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THE TOTAL IN THE CATEGORY OF MODULES

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1. INTRODUCTION

The study of the notions "total", "partially invertible" and "regular" will be based here on a new, more general foundation and further developed. The cases considered in the literature will be included as special cases. Proofs will be given here only if they are not routine or if they are not generalisations from the special cases in [1], [2].

Let S, T be rings with identity and let ${}_SA_T$, ${}_TB_S$ be unitary bimodules. We assume two mappings

$$\sigma : A \times B \to S, \qquad \tau : B \times A \to T$$

with the following properties:

(I) $\sigma(sa,b) = s\sigma(a,b)$, $\sigma(a,bs) = \sigma(a,b)s$,

$\sigma(at,b) = \sigma(a,tb)$, $a \in A$, $b \in B$, $s \in S$, $t \in T$,

similar properties for τ;

(II) Associative laws:

$\sigma(a,b)a_1 = a\tau(b,a_1)$, $\tau(b,a)b_1 = b\sigma(a,b_1)$, $a, a_1 \in A$, $b, b_1 \in B$;

(III) Additivity:

$\sigma(a+a_1,b+b_1) = \sigma(a,b) + \sigma(a,b_1) + \sigma(a_1,b) + \sigma(a_1,b_1)$,

similar for τ.

For abbreviation we write

$$ab := \sigma(a,b), \qquad ba := \tau(b,a), \qquad a \in A,\ b \in B.$$

If we have a meaningful product of elements from A, B, S, T, then by the associative laws we can avoid using brackets. If (I), (II), (III) are satisfied, then these conditions define a Morita context and the mappings σ and τ can be factorised via the tensor products $A \underset{T}{\otimes} B$ resp. $B \underset{S}{\otimes} A$. But we will first assume only (I) and (II), in which case A and B have only to be sets and S and T multiplicative monoids. Then we assume (I), (II), (III) but without further conditions. Only in the last section we have to assume "Morita conditions" since we show that the notions "total", "radicaltotal" and "totalfree" for rings are preserved under Morita equivalence.

To have later the possibility for short quotations, we mention here three examples for a Morita context.

EXAMPLES.

(E1) F. Kasch [1], W. Schneider [2].

Let R be a ring with identity and let M_R, N_R be unitary R-modules.

Denote $\qquad S := \mathrm{End}(N_R), \qquad T := \mathrm{End}(M_R),$

$$_SA_T := \mathrm{Hom}_R(M,N), \qquad _TB_S := \mathrm{Hom}_R(N,M),$$

and $\qquad \sigma(f,g) := fg, \qquad \tau(g,f) := gf, \qquad f \in A,\ g \in B.$

Then (I), (II), (III) are satisfied.

(E2) J. Zelmanowitz [3].

Let T be a ring with identity and let A_T be a unitary T-module (in [3] $1 \in T$ is not assumed).

Denote $\qquad S := \mathrm{End}(A_T), \qquad B := A^* = \mathrm{Hom}_T(A,T).$

Then $_SA_T$, $_TB_S$ are bimodules. For $a \in A$, $g \in B$ define

$$\sigma(a,g) := ag : A \ni x \mapsto ag(x) \in A,$$

(implying $ag \in S$). Further define $\tau(g,a) := g(a)$. Then (I), (II), (III) are satisfied.

By a slight change, this can also be considered as a special case of (E1). To see this, one has to substitute $_SA_T$ by the S-T-isomorphic module $\mathrm{Hom}_T(T,A)$ with the isomorphism

$$\varphi : A \ni a \mapsto (T \ni x \mapsto ax \in A) \in \mathrm{Hom}_T(T,A).$$

By this substitution σ and τ change to the mappings in (E1). It is easy to see that the isomorphism φ preserves all the notions considered in this paper.

(E3) Rings.

For a ring R with $1 \in R$ let $A = B = S = T := R$ and for $r_1, r_2 \in R$, $\sigma(r_1, r_2) = \tau(r_1, r_2) := r_1 r_2$. Then all conditions are satisfied.

2. DEFINITIONS AND MULTIPLICATIVE PROPERTIES

In this section we assume only (I) and (II) but not (III) as mentioned already in the introduction.

2.1 LEMMA (compare [1], 1.1, [2], 6.1). *For $a \in A$ are equivalent:*

(a) $\exists\, b \in B\, [ab$ *is an idempotent* $\neq 0$ *in* $S]$,

(b) $\exists\, b_1 \in B\, [b_1 a$ *is an idempotent* $\neq 0$ *in* $T]$,

(c) $\exists\, c \in B\, [ac$ *is an idempotent* $\neq 0$ *in* $S \wedge$
ca is an idempotent $\neq 0$ *in* $T]$.

2.2 DEFINITIONS. Let $a \in A$.

(1) a is called *partially invertible* = "pi" $:\Leftrightarrow$ the conditions of 2.1 are satisfied.

(2) *Total* of $A = \mathrm{Tot}(A) := \{u \mid u \in A \wedge u$ is not pi$\}$.

(3) a is called *regular* $:\Leftrightarrow \exists\, b \in B\, [aba = a]$.

Similar definitions for the elements in B.

For the elements of Tot(A) we give later (using (III)) a characterisation which justifies the notion *total nonisomorphisms*. The definition (1) means in the special case (E3) that $s \in S$ is pi, iff there exists $s' \in S$ such that ss' is an idempotent $\neq 0$ in S. Attention with the notation: In [1] and [2] (that is in the case (E1)) Tot(M) resp. Tot(M,N) are subsets of End(M) resp. $\mathrm{Hom}_R(M,N)$. Now, Tot(A) is a subset of A itself.

2.3 COROLLARY (compare special cases in [1], [2]). *Assume* $a \in A$,

$b \in B$, $s \in S$, $t \in T$.

(a) *If sat is pi, then s, a, t are pi (in S resp. B resp. T).*

(b) *If ab is pi, then a, b are pi (in B resp. A).*

(c) $S\operatorname{Tot}(A)T = \operatorname{Tot}(A)$, $\operatorname{Tot}(A)B \subset \operatorname{Tot}(S)$, $A\operatorname{Tot}(B) \subset \operatorname{Tot}(S)$, $\operatorname{Tot}(S)A \subset \operatorname{Tot}(A)$, $A\operatorname{Tot}(T) \subset \operatorname{Tot}(A)$, $S\operatorname{Tot}(S)S = \operatorname{Tot}(S)$.

2.4 COROLLARY.

(a) *If $aba = a \neq 0$, then ab and ba are idempotents $\neq 0$. Hence regular elements are pi.*

(b) *If $ab = d = d^2 \neq 0$ resp. $ba = e = e^2 \neq 0$, then da, bd, eb, ae are regular elements.*

(c) *a is pi* $\Leftrightarrow \exists b \in B[bab = b \neq 0]$.

(d) *If $aba = a$, then $a(bab)a = a$, $(bab)a(bab) = bab$.*

2.5 COROLLARY. *If $aba = a$, $d = ab$, $e = ba$, then*

$$Sa \ni sa \mapsto sd \in Sd, \qquad aT \ni at \mapsto et \in eT$$

are isomorphisms, hence Sa resp. aT are projective S- resp. T-modules.

The definitions and corollaries show that Lemma 2.1 is essential for our considerations. Notions, based on this lemma are independent of the side. Moreover, it gives a close connection to regular elements. Finally, it makes it possible to give very simple and short proofs by computations with idempotents.

2.6 THEOREM.

(a) *a is pi* $\Leftrightarrow \exists d \in S,\ d = d^2 \neq 0\ [dS \subset aB \wedge dA \subset aT]$
$\Leftrightarrow \exists e \in T,\ e = e^2 \neq 0\ [Te \subset Ba \wedge Ae \subset Sa]$.

(b) *a is regular* $\Leftrightarrow \exists d \in S,\ d = d^2\ [dS = aB \wedge dA = aT]$
$\Leftrightarrow \exists e \in T,\ e = e^2\ [Te = Ba \wedge Ae = Sa]$.

Proof for (b). $\Rightarrow$: Assume $aba = a$. Denote $d = ab$, then

$$dS = a(bS) \subset aB = ab(aB) \subset dS \Rightarrow dS = aB.$$

Similar in the other cases.

$\Leftarrow$: By assumption, there exist $b \in B$, $a_1 \in A$ such that $d = ab$, $da_1 = a$. Then

$$a = da = aba.$$

Similar for the other cases.

3. ADDITIVE PROPERTIES

Now, we have to use the additivity (III). Further we need the following mappings

$$(-b)a : A \ni x \mapsto (xb)a \in Sa,$$

$$a(b-) : A \ni x \mapsto a(bx) \in aT.$$

$\mathrm{Ke}(\cdots)$ denotes the kernel of the mapping $\cdots$.

3.1 THEOREM. *If* $aba = a$, *then*

$$A = Sa \oplus \mathrm{Ke}((-b)a) = aT \oplus \mathrm{Ke}(a(b-)).$$

If a is regular, then by 2.5 and 3.1, Sa is a projective, direct summand of A. In case (E2) also the opposite direction of this statement is true (see [3]). In general we have the following situation. Since Sa is projective, the epimorphism

$$\rho : S \ni s \mapsto sa \in Sa$$

splits. Therefore there exists an idempotent $d \in S$ such that

$$S = Sd \oplus S(1-d), \qquad a = da.$$

In case there exists $b \in B$ such that $ab = d$, then $aba = da = a$. The existence of such b is easy to see in (E2), but in general it will not exist.

3.2 COROLLARY. *If* $ab = d = d^2$, $ba = e = e^2$, *then*

$$Sda \subsetneq^{\oplus} {}_S A, \qquad Sa = Sda \oplus S(1-d)a,$$

$$aeT \subsetneq^{\oplus} A_T, \qquad aT = aeT \oplus a(1-e)T.$$

Similar for b.

Now, we will give an other characterisation (besides 2.6) for pi and regular elements. We use the following notation: *A operates faithfully* on B iff for all $x \in A$, $x \neq 0$, also $xB \neq 0$.

3.3 THEOREM. *For $a \in A$ the following is true.*

(a) *a is pi* $\Leftrightarrow \exists B_0 \subsetneq^{\oplus} B_S, 0 \neq D \subsetneq^{\oplus} S_S$ $[B_0 \ni y \mapsto ay \in D$ *is an isomorphism*$]$.

(b) *a is regular* $\Rightarrow \exists B_S = B_0 \oplus B_1, D \subsetneq^{\oplus} S_S$ $[B_0 \ni y \mapsto ay \in D$ *is an isomorphism* $\wedge\ aB_1 = 0]$.

If A operates faithfully on B, the converse is also true.

Proof. (a)$\Rightarrow$: If $ab = d = d^2 \neq 0$, then by 2.4, bd is regular and by 3.2, $B_0 := bdS \subsetneq^{\oplus} B$. Denote $D = dS$, then

$$B_0 \ni bds \mapsto abds = ds \in D$$

is an isomorphism.

(a)$\Leftarrow$: Let be $dS = D \neq 0$, $d = d^2$, then there exists $b \in B_0$ with $ab = d \neq 0$.

(b)$\Rightarrow$: By 2.4 (d), we can assume $aba = a$, $bab = b$. Then by 3.1 (for B) we have

$$B = bS \oplus \mathrm{Ke}(b(a-)).$$

Define $B_0 := bS$, $D = dS$, then there exists the same isomorphism as in (a). For $y \in \mathrm{Ke}(b(a-))$ follows

$$ay = (aba)y = a(b(ay)) = 0,$$

hence with $B_1 := \mathrm{Ke}(b(a-))$ the proof is complete.

(b)$\Leftarrow$: $D \subsetneq^{\oplus} S_S$ implies $D = dS$, $d = d^2$. By assumption there exists $b \in B$ with $ab = d$. Then $B_0 = bS$ and together with $aB_1 = 0$ we get

$$aB = abS \oplus aB_1 = abS.$$

Further we have for $s \in S$

$$a(bs) = ds = d^2 s = (aba)bs.$$

Since A operates faithfully on B this implies $a = aba$.

3.4 REMARKS. In (E1) the following is true [1]:

$f \in A = \mathrm{Hom}_R(M, N)$ is pi $\Leftrightarrow$

$\exists 0 \neq U \subsetneq^{\oplus} M,\ V \subsetneq^{\oplus} N\ [U \ni x \mapsto f(x) \in V$ is an isomorphism$]$.

If f is not pi, that is, $f \in \mathrm{Tot}(M,N)$, we therefore used the notation [1] "f is a total nonisomorphism". The total was then defined as the set of all total nonisomorphisms. Now, by 3.3, also in the general case it is justified to denote the elements of Tot(A) as *total nonisomorphisms*. Please note the fact, that for (E1) 3.3 and the result stated before are not the same (but connected).

It is interesting to ask for conditions such that Tot(A) is closed under addition (see [1], [2]). A ring S is called a *total ring* iff Tot(S) is additively closed. Then Tot(S) is a two-sided ideal of S.

3.5 THEOREM. *If S or T is a total ring, then* Tot(A) *and* Tot(B) *are S-T- resp. T-S-submodules of A resp. B.*

In general Tot(A) is not closed under addition but it has always the following closure property. With $\mathrm{Rad}({}_SA)$ we denote the radicals of the S-module ${}_SA$.

3.6 THEOREM. $\mathrm{Rad}({}_SA) + \mathrm{Tot}(A) = \mathrm{Rad}(A_T) + \mathrm{Tot}(A) = \mathrm{Tot}(A)$.

3.7 COROLLARY. $\mathrm{Rad}({}_SA) + \mathrm{Rad}(A_T) \subset \mathrm{Tot}(A)$.

This implies the question: Under which conditions is the radical equal to the total (see results in [1], [2]).

If for a ring S $\mathrm{Rad}(S) = \mathrm{Tot}(S)$ holds, then S is called *radicaltotal* ([2], 4.2). Examples for radicaltotal rings are semi-perfect rings or — more general — F-semi-perfect rings (Oberst, Schneider) = semi-regular rings (Nicholson). We call a ring S *totalfree* if $\mathrm{Tot}(S) = 0$. Then a totalfree ring is radicaltotal. If S is a total ring, then $S/\mathrm{Tot}(S)$ is totalfree ([1], 3.6).

4. MORITA EQUIVALENCE

The main goal in this section is to show that the properties "total", "radicaltotal" and "totalfree" of a ring are preserved under Morita equivalence. If the rings T and S are Morita equivalent, we write $T \approx S$. In this case, there exists a progenerator A_T such that $S \cong \mathrm{End}(A_T)$ and in the Morita context (E2), the mappings σ and τ are surjective. Since our properties are preserved under ringisomorphisms, we can assume $S = \mathrm{End}(A_T)$. The surjectivity of σ and τ means $AA^* = S$, $A^*A = T$. We have not to use always all these assumptions. Without any assumption we have by 3.5, that if T or S is a total ring, then $\mathrm{Tot}(A)$ is closed under addition. In the following we consider only the case (E2).

4.1 LEMMA. *If* $S = \mathrm{End}(A_T)$, $AA^* = S$ *and* $\mathrm{Tot}(A)$ *is additively closed, then* S *is a total ring.*

Proof. Let $s_1, s_2 \in \mathrm{Tot}(S)$ and assume $s_1 + s_2 \notin \mathrm{Tot}(S)$. Then there exists $s \in S$ such that $s(s_1 + s_2) = d = d^2 \neq 0$. Since $ss_1A, ss_2A \subset \mathrm{Tot}(A)$ by 2.3 and $\mathrm{Tot}(A)$ is additively closed, also $dA \subset \mathrm{Tot}(A)$. Then $dAA^* = dS \subset \mathrm{Tot}(S)$, hence $d \in \mathrm{Tot}(S)$, contradiction!

4.2 COROLLARY. *If* $T \approx S$ *and* T *is total, then* S *is total.*

4.3 LEMMA.

(a) *If* A_T *is projective and* T *is radicaltotal, then* $\mathrm{Tot}(A) = \mathrm{Rad}(A_T)$.

(b) *If* A_T *is finitely generated and projective,* $S = \mathrm{End}(A_T)$ *and* $\mathrm{Tot}(A) = \mathrm{Rad}(A_T)$, *then* S *is radicaltotal.*

Proof. (a): By 3.7, it is only to show $\mathrm{Tot}(A) \subset \mathrm{Rad}(A_T)$. Let $a \in A$, then we write a with a dual basis $a = \sum b_i \varphi_i(a)$. For $a \in \mathrm{Tot}(A)$ it follows $\varphi_i(a) \in \mathrm{Tot}(T) = \mathrm{Rad}(T)$, hence $a \in \mathrm{Rad}(A_T)$.

(b): Again, only $\mathrm{Tot}(S) \subset \mathrm{Rad}(S)$ is to prove. For $s \in \mathrm{Tot}(S)$ it follows $sA = \mathrm{Im}(s) \subset \mathrm{Tot}(A) = \mathrm{Rad}(A_T)$. Since A_T is finitely generated, $\mathrm{Rad}(A_T)$ is small in A_T, hence $\mathrm{Im}(s)$ is small in A_T. Since A_T is projective, that implies $s \in \mathrm{Rad}(S)$.

4.4 COROLLARY. *If $T \approx S$ and if T is radicaltotal, then S is radicaltotal.*

Proof. By 4.3.

4.5 COROLLARY. *If $T \approx S$ and if T is totalfree, then S is totalfree.*

Proof. $\mathrm{Tot}(T) = 0$ implies $\mathrm{Rad}(T) = 0$ and T is radicaltotal. Then $\mathrm{Rad}(S) = 0$ and $\mathrm{Rad}(S) = \mathrm{Tot}(S)$.

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