_1(1)) = d_1^2(1) = d_1(1)$$

and

$$d_1(1) + d_2(1) = 1 \quad ,$$

hence $d_2(1) = 1 - d$.

We still have to prove $d \in Sa$ and $1 - d \in S(1-a)$. Denote by π_i the projection of $S_i = S_i' \oplus S_i''$, i = 1,2 onto S_i'' , then we show first

(34) $$w_i \tau \mu = \pi_i \mu_i \quad , \quad i = 1,2 \quad .$$

Let be

$$(as , (1-a)s) = (x',0) + (0,y') + (x'',0) + (0,y'')$$

with $(x',0) \in S_1'$, $(x'',0) \in S_1''$, $(0,y') \in S_2'$, $(0,y'') \in S_2''$,

then

$$w_i \tau \mu(s) = w_i \tau(as,(1-a)s) = w_i((x'',0) + (0,y'')) =$$

$$= \begin{cases} (x'',0) & \text{for } i = 1 \\ (0,y'') & \text{for } i = 2 \end{cases}$$

$$\pi_i \mu_i(s) = \begin{cases} \pi_1(as,0) = (x'',0) & \text{for } i = 1 \\ \pi_2(0,(1-a)s) = (0,y'') & \text{for } i = 2 \end{cases}$$

Therefore we have (34) . With (34) we get

$$d_i = \rho \iota_U \tau^{-1} \iota_i (w_i \tau \mu) = \rho \iota_U \tau^{-1} \iota_i (\pi_i \mu_i) = (\rho \iota_U \tau^{-1} \iota_i \pi_i \delta_i) \delta_i^{-1} \mu_i \quad , \quad i = 1,2$$

The mapping in the bracket is an element in $End(S_S)$, which is induced by leftmultiplication of S by an element $s_i \in S$, i = 1,2 . Therefore

$$d = d_1(1) = s_1(\delta_1^{-1} \mu_1(1)) = s_1 \delta_1^{-1}(a,0) = s_1 a \quad ,$$

$$1 - d = d_2(1) = s_2(\delta_2^{-1} \mu_2(1)) = s_2 \delta_2^{-1}(0,1-a) = s_2(1-a) \quad .$$

$\Leftarrow$: In this direction , we prove the lemma not only for a ring S , but for a R-rightmodule A_R with $S := \mathrm{End}(A_R)$. Consider the situation

$$M = A \oplus B = C_1 \oplus C_2$$

with R-rightmodules A , B , C_1 , C_2 . By e_i $(i = 1,2)$ we denote the projectors on C_i and by π_A resp. ι_A the projection of M to A resp. the inclusion of A in M . Then

$$e_1 + e_2 = 1_M \ .$$

Define $f_i := \pi_A e_i \iota_A$ $(i = 1,2)$, then f_1 , $f_2 \in S$ and $f_1 + f_2 = 1_A$, hence $f_2 = 1_A - f_1$. Now , f_1 is the element a in our assumption .
Then there exist s_1 , s_2 such that

$$d_i := s_i f_i \quad (i = 1,2)$$

are orthogonal idempotents with $d_1 + d_2 = 1_A$.
Finally we define

$$g_i := d_i s_i \pi_A e_i \quad (i = 1,2) \quad , \quad g := g_1 + g_2$$

then it follows easily

$$g_i \iota_A g_j = \delta_{ij} g_i \quad (i = 1,2) \quad ,$$

$$g \iota_A g = g \quad , \quad g \iota_A = 1_A$$

and this equations imply

$$M = \mathrm{Im}(\iota_A) \oplus \mathrm{Ke}(g) = A \oplus (C_1 \cap \mathrm{Ke}(g_1)) \oplus (C_2 \cap \mathrm{Ke}(g_2)) \quad ,$$

what we had to prove .

Now we come to the example . Let K be a field and $R \neq 0$ a subring of K with $1 \in R$, which is not a local ring (for example $K = \mathbb{Q}$, $R = \mathbb{Z}$) . Then the ring S is defined by

$$S := \{ (x_i) \in K^{\mathbb{N}} \mid \exists\ m \in \mathbb{N} , x \in R \quad \forall\ i \geq m \quad [x_i = x] \}$$

with

$$(x_i) + (y_i) = (x_i + y_i) \quad , \quad (x_i)(y_i) = (x_i y_i) \ .$$

Then $(1) = (1\ 1\ 1\ .\ .\ .)$ is the 1-element of S .
First we show : Every element $\neq 0$ in S is pi . If in (x_i) $x_{i_o} \neq 0$, then with $s = (0\ .\ .\ .\ 0\ x_{i_o}^{-1}\ 0\ .\ .\ .)$ follows $(x_i)s = (0\ .\ .\ .\ 0\ 1\ 0\ .\ .\ .)$ and this is an idempotent $\neq 0$ in S . Hence

$$\mathrm{Rad}(S) = \mathrm{Tot}(S) = 0 \quad ,$$

that is , S is radicaltotal .

Then we prove , that S is not a regular ring . Since R is not local , R is not a field . Therefore exists $0 \neq r \in R$, $r^{-1} \notin R$ ($r^{-1} \in K$) . Then $(r) = (r\ r\ r\ \dots)$ is not regular . Assume $(r)(x_i)(r) = (r)$ with $x_i = x$ for $i \geq m$, then

$$r x_i r = rxr = r \quad , \quad i \geq m \quad .$$

Since $r \neq 0$ also $xr \neq 0$. Then follows

$$xr = rx = 1$$

hence $x = r^{-1} \in R$ ↯ .

Finally we show : S is not a 2-EP-ring . Since R is not local , there exist not-invertible a_1 , $a_2 \in R$ such that $a_1 + a_2$ is invertible in R : $(a_1 + a_2)b = 1$, $b \in R$. Then also $r := a_1 b$, $1 - r = a_2 b$ are not invertible in R and $r \neq 0$, $1 - r \neq 0$. Assume S_S is a 2-EP ring , then by 5.1. there exist (x_i) , $(y_i) \in S$ such that

$$(x_i)(r) \quad , \quad (y_i)((1) - (r)) = (y_i)(1 - r)$$

are idempotents with

$$(x_i)(r) + (y_i)(1 - r) = 1 = (1\ 1\ 1\ \dots) \tag{35}$$

Since $(x_i)(r) = (x_i r)$ is an idempotent , for $i \geq m$ with $x_i = x$ holds

$$x_i r x_i r = xrxr = xr \quad .$$

If $xr \neq 0$, then $xr = 1$, hence $x = r^{-1} \in R$ ↯ . Therefore $x_i = 0$, $i \geq m$. Similar for $(y_i)(1 - r)$ $y_i = 0$, $i \geq n$. But then (35) cannot be satisfied ↯ .

III. Direct decompositions

§1. RTE-decompositions

1.1. Definition

Let be M_R a R-module and R a ring with $1 \in R$. Denote $T := \text{End}(M_R)$. Then M_R is called a **LE-** resp. **TE-** resp. **RTE-module** $:\Leftrightarrow$ T is a local resp. total resp. radicaltotal ring . A decomposition

$$(1) \qquad M = \bigoplus_{i \in I} M_i$$

is called a LE- resp. TE- resp. RTE-decomposition $:\Leftrightarrow$ all M_i , $i \in I$ are LE- resp. TE- resp. RTE-modules .

We know already , that direct summands of TE- resp. RTE-modules have again this property (I. 5.6. ; the endomorphismring of a direct summand of M is of the form eEnd(M)e with an idempotent $e \in \text{End}(M)$) . Now we come to the question if for a TE- resp. RTE-decomposition (1) M is a TE- resp. RTE-module . This question is open for TE-decompositions even if I is finite . We are able to show , that the direct sum of a RTE-module and a TE-module is a TE-module . If (1) is a RTE-decomposition , then M is a TE-module and if I in (1) is finite , then M is again a RTE-module . There are also examples that for infinite I M is a RTE-module . For LE-decompositions , this case was already considered by Harada .

For the following proofs , it is useful to put II. 1.2. and II. 3.3. (i) together to the following lemma .

1.2. Lemma

The module M_R is a TE-module $\Leftrightarrow$ every nonzero direct summand of M_R contains a nonzero direct summand , which is a TE-module .

1.3. Proposition

If in

$$M = A \oplus B$$

A is a RTE-module and B a TE-module , then M is a TE-module .

Proof :

We can assume $A \neq 0$. Assume $M = C \oplus D$ with $C \neq 0$. Then we consider the different cases in II. 2.3 .

case (i) : Now , $A \oplus (B \cap C) \cong C$; therefore C contains a direct summand isomorphic to A , hence a RTE-module .

case (ii) : Now , $B \cong C \oplus (B \cap D)$; therefore C is isomorphic to a direct summand of B , hence a TE-module .

case (iii) : Now , $A' \oplus C' \cong C$; therefore C contains a direct summand isomorphic to A' , which is a direct summand $\neq 0$ of A (by the modular law) , hence a RTE-module .

1.4. Theorem

Assume , that (1) is a RTE-decomposition and

$$M = C \oplus D \quad , \quad C \neq 0 \ .$$

Then there exists $i_o \in I$ and

$$0 \neq L_1 \subsetneq^{\oplus} M_{i_o} \quad , \quad L_2 \subsetneq^{\oplus} M_{i_o} \quad , \quad C_o \subsetneq^{\oplus} C$$

such that

$$(2) \qquad M = L_1 \oplus C_o \oplus D = e_C(L_1) \oplus L_2 \oplus (\bigoplus_{\substack{i \in I \\ i \neq i_o}} M_i)$$

Proof :

Let be $c \in C$, $c \neq 0$ and

$$c = c_{i_1} + \ldots + c_{i_m} \quad , \quad c_{i_j} \in M_{i_j}$$

with different $i_1 , \ldots , i_m \in I$. Since in the following we need only the M_{i_j} , $j = 1 , \ldots , m$, we write j for i_j and

$$B_o := \bigoplus_{i \in I \setminus \{i_1, \ldots, i_m\}} M_i \ .$$

Then $c = c_1 + \ldots + c_n$, $c_i \in M_i$. Now , we consider

$$1_{M_j} = \pi_{M_j} e_C \iota_{M_j} + \pi_{M_j} e_D \iota_{M_j} \quad , \quad j = 1, \ldots, m \; .$$

If one $\pi_{M_j} e_C \iota_{M_j}$ is pi , then we have the result by II. 2.1. 1) . Therefore we assume , that all

$$\pi_{M_j} e_C \iota_{M_j} \in \mathrm{Tot}(\mathrm{End}(M_j)) = \mathrm{Rad}(\mathrm{End}(M_j)) \quad , \quad j = 1, \ldots, m$$

and derive a contradiction . By this assumption all

$$\pi_{M_j} e_D \iota_{M_j} = 1_{M_j} - \pi_{M_j} e_C \iota_{M_j} \quad , \quad j = 1, \ldots, m$$

are automorphisms . By II. 2.3. (ii) (with $A = M_1$, $B = \bigoplus_{\substack{i \in I \\ i \neq i_1}} M_i$) we get

(3) $$M = e_D(M_1) \oplus M_2 \oplus \ldots \oplus M_m \oplus B_o \; .$$

Induction over $j = 1, \ldots, m$ with (3) as the case $j = 1$ implies

(4) $$M = e_D(M_1) \oplus \ldots \oplus e_D(M_m) \oplus B_o \; .$$

By II. 2.1. 3) we know that all mappings

(5) $$M_j \ni x \mapsto e_D(x) \in e_D(M_j)$$

are isomorphisms . Since $c = c_1 + \ldots + c_n \in C$ it follows that

$$0 = e_D(c) = e_D(c_1) + \ldots + e_D(c_m) \; .$$

Since (4) is a direct sum , this implies $e_D(c_j) = 0$, $j = 1, \ldots, m$. Then by (5) we have $c_j = 0$, $j = 1, \ldots, m$, hence $c = 0$ ↯ .

In the special case that (1) is a LE-decomposition , 1.4. is the key for the proof of the Krull-Remak-Schmid-Azumaya-theorem . This gives reason for the following question : If (1) is a RTE-decomposition and if $M = \bigoplus_{j \in J} C_j$ is an arbitrary decomposition , do there exist refinements of both decompositions , which are isomorphic ?

<u>1.5. Corollary</u>

Assumptions and notations as in 1.4 .

Further let D be a maximal direct summand of M , then $M = L_1 \oplus D$, $L_1 \cong C$.

That means , that a refinement of (1) complements a maximal direct summand .

1.6. Corollary

Assume , that (1) is a RTE-decomposition . Then :

(i) : Every nonzero direct summand of M contains a direct summand , which is isomorphic to a nonzero direct summand of one of the M_i , $i \in I$.

(ii): M is a TE-module .

Proof :

(i) : From (2) follows $C \cong L_1 \oplus C_o$ and $L_1 \subsetneq^{\oplus} M_{i_o}$.

(ii): By (i) and 1.2 .

For a finite set I in (1) , M is even a RTE-module . To prove this and other interesting facts , we need two lemmas .

1.7. Lemma

If A_R , B_R are R-modules and $f \in Hom_R(A,B)$, $g \in Hom_R(B,A)$, then :

$1_A + gf$ is an automorphism of A $\Leftrightarrow$

$1_B + fg$ is an automorphism of B .

Proof :

It is easy to check , that if $1_A + gf$ is an automorphism , then

$(1_B + fg)^{-1} = 1_B - f(1_A + gf)^{-1}g$;

if $1_B + fg$ is an automorphism , then

$(1_A + gf)^{-1} = 1_A - g(1_B + fg)^{-1}f$.

1.8. Lemma

Assume , that (1) is a RTE-decomposition . Denote $T := End(M_R)$ and by $e_i \in T$ the projector of M onto M_i with respect to (1) . Then for all $i \in I$

$e_i Tot(T) \subset Rad(T)$, $Tot(T)e_i \subset Rad(T)$.

Proof :

Denote by $T_i := End(M_i)$ and by e_i resp. π_i the projector resp. projection of M onto M_i along (1) . Further let be ι_i the inclusion of M_i in M and 1_i the identity of M_i . Then

$$1_i = \pi_i\iota_i \quad , \quad e_i = \iota_i\pi_i \quad , \quad \pi_i e_i = \pi_i \quad , \quad e_i\iota_i = \iota_i \quad .$$

By I. 2.3. follows

$$\pi_i \mathrm{Tot}(T)\iota_i \subset \mathrm{Tot}(T_i) = \mathrm{Rad}(T_i) \quad .$$

By multiplication from the left with ι_i and from the right with π_i follows

$$e_i \mathrm{Tot}(T) e_i \subset \iota_i \mathrm{Rad}(T_i)\pi_i \quad .$$

Now , we show that each element of $\iota_i\mathrm{Rad}(T_i)\pi_i$ is quasi-regular . Let $f \in \mathrm{Rad}(T_i)$, then there exists $g \in T_i$ such that

$$(1_i + f)g = 1_i \Rightarrow (e_i + \iota_i f\pi_i)\iota_i g\pi_i = e_i \Rightarrow$$

$$(1_M - e_i + e_i + e_i\iota_i f\pi_i e_i)(1_M - e_i + e_i\iota_i g\pi_i e_i) = 1_M - e_i + e_i = 1_M \quad .$$

Then also each element of $e_i\mathrm{Tot}(T)e_i$ is quasi-regular . Now , let be $h \in \mathrm{Tot}(T)$, then e_ihT is a right ideal of T contained in $e_i\mathrm{Tot}(T)$. We show that e_ihT is a quasi-regular right ideal , hence $e_ihT \subset \mathrm{Rad}(T)$. We use 1.7 . Since for $t \in T$ $1_M + e_ihte_i$ is an automorphism , also $1_M + e_i(e_iht) = 1_M + e_iht$ is an automorphism (Take in 1.7 $g = e_iht$, $f = e_i$) . We have now $e_i\mathrm{Tot}(T) \subset \mathrm{Rad}(T)$. Similar is the proof for $\mathrm{Tot}(T)e_i \subset \mathrm{Rad}(T)$.

<u>1.9. Corollary</u>

If (1) is a RTE-decomposition and $I = \{1, \ldots, n\}$, then M is a RTE-module .

<u>Proof :</u>

Let be e_i , $i = 1, \ldots, n$ the projectors of M onto M_i and $T := \mathrm{End}(M_R)$. Then $1_M = e_1 + \ldots + e_n$ and 1.8. implies

$$\mathrm{Tot}(T) = \sum_{i=1}^{n} e_i\mathrm{Tot}(T) \subset \mathrm{Rad}(T) \quad ,$$

hence $\mathrm{Tot}(T) = \mathrm{Rad}(T)$.

<u>1.10. Corollary</u>

Assume , that (1) is a RTE-decomposition and $T = \mathrm{End}(M_R)$.

(i) If $\mathrm{Rad}(T) = 0$, then $\mathrm{Tot}(T) = 0$.

(ii) If $f \in \mathrm{Tot}(T)$, then $1_M - f$ is a monomorphism .

Proof :

For $x \in M$, there exists a finite subset $I_o \subset I$, such that $e(x) = x$ for $e := \sum_{i \in I_o} e_i$.
If $f \in Tot(T)$, then $f(x) = fe(x)$, with $fe \in Rad(T)$.
(i) If $Rad(T) = 0$, then $f(x) = fe(x) = 0$, hence $f = 0$.
(ii) Assume $(1_M - f)(x) = 0$. Then $(1_M - f)(x) = (1_M - fe)(x) = 0$. Since $fe \in Rad(T)$ $1_M - fe$ is an automorphism , hence $x = 0$.

1.11. Corollary

If M_R is an artinian or noetherian module , then there are equivalent :
(i) M is a RTE-module ,
(ii) M is a TE-module ,
(iii) M has a finite LE-decomposition .

Proof :

An artinian or noetherian module has a decomposition

$$M = \bigoplus_{i=1}^{n} M_i \quad , \quad M_i \text{ directly indecomposable} .$$

(i) $\Rightarrow$ (ii) : O.k.
(i) $\vee$ (ii) $\Rightarrow$ (iii) : Since a direct summand of a RTE- resp. TE-module is again such a module , (iii) follows by II. 4.2 .
(iii) $\Rightarrow$ (i) : By 1.9 .

§2. Connection with "Harada"-properties

If (1) is a LE-decomposition , then this is a special RTE-decomposition. In the literature - mainly by Harada - there are several interesting characterizations for the case , that $T = End(M)$ is a radicaltotal ring (and not only a total ring) . Harada used a more special definition for Tot(T) (and not our notation) . We show first , that the definition of Harada and our definition are equivalent for LE-decompositions .

We need the following lemma .

2.1. Lemma

If $f : A \to B$, $g : B \to C$ are modulehomomorphisms and $A \neq 0$, B directly indecomposable and gf an isomorphism , then f and g are isomorphisms .

Proof :

Since gf is an isomorphism , f is injective , g is surjective and

$$B = \operatorname{Im}(f) \oplus \operatorname{Ke}(g) .$$

Since $A \neq 0$ and B is directly indecomposable , we have $B = \operatorname{Im}(f)$ and $\operatorname{Ke}(g) = 0$, hence f is also surjective and g injective .

Now we consider LE-decompositions , which are special RTE-decompositions . Assume , that (1) is now a LE-decomposition (with all $M_i \neq 0$) and that

(6) $$N = \bigoplus_{j \in J} N_j \quad , \quad (\text{all } N_j \neq 0)$$

is also a LE-decomposition of R-modules . We use the same notation as in I.6 :

$$T := \operatorname{End}(M_R) \quad , \quad S := \operatorname{End}(N_R) .$$

Further we denote by

$\iota_i : M_i \to M$	the inclusion ,
$\pi_i : M \to M_i$	the projection belonging to (1) ,
$e_i = \iota_i \pi_i$	the projector .

Similar notations for (6)

$\kappa_j : N_j \to N$	the inclusion ,
$\rho_j : N \to N_j$	the projection belonging to (6) ,
$d_j = \kappa_j \rho_j$	the projector .

2.2. Proposition

Assumptions as before and $f \in \operatorname{Hom}_R(M,N)$. Then :

f is pi $\Leftrightarrow$ $\exists\, i \in I$, $j \in J$ [$\rho_j f \iota_i$ is an isomorphism]

Proof :

$\Leftarrow$: Since $\rho_j f \iota_i$ is an isomorphism , we have

$$N = \mathrm{Im}(f\iota_i) \oplus \mathrm{Ke}(\rho_j) = f(M_i) \oplus \mathrm{Ke}(\rho_j)$$

and

$$M_i \ni x \mapsto f(x) \in f(M_i)$$

is an isomorphism . Then I. 6.2 implies , that f is pi .

$\Rightarrow$: By I. 6.2 there exist

$$M = C \oplus D \ , \ C \neq 0 \quad , \quad N = P \oplus Q$$

(C for M_o and P for N_o in I. 6.2) ,

such that

(7) $$C \ni x \mapsto f(x) \in P$$

is an isomorphism . Now , we apply 1.6 (i) ; since the M_i are directly indecomposable , there exist M_{i_o} and a decomposition

$$C = C_1 \oplus C_2 \quad , \quad C_1 \cong M_{i_o}$$

The isomorphism (7) implies

$$P = f(C_1) \oplus f(C_2)$$

and the isomorphism

$$C_1 \ni x \mapsto f(x) \in f(C_1) \quad .$$

Denote by

$$g : M_{i_o} \to C_1 \quad \text{an isomorphism} \ ,$$

$$\iota : C_1 \to M \quad \text{the inclusion}$$

and by

$$\rho : N \to f(C_1)$$

the projection belonging to

$$N = f(C_1) \oplus f(C_2) \oplus Q \quad .$$

Then $\rho f \iota g : M_{i_o} \to f(C_1)$ is an isomorphism . Denote by h the inverse isomorphism , then

$$1_{M_{i_o}} = h \rho f \iota g \quad .$$

For $x \in M_{i_o}$, $x \neq 0$ let be $I_o \subset I$, $J_o \subset J$ finite subsets , such that

$$\sum_{i \in I_o} e_i g(x) = g(x)$$

and

$$\sum_{j \in J_o} d_j f g(x) = f g(x) \quad .$$

Denote $t \in End(M_{i_o})$ by

$$t := 1_{M_{i_o}} - h\rho \sum_{j\in J_o} \sum_{i\in I_o} d_j f e_i \iota g$$

then $t(x) = 0$ and

$$(8) \qquad 1_{M_{i_o}} = h\rho \sum_{j\in J_o} \sum_{i\in I_o} d_j f e_i \iota g + t .$$

Since $End(M_{i_o})$ is local, there must be at least one summand in (8), which is an automorphism. Since $t(x) = 0$, this cannot be t. Assume

$$h\rho d_j f e_i \iota g = h\rho\kappa_j\rho_j f\iota_i\pi_i\iota g$$

is an automorphism. By 1.10 follows, that $h\rho\kappa_j\rho_j f\iota_i$ is an isomorphism and again by 1.10 we get, that $\rho_j f \iota_i$ is an isomorphism, what we had to show.

By 2.2 it is easy to give an example for a LE-decomposition, for which T is not a radicaltotal (but a total) ring.

Assume $R = \mathbb{Z}$, p a primnumber,

$$M_{\mathbb{Z}} := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \quad , \quad T = End(M) .$$

Then $End(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z}$ (since this is a ring with 1-element) and $\mathbb{Z}/p^n\mathbb{Z}$ is local with $Rad(\mathbb{Z}/p^n\mathbb{Z}) = Tot(\mathbb{Z}/p^n\mathbb{Z}) = p\mathbb{Z}/p^n\mathbb{Z}$ and $(\mathbb{Z}/p^n\mathbb{Z})/(p\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Denote by

$\iota_n : \mathbb{Z}/p^n\mathbb{Z} \to M$ the inclusion

$\pi_n : M \to \mathbb{Z}/p^n\mathbb{Z}$ the projection.

Then is $(\iota_n(1 + p^n\mathbb{Z}) \mid n \in \mathbb{N})$ a generating family of M. We define $f \in T$ by

$$f(\iota_n(z + p^n\mathbb{Z})) = \iota_n(pz + p^{n+1}\mathbb{Z}) \quad , \quad z \in \mathbb{Z} .$$

Obviously $\pi_{n+1} f \iota_n$ is not an isomorphism for all $n \in \mathbb{N}$, since it is not surjective. Further $\pi_i f \iota_n = 0$ for $i \neq n+1$. Then by 1.11. $f \in Tot(T)$.
But $f \notin Rad(T)$, since $1_M - f$ is not an automorphism. If
$x = (z_1 + p\mathbb{Z}, z_2 + p^2\mathbb{Z}, \ldots, z_t + p^t\mathbb{Z}, 0, 0, 0, \ldots)$
with $p^t \nmid z_t$ is an element $\neq 0$ of M, then $(1_M - f)(x) =$
$= (z_1 + p\mathbb{Z}, z_2 - pz_1 + p^2\mathbb{Z}, \ldots, z_t - pz_{t-1} + p^t\mathbb{Z}, -pz_t + p^{t+1}\mathbb{Z}, 0, 0, 0, \ldots)$
and this shows, that $\iota_1(1 + p\mathbb{Z}) \notin Im(1_M - f)$.

As mentioned before , in the case , that (1) is a LE-decomposition , there exist several characterizations of the property , that $T = \mathrm{End}(M)$ is radicaltotal . The total was there defined as the set of all $f \in T$, such that $\pi_j f \iota_i$ is not an isomorphism for all $i , j \in I$. We ask here , if these characterizations can also be applied for RTE-decompositions (1) . There is at least one important difference : For a LE-decomposition (1) the M_i are directly indecomposable , but for RTE-decompositions this is not the case ; even more : There exist RTE-modules which are not direct sums of LE-modules (see example after 2.3) .

We consider first locally direct summands of M . These are submodules of M of the form $\bigoplus_{j\in J} B_j$, where for every finite subset $J_o \subset J$ $\bigoplus_{j\in J_o} B_j$ is a direct summand of M .

We use the fact 1.10. (ii) , that for $f \in \mathrm{Tot}(T)$ $1_M - f$ is a monomorphism .

2.3. Proposition

If (1) is a RTE-decomposition and $f \in \mathrm{Tot}(\mathrm{End}(M_R))$, then

(i) $\mathrm{Im}(1_M - f) = \bigoplus_{i\in I} (1_M - f)(M_i)$ is a locally direct summand of M ,

(ii) if every locally direct summand of the form $\mathrm{Im}(1_M - f)$ is a direct summand , then M is a RTE-module .

Proof :

(i) : Since $1_M - f$ is a monomorphism

$$\mathrm{Im}(1_M - f) = \bigoplus_{i\in I} (1_M - f)(M_i) \quad .$$

For a finite $I_o \subset I$ and $e := \sum_{i\in I} e_i$, we have

$$\bigoplus_{i\in I_o} (1_M - f)(M_i) = \bigoplus_{i\in I_o} (1_M - fe)(M_i)$$

and since $1_M - fe$ is an automorphism

$$M = \bigoplus_{i\in I_o} (1_M - f)(M_i) \oplus \Big(\bigoplus_{i\in I\setminus I_o} (1_M - fe)(M_i) \Big) \quad ,$$

hence $\mathrm{Im}(1_M - f)$ is a locally direct summand .

(ii) : If $\mathrm{Im}(1_M - f)$ is a direct summand , then $1_M - f$ has a left inverse $g \in T$ such that

$$g(1_M - f) = 1_M \quad \Rightarrow \quad g = 1_M - (-g)f \quad .$$

Since for $f \in \mathrm{Tot}(T)$ also $(-g)f \in \mathrm{Tot}(T)$ there exists $h \in T$ such that

$$hg = h(1_M - (-g)f) = 1_M \quad .$$

Therefore $g^{-1} = h = 1_M - f \Rightarrow (1_M - f)g = 1_M$, that is , $1_M - f$ is an automorphism and $f \in \mathrm{Rad}(T)$, hence $\mathrm{Tot}(T) = \mathrm{Rad}(T)$.

For a LE-decomposition (1) the following is true : If M is a RTE-module then every locally direct summand of the form

$$\bigoplus_{j \in J} B_j \quad , \quad \mathrm{End}(B_j) \text{ local}$$

is a direct summand . We show by an example , that a similar result is not true for RTE-decompositions . We consider the ring S , defined in II. 5. , for which $\mathrm{Rad}(S) = \mathrm{Tot}(S) = 0$. Then S_S itself is a RTE-decomposition . The ideal $K^{(\mathbb{N})}$ of S has a RTE-decomposition , even a LE-decomposition , and is a locally direct summand of S_S . Since $K^{(\mathbb{N})}$ is obviously large in S_S , it cannot be a direct summand of S_S .

For a LE-decomposition (1) $\mathrm{End}(M)/\mathrm{Tot}(\mathrm{End}(M))$ is a ring with 2-exchange property (= EP) . Here for the ring S we have $\mathrm{End}(S_S) = S$, $\mathrm{Tot}(S) = 0$, but S_S does not have the 2-EP (II. 5.) . Therefore , S_S does not have a LE-decomposition .

§3. Decompositions with duality properties

Already in I. 7.2. and I. 7.3. we had results about direct decompositions with duality properties . We get here some more informations .

Let be

(9) $$M_R = \bigoplus_{i \in I} M_i$$

a decomposition , where we have "total properties" of the M_i resp. M with respect to M_i^* resp. M^* (E3) . Especially $m \in M$ is pi iff there exists $\vartheta \in M^*$ such that $\vartheta(m)$ is an idempotent $\neq 0$ in R and Tot(M) is the set of all not pi elements in M .

3.1. Lemma

(i) If $f \in \mathrm{Hom}_R(M,N)$, then $f(\mathrm{Tot}(M)) \subset \mathrm{Tot}(N)$.

(ii) If in (9) M is total (that is Tot(M) is additively closed) , then

$$\mathrm{Tot}(M) = \bigoplus_{i \in I} \mathrm{Tot}(M_i) .$$

Proof :

(i) : We show : If for $m \in M$ $f(m)$ is pi , then m is pi . If f(m) is pi , then there exists $\gamma \in N^*$ such that $\gamma f(m) = e = e^2 \neq 0$. Then $\gamma f \in M^*$, hence m is pi .
(ii) : For the inclusion $\iota_i : M_i \to M$ resp. the projection $\pi_i : M \to M_i$ follows by (i)

(10) $$\iota_i(\mathrm{Tot}(M_i)) \subset \mathrm{Tot}(M)$$

(11) $$\pi_i(\mathrm{Tot}(M)) \subset \mathrm{Tot}(M_i) \quad , \quad i \in I .$$

For $x , y \in \mathrm{Tot}(M_i)$ follows by (10) and the assumption $x + y \in \mathrm{Tot}(M)$ and by (11) $x + y \in \mathrm{Tot}(M_i)$. Therefore , also the M_i are total . Again by (10) and the assumption follows

$$\bigoplus_{i \in I} \mathrm{Tot}(M_i) \subset \mathrm{Tot}(M) .$$

If $u \in \mathrm{Tot}(M)$ and $u = \Sigma u_i$ $(u_i \in M_i)$ in (9) , then by (11) $u_i \in \mathrm{Tot}(M_i)$, hence also

$$\mathrm{Tot}(M) \subset \bigoplus_{i \in I} \mathrm{Tot}(M_i) .$$

3.2. Corollary

(a) If in (9) all M_i are radicaltotal (that is $\mathrm{Rad}(M_i) = \mathrm{Tot}(M_i)$) , then M is radicaltotal .

(b) If (9) is a RTE-decomposition and all M_i , $i \in I$ are projective , then M is radicaltotal .

Proof :

(a) : Since $Rad(M_i) = Tot(M_i)$, we have

$$\bigoplus_{i \in I} Tot(M_i) = \bigoplus_{i \in I} Rad(M_i) = Rad(M) \subset Tot(M) \ .$$

In the proof of 3.1. we showed (without any assumption) : If $u \in Tot(M)$ and $u = \Sigma u_i$, $u_i \in M_i$, then $u_i \in Tot(M_i)$. This implies

$$Tot(M) \subset \bigoplus_{i \in I} Tot(M_i) = Rad(M) \ ,$$

hence $Rad(M) = Tot(M)$.

(b) : The assumption in (b) and I. 4.8. 1) imply $Rad(M_i) = Tot(M_i)$. Then (b) follows by (a) .

For example , (b) is satisfied if (9) is a projective LE-decomposition .

3.3. Corollary

Assume , that (9) is a RTE-decomposition . Then :

$Tot(M) \neq M \Leftrightarrow \exists \ i_o \in I$ [M_{i_o} has a nonzero , projective direct summand] .

Proof :

By 1.6. (i) M is a TE-module and then by I. 3.7. (for (E3)) M is total . Then by 3.1. (ii)

$$(12) \qquad Tot(M) = \bigoplus_{i \in I} Tot(M_i) \ .$$

$\Rightarrow$: Now assume $Tot(M) \neq M$. Then there must exist at least one $i_o \in I$ with $Tot(M_{i_o}) \neq M_{i_o}$. Then I. 7.2. implies the statement .

$\Leftarrow$: On the other side , if $C \neq 0$ is a projective , direct summand of M_{i_o} , then with M_{i_o} also C is a RTE-module and by I. 8.4. 1) $Tot(C) = Rad(C)$. Since C is a nonzero projective module $Rad(C) \neq C$, hence $Tot(M_{i_o}) \neq M_{i_o}$, hence by (12) $Tot(M) \neq M$.

IV. The relative total in the category of R-modules

§1. Semi-ideals and ideals in the category of R-modules

For a ring R with $1 \in R$ we consider the category Mod-R of all unitary R-right modules . By $\mathfrak{M}_R$ we denote the class of objects of Mod-R .

1.1. Definition

1.) A semi-ideal I of Mod-R is given by a set

(1) $I(M,N) \subset Hom_R(M,N)$ for all $M , N \in \mathfrak{M}_R$,

such that the following property is satisfied :

$Hom_R(N,Y)I(M,N)Hom_R(X,M) \subset I(X,Y)$

for all $M , N , X , Y \in \mathfrak{M}_R$.

2.) A semi-ideal I is called an ideal of Mod-R if further

(2) $I(M,N)$ is additively closed for all $M , N \in \mathfrak{M}_R$.

If for one pair M,N $I(M,N) \neq \emptyset$, then by (1) $0 \in I(X,Y)$ for all $X,Y \in \mathfrak{M}_R$. We add to the definition of a semi-ideal I , that it is not empty .

If I is an ideal , I(M,N) is not only additively closed but by (1) even a subgroup of $Hom_R(M,N)$ and a End(N)-End(M)-bimodule .

If I , J are two semi-ideals , then we write $I \subset J$ resp. $I = J$ resp. $I \supset J$ iff for all $M , N \in \mathfrak{M}_R$

$I(M,N) \subset J(M,N)$ resp. $I(M,N) = J(M,N)$ resp. $I(M,N) \supset J(M,N)$.

The following lemma shows , that a semi-ideal I is uniquely determined by $I(M,M)$ for all $M \in \mathfrak{M}_R$.

1.2. Lemma

For semi-ideals I , J the following is true :

(i) $I \subset J \Leftrightarrow \forall M \in \mathfrak{M}_R \; [I(M,M) \subset J(M,M)]$

(ii) $I = J \Leftrightarrow \forall M \in \mathfrak{M}_R \; [I(M,M) = J(M,M)]$.

Proof :

$\Rightarrow$: Clear .

(i) , $\Leftarrow$: Consider $A , B \in \mathfrak{M}_R$ and $f \in I(A,B)$. Define $M = A \oplus B$ with the inclusions ι_A , ι_B and the projections π_A , π_B . Then by (1) $\iota_B f \pi_A \in I(M,M) \subset J(M,M)$. Then again by (1) $\pi_B \iota_B f \pi_A \iota_A = 1_B f 1_A = f \in J(A,B)$, hence $I \subset J$.

(ii) , $\Rightarrow$: $I(M,M) = J(M,M)$ implies $I \subset J$ and $J \subset I$, hence $I = J$.

Our main goal in this chapter is to define semi-ideals and ideals in Mod-R by using a modified notion of the total relative to certain classes of R-modules .

First we give two examples for ideals in Mod-R .

1. Example :

Denote by Q a (proper or inproper) subring of the centre of R . Then every $Hom_R(M,N)$ is a Q-right-module by the definition

$$fq(x) := f(x)q \quad , \quad x \in M \quad , \quad q \in Q \; .$$

Then it is easy to see , that

$$Rad(Hom_R(M,N)_Q) \quad \text{for all } M , N \in \mathfrak{M}_R$$

is an ideal in Mod-R . For $g \in Hom_R(X,M)$ the mapping

$$\hat{g} : Hom_R(M,N) \ni f \mapsto fg \in Hom_R(X,N)$$

is obviously a Q-module homomorphism , hence

$$Rad(Hom_R(M,N))g \subset Rad(Hom_R(X,N)) \; .$$

The same is true for the other side . That means that (1) holds . (2) is anyway satisfied for a radical .

2. Example :

The Jacobson-radical in Mod-R .

1.3. Definition

$\mathrm{Rad}(M,N) := \{ f \in \mathrm{Hom}_R(M,N) \mid \forall\, g \in \mathrm{Hom}_R(N,M)\ [1_M - gf$ is an invertible element in $T := \mathrm{End}(M)] \}$.

In the following we have to use III. 1.7 :

$1_M - gf$ is invertible in T $\Leftrightarrow$

$1_N - fg$ is invertible in $S := \mathrm{End}(N)$.

Hence the definition of Rad(M,N) can also be given by using $1_N - fg$. For $M = N$, this is the definition of the Jacobson radical for T by using quasi-regularity .

1.4. Corollary

Rad is an ideal in Mod-R .

Proof :

Semi-ideal Rad : For $h \in \mathrm{Hom}_R(N,Y)$ we have to show , that $1_M - ghf$ is invertible in T for all $g \in \mathrm{Hom}_R(Y,M)$. But $gh \in \mathrm{Hom}_R(N,M)$, therefore we have this property by assumption . Similar for the other side .

Ideal Rad : Assume $f_1 , f_2 \in \mathrm{Rad}(M,N)$ and consider

$$1_M - g(f_1 + f_2) = (1_M - gf_1) - gf_2 \quad , \quad g \in \mathrm{Hom}_R(N,M) \quad .$$

By assumption , there exists an inverse $t_1 \in T$ of $1_M - gf_1$. With this follows

$$t_1(1_M - gf_1) - t_1gf_2 = 1_M - t_1gf_2 \; .$$

Since also $t_1g \in \mathrm{Hom}_R(N,M)$, there exists also an inverse $t_2 \in T$ of $1_M - t_1gf_2$:

$$t_2t_1(1_M - g(f_1 + f_2)) = t_2(1_M - t_1gf_2) = 1_M \quad .$$

Since t_1 , t_2 are invertible elements , also t_2t_1 is invertible . Therefore , this is also the right inverse of $1_M - g(f_1 + f_2)$, hence also this element is invertible .

Now , we consider also the radicals of the T- resp. S-modules $\mathrm{Hom}_R(M,N)_T$ resp. ${}_S\mathrm{Hom}_R(M,N)$ and ask for a connection to Rad(M,N) .

1.5. Proposition

$Rad(Hom_R(M,N)_T)$ + $Rad({}_S Hom_R(M,N)) \subset Rad(M,N)$ for all M , N $\in \mathfrak{M}_R$.

Proof :

Since Rad(M,N) is an ideal , we have only to show , that both radicals are contained in Rad(M,N) . With respect to symmetry , we have only to prove $Rad(Hom_R(M,N)_T) \subset Rad(M,N)$. Let be f $\in Rad(Hom_R(M,N)_T)$, that is fT $\subsetneq^\circ$ $Hom_R(M,N)_T$ (" $\subsetneq^\circ$ " means "small submodule" .) . For any g $\in Hom_R(N,M)$ follows

$$gfT \subsetneq^\circ T_T \ ,$$

since the image of a small submodule is small in the image of a homomorphism . Since for any t $\in$ T

$$(1_M - gft)T + gfT = T$$

we get $(1_M - gft)T = T$. Then there exists $t_1 \in T$ with $(1_M - gft)t_1 = 1_M$, hence $t_1 = 1_M - gft(-t_1)$. By the same reason , also this element has a right inverse . Therefore t_1 has a left and a right inverse and then this is the invertible element 1_M - gft . For t = 1 , that means 1_M - gf is invertible , hence f $\in$ Rad(M,N) .

Finally we mention a result which we need later .

1.6. Lemma

If f $\in$ Rad(M,N) , g $\in Hom_R(N,M)$, t $\in$ T , then gftT $\subsetneq^\circ T_T$.

Proof :

Since Rad(M,N) is an ideal , also ft $\in$ Rad(M,N) . We assume now for a right-ideal U of T :

$$gftT + U = T \ .$$

Then there exist $t_1 \in T$, u $\in$ U such that

$$gftt_1 + u = 1_M$$

$$\Rightarrow \quad u = 1_M - gftt_1$$

But since $ftt_1 \in Rad(M,N)$ u is invertible , hence U = T , which means gftT $\subsetneq^\circ T_T$.

§2. Some properties of idempotents and induced isomorphisms

For preparation of the definition of the total in Mod-R , we need some simple results about idempotents and induced isomorphisms .

For $M , N \in \mathfrak{M}_R$ let be $f \in \mathrm{Hom}_R(M,N)$, $g , h \in \mathrm{Hom}_R(N,M)$, $S := \mathrm{End}(N)$, $T := \mathrm{End}(M)$. By $d , d_1 \in S$, $e , e_1 \in T$ we denote nonzero idempotents .

2.1. Lemma

(i) If $fg = d$, then $gdf = e$ and

$$e(M) \ni e(x) \mapsto fe(x) \in d(N)$$

is an isomorphism .

(ii) If $hf = e_1$, then $fe_1h = d_1$ and

$$e_1(M) \ni e_1(x) \mapsto fe_1(x) \in d_1(N)$$

is an isomorphism .

Proof :

(i) : If $fg = d \ (= d^2 \neq 0)$, then

$$(gdf)^2 = g(dfgd)f = gdf \quad .$$

Denote $e := gdf$, then $feg = d \neq 0$, hence $e \neq 0$. Further

(3) $$df = d^2f = fgdf = fe$$

and

$$d(N) = d^2(N) = dfg(N) \subset df(M) \subset d(N)$$

hence

(4) $$d(N) = df(M) \quad .$$

Now , we consider the mapping in (i) . That this is an epimorphism follows from (3) and (4) . Assume $fe(x) = 0$, then $gdfe(x) = e(x) = 0$, hence it is also injective .

(ii) : Similar , we have now $hf = e_1 \ (= e_1{}^2 \neq 0)$, then

$$(fe_1h)^2 = f(e_1hfe_1)h = fe_1h \quad .$$

Denote $d_1 := fe_1h$, then $hd_1f = e_1 \neq 0$, hence $d_1 \neq 0$. Further

$$fe_1 = fe_1{}^2 = fe_1hf = d_1f$$

and

$$d_1(N) = fe_1h(N) \subset fe_1(M) = d_1f(M) \subset d_1(N)$$

hence

$$d_1(N) = d_1f(M) \quad .$$

Similar to the proof of (i) follows , that the mapping in (ii) is an isomorphism .

2.2. Definition

1.) A class k of objects from $\mathfrak{M}_R$ (that is of R-modules) is called a **closed class** iff it is closed with respect to isomorphisms and direct summand .

2.) An idempotent $d \in \mathrm{End}(N)$ is called a **k-idempotent** iff $d(N) \in k$.

2.3. Lemma

Given a closed class k . Then for $f \in \mathrm{Hom}_R(M,N)$ the following conditions are equivalent :

(a) There exists $g \in \mathrm{Hom}_R(N,M)$ such that fg is a nonzero k-idempotent ;

(b) there exists $h \in \mathrm{Hom}_R(M,N)$ such that hf is a nonzero k-idempotent ;

(c) there exist $0 \neq A \subsetneq^{\oplus} M$, $B \subsetneq^{\oplus} N$ with $A , B \in k$, such that the mapping

$$\tilde{f} : A \ni x \mapsto f(x) \in B$$

is an isomorphism ($\tilde{f} = \pi_B f \iota_A$) ;

(d) there exists $0 \neq C \in k$ and homomorphisms $\alpha : C \to M$, $\beta : N \to C$, such that $\beta f \alpha$ is an automorphism .

Proof :

(a) $\Rightarrow$ (b) : By 2.1. (i) with $h = gd$. Since $d(N) \in k$ and k is closed , also $e(M) \in k$.

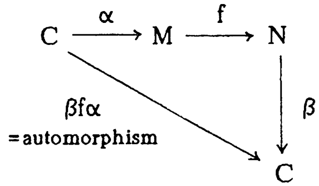
(b) $\Rightarrow$ (c) : By 2.1. (ii) with $A = e_1(M)$, $B = d_1(N)$.

(c) $\Rightarrow$ (a) : If $N = B \oplus B_1$ and d is the projector on B belonging to this decomposition , then $d = d^2$ and $d(N) = B$, that is , d is a nonzero k-idempotent . Define $g \in \mathrm{Hom}_R(N,M)$ by

$$g_{|B} := \tilde{f}^{-1} \quad , \quad g(B_1) = 0 \quad ,$$

then $fg = d$.

(c) ⇒ (d) : Take $C = A$, $\alpha = \iota_A$, $\beta = \tilde{f}^{-1}\pi_B$, then $\beta f\alpha = \tilde{f}^{-1}\pi_B f\iota_A = 1_A$.
(d) ⇒ (c) : Now we have the situation



then

$$M = \mathrm{Im}(\alpha) \oplus \mathrm{Ke}(\beta f)$$
$$N = \mathrm{Im}(f\alpha) \oplus \mathrm{Ke}(\beta)$$

and α is a monomorphism . Take in (c) $A = \mathrm{Im}(\alpha)$ and $B = \mathrm{Im}(f\alpha)$. Since $0 \neq C \in k$ and α is mono , also $0 \neq \mathrm{Im}(\alpha) \in k$. Since $\mathrm{Ke}(f) \subset \mathrm{Ke}(\beta f)$ f induces the isomorphism $\mathrm{Im}(\alpha) \ni x \mapsto f(x) \in \mathrm{Im}(f\alpha) = f\mathrm{Im}(\alpha)$.

§3. k-partially invertible elements and the k-total

3.1. Definition

Given a closed class k .

1) $f \in \mathrm{Hom}_R(M,N)$ is called **k-partially invertible** = k-pi :⇔ the conditions of 2.3 are satisfied .

2) $\mathrm{TOT}_k(M,N) := \{ f \mid f \in \mathrm{Hom}_R(M,N) \wedge f$ is not k-pi $\}$.

This is called the **k-total from M to N** . If k is the class of all R-modules we write TOT(M,N) and call this the **total from M to N** .

Obviously we have then $\mathrm{TOT}(M,N) = \mathrm{Tot}(\mathrm{Hom}_R(M,N))$ in the meaning of (E2) .

In the following k always denotes a closed class of R-modules .

3.2. Lemma

1) TOT_k is a semi-ideal in Mod-R .

2) $\forall\ M,N \in \mathfrak{M}_R \quad \forall\ f_1 \in Rad(M,N) \quad \forall\ f_2 \in TOT(M,N)$

$$f_1 + f_2 \in TOT(M,N) \ ;$$

we write for this :

$$Rad + TOT = TOT \ .$$

Proof :

1) : For $f \in TOT_k(M,N)$, $g \in Hom_R(X,M)$, $h \in Hom_R(N,Y)$ we have $hfg \in TOT_k(X,Y)$ to show . Proof, indirect . Assume hfg is k-pi . Then there exists $p \in Hom_R(Y,X)$ such that $(hfg)p = (hf)(gp) = d = d^2 \neq 0$ with a k-idempotent d . Then by 2.3. there exists $q \in Hom_R(Y,M)$ such that

$$q(hf) = (qh)f$$

is a k-idempotent , hence f is k-pi $\lightning$.

2) : Proof indirect . Assume there exists $g \in Hom_R(M,N)$ such that

$$g(f_1 + f_2) = e = e^2 \neq 0$$

$$\Rightarrow \quad T = gf_1T + gf_2T + (1 - e)T \ .$$

Since by 1.6. $gf_1T \subsetneq^{o} T_T$ we have

$$T = gf_2T + (1 - e)T$$

$$\Rightarrow \quad eT = egf_2T \ ,$$

then there exists $t \in T$ with $e = egf_2t$.

But this is not possible , since TOT is a semi-ideal and $e \notin TOT(M,M)$.

3.3. Remark

If a finite meaningful product of modulehomomorphisms is k-pi , then every factor of this product is k-pi .

Proof :

Since TOT_k is a semi-ideal .

With respect of 3.2. 1) we have now several questions .

1) For which k is TOT_k an ideal ?
2) What are the conditions for closed classes k_1, k_2 such that $TOT_{k_1} = TOT_{k_2}$?
3) Is there some kind of correspondence between closed classes and semi-ideals ?

We give first complete answer to the first two questions .

3.4. Proposition

For a closed class k are equivalent :
(i) TOT_k is an ideal ;
(ii) k is a subclass of the class of all TE-modules (TE-module see III. 1.1.) .

Proof :

(i) $\Rightarrow$ (ii) : Let be $M \in k$. Since k is closed , every direct summand of M is in k and every idempotent of $T = End(M)$ is a k-idempotent . Therefore $TOT_k(M,M) = Tot(T)$; since $TOT_k(M,M)$ is additively closed , T is a total ring and M is a TE-module .

(ii) $\Rightarrow$ (i) : Consider $f, g \in TOT_k(M,N)$ and assume $f + g$ is k-pi . Then there exists $h \in Hom_R(N,M)$ and a k-idempotent $e \in T = End(M)$ such that

$$h(f + g) = e = e^2 \neq 0 .$$

Denote $A := e(M)$ and $\iota : A \to M$ the inclusion and $\pi : M \to A$ the projection along $M = e(M) \oplus (1 - e)(M)$, then $e = \iota\pi$, $1_A = \pi\iota$. From $h(f + g) = e$ follows

$$ehfe + ehge = e = \iota\pi$$

and

$$\pi hf\iota + \pi hg\iota = 1_A .$$

Since $f, g \in TOT_k(M,N)$ and TOT_k is a semi-ideal , also $\pi hf\iota, \pi hg\iota \in TOT_k(A,A)$. Since by assumption A is a TE-module , it follows , that $\pi hf\iota + \pi hg\iota = 1_A$ is in $Tot(End(A))$ $\lightning$.

Now we answer the second question .

3.5. Proposition

Let k_1 , k_2 be closed classes . Then

$$TOT_{k_2} \subset TOT_{k_1} \Leftrightarrow \text{every } A \in k_1 , A \neq 0 \text{ contains a nonzero direct summand } C \in k_2$$

Proof :

$\Rightarrow$: Consider $A \in k_1$, $A \neq 0$, then 1_A is k_1-pi , hence $1_A \notin TOT_{k_1}(A,A)$. Since $TOT_{k_2} \subset TOT_{k_1}$ also $1_A \notin TOT_{k_2}(A,A)$. Then there must exist $0 \neq B \subsetneq^{\oplus} A$, $B \in k_2$, such that 1_A induces the identical isomorphism on B .

$\Leftarrow$: If $f \in Hom_R(M,N)$ and f is k_1-pi , then there exist $0 \neq A \subsetneq^{\oplus} M$, $B \subsetneq^{\oplus} N$, A , B $\in k_1$, such that

$$\tilde{f} : A \ni x \mapsto f(x) \in B$$

is an isomorphism . By assumption there exists $C \subsetneq^{\oplus} A$, $C \neq 0$, $C \in k_2$, and the isomorphism $\tilde{f}$ induces an isomorphism

$$\hat{f} : C \to f(C) \quad ,$$

where $f(C) \subsetneq^{\oplus} B$, hence $f(C) \subsetneq^{\oplus} N$ and since $C \cong f(C)$ also $f(C) \in k_2$. That means , f is also k_2-pi . This implies $TOT_{k_2} \subset TOT_{k_1}$.

3.6. Corollary

1) If k_1 , k_2 are closed classes , then :

$$TOT_{k_2} = TOT_{k_1} \Leftrightarrow TOT_{k_2} \subset TOT_{k_1} \wedge TOT_{k_1} \subset TOT_{k_2}$$

(see conditions in 3.5)

2) If k_1 , k_2 are closed classes and $k_1 \subset k_2$, then $TOT_{k_2} \subset TOT_{k_1}$.

If we denote by

k_o = class of LE-modules

k_1 = class of injective modules

k_2 = class of quasi-injective modules

k_3 = class of 2-EP modules

k_4 = class of RTE-modules

k_5 = class of TE-modules

then we have

$$\left.\begin{matrix} k_o \\ k_1 \subset k_2 \end{matrix}\right\} \subset k_3 \subset k_4 \subset k_5$$

and the ideals (which all contain Rad)

$$TOT_{k_5} \subset TOT_{k_4} \subset TOT_{k_3} \subset \begin{cases} TOT_{k_o} \\ TOT_{k_2} \subset TOT_{k_1} \end{cases} .$$

For a given ring R we would like to know , which of these are different . For example TOT_{k_5} is different from TOT_{k_i} , $i = 0, \dots, 4$, if there exists $0 \neq A \in k_5$, which does not contain a nonzero direct summand in k_i .

We show at least for a certain ring R , that $TOT_{k_3} \subsetneqq TOT_{k_o}$. For a field k we consider

$$R := K^{\mathbb{N}}/K^{(\mathbb{N})} \quad .$$

We prove , that R_R has the 2-EP but does not have a nonzero direct summand with local endomorphismring .

For the prove , that R_R has the 2-EP , we show (II. 5.1) , that for any $a \in R$ there exists an idempotent $d \in R$ such that

$$d \in Ra \quad , \quad 1\text{-}d \in R(1\text{-}a) \quad :$$

Let $\overline{(a_i)} \in R$ with a representative $(a_i) \in K^{\mathbb{N}}$. Then define $(d_i) \in K^{\mathbb{N}}$ by

$$d_i = \begin{cases} 1 & \text{if } a_i = 1 \\ 0 & \text{if } a_i \neq 1 \end{cases}$$

Then (d_i) is idempotent and $(d_i) = (d_i)(a_i)$, hence $\overline{(d_i)} = \overline{(d_i)}\overline{(a_i)} \in R\overline{(a_i)}$.
Further

$$1 - d_i = \begin{cases} 0 & \text{for } a_i = 1 \\ 1 & \text{for } a_i \neq 1 \end{cases}$$

and

$$1 - a_i = \begin{cases} 0 & \text{for } a_i = 1 \\ 1 - a_i \neq 0 & \text{for } a_i \neq 1 \end{cases} ,$$

hence $(1) - (d_i) \in K^{\mathbb{N}}((1) - (a_i))$

and $\overline{(1)} - \overline{(d_i)} \in R(\overline{(1)} - \overline{(a_i)})$.

By this we know , that R_R has the 2-EP .

Assume

$$R_R = A \oplus B \quad , \quad A \neq 0 \quad ,$$

then there exists an idempotent $0 \neq \overline{(e_i)} \in R$ such that $A = \overline{(e_i)}R$ and $\mathrm{End}(A_R) \cong \overline{(e_i)}R\overline{(e_i)}$. Since $\overline{(e_i)}^2 = \overline{(e_i)} \neq 0$, there exists $n \in \mathbb{N}$ such that

$$e_i = \begin{cases} 0 \\ \text{or} \\ 1 \end{cases} \quad \text{for } i \geq n$$

and there are infinitely many $e_i = 1$. Define now (d_i) by substituting in (e_i) every second $e_i = 1$ by 0 ; then $\overline{(d_i)}$ is a nonzero idempotent $\neq \overline{(e_i)}$ in $\overline{(e_i)}R\overline{(e_i)}$. Hence this ring is not local .

We have also an easy example for $TOT_{k_2} \subsetneqq TOT_{k_1}$ for $R = \mathbb{Z}$.

Obviously is $\mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module quasi-injective , since $2\mathbb{Z}/4\mathbb{Z}$ is the only nontrivial submodule and $\mathrm{Hom}_{\mathbb{Z}}(2\mathbb{Z}/4\mathbb{Z}) = \{0 ; \iota\}$ (0 = zero-homomorphism , ι = inclusion) . But $\mathbb{Z}/4\mathbb{Z}_{\mathbb{Z}}$ is not injective , since the homomorphism

$$4\mathbb{Z} \ni 4x \mapsto x + 4\mathbb{Z} \in \mathbb{Z}/4\mathbb{Z}$$

cannot be lifted to a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$. Since $\mathbb{Z}/4\mathbb{Z}_{\mathbb{Z}}$ is directly indecomposable , it does not contain a nonzero injective direct summand .

It would be interesting to give examples for proper containment for all possible cases . Or even to do more : To characterize all rings for which a certain containment $TOT_{k_i} \subset TOT_{k_j}$ $(i > j)$ is proper .

§4. A Galois-correspondence in an arbitrary category

We are mainly interested in the category Mod-R , but the following interesting Galois-correspondence can be described in an arbitrary category . We do not use anything from before but formulate this §4. selfcontained .

Denote by **C** a category and by Obj(**C**) resp. Mor(**C**) the class of objects resp. morphisms of **C** . For A , B $\in$ Obj(**C**) we denote by Mor(A,B) the set of morphisms from A to B . If we write $0 \neq C \in$ Obj(**C**) , then this makes sense only if **C** has zeroelements . If **C** has no zeroelements , then the condition $0 \neq C$ is superflous .

4.1. Definition

1) A nonempty class $k \subset$ Obj(**C**) is called **closed** $:\Leftrightarrow$ $\forall\, M \in k \quad \forall\, C \in$ Obj(**C**) and morphisms $\alpha : C \to M$, $\beta : M \to C$ with $\beta\alpha = 1_C$ also $C \in k$.
2) A semi-ideal I in **C** is given by a set

   $\emptyset \neq I(A,B) \subset Mor(A,B)$ for all A , B $\in$ Obj(**C**)

   such that for all A , B , X , Y $\in$ Obj(**C**) and all $h : X \to A$, $g : B \to Y$, $f \in I(A,B)$

   $gfh \in I(X,Y)$.
3) If I and J are two semi-ideals , we write $I \subset J$ resp. $J \supset I$ $:\Leftrightarrow$

   $I(A,B) \subset J(A,B)$ for all A , B $\in$ Obj(**C**) .
4) Let be $k \subset$ Obj(**C**) , $f \in Mor(A,B)$.

   f is called k-partially invertible (= k-pi) $:\Leftrightarrow$

   $\exists\, C \in k$, $C \neq 0$, $\alpha : C \to A$, $\beta : B \to C$ with $\beta f\alpha = 1_C$.
5) $TOT_k(A,B) := \{ f \in Mor(A,B) \mid f \text{ is not k-pi} \}$ for all A , B $\in$ Obj(**C**) .

   In the case k = Obj(**C**) we write for abbreviation $TOT = TOT_{Obj(\mathbf{C})}$ and $TOT(A,B) = TOT_{Obj(\mathbf{C})}(A,B)$.
6) Let be I a semi-ideal , then

   $K(I) := \{ M \in Obj(\mathbf{C}) \mid TOT(M,M) \supset I(M,M) \}$.

4.2. Corollary

Let be k_1 , k_2 , k closed classes and I_1 , I_2 , I semi-ideals . Then the following properties are satisfied :

(1) If a product of morphisms is k-pi $\Rightarrow$ every factor of the product is k-pi ,

(2) TOT_k is a semi-ideal ,

(3) K(I) is closed ,

(4) $k_1 \subset k_2 \;\Rightarrow\; TOT_{k_2} \subset TOT_{k_1}$

(5) $I_1 \subset I_2 \Rightarrow K(I_2) \subset K(I_1)$

(6) $M \in k \Rightarrow TOT(M,M) = TOT_k(M,M)$

(7) $k \subset K(TOT_k)$

(8) $I \subset TOT_{K(I)}$.

Proof :

(1) By definition of k-pi .

(2) By (1) .

(3) Assume $M \in K(I)$ and $C \xrightarrow{\alpha} M \xrightarrow{\beta} C$ with $\beta\alpha = 1_C$, then

$$\alpha I(C,C)\beta \subset I(M,M) \subset TOT(M,M) \Rightarrow$$

$$I(C,C) \subset \beta\, TOT(M,M)\alpha \subset TOT(C,C) .$$

(4) By def .

(5) By def .

(6) By (4) : $TOT(M,M) \subset TOT_k(M,M)$. Assume $f \in Mor(M,M)$ and $f \notin TOT(M,M) \Rightarrow \exists\, 0 \neq C \in Obj(\mathbf{C})$ and

$$C \xrightarrow{\alpha} M \xrightarrow{f} M \xrightarrow{\beta} C$$

with $\beta f\alpha = 1_C = \beta(f\alpha)$, hence $C \in k$ (since k is closed and $M \in k$) . This means f is k-pi , hence $f \notin TOT_k(M,M)$. Therefore also $TOT_k(M,M) \subset TOT(M,M)$.

(7) If $M \in k$, then by (6) $TOT(M,M) = TOT_k(M,M) \Rightarrow M \in K(TOT_k)$.

(8) Assume $f \in I(A,B)$, $f \notin TOT_{K(I)}(A,B) \Rightarrow$ there exists

$$C \xrightarrow{\alpha} A \xrightarrow{f} B \xrightarrow{\beta} C$$

with $0 \neq C \in K(I)$, $\beta f\alpha = 1_C$. Then $1_C \in I(C,C)$, since I is a semi-ideal . Since $C \in K(I)$ $I(C,C) \subset TOT(C,C)$, hence $1_C \in TOT(C,C)$ $\lightning$.

Now we come to the theorem , which shows , that we have indeed a Galois-correspondence in $\mathbf{C}$.

4.3. Theorem

Let be k a closed class and I a semi-ideal . Then

(i) $TOT_k = TOT_{K(TOT_k)}$.

(ii) $K(I) = K(TOT_{K(I)})$.

Proof :

(i) : By (7) and (4) (from 4.2) follows

$$TOT_{K(TOT_k)} \subset TOT_k$$

and by (8) (with $I = TOT_k$) follows the converse inclusion .

(ii) : By (8) and (5) follows

$$K(TOT_{K(I)}) \subset K(I)$$

and by (7) (with $k = K(I)$) follows the converse inclusion .

Example

G. M. Kelly [4] defined the notion "radix" of a category , which is an equivalencerelation in Mor(A,B) for all A , B in the category . This definition seems to have some connection with our notion "total" . But we show by an example that these are really different notions .

Let be V_K a vector space over a field K with $\dim(V_K) = n > 1$. Denote by $\langle V \rangle$ the category with the objects V^i , $i \in \mathbb{N}$ and all linear mappings as morphisms . Then the following is easily to see :

If k is a closed nonempty subset of $Obj(\langle V \rangle)$, then

$$Tot_k(A,B) = \{ f \in Hom_K(A,B) \mid \dim(Im(f)) < n \}$$

and

$$Tot_{\emptyset}(A,B) = Hom_K(A,B) .$$

Consider a fixed $q \in \mathbb{N}$ and define

$$I_q(A,B) := \{ f \in Hom_K(A,B) \mid \dim(Im(f)) \leq q \} ,$$

then this is a semi-ideal in $\langle V \rangle$.

For $q \leq n - 1$

$$I_q(A,B) \subset TOT(A,B)$$

and $K(I_q) = Obj(\langle V \rangle)$.

For $q \geq n$

$$I_q(A,B) \not\subset TOT(A,B)$$

and $K(I_q) = \emptyset$.

The radix of Kelly gives for $Hom_K(A,A)$ the partitioning in two classes : the units and the nonunits . This shows , that the radix and the total are different notions .

In an additive category , the radical can be defined ([4]) as we did for Mod-R . In $\langle V \rangle$ we have Rad = 0 . Therefore , in $\langle V \rangle$ also the radical is different from the total and the radix .