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Extreme value theory for moving average processes with light-tailed innovations

Claudia Klüppelberg [*] Alexander Lindner[*†]

Abstract

We consider stationary infinite moving average processes of the form

$$Y_n = \sum_{i=-\infty}^{\infty} c_i Z_{n+i}, \quad n \in \mathbb{Z},$$

where $(Z_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables with "light tails" and $(c_i)_{i\in\mathbb{Z}}$ is a sequence of positive and summable coefficients. By light tails we mean that Z_0 has a bounded density

$$f(t) \sim \nu(t)\exp(-\psi(t)),$$

where $\nu(t)$ behaves roughly like a constant as $t \to \infty$ and ψ is strictly convex satisfying certain asymptotic regularity conditions. We show that the iid sequence associated with Y_0 is in the maximum domain of attraction of the Gumbel distribution. Under additional regular variation conditions on ψ, it is shown that the stationary sequence $(Y_n)_{n\in\mathbb{N}}$ has the same extremal behaviour as its associated iid sequence. This generalizes results of Rootzén (1986, 1987), where $f(t) \sim ct^{\alpha}\exp(-t^p)$ for $c > 0$, $\alpha \in \mathbb{R}$ and $p > 1$.

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1 Introduction

The goal of this paper is to study extreme value theory of strictly stationary moving average processes of the form

$$Y_n = \sum_{i=-\infty}^{\infty} c_i Z_{n+i}, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $(Z_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables (rvs) with $E|Z_0| < \infty$ and $(c_i)_{i\in\mathbb{Z}}$ is a sequence of non-negative real coefficients satisfying $\sum_{i=-\infty}^{\infty} c_i < \infty$. The extremal behaviour of such processes can be classified according to the tail behaviour of the innovation sequence $(Z_i)_{i\in\mathbb{Z}}$ and the decrease of the coefficient sequence $(c_i)_{i\in\mathbb{Z}}$. Davis and Resnick (1985) investigated the extremes of such moving average processes for innovations whose distributions have regularly varying tails. In that case Y belongs to the maximum domain of attraction of the Fréchet distribution and the point processes of exceedances of $(Y_n)_{n\in\mathbb{Z}}$ converge to a compound Poisson process; i.e. extremes appear in clusters. Davis and Resnick (1988) also considered innovations in the domain of attraction of the Gumbel distribution, which are convolution equivalent. Here only the multiplicity of the maximum of the coefficients $(c_i)_{i\in\mathbb{Z}}$ determines the cluster size of the limiting compound Poisson process. A summary of results for innovations with subexponential tails can be found in Embrechts *et al.* (1997, Section 5.5). All such innovations have tails which are heavier than exponential.

A different regime was considered in Rootzén (1986, 1987), who investigated innovations whose tails are lighter than exponential. More precisely, he considered innovations with densities of the form $f(t) \sim K t^{\alpha} \exp(-t^p)$ as $t \to \infty$, with $p > 1$. Here $a(t) \sim b(t)$ as $t \to \infty$ means that the quotient of left hand side and right hand side converges to 1 as $t \to \infty$. The present paper can be seen as a generalization of Rootzén's results.

We work under the following conditions on the innovations. Let Z be a generic rv with the same distribution as Z_0. We assume that Z has a bounded probability density and that it satisfies

$$f(t) \sim \nu(t)\, \exp(-\psi(t))\,, \quad t \to \infty\,. \tag{1.2}$$

Here ψ is convex, C^2, with $\psi'' > 0$ and $\psi'(\infty) = \infty$, and the function $\phi = 1/\sqrt{\psi''}$ is *self-neglecting*, i.e.

$$\lim_{t\to\infty} \frac{\phi(t + x\phi(t))}{\phi(t)} = 1 \quad \text{uniformly on bounded } x\text{-intervals.} \tag{1.3}$$

The function ν is measurable and is *flat for* ϕ, i.e.

$$\lim_{t\to\infty} \frac{\nu(t + x\phi(t))}{\nu(t)} = 1 \quad \text{uniformly on bounded } x\text{-intervals,} \tag{1.4}$$

which guarantees that it is more or less flat on intervals of the appropriate length determined by ϕ. Such densities are closed with respect to finite convolutions, which applies to a finite moving average process; see Balkema *et al.* (1993). This is a basic property needed to analyze such light tailed linear models. As the assumptions in Balkema *et al.* (1993) are minimal, our framework is to our knowledge the most general framework possible.

Our paper is organized as follows. In Section 2 we introduce the necessary assumptions, state the main results and conclude with some examples. Assumption (A1) redefines any density (1.2) satisfying (1.3) and (1.4) such that it satisfies certain conditions, which are no restriction, but make calculations easier. Assumption (A2) allows for a generalization of results from the finite moving average to the general model (1.1). Assumption (A2) suffices already to determine the tail behaviour of Y_0 up to a certain order (Theorem 2.1) and to show that Y_0 belongs to the domain of attraction of the Gumbel distribution (Theorem 2.2). To investigate the extremal behaviour of the stationary sequence $(Y_n)_{n\in\mathbb{Z}}$, we have to impose certain regularity conditions on the function ψ. As is natural in extreme value theory we require regular variation or rapid variation of ψ, as given in Assumptions (A3) and (A4). Theorem 2.3 then shows that the extremal behaviour of the moving average process $(Y_n)_{n\in\mathbb{Z}}$ is exactly that of its associated iid sequence; i.e $(Y_n)_{n\in\mathbb{Z}}$ belongs to the domain of attraction of the Gumbel distribution with the same norming constants as the associated iid sequence.

In Section 3 we state some auxiliary results and discuss the assumptions. Section 4 is devoted to the proof of the tail behaviour and domain of attraction of Y_0 as stated in Theorems 2.1 and 2.2, while the extremal behaviour of the stationary sequence $(Y_n)_{n\in\mathbb{Z}}$ as stated in Theorem 2.3 is proved in Section 5. Applications of the results to financial time series such as stochastic volatility models or the EGARCH model are considered in Section 6. Finally, in Section 7 we give some extensions of our results, treating for example the case of positive and negative coefficients.

2 Assumptions and main results

We make the general assumptions of the Introduction more precise, introduce the necessary notation, state our main results and give some examples. Throughout the paper we shall assume the following condition (such a representation can always be found for the class of densities introduced in Section 1).

Assumption (A1): The rv Z has finite expectation and a bounded density f, which satisfies

$$f(t) = \nu(t)\,\exp(-\psi(t)), \quad t \geq t_0, \tag{2.1}$$

for some $t_0 \in \mathbb{R}$ and functions $\nu, \psi : [t_0, \infty) \to \mathbb{R}$, where ψ is C^2, $\psi'(t_0) = 0$, $\psi'(\infty) = \infty$, ψ'' is strictly positive on $[t_0, \infty)$ and $1/\sqrt{\psi''}$ is self-neglecting. The function ν is measurable and flat for $1/\sqrt{\psi''}$.

The function ψ' is continuous and strictly increasing on $[t_0, \infty)$ with range $[0, \infty)$. Therefore, for any $\tau \in [0, \infty)$ and the non-negative summable sequence $(c_i)_{i\in\mathbb{Z}}$ we can define

$$\begin{aligned} q(\tau) &:= \psi'^{\leftarrow}(\tau), \\ S^2(\tau) &:= q'(\tau) = 1/\psi''(q(\tau)), \\ q_i(\tau) &:= c_i q(c_i\tau), \\ \sigma_i^2(\tau) &:= q_i'(\tau) = c_i^2 S^2(c_i\tau), \end{aligned}$$

where $\psi'^{\leftarrow}$ denotes the inverse of ψ'. Note that $q(0) = t_0$, q is C^1 on $[t_0, \infty)$ and strictly increasing with $q(\infty) = \infty$. Furthermore, on any compact interval of the form $[t_0, s]$ for $s \in [t_0, \infty)$, $S^2 = q'$ is bounded above and bounded away from zero.

Then, by the previous considerations,

$$Q(\tau) := \sum_{i=-\infty}^{\infty} q_i(\tau) \quad \text{and} \quad \sigma_\infty^2(\tau) := \sum_{i=-\infty}^{\infty} \sigma_i^2(\tau)$$

can be defined pointwise for any $\tau \geq 0$. The sum defining σ_∞^2 converges uniformly on any compact interval $[0, s]$ $(s > 0)$, which then implies that the sum defining Q converges uniformly on compacts, and that Q is C^1 satisfying

$$Q'(\tau) = \sigma_\infty^2(\tau) = \sum_{i=-\infty}^{\infty} q_i'(\tau), \quad \tau \geq 0. \tag{2.2}$$

Furthermore, Q is strictly increasing and maps $[0, \infty)$ onto $[t_0 \sum_{i=-\infty}^{\infty} c_i, \infty)$. Set $S := \sqrt{S^2}$, $\sigma_i := \sqrt{\sigma_i^2}$, $\sigma_\infty := \sqrt{\sigma_\infty^2}$. To describe the tail behaviour of Y_0, we will need further conditions on the speed of convergence of the sum defining σ_∞^2. More precisely, we will impose:

Assumption (A2): $(c_i)_{i\in\mathbb{Z}}$ is a summable sequence of non-negative real numbers, not all zero, and the following two conditions hold:

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{|j|>m} \sigma_j^2(\tau)}{\sigma_\infty^2(\tau)} = 0, \tag{2.3}$$

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{|j|>m} \sigma_j(\tau)}{\sigma_\infty(\tau)} = 0. \tag{2.4}$$

Clearly, (A2) is satisfied if all but finitely many of the c_i are zero. Assumptions (A1) and (A2) allow us to obtain the tail behaviour of Y_0. Denote by Φ the moment generating function of Y_0, which in Lemma 4.1 will be shown to exist under (A1) and (A2). Then with the aid of Φ we can express the exact tail behaviour of Y_0, and without using Φ we obtain the tail behaviour of Y_0 up to a certain order:

Theorem 2.1. *Suppose that (A1) and (A2) hold. Then*

$$P(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)) \sim \frac{1}{\sqrt{2\pi}\tau\sigma_\infty(\tau)} e^{-\tau Q(\tau)} \Phi(\tau), \quad \tau \to \infty. \tag{2.5}$$

Furthermore, there is a function $\rho(\tau) = o(1/\sigma_\infty(\tau))$, $\tau \to \infty$, *such that*

$$P(\sum_{i=-\infty}^{\infty} c_i Z_i > t) \sim \frac{1/\sqrt{2\pi}}{Q^{\leftarrow}(t)\sigma_\infty(Q^{\leftarrow}(t))} \exp\left(-\int_{t_0 \sum c_i}^{t} (Q^{\leftarrow}(v) + \rho(Q^{\leftarrow}(v)))\, dv\right), \quad t \to \infty, \tag{2.6}$$

and $1/\sigma_\infty(\tau) = o(\tau)$, $\tau \to \infty$, *so the first term in the integral is the leading term.*

As Y_0 is light-tailed, it is no surprise that Y_0 belongs to the domain of attraction of the Gumbel distribution; we write $Y_0 \in \mathrm{MDA}(\Lambda)$. We also say that the associated iid sequence to $(Y_n)_{\in\mathbb{Z}}$ belongs to $\mathrm{MDA}(\Lambda)$; this is a sequence $(\widetilde{Y}_n)_{n\in\mathbb{Z}}$ of iid rvs all with the stationary distribution. Then $Y_0 \in \mathrm{MDA}(\Lambda)$ means that there exist norming constants $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $a_n > 0$, $b_n \in \mathbb{R}$, and

$$\lim_{n\to\infty} P(a_n(\max_{j=1,\ldots,n} \widetilde{Y}_j - b_n) \le x) = \Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R}.$$

For more details on classical extreme value theory we refer to Embrechts *et al.* (1997), Leadbetter *et al.* (1983) or Resnick (1987).

Theorem 2.2. *Assume conditions (A1) and (A2). Then*

$$\lim_{t\to\infty} \frac{P(Y_0 > t + \frac{x}{Q^{\leftarrow}(t)})}{P(Y_0 > t)} = e^{-x}, \quad x \in \mathbb{R}. \tag{2.7}$$

The iid sequence associated with $(Y_n)_{n\in\mathbb{Z}}$ *belongs to* $\mathrm{MDA}(\Lambda)$, *with norming constants* a_n *and* b_n *given by the equations*

$$\lim_{n\to\infty} nP(Y_0 > b_n) = 1 \quad \text{and} \quad a_n := Q^{\leftarrow}(b_n). \tag{2.8}$$

It does not seem to be too restrictive to impose further regular variation conditions on ψ. We shall denote the class of functions regularly varying in infinity with index β by RV_β; for definitions and results we refer to the monograph by Bingham *et al.* (1987).

Assumption (A3): Suppose that $\psi'' \in \mathrm{RV}_\beta$ for $\beta \in [-1, \infty]$. For $\beta = \infty$, which corresponds to the class of rapidly varying functions, we require additionally that ψ'' is

ultimately absolutely continuous on compacts (i.e. there is T such that ψ'' is absolutely continuous on $[T, T+x]$ for any $x > 0$) and that $\lim_{t\to\infty} \frac{d}{dt}\frac{\psi'(t)}{\psi''(t)} = 0$.
Define β' such that $1+\beta' = 1/(1+\beta)$ with the convention that the left hand side is equal to 0 for $\beta = \infty$ and equal to ∞ if $\beta = -1$.
Furthermore, suppose there exists $\theta \in [0,2)$ such that $\theta + \beta' > 0$ and $\sum_{i=-\infty}^{\infty} c_i^{1-\theta/2} < \infty$, where $(c_i)_{i\in\mathbb{Z}}$ is a sequence of non-negative real numbers, not all zero.

In Proposition 3.2 it will be shown that (A3) together with (A1) already imply (A2). Under the slightly stronger condition (A4) given below we will show that the extremal behaviour of the moving average process $(Y_n)_{n\in\mathbb{Z}}$ is the same as the extremal behaviour of its associated iid sequence: the dependence vanishes in the extremes.

Assumption (A4): Suppose that ψ, β and β' are as in (A3).
Furthermore, suppose there is some constant $\vartheta > \max\{1, 2/(2+\beta')\}$ such that $c_i = O(|i|^{-\vartheta})$, $i \to \infty$, where $(c_i)_{i\in\mathbb{Z}}$ is a sequence of non-negative real numbers, not all zero.
Finally, suppose that Z has finite variance.

Condition (A4) implies (A3): if we choose $\theta \in [0, 2-2/\vartheta)$ such that $\theta + \beta' > 0$, then (A3) follows, since $(1-\theta/2)\vartheta > 1$. The extremal behaviour of the stationary $(Y_n)_{n\in\mathbb{Z}}$ can now be described as follows:

Theorem 2.3. *Suppose that (A1) and (A4) hold. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ as given in* (2.8) *be norming constants of the iid sequence associated with Y_0. Then $(Y_n)_{n\in\mathbb{N}}$ belongs to* MDA(Λ) *with the same norming constants, i.e.*

$$\lim_{n\to\infty} P\left(a_n(\max_{j=1,\ldots,n} Y_j - b_n) \le x\right) = \exp(-e^{-x}), \quad x \in \mathbb{R}.$$

In the course of proving our results, we will use the following
Notation: For any summable sequence $(c_i)_{i\in\mathbb{Z}}$ of non-negative real numbers let i_0 be an index such that $c_{i_0} = \max\{c_i : i \in \mathbb{Z}\}$. Let c and d be strictly positive real numbers, and let $0 \le \theta < 2$. Denote by $\mathcal{G}_{c,d,\theta}$ the set of all non-negative sequences $(c_i)_{i\in\mathbb{Z}}$ such that $\sum_{i=-\infty}^{\infty} c_i \le d$, $\sum_{i=-\infty}^{\infty} c_i^{2-\theta} \le d$, $\sum_{i=-\infty}^{\infty} c_i^{1-\theta/2} \le d$, and $\frac{c}{2} \le c_{i_0} \le c$.
If in the following limits of summation are missing, then it is understood that summation is over $\mathbb{Z}$. Convergence in distribution will be denoted by $\xrightarrow{d}$, and convergence in probability by $\xrightarrow{P}$.

We conclude this section with some examples.

Example 2.4. (a) Let $\psi(t) := \frac{1}{\beta+2}t^{\beta+2}$, where $\beta \in (-1,\infty)$. Then $\psi'' \in \mathrm{RV}_\beta$ and ψ satisfies (A1) with $t_0 = 0$. An example for a flat function ν for $1/\sqrt{\psi''}$ would be any function behaving asymptotically like a rational function, or also $\nu(t) = e^t$ if $\beta > 0$. Put $\beta' := (1+\beta)^{-1} - 1$ and suppose that $c_i = O(|i|^{-\vartheta})$ for some $\vartheta > \max(1, 2/(2+\beta'))$. If Z is then such that it has finite variance and bounded density f as in (2.1), then (A1) and (A4) hold and Theorems 2.1 – 2.3 can be applied. In particular, since $Q^{\leftarrow}(t) = (t/\sum c_i^{2+\beta'})^{1+\beta}$ and $Q'(Q^{\leftarrow}(t)) = ct^{-\beta}$ for some constant c, (2.6) gives

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) = \exp\left(-(2+\beta)^{-1}(\sum c_i^{2+\beta'})^{-1-\beta}\, t^{2+\beta} + o(t^{1+\beta/2})\right), \quad t \to \infty.$$

This agrees with Theorem 6.1 in Rootzén (1987); however, focusing on this example and under an additional smoothness condition, Rootzén obtains the estimate $O(t^{(1+\beta)/\vartheta})$ for the remaining term (as $t \to \infty$), which can be seen to be slightly better than our estimate, since $\vartheta > 2/(2+\beta')$ implies $(1+\beta)/\vartheta < 1 + \beta/2$.
(*b*) Let $\psi : [1,\infty) \to (0,\infty)$ be given by $\psi(t) = t\log t - t$. Then $\psi''(t) = 1/t \in \mathrm{RV}_{-1}$ and ψ satisfies (A1) with $t_0 = 1$. Any rational function would then be flat for $1/\sqrt{\psi''}$. Let $c_i = O(|i|^{-\vartheta})$ for some $\vartheta > 1$. For simplicity, assume that $c_{i_0} = 1$, and that this maximum c_{i_0} is taken with multiplicity N. Let $c' := \max\{c_i : i \in \mathbb{Z}, c_i \neq 1\} < 1$. Assume that Z also satisfies all other properties of (A1) and (A4). Then Theorems 2.1 – 2.3 are applicable. For the tail, note that $q(\tau) = e^\tau$, $Q(\tau) = Ne^\tau + O(e^{c'\tau})$, $\tau \to \infty$, and approximate inversion shows

$$Q^{\leftarrow}(t) = \log t - \log N + O(t^{c'-1}), \quad t \to \infty.$$

Since $Q'(\tau) \sim Ne^\tau$, $\tau \to \infty$, it follows that $\sigma_\infty^{-1}(Q^{\leftarrow}(\tau)) \sim t^{-1/2}$, so that (2.6) gives

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) = \exp\left(-t\log t + t(1+\log N) + O(t^{\max\{c',1/2\}})\right), \quad t \to \infty.$$

(*c*) Examples where ψ'' is in RV_∞ and satisfies the additional condition in (A3) are $\psi(t) = e^t$ or $\psi(t) = \exp(e^t)$ for large t. If then $c_i = O(|i|^{-\vartheta})$ for some $\vartheta > 2$ and the additional conditions in (A1) and (A4) are satisfied (a flat function could be a rational function, or also $\nu(t) = e^t$), then Theorems 2.1 – 2.3 can be applied. We consider one example in more detail. Let $\psi : [0,\infty) \to (0,\infty)$ be given by $\psi(t) = et^2/2$ for $t \in [0,1]$ and $\psi(t) = e^t - e/2$ for $t > 1$. Let $\theta \in (1,2)$ such that $\sum c_i^{1-\theta/2} < \infty$. For simplicity, assume that $\sum c_i = 1$. Then $q(\tau) = \tau/e$ for $0 \le \tau \le e$ and $q(\tau) = \log \tau$ for $\tau \ge e$. This shows

$$Q(\tau) = \sum_{i=-\infty}^{\infty} c_i \log c_i + \log \tau + \sum_{i:c_i\tau<e} \left(\frac{c_i^2\tau}{e} - c_i \log(c_i\tau)\right),$$

where

$$\sum_{i:c_i\tau<e} \left(\frac{c_i^2\tau}{e} - c_i\log(c_i\tau)\right) = \tau^{-\theta/2} \sum_{i:c_i\tau<e} c_i^{1-\theta/2}\left(\frac{(c_i\tau)^{1+\theta/2}}{e} - (c_i\tau)^{\theta/2}\log(c_i\tau)\right) = o(\tau^{-\theta/2}),$$

as $\tau \to \infty$. Approximate inversion yields

$$Q^{\leftarrow}(t) = e^{t-\sum c_i \log c_i} + o(e^{t(1-\theta/2)}), \quad t \to \infty.$$

Furthermore, it holds

$$Q'(\tau) = \frac{1}{\tau}\left(\sum_{i:c_i\tau\geq e} c_i + \sum_{c_i\tau<e} (c_i\tau)\frac{c_i}{e}\right) \sim \frac{1}{\tau}, \quad \tau \to \infty,$$

so that $\sigma_\infty^{-1}(Q^{\leftarrow}(t)) = O(e^{t/2})$, $t \to \infty$. An application of (2.6) then shows

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) = \exp\left(-e^{t-\sum c_i \log c_i} + O(e^{t/2})\right), \quad t \to \infty.$$

3 Auxiliary results

3.1 Exponential families

A basic role in our proofs will be played by exponential families. Let X be a rv whose moment generating function $Ee^{\tau X}$ exists for all $\tau \in [0, \infty)$. Then the *exponential family* $(\overline{X}_\tau)_{\tau\geq 0}$ is defined to be a family of rvs such that

$$F_{\overline{X}_\tau}(dz) = \frac{e^{\tau z} F_X(dz)}{Ee^{\tau X}}, \quad \tau \geq 0,$$

where F_X and $F_{\overline{X}_\tau}$ denote the distribution function of X and $\overline{X}_\tau$, respectively. Exponential families have the following useful properties, which follow by standard calculations; see e.g. Rootzén (1987, Section 3):

$$P(X \in A) = E(e^{-\tau \overline{X}_\tau} 1_{\overline{X}_\tau \in A}) Ee^{\tau X}, \quad \tau \geq 0, \quad A \text{ a Borel set}, \tag{3.1}$$

$$\overline{(cX)}_\tau \stackrel{d}{=} c\overline{X}_{c\tau}, \quad c, \tau \geq 0. \tag{3.2}$$

We will consider the exponential families of the random variables $X_i := c_i Z_i$. Denote by Φ_i the moment generating function of X_i, which by (A1) exists and is finite for all $\tau \geq 0$, as shown in Balkema *et al.* (1993, Prop. 5.11). Denote the density of X_i by f_i, and the exponential family associated with X_i by $(\overline{X}_{i,\tau})_{\tau\geq 0}$. Assume throughout that the exponential families are taken such that $(\overline{X}_{i,\tau})_{i\in\mathbb{Z}}$ are mutually independent for any $\tau \geq 0$. The exponential family associated with the generic rv Z will be denoted by $(\overline{Z}_\tau)_{\tau\geq 0}$. In Lemma 4.1 it will be shown that the moment generating function Φ of $\sum X_i$ exists and is finite for every argument $\tau \geq 0$, and that $\sum_{i=-\infty}^{\infty} \overline{X}_{i,\tau}$ converges almost surely for any $\tau \geq 0$. In particular, the exponential family of $\sum X_i$ exists, and since taking exponential families commutes with taking convolution (see e.g. Rootzén 1987, equation (3.4)), this exponential family is given by $(\sum_{i=-\infty}^{\infty} \overline{X}_{i,\tau})_{\tau\geq 0}$.

3.2 ANET convergence

A family $(W_\tau)_{\tau\ge 0}$ of rvs with densities w_τ is called *asymptotically normal with exponential tails* (ANET), if $w_\tau(x)$ converges locally uniformly in x to the density $\varphi(x)=e^{-x^2/2}/\sqrt{2\pi}$ of the standard normal distribution as $\tau\to\infty$, and if for any $\varepsilon>0$ there exist τ_ε and a constant $M_\varepsilon>1$, such that

$$w_\tau(x)\le e^{-|x|/\varepsilon}\quad \forall\,|x|\ge M_\varepsilon,\tau\ge\tau_\varepsilon.$$

If a sequence is ANET, it is known that the moment generating functions and the (absolute) moments of all orders converge to the corresponding moment generating function and (absolute) moments of the standard normal distribution, and that W_τ converges in distribution to $N(0,1)$, see Balkema *et al.* (1993, Proposition 6.3).

In Balkema *et al.* (1993, Theorem 6.6) it is shown that under the assumption (A1), a suitable centering and normalization transforms the exponential family associated with Z into an ANET sequence. More precisely, the sequence $\big((\overline{Z}_\tau-q(\tau))/S(\tau)\big)_{\tau\ge0}$ is ANET. Since the set of random variables satisfying (A1) is closed under finite convolution, as shown in Balkema *et al.* (1993, Theorem 1.1), it follows that for any $m\in\mathbb{N}_0$ such that at least one of the c_i for $|i|\le m$ is non-zero, the exponential family associated with $\sum_{i=-m}^m X_i$ can be transformed into an ANET sequence. More precisely, the sequence $\left(\sum_{i=-m}^m(\overline{X}_{i,\tau}-q_i(\tau))/\sqrt{\sum_{i=-m}^m\sigma_i^2(\tau)}\right)_{\tau\ge0}$ is ANET, see Balkema *et al.* (1993, p. 586). See also Barndorff-Nielsen and Klüppelberg (1992) for further calculations.

3.3 Discussion of the assumptions

Recall that a function $g:[0,\infty)\to\mathbb{R}$ is in RV_β ($\beta\in\mathbb{R}$) if and only if there are constants $a,c>0$, a measurable function $c(\cdot)$ and a locally Lebesgue integrable function ε on $[a,\infty)$ such that $\lim_{x\to\infty}c(x)=c$, $\lim_{x\to\infty}\varepsilon(x)=0$, and

$$g(x)=x^\beta\,c(x)\,\exp\left(\int_a^x\frac{\varepsilon(u)}{u}\,du\right)\,,\quad x\ge a. \tag{3.3}$$

If the function $c(\cdot)$ in (3.3) can be taken as a constant, then g is said to be *normalized regularly varying with index* β; we write $g\in\mathrm{NRV}_\beta$.

The following lemma clarifies condition (A3). In particular, $\lim_{t\to\infty}\frac{d}{dt}\frac{\psi'(t)}{\psi''(t)}=0$ means nothing else than $q'\in\mathrm{NRV}_{-1}$, which already implies that $\psi''\in\mathrm{RV}_\infty$.

Lemma 3.1. *Suppose that* $\psi:[t_0,\infty)\to\mathbb{R}$ *is* C^2, $\psi'(\infty)=\infty$ *and that* $\psi''>0$. *Let* $q=\psi'^{\leftarrow}$, *and for* $\beta\in[-1,\infty]$ *define* β' *through* $1+\beta'=(1+\beta)^{-1}$.
(*a*) *For all* $\beta\in[-1,\infty]$ *we have* $\psi'\in\mathrm{RV}_{1+\beta}$ *if and only if* $q\in\mathrm{RV}_{1+\beta'}$.
(*b*) *If* $\psi''\in\mathrm{RV}_\beta$ *where* $\beta\in\mathbb{R}$, *then* $\beta\ge-1$, $\psi'\in\mathrm{RV}_{1+\beta}$, $1/\sqrt{\psi''}$ *is self-neglecting, and*

$q' \in \mathrm{RV}_{\beta'}$. *If* $\beta \in (-1, \infty)$, *then* $\psi'' \in \mathrm{RV}_\beta$ *if and only if* $q' \in \mathrm{RV}_{\beta'}$.
(c) Let $\beta' \in [-1, \infty)$. *Then* ψ'' *is ultimately absolutely continuous on compacts and satisfies* $\lim_{t\to\infty} \frac{d}{dt} \frac{\psi'(t)}{\psi''(t)} = 1 + \beta'$ *if and only if* $q' \in \mathrm{NRV}_{\beta'}$.
(d) If $q' \in \mathrm{RV}_{-1}$, *then* $1/\sqrt{\psi''}$ *is self-neglecting and* $\psi'' \in \mathrm{RV}_\infty$.
(e) $1/\sqrt{\psi''}$ *is self-neglecting if and only if* $1/\sqrt{q'}$ *is self-neglecting.*

Proof. (a) This follows from Proposition 1.5.15 and Theorem 2.4.7 of Bingham *et al.* (1987).

(b) Since $\psi'(\infty) = \infty$ and $\psi'' \in \mathrm{RV}_\beta$, it follows from l'Hospital's rule that $\psi' \in \mathrm{RV}_{1+\beta}$ and further that $1 + \beta \geq 0$. Since $q'(\tau) = 1/\psi''(q(\tau))$, by composition it follows that $q' \in \mathrm{RV}_{\beta'}$ if $\beta \neq -1$, and the converse follows similarly. If $\beta = -1$, then $\psi' \in \mathrm{RV}_0$, hence $q \in \mathrm{RV}_\infty$. By the monotone equivalence theorem (Bingham *et al.* 1987, Theorem 1.5.3), ψ'' is asymptotically equivalent to a decreasing function h, say. Then if $c \in (0,1)$, for any $\varepsilon > 0$ there exists τ_ε such that $q(c\tau) < \varepsilon q(\tau)$ for $\tau \geq \tau_\varepsilon$, since $q \in \mathrm{RV}_\infty$. This then implies

$$\frac{q'(c\tau)}{q'(\tau)} \sim \frac{h(q(\tau))}{h(q(c\tau))} \leq \frac{h(q(\tau))}{h(\varepsilon q(\tau))} \to \varepsilon, \quad \tau \to \infty,$$

showing that $q' \in \mathrm{RV}_\infty$. To show that $1/\sqrt{\psi''}$ is self-neglecting note that

$$\lim_{t\to\infty} \frac{t + x/\sqrt{\psi''(t)}}{t} = 1 + \lim_{t\to\infty} \frac{x}{t\sqrt{\psi''(t)}} = 1$$

uniformly in $x \in \mathbb{R}$, since $t \mapsto t\sqrt{\psi''(t)}$ is in $\mathrm{RV}_{1+\beta/2}$.

(c) Note that ψ'' is ultimately absolutely continuous on compacts and satisfies the relation $\lim_{t\to\infty} \frac{d}{dt} \frac{\psi'(t)}{\psi''(t)} = 1 + \beta'$ if and only if q' is ultimately absolutely continuous on compacts and satisfies

$$\lim_{\tau\to\infty} \frac{\tau q''(\tau)}{q'(\tau)} = \lim_{\tau\to\infty} \frac{-\psi'(q(\tau))\psi'''(q(\tau))}{\psi''(q(\tau))^2} = \lim_{t\to\infty} \frac{-\psi'(t)\psi'''(t)}{\psi''(t)^2} = \beta'.$$

But this is equivalent to q' being ultimately absolutely continuous on compacts and satisfying

$$\lim_{\tau\to\infty} \frac{\tau \frac{d}{d\tau}\left(\tau^{-\beta'} q'(\tau)\right)}{\tau^{-\beta'} q'(\tau)} = 0,$$

which is equivalent to $q' \in \mathrm{NRV}_{-1}$, see Bingham *et al.* (1987, p. 15).

The proof of (d) is similar to the proof of (b), using (e) to show that $1/\sqrt{\psi''}$ is self-neglecting.

(e) itself is proved in Balkema *et al.* (1993, Theorem 5.3). ☐

Next we show that (A1) and (A3) already imply (A2):

Proposition 3.2. *Suppose the assumptions (A1) and (A3) are satisfied. Then (A2) holds. Furthermore, there exists a positive constant D, depending only on ψ and on θ, such that for every constant c bounding $(c_i)_{i\in\mathbb{Z}}$ from above, it holds*

$$\sigma_\infty^2(\tau) \le D \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c}\right)^{2-\theta} c^2 q'(c\tau)\,, \quad \tau \ge 0. \tag{3.4}$$

Proof. Note that $q' \in \mathrm{RV}_{\beta'}$ by Lemma 3.1. Define $p_1(\tau) := \tau^\theta q'(\tau)$ for $\tau \ge 0$. Then there exists an increasing function $p_2 : [0,\infty) \to \mathbb{R}$ such that $p_1(\tau) \le p_2(\tau)$ for any $\tau \ge 0$, and $p_1(\tau) \sim p_2(\tau)$ as $\tau \to \infty$. For $\beta' \ne \infty$, this follows from the monotone equivalence theorem (Bingham *et al.* 1987, Theorem 1.5.3), and for $\beta' = \infty$ from $q'(\tau) = 1/\psi''(q(\tau))$, the monotonicity of q and an application of the monotone equivalence theorem to $1/\psi'' \in \mathrm{RV}_1$. We conclude that there exists a positive constant d_1 such that $p_2(\tau) \le d_1 p_1(\tau)$ for all $\tau \ge 1$. Let $c \ge \max\{c_i : i \in \mathbb{Z}\}$. Then if $c\tau \ge 1$, we have

$$p_1(c_i\tau) \le p_2(c_i\tau) \le p_2(c\tau) \le d_1 p_1(c\tau).$$

Since q' is continuous and strictly positive on $[0,1]$, there exists some $d_2 > 0$ such that $q'(x) \le d_2 q'(y)$ for every $x, y \in [0,1]$. In particular, for $c\tau \le 1$, $q'(c_i\tau) \le d_2 q'(c\tau)$. Then, with $D := \max(d_1, d_2)$, it follows

$$c_i^\theta q'(c_i\tau) \le D c^\theta q'(c\tau)\,, \quad \tau \ge 0, \tag{3.5}$$

giving (3.4). Since $\sum c_i^{1-\theta/2} < \infty$, it follows from (3.5), the dominated convergence theorem and the fact that $p_1 \in \mathrm{RV}_{\beta'+\theta}$, that

$$\lim_{\tau\to\infty} \frac{\sum_{i=-\infty}^{\infty} c_i \sqrt{q'(c_i\tau)}}{c\sqrt{q'(c\tau)}} = \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c}\right)^{1-\theta/2} \lim_{\tau\to\infty} \sqrt{\frac{c_i^\theta \tau^\theta q'(c_i\tau)}{c^\theta \tau^\theta q'(c\tau)}} = \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c}\right)^{1+\beta'/2},$$

where the right hand side has to be interpreted as $\mathrm{card}\{i : c_i = c\}$ if $\beta' = \infty$. Similarly, for any $m > 0$,

$$\lim_{\tau\to\infty} \frac{\sum_{|i|>m} c_i \sqrt{q'(c_i\tau)}}{c\sqrt{q'(c\tau)}} = \sum_{|i|>m} \left(\frac{c_i}{c}\right)^{1+\beta'/2},$$

and (2.4) follows. The limit relation (2.3) follows similarly. □

Remark 3.3. The proof shows that the condition $\lim_{t\to\infty} \frac{d}{dt} \frac{\psi'(t)}{\psi''(t)} = 0$ (for the case $\psi'' \in \mathrm{RV}_\infty$), which by Lemma 3.1 is equivalent to $q' \in \mathrm{NRV}_{-1}$, can be slightly relaxed to $q' \in \mathrm{RV}_{-1}$, and (A2) still follows.

There are also many examples when (A1) and (A2) hold, but (A3) does not:

Example 3.4. Let $\psi : [0,\infty) \to (0,\infty)$ such that $\psi'(0) = 0$ and $\psi''(t) = (2+\cos(\pi\sqrt{t}))^{-2}$. Then the derivative of $1/\sqrt{\psi''(t)}$ tends to 0 as $t \to \infty$, and the mean value theorem implies that $1/\sqrt{\psi''}$ is self-neglecting. A flat function ν would be any rational function or $\nu(t) = \exp(t^\alpha)$ for $\alpha \in [0,1)$. If then Z has finite expectation and bounded density f satisfying (2.1), then (A1) holds. If furthermore $(c_i)_{i\in\mathbb{Z}}$ is a summable sequence of non-negative numbers, then it is easy to see that (A2) holds, too. Note, however, that (A3) is not satisfied for this example.

4 Proof of Theorems 2.1 and 2.2

In this section we shall prove the tail behaviour of Y_0 as stated in Theorem 2.1 and then use this result to prove Theorem 2.2, i.e. that the associated iid sequence is in MDA(Λ). The proofs will be split up into several lemmas, and exponential families will play an important role. We will also give some uniform estimates under the extra assumption (A3) and for coefficient sequences in $\mathcal{G}_{c,d,\theta}$. These will be used in Section 5 when proving Theorem 2.3. Recall the notations of Section 3.1.

Lemma 4.1. *Under the assumptions (A1) and (A2), the moment generating function* Φ *of* $\sum X_i = \sum c_i Z_i$ *exists and is finite for all* $\tau \geq 0$*, and it holds*

$$\Phi(\tau) = \prod_{i=-\infty}^{\infty} \Phi_i(\tau), \quad \tau \geq 0,$$

as well as

$$\frac{d}{d\tau}\log\Phi(\tau) = \sum_{i=-\infty}^{\infty} \frac{d}{d\tau}\log\Phi_i(\tau) = \sum_{i=-\infty}^{\infty} E\overline{X}_{i,\tau}, \quad \tau \geq 0, \tag{4.1}$$

where the sum and the product converge uniformly on compact subsets of $[0,\infty)$*. The exponential family associated with* $\sum X_i$ *is* $(\sum_{i=-\infty}^{\infty} \overline{X}_{i,\tau})_{\tau\geq 0}$*, where the sum converges a.s. absolutely.*

Proof. By the definition of the exponential family,

$$E\overline{X}_{i,\tau} = \frac{EX_i e^{\tau X_i}}{\Phi_i(\tau)} = \frac{\int_{-\infty}^{\infty} f_i(t)\, t\, e^{\tau t}\, dt}{\Phi_i(\tau)} = \frac{\frac{d}{d\tau}\Phi_i(\tau)}{\Phi_i(\tau)} = \frac{d}{d\tau}\log\Phi_i(\tau),$$

where we used the differentiation lemma for the third equality. Furthermore, we see (since $E|X_i| < \infty$) that $[0,\infty) \to \mathbb{R}$, $\tau \mapsto E|\overline{X}_{i,\tau}|$ is continuous. Since $(\overline{Z}_\tau - q(\tau))/S(\tau))_{\tau\geq 0}$ is ANET as noted in Section 3.2, it follows that the absolute moment $E|(\overline{Z}_\tau - q(\tau))/S(\tau)|$ converges to the absolute moment of $N(0,1)$ as $\tau \to \infty$. Furthermore, $q(\tau)$, $1/S(\tau)$ and

$E|\overline{Z}_\tau|$ are bounded on compact subintervals of $[0,\infty)$. This shows that there is a constant C, such that $E|\overline{Z}_\tau - q(\tau)| \le CS(\tau)$ for all $\tau \ge 0$. Using (3.2), this implies that

$$E|\overline{X}_{i,\tau} - q_i(\tau)| \le C\sigma_i(\tau) \quad \forall \tau \ge 0 \quad \forall i \in \mathbb{Z}. \tag{4.2}$$

In particular, it follows that for any $s > 0$,

$$\sup_{0\le\tau\le s} E|\overline{X}_{i,\tau}| \le C \sup_{0\le\tau\le s} \sigma_i(\tau) + \sup_{0\le\tau\le s} |q_i(\tau)|,$$

implying absolute and uniform convergence on compacts of $\sum_{i=-\infty}^\infty E\overline{X}_{i,\tau}$. The convergence of $\sum_{i=-\infty}^\infty E|\overline{X}_{i,\tau}|$ gives almost sure convergence of $\sum_{i=-\infty}^\infty \overline{X}_{i,\tau}$. Note that uniform convergence on compacts of $\sum \frac{d}{d\tau}\log\Phi_i(\tau)$ implies uniform convergence on compacts of $\sum \log\Phi_i(\tau)$ and hence of $\prod_{i=-\infty}^\infty \Phi_i(\tau)$. That the limit is in fact $\Phi(\tau)$ follows from the dominated convergence theorem. For application of the latter, construct a random variable $\widetilde{Z}$ such that $\widetilde{Z} = Z$ if $Z \ge 0$, and $\widetilde{Z} \in [0,1]$ if $Z < 0$, and such that $\widetilde{Z}$ has a bounded density. Then if $(\widetilde{Z}_i)_{i\in\mathbb{Z}}$ is an iid sequence with distribution $\widetilde{Z}$, then the same calculations as before show that $\prod_{i=-\infty}^\infty e^{c_i\widetilde{Z}_i}$ is an integrable majorant. That the exponential family associated with $\sum X_i$ is indeed $(\sum_{i=-\infty}^\infty \overline{X}_{i,\tau})_{\tau\ge0}$ was already noted in Section 3.1. □

Lemma 4.2. *Under the assumptions (A1) and (A2),*

$$\frac{1}{\sigma_\infty(\tau)} \sum_{i=-\infty}^{\infty} (\overline{X}_{i,\tau} - q_i(\tau)) \quad \xrightarrow{d} \quad N(0,1), \quad \tau\to\infty\,. \tag{4.3}$$

Proof. For $\tau \ge 0$ and $m \in \mathbb{N}$ such that not all of the $(c_i)_{|i|\le m}$ are zero define

$$\begin{aligned} A_{m\tau} &:= \sum_{i=-m}^{m} (\overline{X}_{i,\tau} - q_i(\tau)) \left(\frac{1}{\sigma_\infty(\tau)} - \frac{1}{\left(\sum_{j=-m}^m \sigma_j^2(\tau)\right)^{1/2}} \right) \\ B_{m\tau} &:= \frac{\sum_{|i|>m}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)}. \end{aligned}$$

Then

$$\frac{\sum_{i=-\infty}^\infty(\overline{X}_{i,\tau} - q_i(\tau))}{\left(\sum_{i=-\infty}^\infty \sigma_i^2(\tau)\right)^{1/2}} - \frac{\sum_{i=-m}^m(\overline{X}_{i,\tau} - q_i(\tau))}{\left(\sum_{i=-m}^m \sigma_i^2(\tau)\right)^{1/2}} = A_{m\tau} + B_{m\tau}.$$

By the ANET property,

$$\frac{\sum_{|i|\le m}(\overline{X}_{i,\tau} - q_i(\tau))}{\left(\sum_{|i|\le m} \sigma_i^2(\tau)\right)^{1/2}} \quad \xrightarrow{d} \quad N(0,1), \quad \tau\to\infty\,.$$

Then (4.3) follows from a variant of Slutsky's Theorem (see Billingsley 1999, Theorem 3.2), provided that for any $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} P(|A_{m\tau}| > \varepsilon) = 0 = \lim_{m\to\infty} \limsup_{\tau\to\infty} P(|B_{m\tau}| > \varepsilon). \tag{4.4}$$

To show (4.4), write

$$A_{m\tau} = \frac{\sum_{i=-m}^{m}(\overline{X}_{i,\tau} - q_i(\tau))}{\left(\sum_{i=-m}^{m}\sigma_i^2(\tau)\right)^{1/2}} \left(\left(\frac{\sum_{j=-m}^{m}\sigma_j^2(\tau))}{\sigma_\infty^2(\tau))} \right)^{1/2} - 1 \right).$$

Since $\lim_{\tau\to\infty} E|\sum_{i=-m}^{m}(\overline{X}_{i,\tau} - q_i(\tau))/(\sum_{i=-m}^{m}\sigma_i^2(\tau))^{1/2}| = \sqrt{2/\pi}$, it follows from (2.3) that

$$\limsup_{m\to\infty}\limsup_{\tau\to\infty} E(|A_{m\tau}|) \le \sqrt{\frac{2}{\pi}} \limsup_{m\to\infty}\limsup_{\tau\to\infty} \left(1 - \left(\frac{\sum_{j=-m}^{m}\sigma_j^2(\tau))}{\sigma_\infty^2(\tau))} \right)^{1/2} \right) = 0,$$

implying the left-hand equality of (4.4) by Markov's inequality. The right-hand side of (4.4) follows similarly from (2.4), noting that

$$E|B_{m\tau}| \le \frac{\sum_{|i|>m} E|\overline{X}_{i,\tau} - q_i(\tau)|}{\sigma_\infty(\tau)} \le \frac{C\sum_{|i|>m}\sigma_i(\tau)}{\sigma_\infty(\tau)}$$

by (4.2). □

Lemma 4.3. (*a*) *Assume conditions (A1) and (A2). Then* $\sigma_\infty(\tau)^{-1}\sum(\overline{X}_{i,\tau} - q_i(\tau))$ *has a density, denoted by* $r_\tau(x)$*, which converges locally uniformly to the density* $\varphi(x)$ *of the standard normal distribution, as* $\tau \to \infty$*. Furthermore, the densities* r_τ *are uniformly bounded by the same constant for sufficiently large* τ*.*
(*b*) *Suppose that (A1) holds and that* ψ *and* θ *are as in (A3). Let* c, d *be positive constants. Then there are positive constants* τ_0*,* D_0*, such that for any coefficient sequence in* $\mathcal{G}_{c,d,\theta}$ *the density* r_τ *is bounded by* D_0 *for any* $\tau \ge \tau_0$*.*

Proof. (a) By (2.3), there is some $m \in \mathbb{N}_0$ such that

$$\frac{1}{2} \le \frac{1}{\sigma_\infty(\tau)}\sqrt{\sum_{|i|\le m}\sigma_i^2(\tau)} \le 1 \quad \text{for large } \tau. \tag{4.5}$$

Denote by g_τ the density of $\sum_{|i|\le m}(\overline{X}_{i,\tau} - q_i(\tau))/\sqrt{\sum_{|i|\le m}\sigma_i^2(\tau)}$. By the ANET-property, $g_\tau(x)$ converges locally uniformly to $\varphi(x)$ as $\tau \to \infty$, and $|g_\tau(x)| \le e^{-|x|}$ for large x and τ. This implies that for any $\varepsilon > 0$ there are $\delta_{1,\varepsilon} > 0$ and $\tau_{1,\varepsilon}$ such that

$$|g_\tau(x) - g_\tau(y)| \le \varepsilon \quad \forall\,\tau \ge \tau_{1,\varepsilon}\ \forall\, x, y \in \mathbb{R} : |x-y| \le \delta_{1,\varepsilon}.$$

The density of $\sum_{|i|\le m}(\overline{X}_{i,\tau} - q_i(\tau))/\sigma_\infty(\tau)$ is given by

$$x \mapsto g_\tau\left(\frac{\sigma_\infty(\tau)}{\sqrt{\sum_{|i|\le m}\sigma_i^2(\tau)}}\, x \right) \frac{\sigma_\infty(\tau)}{\sqrt{\sum_{|i|\le m}\sigma_i^2(\tau)}} =: h_\tau(x).$$

By (4.5) there are $\delta_{2,\varepsilon} > 0$ and $\tau_{2,\varepsilon}$ such that

$$|h_\tau(x) - h_\tau(y)| \leq \varepsilon \quad \forall\, \tau \geq \tau_{2,\varepsilon}\; \forall\, x, y \in \mathbb{R} : |x - y| \leq \delta_{2,\varepsilon}.$$

Denote by H_τ the distribution function of $\sum_{|i|>m}(\overline{X}_{i,\tau} - q_i(\tau))/\sigma_\infty(\tau)$. Then

$$\frac{\sum_{i=-\infty}^{\infty}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)} = \frac{\sum_{|i|\leq m}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)} + \frac{\sum_{|i|>m}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)}$$

has a density, say $r_\tau(x)$ (since the first summand has a density), which satisfies

$$|r_\tau(x) - r_\tau(y)| = \left| \int_{-\infty}^{\infty} (h_\tau(x - t) - h_\tau(y - t))\, dH_\tau(t) \right| \leq \int_{-\infty}^{\infty} \varepsilon\, dH_\tau(t) = \varepsilon \qquad (4.6)$$

for all $\tau \geq \tau_{2,\varepsilon}$ and $x, y \in \mathbb{R}$ such that $|x - y| \leq \delta_{2,\varepsilon}$. Similarly, one obtains that the r_τ are uniformly bounded for large τ. Now assume that $r_\tau(x)$ does not converge to $\varphi(x)$ as $\tau \to \infty$ for all $x \in \mathbb{R}$. Without loss of generality assume that

$$\varphi(x_0) + 3\varepsilon \leq \limsup_{\tau\to\infty} r_\tau(x_0)$$

in some x_0 and for sufficiently small $\varepsilon > 0$. Then there is a subsequence $(\tau_n)_{n\in\mathbb{N}}$ tending to ∞ such that $\lim_{n\to\infty} r_{\tau_n}(x_0) = \limsup_{\tau\to\infty} r_\tau(x_0)$. By (4.6) this implies that there is some $\delta > 0$ such that for sufficiently large n,

$$r_{\tau_n}(y) \geq \varphi(y) + \varepsilon \quad \forall\, y \in [x_0 - \delta, x_0 + \delta].$$

It follows

$$\lim_{n\to\infty} \int_{x_0-\delta}^{x_0+\delta} r_{\tau_n}(y)\, dy \geq \int_{x_0-\delta}^{x_0+\delta} (\varphi(y) + \varepsilon)\, dy,$$

contradicting Lemma 4.2. This shows that $r_\tau(x)$ converges to $\varphi(x)$ in any $x \in \mathbb{R}$ as $\tau \to \infty$, and by (4.6) we see that this convergence is locally uniform.

(b) By Proposition 3.2, there is a constant $D_1 > 0$ such that for any $(c_i)_{i\in\mathbb{Z}} \in \mathcal{G}_{c,d,\theta}$, $D_1 \leq \sigma_{i_0}(\tau)/\sigma_\infty(\tau) \leq 1$ for $\tau \geq 0$. Denote by g_τ the density of $(\overline{X}_{i_0,\tau} - q_{i_0}(\tau))/\sigma_{i_0}(\tau) \overset{d}{=} (\overline{Z}_{c_{i_0}\tau} - q(c_{i_0}\tau))/S(c_{i_0}\tau)$. Since $c/2 \leq c_{i_0}$, it follows from the ANET property of $((\overline{Z}_\tau - q(\tau))/S(\tau))_{\tau\geq 0}$ that there are τ_0, D_2, depending only on f, ψ and c, such that g_τ is bounded by D_2 for $\tau \geq \tau_0$. The density h_τ of $(\overline{X}_{i_0,\tau} - q_{i_0}(\tau))/\sigma_\infty(\tau)$ is then bounded by $D_0 := D_2/D_1$ for $\tau \geq \tau_0$. Similarly to (4.6), this then implies that r_τ is bounded by D_0 for $\tau \geq \tau_0$. $\square$

We are now able to prove the first part of Theorem 2.1:

Proof of (2.5) *in Theorem 2.1(a).* Using (3.1) it follows

$$\begin{aligned}
&P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)\right)\\
&\quad = E\left(e^{-\tau\sum \overline{X}_{i,\tau}} 1_{\sum \overline{X}_{i,\tau}>Q(\tau)}\right)\,\Phi(\tau)\\
&\quad = E\left(e^{-\tau\sigma_\infty(\tau)\sum(\overline{X}_{i,\tau}-q_i(\tau))/\sigma_\infty(\tau)}\, 1_{\sum(\overline{X}_{i,\tau}-q_i(\tau))/\sigma_\infty(\tau)>0}\right)\, e^{-\tau Q(\tau)}\Phi(\tau)\\
&\quad = e^{-\tau Q(\tau)}\Phi(\tau)\int_0^\infty e^{-\tau\sigma_\infty(\tau)x} r_\tau(x)\,dx.
\end{aligned}$$

Noting that

$$\lim_{\tau\to\infty}\tau^2 q'(\tau) = \lim_{\tau\to\infty}\frac{\tau^2}{\psi''((\psi')^{\leftarrow}(\tau))} = \lim_{t\to\infty}\frac{\psi'(t)^2}{\psi''(t)},$$

where the last limit was shown to equal ∞ in Balkema *et al.* (1993, Proposition 5.8), it follows

$$\lim_{\tau\to\infty}\tau\sigma_\infty(\tau) = \infty. \tag{4.7}$$

Then using dominated convergence and Lemma 4.3(a) gives

$$\begin{aligned}
&\tau\sigma_\infty(\tau)\int_0^\infty e^{-\tau\sigma_\infty(\tau)x} r_\tau(x)\,dx\\
&\quad = \int_0^\infty e^{-z}\, r_\tau(z/(\tau\sigma_\infty(\tau))\,dz\\
&\quad \to \int_0^\infty e^{-z}\,\frac{1}{\sqrt{2\pi}}\,dz \;=\; \frac{1}{\sqrt{2\pi}},\quad \tau\to\infty,
\end{aligned}$$

implying (2.5). □

With exactly the same proof, but now using Lemma 4.3(b) instead of (a), we get the following uniform estimate, which will be used in Lemma 4.6:

Lemma 4.4. *Suppose that (A1) holds and that ψ and θ are as in (A3). Let c, d be positive constants. Then there are positive constants τ_0, D_0, such that for any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$,*

$$P(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)) \le \frac{D_0}{\tau\sigma_\infty(\tau)}\, e^{-\tau Q(\tau)}\Phi(\tau)\,,\quad \tau\ge\tau_0. \tag{4.8}$$

In order to derive the approximation for the tail behaviour of Y_0 as stated in Theorem 2.1(b), we need estimates for Φ, which are derived in the following lemma:

Lemma 4.5. (*a*) *Suppose that (A1) and (A2) hold. Then for $\tau\ge 0$,*

$$\frac{d}{d\tau}\log\left(e^{-\tau Q(\tau)}\Phi(\tau)\right) = -\tau\sigma_\infty^2(\tau) + \sum_{i=-\infty}^{\infty}(E\overline{X}_{i,\tau} - q_i(\tau)) = -\tau\sigma_\infty^2(\tau) + o(\sigma_\infty(\tau)),\ \tau\to\infty.$$

(*b*) *Suppose that (A1) holds and that* ψ *and* θ *are as in (A3). Let* c, d *be positive constants. Then there exists a positive constant* D*, such that for any coefficient sequence* $(c_i)_{i\in\mathbb{Z}}$ *in* $\mathcal{G}_{c,d,\theta}$ *it holds*

$$\sum_{i=-\infty}^{\infty} |E\overline{X}_{i,\tau} - q_i(\tau)| \le D\sigma_\infty(\tau)\,, \quad \tau \ge 0. \tag{4.9}$$

Proof. (a) From Lemma 4.1 and (2.2) follows that for any $\tau \ge 0$,

$$\frac{d}{d\tau}(-\tau Q(\tau)+\log\Phi(\tau)) = -\tau Q'(\tau)-Q(\tau)+\sum_{i=-\infty}^{\infty} E\overline{X}_{i,\tau} = -\tau\sigma^2_\infty(\tau)+\sum_{i=-\infty}^{\infty}(E\overline{X}_{i,\tau}-q_i(\tau)).$$

Let $\varepsilon > 0$. By (4.2) and (2.4), there exists an $m_\varepsilon \in \mathbb{N}$ such that

$$\limsup_{\tau\to\infty} E\sum_{|i|>m_\varepsilon}\left|\frac{\overline{X}_{i,\tau}-q_i(\tau)}{\sigma_\infty(\tau)}\right| \le \varepsilon.$$

Furthermore, from the ANET–property of $\sum_{i=-m_\varepsilon}^{m_\varepsilon}(\overline{X}_{i,\tau}-q_i(\tau))/\sqrt{\sum_{i=-m_\varepsilon}^{m_\varepsilon}\sigma_i^2(\tau)}$ follows

$$\limsup_{\tau\to\infty}\left|\frac{E\sum_{i=-m_\varepsilon}^{m_\varepsilon}(\overline{X}_{i,\tau}-q_i(\tau))}{\sigma_\infty(\tau)}\right| \le \limsup_{\tau\to\infty}\left|\frac{E\sum_{i=-m_\varepsilon}^{m_\varepsilon}(\overline{X}_{i,\tau}-q_i(\tau))}{\sqrt{\sum_{i=-m_\varepsilon}^{m_\varepsilon}\sigma_i^2(\tau)}}\right| = 0.$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows.

(b) From (4.2) follows that there is a positive constant C, depending only on the density f and ψ, such that $|E\overline{X}_{i,\tau} - q_i(\tau)| \le C\sigma_i(\tau)$ for $\tau \ge 0$. By (3.5), there exists a constant C_1, depending only on ψ and θ, such that for any coefficient sequence in $\mathcal{G}_{c,d,\theta}$,

$$\sum_{i=-\infty}^{\infty}\sigma_i(\tau) \le \sqrt{C_1}\sum_{i=-\infty}^{\infty} c_i^{1-\theta/2}c_{i_0}^{\theta/2-1}c_{i_0}\sqrt{q'(c_{i_0}\tau)} \le \sqrt{C_1}\,d\,(c/2)^{\theta/2-1}\sigma_{i_0}(\tau), \quad \tau\ge 0,$$

giving (4.9). □

Now we are able to complete the proof of Theorem 2.1:

Proof of (2.6) *in Theorem 2.1.* By (2.5) and Lemma 4.5(a), there is a function $\zeta(\tau) = o(\sigma_\infty(\tau))$, $\tau\to\infty$, such that

$$P\left(\sum_{i=-\infty}^{\infty} c_iZ_i > Q(\tau)\right) \sim \frac{1}{2\pi\tau\sigma_\infty(\tau)}\exp\left(-\int_0^\tau (uQ'(u)+\zeta(u))\,du\right), \quad \tau\to\infty. \tag{4.10}$$

Setting $t = Q(\tau)$ and $\rho(\tau) := \zeta(\tau)/\sigma^2_\infty(\tau) = o(1/\sigma_\infty(\tau))$, $\tau\to\infty$, (2.6) follows from

$$\int_0^{Q^{\leftarrow}(t)}\left(uQ'(u)+\frac{\zeta(u)}{Q'(u)}\,Q'(u)\right)du = \int_{t_0\sum c_i}^{t}(Q^{\leftarrow}(v)+\rho(Q^{\leftarrow}(v)))\,dv.$$

That $1/\sigma_\infty(\tau) = o(\tau)$, $\tau\to\infty$, follows from (4.7). □

In Section 5 we will need uniform estimates for the tail behaviour, which are derived in the following lemma:

Lemma 4.6. *Suppose that (A1) holds and that ψ and θ are as in (A3). Let c, d be positive constants. Then there are positive constants D_1, D_2, t_1 such that for any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$,*

$$P(\sum_{i=-\infty}^{\infty} c_i Z_i > t) \leq D_1 \exp\left(-\int_{t_0 \sum c_i}^{t} \left(Q^{\leftarrow}(v) - \frac{D_2}{\sigma_\infty(Q^{\leftarrow}(v))}\right) dv\right), \quad t \geq t_1. \quad (4.11)$$

Furthermore, for any fixed sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$, there exist positive constants D_3, D_4, t_2 such that

$$P(\sum_{i=-\infty}^{\infty} c_i Z_i > t) \geq D_3 \exp\left(-\int_{t_0 \sum c_i}^{t} \left(Q^{\leftarrow}(v) + \frac{D_4}{\sigma_\infty(Q^{\leftarrow}(v))}\right) dv\right), \quad t \geq t_2. \quad (4.12)$$

Proof. Similar to (4.10), but now using Lemma 4.4 and Lemma 4.5(b), there are $\tau_0, D_0 > 0$ such that

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)\right) \leq \frac{D_0}{\tau\sigma_\infty(\tau)} \exp\left(-\int_0^{\tau} (uQ'(u) + \zeta(u))\, du\right), \quad (4.13)$$

for $\tau \geq \tau_0$ and any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$. Further, $|\zeta(\tau)| \leq D\sigma_\infty(\tau)$ for $\tau \geq 0$, with D from Lemma 4.5. Choosing $\tau_1 \geq \tau_0$ such that $q(c\tau_1) \geq 0$ and using the monotonicity of q, it follows that for $t \geq t_1 := d\, q(c\tau_1)$,

$$t \geq d\, q(c\tau_1) \geq \sum_{i=-\infty}^{\infty} c_i q(c\tau_1) \geq \sum_{i=-\infty}^{\infty} c_i q(c_i\tau_1) = Q(\tau_1). \quad (4.14)$$

This shows that (4.13) holds for any $t = Q(\tau) \geq t_1$, and t_1 is independent of the specific coefficient sequence in $\mathcal{G}_{c,d,\theta}$. Since $\tau^2\sigma_\infty^2(\tau) \geq \tau^2 c_{i_0}^2 q'(c_{i_0}\tau)$, it follows as in the proof of (2.5) that (4.7) holds uniformly for the sequences in $\mathcal{G}_{c,d,\theta}$, hence $D_0/(\tau\sigma_\infty(\tau))$ in (4.13) can be replaced by some D_1. Then (4.11) follows as in the proof of (2.6).

For the proof of (4.12), for a fixed coefficient sequence, note that (4.10) implies that the inequality in (4.13) can be reversed, by replacing D_0 by $1/3 < 1/\sqrt{2\pi}$. Once it is shown that for large τ,

$$\tau\sigma_\infty(\tau) \leq \exp\left(\int_0^{\tau} \sigma_\infty(v)\, dv\right), \quad (4.15)$$

relation (4.12) follows similarly to (4.11). From (3.4) and the dominated convergence theorem follows that there is a $C > 0$ such that $\sigma_\infty(\tau) \sim C\sqrt{q'(c_{i_0}\tau)}$, $\tau \to \infty$. Now if $\beta \in (-1, \infty]$, i.e. $q' \in \mathrm{RV}_{\beta'}$ with $\beta' \in [-1, \infty)$, then $\tau\sigma_\infty(\tau)/\int_0^\tau \sigma_\infty(u)\, du \to 1 + \beta'/2$, $\tau \to \infty$, by Karamata's Theorem (see e.g. Bingham *et al.* 1987, Theorem 1.5.11), clearly implying (4.15) for large τ. If $\psi'' \in \mathrm{RV}_{-1}$, then $q' \in \mathrm{RV}_\infty$, and by Proposition 3.2, $\tau\sigma_\infty(\tau) \leq (q'(c_{i_0}\tau))^{2/3}$ for large τ. For simplicity, assume that $c_{i_0} = 1$. With $s := q(\tau)$

it follows for large s that $q^{\leftarrow}(s)\sigma_\infty(q^{\leftarrow}(s)) \le (q'(q^{\leftarrow}(s))^{2/3} = (1/\psi''(s))^{2/3}$, and the latter function is in $\mathrm{RV}_{2/3}$. On the other hand,

$$\int_0^{q^{\leftarrow}(s)} \sigma_\infty(v)\,dv \ge \int_0^{q^{\leftarrow}(s)} \sqrt{q'(v)}\,dv = \int_{t_0}^{s} \frac{1}{\sqrt{q'(q^{\leftarrow}(u))}}\,du = \int_{t_0}^{s} \sqrt{\psi''(u)}\,du,$$

which (as a function in s) is in $\mathrm{RV}_{1/2}$. But this then clearly implies (4.15) for large $s = q(\tau)$. □

Now we are able to show that the iid sequence associated with Y_0 is in $\mathrm{MDA}(\Lambda)$:

Proof of Theorem 2.2. Once (2.7) has been shown, it follows readily that

$$\lim_{n\to\infty} nP\left(Y_0 > b_n + \frac{x}{a_n}\right) = \lim_{n\to\infty} \frac{P(Y_0 > b_n + \frac{x}{Q^{\leftarrow}(b_n)})}{P(Y_0 > b_n)} = e^{-x}\,, \quad x \in \mathbb{R},$$

showing that the associated iid sequence is in $\mathrm{MDA}(\Lambda)$ with norming constants a_n and b_n, see e.g. Embrechts *et al.* (1997, Proposition 3.3.2). Thus, it only remains to show (2.7). Let

$$\tau := Q^{\leftarrow}(t) \quad \text{and} \quad \tau^* := Q^{\leftarrow}(t + \frac{x}{Q^{\leftarrow}(t)}).$$

Then by (2.5),

$$\lim_{t\to\infty} \frac{P(Y_0 > t + \frac{x}{Q^{\leftarrow}(t)})}{P(Y_0 > t)} = \lim_{t\to\infty} \frac{P(Y_0 > Q(\tau^*))}{P(Y_0 > Q(\tau))} = \lim_{t\to\infty} \frac{\tau\sigma_\infty(\tau)}{\tau^*\sigma_\infty(\tau^*)}\,\frac{e^{-\tau^* Q(\tau^*)}\Phi(\tau^*)}{e^{-\tau Q(\tau)}\Phi(\tau)}.$$

Thus (2.7) will follow once we have shown that

$$\lim_{t\to\infty} \frac{Q^{\leftarrow}(t)}{Q^{\leftarrow}(t + x/Q^{\leftarrow}(t))} = 1 = \lim_{t\to\infty} \frac{Q'(Q^{\leftarrow}(t))}{Q'(Q^{\leftarrow}(t + x/Q^{\leftarrow}(t)))} \tag{4.16}$$

and

$$\lim_{t\to\infty} \int_\tau^{\tau^*} \frac{d}{du} \log\left(e^{-uQ(u)}\Phi(u)\right) du = -x. \tag{4.17}$$

By (2.3), for any $\varepsilon > 0$ there is $m = m_\varepsilon$ in $\mathbb{N}$ and $u_\varepsilon \in \mathbb{R}$ such that

$$P_m'(u) \le Q'(u) \le (1+\varepsilon)P_m'(u) \quad \forall\ u \ge u_\varepsilon, \tag{4.18}$$

where $P_m(u) := \sum_{|i|\le m} c_i q(c_i u)$. But in Balkema *et al.* (1993, Theorem 1.1) it is shown that $\sqrt{P_m'(P_m^{\leftarrow})}$ is self-neglecting. By Lemma 3.1(e) this implies that $1/\sqrt{P_m'}$ is self-neglecting. In particular,

$$\lim_{u\to\infty} \frac{P_m'(u + x/\sqrt{P_m'(u)})}{P_m'(u)} = 1,$$

uniformly on bounded x-intervals. But

$$\frac{1}{1+\varepsilon}\,\frac{P_m'(u + x/\sqrt{Q'(u)})}{P_m'(u)} \le \frac{Q'(u + x/\sqrt{Q'(u)})}{Q'(u)} \le (1+\varepsilon)\,\frac{P_m'(u + x/\sqrt{Q'(u)})}{P_m'(u)}$$

uniformly in bounded x for large u by (4.18) and (4.7). Since $P'_m \le Q'$ and $1/\sqrt{P'_m}$ is self-neglecting, we estimate

$$\frac{1}{1+\varepsilon} \le \liminf_{u\to\infty} \frac{Q'(u + x/\sqrt{Q'(u)})}{Q'(u)} \le \limsup_{u\to\infty} \frac{Q'(u + x/\sqrt{Q'(u)})}{Q'(u)} \le 1+\varepsilon$$

uniformly in bounded x-intervals, showing that $1/\sqrt{Q'}$ is self-neglecting and hence so is $\sigma_\infty(Q^{\leftarrow})$ by Lemma 3.1(e). But this then implies the right-hand side of (4.16), since $1/Q^{\leftarrow}(t)$ is smaller than $\sigma_\infty(Q^{\leftarrow}(t))$ for large t by (4.7). The left-hand side of (4.16) follows from Resnick (1987, Lemma 1.3), noting that $\frac{d}{dt}\frac{1}{Q^{\leftarrow}(t)} = -(Q^{\leftarrow}(t))^{-2}\,\sigma_\infty^{-2}(Q^{\leftarrow}(t)) \to 0$ as $t \to \infty$ by (4.7). For the proof of (4.17), note that by Lemma 4.5 and (4.7),

$$\frac{d}{du}\log\left(e^{-uQ(u)}\Phi(u)\right) = -u\sigma_\infty^2(u) + o(u\sigma_\infty^2(u)), \qquad u \to \infty.$$

Now

$$\int_\tau^{\tau^*} u\sigma_\infty^2(u)\,du = \int_{Q^{\leftarrow}(t)}^{Q^{\leftarrow}(t+x/Q^{\leftarrow}(t))} uQ'(u)\,du = \int_t^{t+x/Q^{\leftarrow}(t)} Q^{\leftarrow}(v)\,dv = \frac{x}{Q^{\leftarrow}(t)}\,Q^{\leftarrow}(\xi)$$

with some ξ between t and $t + x/Q^{\leftarrow}(t)$. As $t \to \infty$, the last expression converges to x since $\tau^*/\tau \to 1$ and by monotonicity of Q. This implies (4.17), completing the proof. □

5 Proof of Theorem 2.3

In this section we will prove Theorem 2.3, stating that the extremal behaviour of the moving average process is the same as the behaviour of the associated iid process. This will be achieved by verifying Leadbetter's $D(u_n)$ and $D'(u_n)$ conditions. For definitions and results we refer to Embrechts *et al.* (1997, Section 4.4) or Leadbetter *et al.* (1983, Chapter 3). The condition $D(u_n)$ is a mixing condition, $D'(u_n)$ can be interpreted as an anti-clustering condition. We shall show that both conditions hold for $(Y_n)_{n\in\mathbb{N}}$, which implies then that its extremal behaviour is exactly as for the associated iid sequence. We need the following result of Rootzén (1986, Lemmas 3.1 and 3.2):

Proposition 5.1. *Suppose that the iid sequence associated with* $(Y_n)_{n\in\mathbb{N}}$, *given by* (1.1), *is in* MDA(Λ) *with norming constants* a_n *and* b_n, *and that* $u_n := x/a_n + b_n$.
(a) If $EZ^2 < \infty$, $|c_i| = O(|i|^{-\vartheta})$ *for some* $\vartheta > 1$ *as* $|i| \to \infty$, *and* $a_n = O((\log n)^\alpha)$ *for some* $\alpha > 0$ *as* $n \to \infty$, *then* $D(u_n)$ *holds.*
(b) If in addition to the conditions of (a) for some constant $\gamma_0 \in (0,1]$ *for* $n' := \lfloor n^{\gamma_0} \rfloor$ *as* $n \to \infty$ *it holds*

$$n \sum_{m=1}^{2n'} P(Y_0 + Y_m > 2u_n) \to 0, \tag{5.1}$$

$$n^2 P\left(a_n \sum_{i=n'+1}^{\infty} c_i Z_i > 1\right) \to 0, \quad n^2 P\left(a_n \sum_{i=-\infty}^{-n'-1} c_i Z_i > 1\right) \to 0, \tag{5.2}$$

$$a_n \sum_{i=n'+1}^{\infty} c_i Z_i \xrightarrow{P} 0, \quad a_n \sum_{i=-\infty}^{-n'-1} c_i Z_i \xrightarrow{P} 0, \tag{5.3}$$

then $D'(u_n)$ holds.

In order to verify (5.1) under conditions (A1) and (A4) we shall need Lemma 5.3. We shall see that we have to consider two different regimes, one corresponding to the case $\beta = \infty$, i.e. $\psi'' \in \mathrm{RV}_\infty$, which implies $\psi \in \mathrm{RV}_\infty$, the other case being $\beta \in [-1, \infty)$; i.e. $\psi \in \mathrm{RV}_\alpha$ for some $\alpha \in [1, \infty)$. We split up the proof into the cases $\beta \in [-1, \infty)$ and $\beta = \infty$, and for the latter case we need some preparation:

Lemma 5.2. *Suppose (A1), that ψ'' is ultimately absolutely continuous on compacts and that $\lim_{t\to\infty} \frac{d}{dt}\frac{\psi'(t)}{\psi''(t)} = 0$. Then there is a constant $\tau_1 \geq 0$ and a C^1-function $p : [0, \infty) \to (0, \infty)$ which is (almost everywhere) twice differentiable, satisfies*

$$p(\tau) = q(\tau)\,, \quad \tau \geq \tau_1,$$

$p'(\tau) > 0$ for all $\tau \geq 0$, $p''(\tau) \leq 0$ for $\tau \geq 0$ (a.e.), and for any constants $c_2 \geq c_1 \geq 0$ it holds

$$c_1 p(c_1\tau) + c_2 p(c_2\tau) - (c_1 + c_2) p(\frac{c_1 + c_2}{2}\tau) \geq \frac{3(c_2 - c_1)^2}{32} \tau\, p'((\frac{c_1}{4} + \frac{3c_2}{4})\tau) \ \geq 0\,, \quad \tau \geq 0. \tag{5.4}$$

Proof. From Lemma 3.1(c) and its proof it follows that q' is in NRV_{-1} and that $q''(\tau) \sim -q'(\tau)/\tau$ as $\tau \to \infty$ (where q'' exists a.e.). In particular, there is τ_1 such that $q''(\tau_1)$ exists and that

$$-\frac{3}{4} q'(\tau) \geq \tau q''(\tau) \geq -\frac{5}{4} q'(\tau)\,, \quad \tau \geq \tau_1 \text{ (a.e.)}.$$

Set $\mu := -\tau_1 q''(\tau_1)/q'(\tau_1)$. Then $3/4 \leq \mu \leq 5/4$. Define the function p through

$$p(\tau) := \begin{cases} q(\tau) & \text{for } \tau \geq \tau_1, \\ q(\tau_1) - q'(\tau_1) e^{\mu} \int_\tau^{\tau_1} e^{-\mu t/\tau_1}\, dt & \text{for } 0 \leq \tau < \tau_1. \end{cases}$$

Then p is C^1 and (almost everywhere) twice differentiable, and for $0 \leq \tau \leq \tau_1$,

$$p'(\tau) = q'(\tau_1) e^{\mu} e^{-\mu\tau/\tau_1}, \quad p''(\tau) = -\mu\, p'(\tau)/\tau_1\,,$$

hence for $0 \leq \tau \leq \tau_1$ it holds

$$\tau p''(\tau) = -\mu \frac{\tau}{\tau_1} p'(\tau) \geq -\mu p'(\tau) \geq -\frac{5}{4} p'(\tau).$$

Thus p satisfies $p'(\tau) > 0$ for $\tau \geq 0$, and $p''(\tau) < 0$ as well as $\tau p''(\tau) \geq -5/4\, p'(\tau)$ for $\tau \geq 0$ (a.e.). For the positivity of p, note that $p(0) \geq q(\tau_1) - e^{\mu}\tau_1 q'(\tau_1)$, which is positive for large enough τ_1, since $\lim_{\tau\to\infty} \tau q'(\tau)/q(\tau) = 0$ by Karamata's theorem, see e.g. Bingham *et al.* (1987, p. 26).

Now let $0 \leq c_1 < c_2$, set $c := c_1 + c_2$ and $c_0 := \frac{3}{4}c_1 + \frac{1}{4}c_2$. For fixed $\tau > 0$ define the function

$$k : [0, c] \to \mathbb{R}, \quad a \mapsto k(a) := a\, p(a\tau) + (c-a)\, p((c-a)\tau).$$

Then

$$\begin{aligned}
k'(a) &= a\tau p'(a\tau) + p(a\tau) - p((c-a)\tau) - (c-a)\tau p'((c-a)\tau),\\
k''(a) &= \tau[a\tau p''(a\tau) + 2p'(a\tau) + (c-a)\tau p''((c-a)\tau) + 2p'((c-a)\tau)]\\
&\geq 3/4\,\tau[p'(a\tau) + p'((c-a)\tau)] > 0 \ \text{(a.e.)}.
\end{aligned}$$

This shows that k' is strictly increasing on $[0, c]$. Since $k'(c/2) = 0$, it follows that k has an absolute minimum at $a = c/2$. To estimate $k(c_1) - k(c/2)$, note that $c_1 < c_0 < \frac{1}{2}c < \frac{1}{4}c_1 + \frac{3}{4}c_2 < c$. Using the mean value theorem, we see that

$$k(c_1) - k(c/2) \geq k(c_1) - k(c_0) = (c_0 - c_1)|k'(\xi)| \geq \frac{c_2 - c_1}{4}|k'(c_0)|,$$

where ξ is between c_1 and c_0. Using $k'(c/2) = 0$, we proceed

$$\begin{aligned}
|k'(c_0)| &= \int_{c_0}^{c/2} k''(a)\, da\\
&\geq \frac{3}{4}\tau \int_{c_0}^{c/2} (p'(a\tau) + p'((c-a)\tau))\, da\\
&= \frac{3}{4}\Big(p(\frac{c}{2}\tau) - p(c_0\tau) - p(\frac{c}{2}\tau) + p((c-c_0)\tau)\Big).
\end{aligned}$$

Using the mean value theorem and the fact that p' decreases, it then follows that

$$k(c_1) - k(c/2) \geq \frac{3(c_2 - c_1)}{16}[p((c-c_0)\tau) - p(c_0\tau))] \geq \frac{3(c_2-c_1)^2\tau}{32}p'((c-c_0)\tau),$$

which proves the assertion. □

The following lemma is the crucial step in showing (5.1). If below m_0 can be chosen to be equal to 1, then (5.6) is redundant and the stronger assertion (5.5) holds for all positive m:

Lemma 5.3. *Suppose that (A1) and (A4) hold. Then there is a constant $\gamma_0 \in (0, 1]$, a positive integer m_0, a constant $t_3 \geq t_0$ and a family $(B_t)_{t\geq t_3}$ of non-negative real numbers,* *tending to zero as $t \to \infty$, such that*

$$\begin{aligned}
\frac{P(\sum_{i=-\infty}^{\infty} \frac{1}{2}(c_i + c_{i-m})Z_i > t)}{(P(\sum_{i=-\infty}^{\infty} c_i Z_i > t))^{1+\gamma_0}} &\leq B_t \quad \forall\, t \geq t_3 \quad \forall\, m \geq m_0, &(5.5)\\
\lim_{t\to\infty} \frac{P(\sum_{i=-\infty}^{\infty} \frac{1}{2}(c_i + c_{i-m})Z_i > t)}{P(\sum_{i=-\infty}^{\infty} c_i Z_i > t)} &= 0 \quad \forall\, m \in \{1, \ldots, m_0 - 1\}. &(5.6)
\end{aligned}$$

Proof. Define $c := c_{i_0} = \max\{c_i : i \in \mathbb{Z}\}$. Choose $\theta \in [0, 2 - 2/\vartheta)$ such that $\theta + \beta' > 0$. For any $m \in \mathbb{N}_0$, define the sequence $(c_{i,m})_{i\in\mathbb{Z}}$ by $c_{i,m} := (c_i + c_{i-m})/2$. Then $c_{i,0} = c_i$ for all i. The corresponding quantities associated with the sequence $(c_{i,m})_{i\in\mathbb{Z}}$ will be denoted by Q_m and $\sigma_{\infty,m}$, respectively. In particular,

$$Q_m(\tau) = \sum_{i=-\infty}^{\infty} \frac{c_i + c_{i-m}}{2} q(\frac{c_i + c_{i-m}}{2}\tau).$$

For $m = 0$ we usually omit the index $m = 0$, so that $Q_0 = Q$ and $\sigma_{\infty,0} = \sigma_\infty$.

By assumption, it follows that there is $d > 0$ such that $(c_{i,m})_{i\in\mathbb{Z}} \in \mathcal{G}_{c,d,\theta}$ for all $m \in \mathbb{N}_0$. Then it follows from (4.11) and (4.12) that there are positive constants $t_3, D_1, \ldots, D_4$ such that for every $m \in \mathbb{N}_0$, $\gamma \geq 0$ and $t \geq t_3$,

$$\begin{aligned}
&\frac{P(\sum_{i=-\infty}^{\infty} c_{i,m} Z_i > t)}{(P(\sum_{i=-\infty}^{\infty} c_i Z_i > t))^{1+\gamma}}\\
&\leq \frac{D_1}{D_3^{1+\gamma}} \exp\left(-\int_{t_0\sum c_i}^{t} \left(Q_m^{\leftarrow}(v) - (1+\gamma)Q^{\leftarrow}(v) - \frac{D_2}{\sigma_{\infty,m}(Q_m^{\leftarrow}(v))} - \frac{D_4(1+\gamma)}{\sigma_\infty(Q^{\leftarrow}(v))}\right) dv\right).
\end{aligned}$$

The assertion will then follow once we have shown that there are $m_0 \in \mathbb{N}$ and $\gamma_0 \in (0, 1]$ such that

$$\begin{aligned}
\lim_{t\to\infty} \inf_{m\geq m_0} \int_{t_0\sum c_i}^{t} (Q_m^{\leftarrow}(v) - (1+\gamma_0)Q^{\leftarrow}(v))\, dv &= \infty, &(5.7)\\
\lim_{v\to\infty} \sup_{m\geq m_0} \frac{\sigma_{\infty,m}^{-1}(Q_m^{\leftarrow}(v)) + \sigma_\infty^{-1}(Q^{\leftarrow}(v))}{Q_m^{\leftarrow}(v) - (1+\gamma_0)Q^{\leftarrow}(v)} &= 0, &(5.8)\\
\lim_{t\to\infty} \int_{t_0\sum c_i}^{t} (Q_m^{\leftarrow}(v) - Q^{\leftarrow}(v))\, dv &= 0 \quad \forall\, m \in \{1, \ldots, m_0 - 1\}, &(5.9)\\
\lim_{v\to\infty} \frac{\sigma_{\infty,m}^{-1}(Q_m^{\leftarrow}(v)) + \sigma_\infty^{-1}(Q^{\leftarrow}(v))}{Q_m^{\leftarrow}(v) - Q^{\leftarrow}(v)} &= 0 \quad \forall\, m \in \{1, \ldots, m_0 - 1\}. &(5.10)
\end{aligned}$$

For showing (5.7) – (5.10), we will distinguish between the cases where $\beta = \infty$ and $\beta \in [-1, \infty)$. Note that (5.9) and (5.10) are redundant if m_0 can be chosen to be 1.

(a) Suppose that $\beta = \infty$, i.e. $\beta' = -1$. Set $m_0 := 1$. Since modifications of q on bounded intervals can be compensated by the function ν appearing in (A1), we can assume that q has already the properties of p as stated in Lemma 5.2. In particular, q is strictly positive on $[0, \infty)$, and from the definitions of Q and Q_m we see that $Q(\tau) \le Q_m(2\tau)$ for $\tau \ge 0$ and $m \in \mathbb{N}$. Furthermore, it is easy to see that for any $m \in \mathbb{N}$ there is $j = j(m) \in \mathbb{Z}$ such that $\inf_{m\in\mathbb{N}}(c_{j(m)} - c_{j(m)-m}) > 0$. It then follows from (5.4) that there are positive constants b_1, b_2, such that

$$Q(\tau) - Q_m(\tau) \ge b_1 \tau q'(b_2\tau) \quad \forall\, \tau \ge 0 \quad \forall\, m \in \mathbb{N}.$$

Thus we have

$$Q^{\leftarrow}(t) \le Q_m^{\leftarrow}(t) \le 2Q^{\leftarrow}(t) \quad \forall\, t \ge t_0 \sum c_i \quad \forall\, m \in \mathbb{N}. \tag{5.11}$$

Using the mean value theorem, for fixed t we find some $\xi_m \in [t, Q(Q_m^{\leftarrow}(t))]$ such that

$$\begin{aligned} Q_m^{\leftarrow}(t) - Q^{\leftarrow}(t) &= Q^{\leftarrow}(Q(Q_m^{\leftarrow}(t))) - Q^{\leftarrow}(t) \\ &= \frac{Q(Q_m^{\leftarrow}(t)) - Q_m(Q_m^{\leftarrow}(t))}{Q'(Q^{\leftarrow}(\xi_m))} \ge \frac{b_1 Q_m^{\leftarrow}(t) q'(b_2 Q_m^{\leftarrow}(t))}{Q'(Q^{\leftarrow}(\xi_m))}. \end{aligned}$$

Since $Q^{\leftarrow}(\xi) \in [Q^{\leftarrow}(t), Q_m^{\leftarrow}(t)]$, it follows from (3.4) and the fact that q' is decreasing that there are $b_3, b_4 > 0$ such that

$$Q'(Q^{\leftarrow}(\xi_m)) \le b_3 q'(b_4 Q^{\leftarrow}(\xi_m)) \le b_3 q'(b_4 Q^{\leftarrow}(t)).$$

Since $q' \in \mathrm{RV}_{-1}$ it follows from (5.11) that there are $d_1, d_2, t_4 > 0$ such that

$$d_1 \le \frac{q'(b_2 Q_m^{\leftarrow}(t))}{q'(b_4 Q^{\leftarrow}(t))} \le d_2 \quad \forall\, t \ge t_4 \quad \forall\, m \in \mathbb{N}.$$

Then it follows from the previous estimates and (5.11) that there is $d_3 > 0$ such that

$$Q_m^{\leftarrow}(t) - Q^{\leftarrow}(t) \ge d_3\, Q^{\leftarrow}(t) \quad \forall\, t \ge t_4 \quad \forall\, m \in \mathbb{N}.$$

This then clearly implies (5.7) with $\gamma_0 := \min\{d_3/2, 1\}$. For the proof of (5.8), observe that with the same arguments as above, there are constants $t_5 > 0$, $b_5 > 0$ such that for any $m \in \mathbb{N}_0$ and $v \ge t_5$,

$$(Q^{\leftarrow}(v))^2 \sigma^2_{\infty,m}(Q_m^{\leftarrow}(v)) \ge c^2_{i_0,m}(Q^{\leftarrow}(v))^2 q'(c_{i_0,m} Q_m^{\leftarrow}(v)) \ge b_5 (Q^{\leftarrow}(v))^2 q'(Q^{\leftarrow}(v)),$$

and the latter tends to ∞ by (4.7).

(*b*) Now suppose that $\beta \in [-1, \infty)$, i.e. $\beta' \in (-1, \infty]$. Again, it is no restriction to modify q such that $q(0) = t_0 > 0$. Firstly, we show that there are constants $0 < A_1 < A_2$, and $\tau_2 > 0$, such that

$$Q_m(\tau) \le A_1 q(c_{i_0}\tau) < A_2 q(c_{i_0}\tau) \le Q(\tau) \quad \forall\, \tau \ge \tau_2 \quad \forall\, m \ge 1, \tag{5.12}$$

and if $\beta' = \infty$ that additionally there is $m_0 \geq 1$, $\tau_3 \geq 0$ and a constant $c' < c = c_{i_0}$ such that

$$Q_m(\tau) \leq A_1 q(c'\tau) \quad \forall\, \tau \geq \tau_3 \quad \forall\, m \geq m_0. \tag{5.13}$$

To show (5.12), note that

$$Q(\tau) = \sum_{i=-\infty}^{\infty} c_i q(c_i\tau) \sim \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c_{i_0}}\right)^{2+\beta'} c_{i_0} q(c_{i_0}\tau), \quad \tau \to \infty,$$

by dominated convergence. Here, $\sum (c_i/c_{i_0})^{2+\beta'}$ has to be interpreted as card $\{i \in \mathbb{Z} : c_i = c_{i_0}\}$ if $\beta' = \infty$. Similarly,

$$Q_m(\tau) \sim \sum_{i=-\infty}^{\infty} \left(\frac{c_{i,m}}{c_{i_0}}\right)^{2+\beta'} c_{i_0} q(c_{i_0}\tau), \quad \tau \to \infty,$$

if $\beta' \neq \infty$, or if $\beta' = \infty$ and $c_{i_m,m} = c_{i_0}$, where i_m is defined to be an index such that $c_{i_m,m} = \max\{c_{i,m} : i \in \mathbb{Z}\}$. It is easy to check (e.g. with methods similar to those used in the proof of Lemma 5.2) that

$$A_3 := c_{i_0} \sup_{m\in\mathbb{N}} \sum_{i\in\mathbb{Z}} \left(\frac{c_{i,m}}{c_{i_0}}\right)^{2+\beta'} < c_{i_0} \sum_{i\in\mathbb{Z}} \left(\frac{c_i}{c_{i_0}}\right)^{2+\beta'} =: A_4.$$

Let $M \subset \mathbb{Z}$ be a finite subset such that $\sum_{i\notin M} c_i \leq (A_4 - A_3)/4$, and put $M_m := M \cup (M+m)$. Then $\sum_{i\notin M_m} c_{i,m} q(c_{i,m}\tau) \leq (A_4 - A_3) q(c_{i_0}\tau)/4$. Furthermore, since M is finite, it follows from the uniform convergence theorem for RV-functions (see e.g. Bingham *et al.* 1987, Theorems 1.5.2 and 2.4.1) that

$$\lim_{\tau\to\infty} \sum_{i\in M_m} \left(\frac{c_{i,m} q(c_{i,m}\tau)}{c_{i_0} q(c_{i_0}\tau)} - \left(\frac{c_{i,m}}{c_{i_0}}\right)^{2+\beta'} \right) = 0,$$

uniformly in $m \in \mathbb{N}$. Thus there is τ_2, such that for any $m \in \mathbb{N}$, and any $\tau \geq \tau_2$,

$$Q_m(\tau) \;\leq\; \frac{A_4 - A_3}{4} q(c_{i_0}\tau) + \left(A_3 + \frac{A_4 - A_3}{4}\right) q(c_{i_0}\tau) = \frac{A_4 + A_3}{2} q(c_{i_0}\tau).$$

(5.12) then follows with $A_1 := (A_4 + A_3)/2$ and $A_2 := 1/4\, A_3 + 3/4\, A_4$. The proof of (5.13) is similar, choosing m_0 and c' such that

$$\sup_{m\geq m_0} c_{i_m,m} < c' < c_{i_0}. \tag{5.14}$$

Since $Q_m(\tau) \leq \sum c_i q(c_{i_0}\tau)$ for any $\tau \geq 0$, we have $Q_m^{\leftarrow}(t) \geq \frac{1}{c_{i_0}} q^{\leftarrow}(\frac{t}{\sum c_i})$, which as $t \to \infty$ converges uniformly in m to ∞. Thus we can invert (5.12) uniformly in m and obtain a constant $t_6 > 0$ such that

$$Q^{\leftarrow}(t) \leq \frac{1}{c_{i_0}} q^{\leftarrow}(\frac{t}{A_2}) < \frac{1}{c_{i_0}} q^{\leftarrow}(\frac{t}{A_1}) \leq Q_m^{\leftarrow}(t) \quad \forall\, t \geq t_6 \quad \forall\, m \geq 1.$$

If $\beta' \neq \infty$, i.e. $\beta \neq -1$, set $m_0 := 1$ and choose $\gamma_0 \in (0,1]$ such that there is $A_5 \in (A_1, A_2)$ such that $(1+\gamma_0)\psi'(t/A_2) \leq \psi'(t/A_5)$ for $t \geq t_6$. Then for $t \geq t_6$ and $m \in \mathbb{N}$,

$$Q_m^{\leftarrow}(t) - (1+\gamma_0)Q^{\leftarrow}(t) \geq \frac{1}{c_{i_0}}\left(\psi'(\frac{t}{A_1}) - \psi'(\frac{t}{A_5})\right) = \frac{1}{c_{i_0}}\left(\frac{1}{A_1} - \frac{1}{A_5}\right) t\psi''(\xi)\,, \quad (5.15)$$

where $\xi \in [t/A_5, t/A_1]$. If $\beta' = \infty$, set m_0 as in (5.14), and $A_5 := A_2$. Then there is a constant t_7 such that

$$Q_m^{\leftarrow}(t) - Q^{\leftarrow}(t) \geq \frac{1}{c_{i_0}}\left(\frac{1}{A_1} - \frac{1}{A_5}\right) t\psi''(\xi) \quad \forall\, t \geq t_7 \quad \forall\, m \in \{1, \ldots, m_0 - 1\}, \quad (5.16)$$

with $\xi \in [t/A_5, t/A_1]$; choosing $0 < \gamma_0 < \min\{\frac{c_{i_0}}{c'} - 1, 1\}$, it follows by inversion of (5.13) that there is a constant t_8 such that for $t \geq t_8$ and $m \geq m_0$,

$$Q_m^{\leftarrow}(t) - (1+\gamma_0)Q^{\leftarrow}(t) \geq \frac{1}{c'}\psi'(\frac{t}{A_1}) - \frac{1+\gamma_0}{c_{i_0}}\psi'(\frac{t}{A_5}) \geq \frac{1+\gamma_0}{c_{i_0}}\left(\frac{1}{A_1} - \frac{1}{A_5}\right) t\psi''(\xi)\,, \quad (5.17)$$

$\xi \in [t/A_5, t/A_1]$. Since $\psi'' \in \mathrm{RV}_\beta$ where $\beta \geq -1$, we have $\lim_{t\to\infty} t^2\psi''(t) = \infty$, and (5.7) and (5.9) are then implied by (5.15) – (5.17). To show (5.8) and (5.10), note that for $m \geq 0$,

$$Q_m'(Q_m^{\leftarrow}(t)) \geq c_{i_m,m}^2 q'(c_{i_m,m}Q_m^{\leftarrow}(t))\,, \quad t \geq t_0.$$

Since

$$\frac{c_{i_0}}{2} q(c_{i_m,m}\tau) \leq Q_m(\tau) \leq \sum_{i=-\infty}^{\infty} c_i q(c_{i_m,m}\tau)\,, \quad \tau \geq 0,$$

it follows that

$$\frac{1}{c_{i_m,m}} q^{\leftarrow}(\frac{t}{\sum c_i}) \leq Q_m^{\leftarrow}(t) \leq \frac{1}{c_{i_m,m}} q^{\leftarrow}(\frac{2t}{c_{i_0}})\,, \quad t \geq t_0.$$

Thus, there is $\eta_m \in [\frac{t}{\sum c_i}, \frac{2}{c_{i_0}}t]$ such that $c_{i_m,m}Q_m^{\leftarrow}(t) = q^{\leftarrow}(\eta_m)$, implying

$$Q_m'(Q_m^{\leftarrow}(t)) \geq \left(\frac{c_{i_0}}{2}\right)^2 q'(q^{\leftarrow}(\eta_m)) = \left(\frac{c_{i_0}}{2}\right)^2 \frac{1}{\psi''(\eta_m)}.$$

Then (5.15) – (5.17) imply (5.8) and (5.10), since $\lim_{t\to\infty} t^2 \frac{(\psi''(\xi))^2}{\psi''(\eta_m)} = \infty$ uniformly in m, using regular variation of ψ''. □

Now we can use Proposition 5.1 to show Theorem 2.3.

Proof of Theorem 2.3. Set $u_n := x/a_n + b_n$. By (4.7), (4.11) and (4.12),

$$P(\sum_{i=-\infty}^{\infty} c_i Z_i > t) = \exp\left(-\int_{t_0\sum c_i}^{t} Q^{\leftarrow}(v)\,dv + o\left(\int_{t_0\sum c_i}^{t} Q^{\leftarrow}(v)\,dv\right)\right), \quad t \to \infty.$$

Since b_n is such that $P(\sum c_i Z_i > b_n) \sim n^{-1}$ as $n \to \infty$, this implies

$$\log n = \int_{t_0 \sum c_i}^{b_n} Q^{\leftarrow}(v)\, dv + o\left(\int_{t_0 \sum c_i}^{b_n} Q^{\leftarrow}(v)\, dv \right), \quad n \to \infty.$$

Dividing by $\int_{t_0 \sum c_i}^{b_n} Q^{\leftarrow}(v)\, dv$ gives $\left(\int_{t_0 \sum c_i}^{b_n} Q^{\leftarrow}(v)\, dv \right) / (\log n) \to 1$ as $n \to \infty$. Since $a_n = Q^{\leftarrow}(b_n)$, i.e. $b_n = Q(a_n)$, there exists $\tau_2 > 0$ and $C_1 > 0$ such that for large n,

$$\begin{aligned} \int_{t_0 \sum c_i}^{b_n} Q^{\leftarrow}(v)\, dv &= \int_0^{a_n} u Q'(u)\, du \\ &\geq \int_0^{a_n} c_{i_0}^2 u^{3/2} q'(c_{i_0} u) u^{-1/2}\, du \\ &\geq C_1 \int_{\tau_2}^{a_n} u^{-1/2} du = 2C_1(\sqrt{a_n} - \sqrt{\tau_2}), \end{aligned}$$

since $\lim_{u\to\infty} u^{3/2} q'(c_{i_0} u) = \infty$ since $\beta' \geq -1$. But this shows that $a_n/(\log n)^2$ is bounded as $n \to \infty$, showing that $D(u_n)$ holds by Proposition 5.1.

For the proof of $D'(u_n)$, we will verify conditions (5.1) – (5.3). Let γ_0, m_0 and $(B_t)_{t \geq t_3}$ be as in Lemma 5.3 and set $n' := \lfloor n^{\gamma_0} \rfloor$. Since $\lim_{n\to\infty} nP(Y_0 > u_n) = e^{-x}$, it follows from (5.6) that

$$n \sum_{m=1}^{m_0 - 1} P(Y_0 + Y_m > 2u_n) \sim \sum_{m=1}^{m_0 - 1} \frac{P(Y_0 + Y_m > 2u_n)}{P(Y_0 > u_n)} e^{-x} \to 0, \quad n \to \infty.$$

On the other hand, (5.5) gives for large n

$$n \sum_{m=m_0}^{2n'} P(Y_0 + Y_m > 2u_n) \leq \frac{(e^{-x}+1)^{1+\gamma_0}}{n^{\gamma_0}} \sum_{m=m_0}^{2n'} \frac{P(Y_0 + Y_m > 2u_n)}{P(Y_0 > u_n)^{1+\gamma_0}} \leq (e^{-x}+1)^{1+\gamma_0}\, 2B_{u_n},$$

and the latter converges to 0 as $n \to \infty$, showing (5.1).

Consider the exponential families $(\overline{Z}_\tau)_{\tau \geq 0}$ and $(\overline{X}_{i,\tau})_{\tau \geq 0}$ as defined in Section 3.1. By (3.2), $E\overline{X}_{i,\tau} = c_i E\overline{Z}_{c_i \tau}$. Since $|c_i| \leq C_2 |i|^{-\vartheta}$ for $i \neq 0$, with some constant C_2, it follows that for any $n \in \mathbb{N}$,

$$|c_i \tau| \leq C_2 \quad \text{for } \tau \leq n^{\vartheta} \quad \text{and} \quad |i| \geq n.$$

Since $[0, C_2] \to \mathbb{R}$, $s \mapsto E\overline{Z}_s$ is a continuous function, it follows that there is some constant $C_3 > 0$ such that

$$|E\overline{X}_{i,\tau}| \leq c_i C_3 \quad \text{for all } \tau \leq n^{\vartheta} \quad \text{and} \quad |i| \geq n.$$

This implies for any $\tau \leq n^{\vartheta}$,

$$\sum_{i=n+1}^{\infty} |E\overline{X}_{i,\tau}| \leq C_2 C_3 \sum_{i=n+1}^{\infty} |i|^{-\vartheta} \leq C_4 n^{1-\vartheta} \tag{5.18}$$

for some constant $C_4 > 0$. Let $\overline{\Phi}_n$ be the moment generating function of $\sum_{i=n+1}^{\infty} c_i Z_i$. As in the proof of Lemma 4.1 follows

$$\frac{d}{d\tau} \log \overline{\Phi}_n(\tau) = \sum_{i=n+1}^{\infty} E\overline{X}_{i,\tau}, \quad \tau \geq 0,$$

implying

$$\overline{\Phi}_n(\tau) = \exp\left(\int_0^{\tau} \sum_{i=n+1}^{\infty} E\overline{X}_{i,v}\, dv \right),$$

since $\overline{\Phi}_n(0) = 1$. Using (5.18), we have

$$\overline{\Phi}_n(\tau) \leq \exp\left(C_4 n^{1-\vartheta}\tau\right) \quad \text{for } \tau \leq n^{\vartheta}.$$

Using Markov's inequality and replacing n by n' and setting $\tau := (n')^{\vartheta}$, we obtain

$$\begin{aligned} P\left(\sum_{i=n'+1}^{\infty} c_i Z_i > 1/a_n \right) &\leq \overline{\Phi}_{n'}((n')^{\vartheta}) \exp(-(n')^{\vartheta}/a_n) \\ &\leq \exp(C_4 n' - (n')^{\vartheta}/a_n) = o(n^{-2}), \quad n \to \infty, \end{aligned}$$

since $a_n = O((\log n)^2)$. This is the left hand side of (5.2). The right hand side of (5.2), as well as (5.3) are obtained similarly. Thus it follows that $D'(u_n)$ holds, giving the assertion, see e.g. Embrechts *et al.* (1997, Theorem 4.4.6) or Leadbetter *et al.* (1983, Theorem 3.5.2). □

6 Applications to financial time series

Financial variables such as stock returns are often modeled using a stochastic volatility process. Prominent models are ARCH and GARCH models as introduced by Engle (1982) and Bollerslev (1986), stochastic volatility models as in Taylor (1986) or the EGARCH model by Nelson (1991). GARCH models have generally heavy tails, so we shall concentrate on stochastic volatility and EGARCH models.

An example of a (discrete time) stochastic volatility model $(\xi_n)_{n\in\mathbb{Z}}$ with volatility process $(\sigma_n)_{n\in\mathbb{Z}}$ is given by

$$\xi_n = \sigma_n \eta_n, \quad n \in \mathbb{Z}, \tag{6.1}$$

$$\log \sigma_n^2 = \sum_{i=1}^{\infty} \alpha_i Z_{n-i}, \quad n \in \mathbb{Z}. \tag{6.2}$$

Here, $(Z_i)_{i\in\mathbb{Z}}$ is a sequence of iid rvs, the coefficient sequence $(\alpha_i)_{i\in\mathbb{N}}$ is such that the sum in (6.2) converges absolutely almost surely, and $(\eta_n)_{n\in\mathbb{Z}}$ is independent of $(Z_i)_{i\in\mathbb{Z}}$, hence of $(\sigma_n)_{n\in\mathbb{Z}}$. Typically, η_0 is Gaussian and Z_0 has light left and right tails, or is assumed to be Gaussian. Extreme value theory for such stochastic volatility models $(\xi_n)_{n\in\mathbb{Z}}$ with Gaussian noise has been carried out by Breidt and Davis (1998). Much information is already contained in the volatility process $(\sigma_n)_{n\in\mathbb{Z}}$, and Theorems 2.1 – 2.3 provide extreme value theory for the process $(\log\sigma_n^2)_{n\in\mathbb{Z}}$ under mild conditions on Z_0 and non-negative coefficient sequences. A simple monotone transformation then yields extremal results for the volatility process $(\sigma_n)_{n\geq 0}$. In particular, from Theorem 2.2 follows that $\log\sigma_0^2$ and hence σ_0 are in MDA(Λ), and Theorem 2.3 shows that extremes of the log-volatility process and hence of the volatility process do not cluster. The restriction of the coefficients being non-negative can be relaxed to a great extent, as follows from Theorems 7.1 and 7.2 and their discussion in the next section.

The EGARCH model $(\xi_n)_{n\in\mathbb{Z}}$ has a similar structure, given by

$$\begin{aligned} \xi_n &= \sigma_n Z_n, \quad n\in\mathbb{Z}, \\ \log\sigma_n^2 &= \mu+\sum_{i=1}^{\infty}\alpha_i g(Z_{n-i}), \quad n\in\mathbb{Z}. \end{aligned} \tag{6.3}$$

Here, μ is a real constant, the coefficient sequence $(a_i)_{i\in\mathbb{N}}$ is as before, g is typically a deterministic piecewise affine linear function (allowing for asymmetry in negative and positive innovations), and $(Z_n)_{n\in\mathbb{Z}}$ is an iid innovation sequence, typically Gaussian. The main difference to the stochastic volatility model considered before is that ξ_n is defined in terms of the innovation sequence $(Z_n)_{n\in\mathbb{Z}}$ only, while the stochastic volatility model is defined in terms of a second independent driving noise sequence $(\eta_n)_{n\in\mathbb{Z}}$. For the extreme value theory of $(\log\sigma_n^2)_{n\in\mathbb{Z}}$ and hence $(\sigma_n)_{n\in\mathbb{Z}}$, however, this is irrelevant, and Theorems 2.1 – 2.3 can be applied for fairly general light noise terms, similar to the stochastic volatility model discussed before. The extreme value behaviour of the price process $(\xi_n)_{n\in\mathbb{Z}}$ itself for Gaussian innovations and a finite coefficient sequence $(\alpha_i)_{i=1,\dots,N}$ has been investigated in Lindner and Meyer (2002).

7 Extensions

The proofs of Theorems 2.1 and 2.2 can be easily generalized to cover independent finite sums of infinite moving average processes. Let $K\in\mathbb{N}$. For $k=1,\dots,K$, let $Z^{(k)}$ be a generic rv which satisfies (A1) with $\nu^{(k)}$, $\psi^{(k)}$ and $t_0^{(k)}$. Suppose that for each k, $(Z_i^{(k)})_{i\in\mathbb{Z}}$ is iid with the distribution of $Z^{(k)}$, and that $(Z_i^{(k)})_{i\in\mathbb{Z},k=1,\dots,K}$ is independent. Let $(c_i^{(k)})_{i\in\mathbb{Z},k=1,\dots,K}$ be a summable sequence of non-negative coefficients and define

$$Y_0 := \sum_{k=1}^{K}\sum_{i=-\infty}^{\infty} c_i^{(k)} Z_{n+i}^{(k)}. \tag{7.1}$$

Set $q^{(k)}(\tau) := (\psi^{(k)})'^{\leftarrow}(\tau)$, $(\sigma_i^{(k)})^2(\tau) := (c_i^{(k)})^2(q^{(k)})'(c_i^{(k)}\tau)$, $Q(\tau) := \sum_{k=1}^K \sum_{i=-\infty}^\infty c_i^{(k)} q^{(k)}(c_i^{(k)}\tau)$, and $\sigma_\infty^2(\tau) := Q'(\tau)$. Instead of (A2) suppose that

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{k=1}^K \sum_{|j|>m} (\sigma_j^{(k)})^2(\tau)}{\sigma_\infty^2(\tau)} = 0,$$

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{k=1}^K \sum_{|j|>m} \sigma_j^{(k)}(\tau)}{\sigma_\infty(\tau)} = 0.$$

Denote by Φ the moment generating function of Y_0. Then we have the following extension of Theorems 2.1 and 2.2:

Theorem 7.1. *Under the assumptions and with the notations above, the assertions of Theorems 2.1 and 2.2 hold, with Y_0 as in* (7.1) *replacing $\sum_{i=-\infty}^\infty c_i Z_i$ in* (2.5) *and* (2.6), *and $\sum_{k=1}^K t_0^{(k)} \sum_{i=-\infty}^\infty c_i^{(k)}$ replacing the lower integration limit $t_0 \sum c_i$ in* (2.6).

Theorem 7.1 can be used to cover infinite moving average processes with negative and positive coefficients. This can be achieved by splitting the sum in (1.1) up into $Y_n = \sum_{c_i\geq 0} c_i Z_{n+i} + \sum_{c_i<0}(-c_i)(-Z_{n+i})$. If then (A1) and (A2) are valid for each of the two sums (posing conditions on the left as well on the right tail behaviour of the density f of Z), then Theorems 2.1 and 2.2 hold.

Theorem 7.1 can also be used to derive further results for the stochastic volatility model and EGARCH model of the previous section. Not only does it allow for positive and negative terms in the coefficient sequence, but from (6.1) and (6.3) follows that $\log \xi_n^2 = \log \sigma_n^2 + \log \eta_n^2$ and $\log \xi_n^2 = \log \sigma_n^2 + \log Z_n^2$, respectively. Then $\log \xi_0^2$ has the general form (7.1), and Theorem 7.1 allows to derive the tail behaviour of $\log \xi_0^2$ (and hence of $|\xi_0|$) and to show that $\log \xi_0^2 \in \mathrm{MDA}(\Lambda)$, under mild conditions on the light tail behaviour of the noise sequences.

There is also an extension of Theorem 2.3 to moving average processes with negative and positive coefficients; its proof follows by slight modifications of the proof of Theorem 2.3:

Theorem 7.2. *Suppose that Z as well as $-Z$ satisfy (A1) and (A4) with functions ψ_+ and ψ_- and regular (rapid) variation index β_+ and β_-, respectively. Define β'_+ and β'_- as in (A3), and suppose that the real coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ satisfies $|c_i| = O(|i|^{-\vartheta})$ as $|i| \to \infty$, for some $\vartheta > \max\{1, 2/(2+\beta'_+), 2/(2+\beta'_-)\}$. Suppose that $\beta_+ \neq \beta_-$, or that $\psi_+ = \psi_-$. Then the assertion of Theorem 2.3 holds for $(Y_n)_{n\in\mathbb{Z}}$ as defined in* (1.1).

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