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## A Limit Theorem for Copulas

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# A Limit Theorem for Copulas

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## Abstract

We characterize convergence of a sequence of  $d$ -dimensional random vectors by convergence of the one-dimensional margins and of the copula. The result is applied to the approximation of portfolios modelled by  $t$ -copulas with large degrees of freedom, and to the convergence of certain dependence measures of bivariate distributions.

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# 1 Introduction

Copula functions are widely applied in statistics and econometrics, especially in finance. For example, Bluhm et al [2] and Li [9] apply copula functions for credit risk modelling, and Rosenberg [11] studies the pricing of exchange rate derivatives using copulas. Besides this, copulas in the context of risk management are emphasised by Embrechts et al [7]. In many applications, the asymptotic behaviour of copulas is of interest for approximation and convergence issues. For example, in order to characterize the limiting behaviour of multivariate extremes, Deheuvels [3, Théorème 2.3, Lemma 4.1] has shown that if  $X = (X^{(1)}, \dots, X^{(d)})$  is a random vector with continuous margins, then a sequence of random vectors converges weakly to  $X$  if and only if the one-dimensional margins of the sequence converge weakly to the margins  $X^{(j)}$ , and if additionally the copulas converge pointwise (and hence uniformly) to the copula of  $X$  on  $[0, 1]^d$ . See also Deheuvels [5, p. 261], [6, Lemma 2]. In the present paper, we shall generalize Deheuvel's result to the case where  $X$  is not assumed to have continuous margins. Since in that case the copula of  $X$  does not need to be unique, convergence of the copulas on  $[0, 1]^d$  cannot be expected. However, we shall show that the copulas converge uniformly on the product of the ranges of the one-dimensional distribution functions of  $X$ . As we recently found out, such a result was already anticipated by Deheuvels in [4, Théorème 4]. However, a proof was given only for the case when  $X$  has continuous margins. Also, due to the increasing importance of copulas in applications and the fact that some of the literature [3] – [6] may be difficult to access it seems justified to give a full proof of this result in the general case.

## 2 Main result

An axiomatic definition of copulas is to be found in Joe [8] and Nelsen [10]. According to this a function  $C : [0, 1]^d \rightarrow [0, 1]$  is a (*d-dimensional*) *copula* if  $C$  is a  $d$ -dimensional distribution function on  $[0, 1]^d$  having *uniform margins*, i.e.  $C(1, \dots, 1, u^{(j)}, 1, \dots, 1) = u^{(j)}$  for  $u^{(j)} \in [0, 1]$ .

Let  $X = (X^{(1)}, \dots, X^{(d)})$  be a  $d$ -dimensional random vector with distribution function  $F$  and marginal distribution functions  $F^{(1)}, \dots, F^{(d)}$ . Then a copula  $C$  is *associated with*  $X$  if it satisfies

$$F(x^{(1)}, \dots, x^{(d)}) = C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)})) \quad \forall x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d.$$

By Sklar's Theorem, an associated copula always exists and is unique on  $\overline{\text{Ran}F^{(1)}} \times \dots \times \overline{\text{Ran}F^{(d)}}$ . On  $\text{Ran}F^{(1)} \times \dots \times \text{Ran}F^{(d)}$  it is given by

$$C(u^{(1)}, \dots, u^{(d)}) = F((F^{(1)})^{\leftarrow}(u^{(1)}), \dots, (F^{(d)})^{\leftarrow}(u^{(d)})),$$

where  $(F^{(j)})^{\leftarrow}(u^{(j)}) := \inf\{y \in \mathbb{R} : F^{(j)}(y) \geq u^{(j)}\}$  denotes the left inverse of the increasing function  $F^{(j)}$ ,  $j \in \{1, \dots, d\}$ .

Now we can proof the limit result for copulas:

**Theorem 2.1.** *Let  $N$  be an ordered index set with limit point  $n_\infty$ . Let  $(X_n)_{n \in N}$  and  $X$  be  $d$ -dimensional random vectors, where  $X_n = (X_n^{(1)}, \dots, X_n^{(d)})$  and  $X = (X^{(1)}, \dots, X^{(d)})$ . Then  $X_n$  converges weakly to  $X$  as  $n \rightarrow n_\infty$ , if and only if the margins  $X_n^{(j)}$  converge weakly to  $X^{(j)}$  as  $n \rightarrow n_\infty$ , for  $j = 1, \dots, d$ , and if the copulas  $C_n$  of  $X_n$  converge pointwise to the copula  $C$  of  $X$  on  $\text{Ran } F^{(1)} \times \dots \times \text{Ran } F^{(d)}$  as  $n \rightarrow n_\infty$ , where  $F^{(j)}$  denotes the distribution function of  $X^{(j)}$ . In that case, the convergence is uniform on  $\overline{\text{Ran } F^{(1)}} \times \dots \times \overline{\text{Ran } F^{(d)}}$ .*

*Proof.* Denote the distribution function of  $X$  and  $X_n$  by  $F$  and  $F_n$ , respectively, and the distribution function of  $X^{(j)}$  and  $X_n^{(j)}$  by  $F^{(j)}$  and  $F_n^{(j)}$ , respectively. Note that any copula  $D$  is Lipschitz continuous, more precisely it holds

$$|D(u) - D(v)| \leq \sum_{j=1}^d |u^{(j)} - v^{(j)}| \quad \forall u = (u^{(1)}, \dots, u^{(d)}), v = (v^{(1)}, \dots, v^{(d)}) \in [0, 1]^d, \quad (1)$$

see Nelsen [10, Theorem 2.10.7]. Suppose that  $X_n \xrightarrow{w} X$  as  $n \rightarrow n_\infty$ , where  $\xrightarrow{w}$  denotes weak convergence. Then  $X_n^{(j)} \xrightarrow{w} X^{(j)}$  as  $n \rightarrow n_\infty$  by the continuous mapping theorem. For the convergence of the copulas, define  $\mathcal{M}^{(j)}$  to be the set of all  $u^{(j)} \in [0, 1]$  such that there exist  $x_{u,j} \in \mathbb{R}$  such that  $u^{(j)} = F^{(j)}(x_{u,j})$  and such that  $F^{(j)}$  is continuous in  $x_{u,j}$ . Let  $(u^{(1)}, \dots, u^{(d)}) \in \mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(d)}$ , and let  $x_{u,j}$  be points as appearing in the definition of  $\mathcal{M}^{(j)}$ . Then (1) gives

$$\begin{aligned} & |C_n(u^{(1)}, \dots, u^{(d)}) - C(u^{(1)}, \dots, u^{(d)})| \\ &= |C_n(F^{(1)}(x_{u,1}), \dots, F^{(d)}(x_{u,d})) - C(F^{(1)}(x_{u,1}), \dots, F^{(d)}(x_{u,d}))| \\ &\leq |C_n(F^{(1)}(x_{u,1}), \dots, F^{(d)}(x_{u,d})) - C_n(F_n^{(1)}(x_{u,1}), \dots, F_n^{(d)}(x_{u,d}))| \\ &\quad + |C_n(F_n^{(1)}(x_{u,1}), \dots, F_n^{(d)}(x_{u,d})) - C(F^{(1)}(x_{u,1}), \dots, F^{(d)}(x_{u,d}))| \\ &\leq |F^{(1)}(x_{u,1}) - F_n^{(1)}(x_{u,1})| + \dots + |F^{(d)}(x_{u,d}) - F_n^{(d)}(x_{u,d})| \\ &\quad + |F_n(x_{u,1}, \dots, x_{u,d}) - F(x_{u,1}, \dots, x_{u,d})|. \end{aligned}$$

Since the  $x_{u,j}$  are continuity points of  $F^{(j)}$ , it follows that  $F_n^{(j)}(x_{u,j})$  converges to  $F^{(j)}(x_{u,j})$  as  $n \rightarrow n_\infty$ , and that  $P(X \in \partial\{(y^{(1)}, \dots, y^{(d)}) \in \mathbb{R}^d : y^{(j)} \leq x_{u,j}, j = 1, \dots, d\}) = 0$ . By assumption, this implies convergence of  $F_n(x_{u,1}, \dots, x_{u,d})$  to  $F(x_{u,1}, \dots, x_{u,d})$ . Thus,  $C_n$  converges pointwise to  $C$  on  $\mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(d)}$ , as  $n \rightarrow n_\infty$ . To show uniform convergence, let  $\varepsilon > 0$ , choose an integer  $m \geq 3d/\varepsilon$ , and for  $k = (k^{(1)}, \dots, k^{(d)}) \in \{0, \dots, m-1\}^d$  set

$$A_k := \left\{ u = (u^{(1)}, \dots, u^{(d)}) \in [0, 1]^d : \frac{k^{(j)}}{m} \leq u^{(j)} \leq \frac{k^{(j)} + 1}{m}, j = 1, \dots, d \right\}.$$

Denote by  $K$  the set of all  $k \in \{0, \dots, m-1\}^d$  such that  $A_k \cap (\mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(d)})$  is nonempty. Choose  $u_k \in A_k \cap (\mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(d)})$  for each  $k \in K$ . Then there exists  $n_0 \in \mathbb{N}$ , such that

$$|C_n(u_k) - C(u_k)| \leq \frac{\varepsilon}{3} \quad \forall k \in K, n \geq n_0.$$

Then for any  $k \in K$  and  $u \in A_k$ , (1) gives for  $n \geq n_0$ ,

$$\begin{aligned} |C_n(u) - C(u)| &\leq |C_n(u) - C_n(u_k)| + |C_n(u_k) - C(u_k)| + |C(u_k) - C(u)| \\ &\leq \frac{d}{m} + \frac{\varepsilon}{3} + \frac{d}{m} \leq \varepsilon. \end{aligned}$$

Since  $\mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(d)}$  is dense in  $\overline{\text{Ran } F^{(1)}} \times \dots \times \overline{\text{Ran } F^{(d)}}$ , this implies uniform convergence of  $C_n$  to  $C$  on  $\overline{\text{Ran } F^{(1)}} \times \dots \times \overline{\text{Ran } F^{(d)}}$ , as  $n \rightarrow n_\infty$ .

For the converse, suppose that  $X_n^{(j)} \xrightarrow{w} X^{(j)}$  for all  $j = 1, \dots, d$ , and that  $C_n$  converges pointwise to  $C$  on  $\mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(d)}$ , as  $n \rightarrow n_\infty$ . Let  $\mathcal{Q}$  be the set of all  $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$  such that  $F^{(j)}$  is continuous in  $x^{(j)}$  for all  $j = 1, \dots, d$ . Then (1) gives for any  $x \in \mathcal{Q}$ ,

$$\begin{aligned} &|F_n(x^{(1)}, \dots, x^{(d)}) - F(x^{(1)}, \dots, x^{(d)})| \\ &= |C_n(F_n^{(1)}(x^{(1)}), \dots, F_n^{(d)}(x^{(d)})) - C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))| \\ &\leq |C_n(F_n^{(1)}(x^{(1)}), \dots, F_n^{(d)}(x^{(d)})) - C_n(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))| \\ &\quad + |C_n(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)})) - C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))| \\ &\leq |F_n^{(1)}(x^{(1)}) - F^{(1)}(x^{(1)})| + \dots + |F_n^{(d)}(x^{(d)}) - F^{(d)}(x^{(d)})| \\ &\quad + |C_n(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)})) - C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))|, \end{aligned}$$

and the latter converges to 0 as  $n \rightarrow n_\infty$ . Thus  $F_n$  converges to  $F$  in any  $x \in \mathcal{Q}$ , which then implies weak convergence of  $X_n$  to  $X$  (e.g. by an obvious modification of the proof of Theorem 29.1 in Billingsley [1]).  $\square$

It should be noted that in the case where margins of the limiting vector are supposed to be continuous and strictly increasing, a simpler proof can be given. In fact, then weak convergence of  $X_n$  to  $X$  implies uniform convergence of  $(F_n^{(j)})^\leftarrow$  to  $(F^{(j)})^\leftarrow$  and of  $F_n$  to  $F$ , so that the copulas converge uniformly, too. In the general case, however, more care has to be taken. Also, convergence of the copulas on the whole unit cube  $[0, 1]^d$  cannot be expected, as is shown by the following example:

**Example 2.2.** Let  $X$  and  $Y$  be two random vectors in  $\mathbb{R}^d$  with different copulas. Set  $X_n := X/n$  if  $n$  is odd and  $X_n := Y/n$  if  $n$  is even. Then  $X_n$  converges weakly to  $\mathbf{0}$  as  $n \rightarrow \infty$ , while the copula  $C_n$  of  $X_n$  is equal to the copula of  $X$  or  $Y$ , depending whether  $X$  is odd or even. Thus  $C_n$  cannot converge on  $[0, 1]^d$ . However, it converges on  $\times_{j=1}^d \{0, 1\}$ , which is the product of the ranges of the marginal distribution functions.

### 3 Applications

In this section we give two applications of Theorem 2.1. The first application is concerned with  $t$ -copulas with increasing degrees of freedom.

#### 3.1 Credit Risk and $t$ -Copula

In credit risk theory, modelling portfolios by  $t$ -copulas presents a common approach away from multivariate normal models, see e.g. Bluhm et. al. [2], Chapter 2.6. Let  $\Sigma$  be a positive definite  $(d \times d)$ -matrix with entries 1 on the diagonal and let  $n \in \mathbb{N}$ . Then the *Gaussian Copula*  $C_{\Sigma}^{Ga}$  is defined to be the copula of an  $N(0, \Sigma)$  distributed vector  $Y$ , and the  *$t$ -Copula*  $C_{n, \Sigma}^t$  is the copula of a multivariate  $t$ -distributed vector  $X_{n, \Sigma} = \sqrt{n/S}Y$ , where  $S$  is  $\chi_n^2$ -distributed and independent of  $Y$ . Since  $X_{n, \Sigma}$  converges weakly to  $Y$  as  $n \rightarrow \infty$ , Theorem 2.1 implies that the  $t$ -copulas  $C_{n, \Sigma}^t$  converge uniformly to the Gaussian copula  $C_{\Sigma}^{Ga}$  as the degree of freedom  $n$  tends to  $\infty$ . Then if  $(Z_n)_{n \in \mathbb{N}}$  is a sequence of random vectors with  $t$ -copula  $C_{n, \Sigma}^t$  and if the margins of  $(Z_n)$  converge to some random variables with distribution function  $F^{(j)}$ ,  $j = 1, \dots, d$ , then  $(Z_n)$  converges as  $n \rightarrow \infty$  to a random variable  $Z$  with distribution function  $C_{\Sigma}^{Ga}(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))$ . In particular, a portfolio which is modelled by a  $t$ -Copula with large degrees of freedom can be approximated by a model using a Gaussian copula and the same margins.

#### 3.2 Kendall's Tau, Spearman's Rho, and Tail Dependence

The next application discusses the convergence of three dependence measures of bivariate distributions, namely Kendall's tau, Spearman's rho and tail dependence.

Let  $(X^{(1)}, X^{(2)})$ ,  $(Y^{(1)}, Y^{(2)})$  and  $(Z^{(1)}, Z^{(2)})$  be three independent and identically distributed random vectors with continuous margins and copula  $C$ . Then *Kendall's tau*,  $\tau$ , and *Spearman's rho*,  $\rho$ , are given by

$$\begin{aligned} \tau &:= P((X^{(1)} - Y^{(1)})(X^{(2)} - Y^{(2)}) > 0) - P((X^{(1)} - Y^{(1)})(X^{(2)} - Y^{(2)}) < 0), \\ \rho &:= 3(P((X^{(1)} - Y^{(1)})(X^{(2)} - Z^{(2)}) > 0) - P((X^{(1)} - Y^{(1)})(X^{(2)} - Z^{(2)}) < 0)). \end{aligned}$$

From this follows readily that bivariate weak convergence implies convergence of Kendall's tau and Spearman's rho. Another proof of this follows immediately from Theorem 2.1, since  $\tau$  and  $\rho$  can be expressed in terms of the copula  $C$  via

$$\tau = 4 \int_0^1 \int_0^1 C(u^{(1)}, u^{(2)}) dC(u^{(1)}, u^{(2)}) - 1, \quad \rho = 12 \int_0^1 \int_0^1 C(u^{(1)}, u^{(2)}) du^{(1)} du^{(2)} - 3,$$

see e.g. Nelsen [10], Theorems 5.1.3 and 5.1.6. Convergence of the lower and upper tail dependence parameter,  $\lambda_L$  and  $\lambda_U$ , however does not follow from bivariate convergence.

For example, the lower tail dependence parameter is given (if it exists) by

$$\lambda_L = \lim_{u \rightarrow 0} \frac{C(u, u)}{u} = \lim_{u \rightarrow 0} P(X^{(2)} \leq (F^{(2)})^{-}(u) | X^{(1)} \leq (F^{(1)})^{-}(u)),$$

see Joe [8], p. 33. Then if the vector  $(X_n^{(1)}, X_n^{(2)})$  has the copula

$$C_n(u^{(1)}, u^{(2)}) := \begin{cases} \min\{u^{(1)}, u^{(2)}\}, & \text{for } \max\{u^{(1)}, u^{(2)}\} \geq 1/n, \\ n u^{(1)} u^{(2)}, & \text{for } \max\{u^{(1)}, u^{(2)}\} < 1/n, \end{cases}$$

then  $C_n$  converges uniformly to the copula  $C(u^{(1)}, u^{(2)}) = \min(u^{(1)}, u^{(2)})$ . However, the lower tail dependence parameter of  $C_n$  is 0, while that of  $C$  is 1. So uniform convergence of  $C_n$  is not enough to ensure convergence of  $\lambda_L$ . A sufficient condition ensuring convergence of  $\lambda_L$  would be that there is some  $\varepsilon > 0$  such that  $(C_n(u, u) - C(u, u))/u$  converges uniformly in  $u \in (0, \varepsilon]$  to 0 as  $n \rightarrow \infty$ , provided the lower tail dependence parameter of  $C_n$  and  $C$  exist.

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