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Multivariate Tail Copula: Modeling and Estimation

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Summary

In general, risk of an extreme outcome in financial markets can be expressed as a function of the tail copula of a high-dimensional vector after standardizing marginals. Hence it is of importance to model and estimate tail copulas. Even for moderate dimension, nonparametrically estimating a tail copula is very inefficient and fitting a parametric model to tail copulas is not robust. In this paper we propose a semi-parametric model for tail copulas via an elliptical copula. Based on this model assumption, we propose a novel estimator for the tail copula, which proves favourable compared to the empirical tail copula, both theoretically and empirically.

Keywords: Asymptotic normality, Dependence modeling, Elliptical copula, Elliptical distribution, Multivariate modeling, Regular variation, Tail copula.

1 Introduction

Risk management is a discipline for living with the possibility that future events may cause adverse effects. An important issue for risk managers is how to quantify different types of risk such as market risk, credit risk, operational risk, etc. Due to the multivariate nature of risk, i.e., risk depending on high dimensional vectors of some underlying risk factors, a particular concern for a risk manager is how to model the dependence between extreme outcomes although those extreme outcomes occur rarely. A mathematical formulation of this question is as follows.

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Let $X = (X_1, \ldots, X_d)^T$ be a random vector with distribution function F and continuous marginals $F_1, \ldots, F_d$. Then the dependence is completely determined by the *copula* C of X given by Sklar's representation (cf. Nelsen (1998) or Joe (1997))

$$F(x) = C(F_1(x_1), \ldots, F_d(x_d)), \quad x = (x_1, \cdots, x_d)^T \in \mathbb{R}^d .$$

Moreover, the copula alone allows us to describe dependence on extreme outcomes. As C is a multivariate uniform distribution on $[0,1]^d$, extreme values are near the boundaries and extreme dependence happens around the points $(0, \ldots, 0)$ and $(1, \ldots, 1)$. This motivates the definition of the *tail copula* of X as

$$\lambda^X(x_1, \ldots, x_d) \quad = \quad \lim_{t \to 0} t^{-1} P\left(1 - F_1(X_1) \leq tx_1, \ldots,\ 1 - F_d(X_d) \leq tx_d\right), \qquad (1.1)$$

where $x_1, \ldots, x_d \geq 0$. The bivariate case, when $d = 2$, has been thoroughly investigated and $\lambda^X(1,1)$ is called the *upper tail dependence coefficient* of X_1 and X_2, see Joe (1997). It models dependence along the 45 degree line, where the bivariate dependence effects are mostly concentrated. For $x, y \in [0,1]^2$ the function $x + y - \lambda^X(x,y)$ is called the *tail dependence function* of X_1 and X_2 by Huang (1992); such notions go back to Gumbel (1960), Pickands (1981) and Galambos (1987), and they represent the full dependence structure of the model.

The approach via a dependence function yields that the risk of an extreme outcome in financial markets can be expressed as a function of the tail copula $\lambda^X(x_1, \ldots, x_d)$ after standardizing marginals. When $d = 2$, the tail copula $\lambda^X(x,y)$ or the tail dependence function $x + y - \lambda^X(x,y)$ can be estimated nonparametrically via bivariate extreme value theory; see Einmahl, de Haan and Piterbarg (2001) and references therein. Also parametric models for the tail dependence function have been suggested and estimated, see Tawn (1988), Ledford and Tawn (1997) and Coles (2001) for examples and further references. The application of both, nonparametric and parametric estimation of tail dependence functions has almost only been investigated for the case $d = 2$ although theoretically both methods are applicable to the case $d > 2$. For an approach to nonparametric estimation of tail dependence in higher dimensions see Hsing, Klüppelberg and Kuhn (2004). Recently, Heffernan and Tawn (2004) proposes a conditional approach to model multivariate extremes via investigating the limits of normalized conditional distributions. Obviously, nonparametric estimation severely suffers from the curse of dimensionality, when d becomes large, and fitting parametric models for large d is not robust in general.

In this paper, we concentrate on the dependence structure only, which means we work in the tradition of estimating a dependence function. However, we neither work with purely nonparametric estimates nor do we specify a parametric model. Instead we propose to model the tail copula via an elliptical copula, a novel approach, which may be viewed as a semi-parametric approach. For the applications of copulas and elliptical copulas to risk management, we refer to Frey, McNeil and Nyfeler (2001) and Embrechts, Lindskog and McNeil (2003). Recently, Demarta and McNeil (2005) study some parameterized elliptical copulas. One of the advantages in employing elliptical copulas is the simplicity of simulating multivariate extremes.

Recall that the random vector $Z = (Z_1, \ldots, Z_d)^T$ has an elliptical distribution,

$$Z \stackrel{\mathrm{d}}{=} GAU, \tag{1.2}$$

where $G > 0$ is a random variable, A is a deterministic $d \times d$ matrix with $AA^T := \Sigma = (\sigma_{ij})_{1 \le i,j \le d}$ and $\mathrm{rank}(\Sigma) = d$, U is a d-dimensional random vector uniformly distributed on the unit hyper-sphere $\mathcal{S}_d := \{z \in \mathbb{R}^d : z^T z = 1\}$, and U is independent of G. Representation (1.2) implies that the elliptical distribution is uniquely defined by the matrix Σ and the random variable G. For a detailed discription of elliptical distributions, we refer to Fang, Kotz and Ng (1987). Then, an *elliptical copula* is defined as the copula of an elliptical distribution.

Define the *linear correlation* between Z_i and Z_j as $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$ and denote by $R := (\rho_{ij})_{1 \le i,j \le d}$ the correlation matrix. Note that ρ_{ij} exists for any elliptical distribution; if finite second moments exist it coincides with the usual correlation. Hult and Lindskog (2002) showed in their Theorem 4.3 under weak regularity conditions that regular variation of $P(G > \cdot)$ with index $\alpha > 0$ (notation: $P(G > \cdot) \in RV_{-\alpha}$) is equivalent to multivariate regular variation of Z with the same index α. We refer to Resnick (1987) for the definition and properties of multivariate regular variation. This implies, in particular, that the correlation matrix and the index α of regular variation are copula parameters.

Further, we denote the upper tail dependence coefficient between Z_i and Z_j as

$$\lambda_{ij}^Z(1,1) = \left(\int_{(\pi/2 - \arcsin \rho_{ij})/2}^{\pi/2} (\cos\phi)^\alpha \, d\phi \right) \Big/ \left(\int_0^{\pi/2} (\cos\phi)^\alpha \, d\phi \right) \tag{1.3}$$

when $P(G > \cdot) \in RV_{-\alpha}$; in this case it is positive (cf. Hult and Lindskog (2002), Theorem 4.3).

For illustration of our methodology, we focus on the case $d = 2$ from now on and the extension to $d > 2$ is given in section 5. Klüppelberg, Kuhn and Peng (2005) studied two estimators for estimating the tail copula $\lambda^X(x,y)$ as defined in (1.1), when observations have an elliptical distribution; i.e., $X \stackrel{\mathrm{d}}{=} Z$ with Z defined in (1.2) and $P(G > \cdot) \in RV_{-\alpha}$ for some $\alpha > 0$. One estimator is based on extreme value theory, another one on an extended version of (1.3); i.e., denoting $\lambda^Z_{12} = \lambda^Z$ and $\rho_{12} = \rho$,

$$\lambda^Z(x,y) = \left(\int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2} x(\cos\phi)^\alpha \,\mathrm{d}\phi + \int_{-\arcsin\rho}^{g\left((x/y)^{1/\alpha}\right)} y\left(\sin(\phi + \arcsin\rho)\right)^\alpha \,\mathrm{d}\phi \right) \times \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi \right)^{-1} := \lambda(\alpha; x, y, \rho), \tag{1.4}$$

where $g(t) := \arctan\left((t-\rho)/\sqrt{1-\rho^2}\right) \in [-\arcsin\rho, \pi/2]$ for $t > 0$. Note that in this setup α can be estimated from observations.

Here we propose to model only the copula C (not the full distribution) of X by the copula of Z with $P(G > \cdot\,) \in RV_{-\alpha}$, i.e.,

$$P\left(F_1(X_1) \le x, F_2(X_2) \le y\right) = P\left(F_1^Z(Z_1) \le x, F_2^Z(Z_2) \le y\right), \tag{1.5}$$

where F_1^Z and F_2^Z denote the marginal distributions of Z.

In our approach, the copula C is not completely determined, since we only work with the tail information (the regular variation) of G. Without doubt, how to test the above model assumptions is important, and will be investigated in a separate paper. In the present paper, we focus on the estimation issue, i.e., seeking a way to improve the *empirical tail copula estimator*. For iid data $X_i = (X_{i1}, X_{i2})$ for $i = 1, \ldots, n$, with unknown distribution function F and tail copula as in (1.1) the empirical tail copula estimator is defined as

$$\widehat{\lambda}^{\mathrm{emp}}(x,y;k) = \frac{1}{k}\sum_{i=1}^n I\left(1 - \widehat{F}_1(X_{i1}) \le \frac{k}{n}x,\ 1 - \widehat{F}_2(X_{i2}) \le \frac{k}{n}y\right), \tag{1.6}$$

where $\widehat{F}_j$ denotes the empirical distribution function of $\{X_{ij}\}_{i=1}^n$ for $j = 1, 2$ and we consider $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$.

A natural way to improve the empirical tail copula estimator is to employ (1.4) like Klüppelberg, Kuhn and Peng (2005). However, α can not be estimated directly from the observations under the model assumptions. Hence, we propose to estimate α first by using (1.4) with the empirical tail copula and an estimator for ρ. Then we estimate the tail copula λ by plugging in the estimators for α and ρ; see section 2 for details. Some theoretical comparisons are provided in section 3. We present a simulation study in section 4. The generalization to higher dimension is discussed in section 5. Finally, all proofs are summarized in section 6.

2 Methodologies and Main Results

Throughout this section we assume that $d = 2$. Because of (1.5), we can estimate $\lambda^Z(x,y)$ by $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$. It follows from Lindskog, McNeil and Schmock (2003) that condition

$|\rho| < 1$ implies $\tau = \frac{2}{\pi} \arcsin \rho$, where τ is *Kendall's tau*, i.e.

$$\tau \;=\; P\left(\left(X_{11}-X_{21}\right)\left(X_{12}-X_{22}\right)>0\right) - P\left(\left(X_{11}-X_{21}\right)\left(X_{12}-X_{22}\right)<0\right).$$

Hence we can estimate ρ by $\widehat{\rho} = \sin\left(\frac{\pi}{2}\widehat{\tau}\right)$, where

$$\widehat{\tau} \;=\; \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \operatorname{sign}\left((X_{i1}-X_{j1})(X_{i2}-X_{j2})\right). \tag{2.7}$$

In order to estimate α via (1.4), we need to solve this equation as a function of α.

Theorem 2.1. *For any fixed $x, y > 0$ and $|\rho| < 1$, define $\alpha^* := |\ln(x/y)/\ln(\rho \vee 0)|$. Then $\lambda(\alpha; x, y, \rho)$ is strictly decreasing in α for all $\alpha > \alpha^*$.*

Based on the above theorem, we are able to define an estimator for α as follows. Let $\lambda^{\leftarrow}(\,\cdot\,; x, y, \rho)$ denote the inverse of $\lambda(\alpha; x, y, \rho)$ with respect to α, if it exists. By Theorem 2.1, we know that $\lambda^{\leftarrow}(\,\cdot\,; 1, 1, \rho)$ exists for all $\alpha > 0$. Hence, an obvious estimator for α is $\widetilde{\alpha}(1,1,k) := \lambda^{\leftarrow}(\widehat{\lambda}^{\mathrm{emp}}(1,1;k); 1, 1, \widehat{\rho})$ for any estimator $\widehat{\rho}$ of ρ. Since this estimator only employs information at $x = y = 1$, it may not be efficient.

Next we define an estimator which takes also $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)$ for other values $(x, y) \in \mathbb{R}_+^2$ into account. Based on Theorem 2.1 we define corresponding ranges for $y/x = \tan\theta$. To ensure that $(x, y) = (1, 1)$ is taken into account, we look at $(x, y) = (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$ for different angles θ. Note that $\widehat{\lambda}^{\mathrm{emp}}(x, y;\, k) = \widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta;\, k^*)$ for $\theta = \arctan(y/x)$ and some k^*, hence it is sufficient not to consider all $(x, y) \in \mathbb{R}_+^2$ but only $(x, y) = (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$. Define

$$\begin{aligned}
\widehat{Q} \;&:=\; \Big\{\theta \in \Big(0, \frac{\pi}{2}\Big) : \widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k) < \\
&\qquad\qquad < \lambda\left(\left|\frac{\ln(\tan\theta)}{\ln(\widehat{\rho}\vee 0)}\right|;\, \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho}\right)\Big\}, \\
\widehat{Q}^* \;&:=\; \Big\{\theta \in \Big(0, \frac{\pi}{2}\Big) : |\ln(\tan\theta)| < \widetilde{\alpha}(1,1;k)\left(1 - k^{-1/4}\right)|\ln(\widehat{\rho}\vee 0)|\Big\} \quad \text{and} \\
Q^* \;&:=\; \Big\{\theta \in \Big(0, \frac{\pi}{2}\Big) : |\ln(\tan\theta)| < \alpha\,|\ln(\rho\vee 0)|\Big\}.
\end{aligned}$$

It follows from Theorem 1 that there exists a unique $\alpha_1 > |\ln(\tan\theta)/\ln(\widehat{\rho}\vee 0)|$ such that

$$\lambda(\alpha_1; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho}) = \widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k)\,, \quad \theta \in \widehat{Q}\,.$$

Therefore, for $\theta \in \widehat{Q}$ we can define the inverse function of $\lambda(\,\cdot\,; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho})$ giving

$$\widetilde{\alpha}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k) \;=\; \lambda^{\leftarrow}\left(\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k); \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho}\right). \tag{2.8}$$

Next we have to ensure consistency of this estimator. This can be done by further requiring $\theta \in \widehat{Q}^*$, which implies that the true value of α is larger than $|\ln(\tan\theta)/\ln(\widehat{\rho}\vee 0)|$ with probability tending to one. Thus, our estimator for α is defined as a smoothed version of $\widetilde{\alpha}$. That is, for an arbitrary nonnegative weight function w we define

$$\widehat{\alpha}(k,w) = \frac{1}{W\left(\widehat{Q}\cap\widehat{Q}^*\right)}\int_{\theta\in\widehat{Q}\cap\widehat{Q}^*}\widetilde{\alpha}(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta;k)\,W(d\theta)\,, \tag{2.9}$$

where W is the measure defined by w.

Before we give the asymptotic normality of $\widehat{\alpha}$, we list the following regularity conditions:

(C1) X satisfies relation (1.5) and Z has tail dependence function (1.4) and $P(G>\cdot)\in RV_{-\alpha}$ for some $\alpha>0$ and $|\rho|<1$.

(C2) There exists $A(t)\to 0$ such that

$$\lim_{t\to 0}\frac{t^{-1}P(1-F_1(X_1)\le tx, 1-F_2(X_2)\le ty)-\lambda^X(x,y)}{A(t)} \;=\; b_{(C2)}(x,y)$$

uniformly on $\mathcal{S}_2$, where $b_{(C2)}(x,y)$ is not a multiple of $\lambda^X(x,y)$.

(C3) $k=k(n)\to\infty$, $k/n\to 0$ and $\sqrt{k}A(k/n)\to b_{(C3)}\in(-\infty,\infty)$ as $n\to\infty$.

The following theorem gives the asymptotic normality of $\widehat{\alpha}$.

Theorem 2.2. *Suppose that (C1)-(C3) hold, and that w is a positive weight funtion satisfying $\sup_{\theta\in Q^*}w(\theta)<\infty$. Then, denoting by W the measure defined by w, as $n\to\infty$,*

$$\sqrt{k}\left(\widehat{\alpha}(k,w)-\alpha\right)$$

$$\xrightarrow{d}\;\frac{1}{W(Q^*)}\int_{\theta\in Q^*}\frac{b_{(C3)}b_{(C2)}(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)+\widetilde{B}(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)}{\lambda'\left(\alpha;\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,\rho\right)}W(d\theta),$$

where $\lambda'(\alpha;x,y,\rho):=\dfrac{\partial}{\partial\alpha}\lambda(\alpha;x,y,\rho)$,

$$\widetilde{B}(x,y) \;=\; B(x,y)-B(x,0)\left(1-\frac{\partial}{\partial x}\lambda(x,y)\right)-B(0,y)\left(1-\frac{\partial}{\partial y}\lambda(x,y)\right)$$

and $B(x,y)$ is a Brownian motion with zero mean and covariance structure

$$\begin{aligned}E\left(B(x_1,y_1)B(x_2,y_2)\right) \;=\;& x_1\wedge x_2+y_1\wedge y_2-\lambda(x_1\wedge x_2,y_1)-\lambda(x_1\wedge x_2,y_2)\\ &-\lambda(x_1,y_1\wedge y_2)-\lambda(x_2,y_1\wedge y_2)+\lambda(x_1,y_2)+\lambda(x_2,y_1)+\lambda(x_1\wedge x_2,y_1\wedge y_2).\end{aligned}$$

Next, like in Klüppelberg, Kuhn and Peng (2005), we estimate $\widehat{\rho}$ via the identity $\tau=\frac{2}{\pi}\arcsin\rho$ and the estimator (2.7) and obtain an estimator for $\lambda(x,y)$ by

$$\widehat{\lambda}(x,y;k,w) \;=\; \lambda\left(\widehat{\alpha}(k,w);x,y,\widehat{\rho}\right). \tag{2.10}$$

We derive the asymptotic normality of this new estimator $\widehat{\lambda}(x,y;k,w)$ as follows.

Theorem 2.3. *Suppose that the conditions of Theorem 2.2 hold. Then, for $T > 0$, we have as $n \to \infty$,*

$$\sup_{0\le x,y\le T}\left|\sqrt{k}\left(\widehat{\lambda}(x,y;k,w)-\lambda^X(x,y)\right)-\lambda'(\alpha;x,y,\rho)\frac{1}{W(Q^*)}\right.$$
$$\left.\times\int_{\theta\in Q^*}\frac{b_{(C3)}b_{(C2)}(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)+\widetilde{B}(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,t)}{\lambda'(\alpha;\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,\rho)}W(d\theta)\right| = o_p(1).$$

3 Theoretical Comparisons

The following corollary gives the optimal choice of the sample fraction k for $\widehat{\alpha}$ in terms of the *asymptotic mean squared error*. First, denote

$$\mathrm{abias}_\alpha(w) = \frac{1}{W(Q^*)}\int_{\theta\in Q^*}\frac{b_{(C2)}(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)}{\lambda'(\alpha;\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,\rho)}W(d\theta)$$

and

$$\mathrm{avar}_\alpha(w) = \frac{1}{(W(Q^*))^2}\times$$
$$\int_{\theta_1\in Q^*}\int_{\theta_2\in Q^*}\frac{E\left(\widetilde{B}(\sqrt{2}\cos\theta_1,\sqrt{2}\sin\theta_1)\widetilde{B}(\sqrt{2}\cos\theta_2,\sqrt{2}\sin\theta_2)\right)}{\lambda'(\alpha;\sqrt{2}\cos\theta_1,\sqrt{2}\sin\theta_1,\rho)\lambda'(\alpha;\sqrt{2}\cos\theta_2,\sqrt{2}\sin\theta_2,\rho)}W(d\theta_2)W(d\theta_1).$$

Corollary 3.1. *Assume that (C1)-(C3) hold and $A(t) \sim ct^\beta$ as $t \to 0$ for some $c \neq 0$ and $\beta > 0$. Then the asymptotic mean squared error of $\widehat{\alpha}(k,w)$ is*

$$\mathrm{amse}_\alpha(k,w) = c^2(k/n)^{2\beta}\left(\mathrm{abias}_\alpha(w)\right)^2 + k^{-1}\mathrm{avar}_\alpha(w).$$

By minimizing the above asymptotic mean squared error, we obtain the optimal choice of k as

$$k_0(w) = \left(\frac{\mathrm{avar}_\alpha(w)}{2\beta c^2(\mathrm{abias}_\alpha(w))^2}\right)^{1/(2\beta+1)} n^{2\beta/(2\beta+1)}.$$

Hence the optimal asymptotic mean squared error of $\widehat{\alpha}$ is

$$\mathrm{amse}_\alpha(k_0(w),w) = \left(\left(\frac{\mathrm{avar}_\alpha(w)}{n}\right)^\beta \mathrm{abias}_\alpha(w)c\sqrt{2\beta}\right)^{2/(2\beta+2)}\left(1+\frac{1}{2\beta}\right).$$

Firstly, we compare $\widehat{\alpha}(k,w)$ with $\widetilde{\alpha}(1,1;k)$. As a first weight function we choose $w_0(\theta)$ equal to one if $\theta = \pi/4$, and equal to zero otherwise. Since $\widetilde{\alpha}(1,1;k) = \widehat{\alpha}(k,w_0)$, the asymptotic variance and optimal asymptotic mean squared error of $\widetilde{\alpha}(1,1;k)$ are

$$\mathrm{avar}_\alpha(w_0) = k^{-1}\mathrm{avar}_\alpha(w_0) \quad \text{and} \quad \mathrm{amse}_\alpha(w_0) = \mathrm{amse}_\alpha(k_0(w_0),w_0)\,.$$

For simplicity, we only compare $\widehat{\alpha}(k,w_0)$ and $\widehat{\alpha}(k,w_1)$ with the weight function

$$w_1(\theta) \quad = \quad 1-\left(\frac{\theta}{\pi/4}-1\right)^2, \quad 0\le\theta\le\frac{\pi}{2}. \tag{3.11}$$

In Figure 1, we plot the ratio $\mathrm{ratio}_{\mathrm{var},\alpha}=\mathrm{avar}_\alpha(w_1)/\mathrm{avar}_\alpha(w_0)$ against α for $\rho\in\{0.3,0.7\}$, which shows that $\widehat{\alpha}(k,w_1)$ has a smaller variance than $\widetilde{\alpha}(1,1;k)$ in many cases, especially when α is large or ρ is small. Hence $\widehat{\alpha}(k,w_1)$ is better than $\widetilde{\alpha}(1,1;k)$ in terms of asymptotic variance. Without doubt, the weight function w_1 is not an optimal one. Seeking an optimal weight function is important, but difficult.

Secondly, we compare $\widehat{\lambda}(x,y;k,w)$ with $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$. It follows from Theorem 2.3 that the asymptotic variance and the asymptotic mean squared error of $\widehat{\lambda}(x,y;k,w)$ are

$$(\lambda'(\alpha;x,y,\rho))^2\,\mathrm{avar}_\alpha(k,w) \quad \text{and} \quad (\lambda'(\alpha;x,y,\rho))^2\,\mathrm{amse}_\alpha(k,w),$$

respectively. As before, we obtain the optimal asymptotic mean squared error of $\widehat{\lambda}(x,y;k,w)$ as $(\lambda'(\alpha;x,y,\rho))^2\,\mathrm{amse}_\alpha(k_0(w),w)$. Put

$$\begin{aligned} k_{\mathrm{emp}} &= \left(\frac{E(B^2(x,y))}{2\beta c^2(b_{(C2)}(x,y))^2}\right)^{1/(2\beta+1)} n^{2\beta/(2\beta+1)} \quad \text{and} \\ \mathrm{amse}_{\mathrm{emp}}(k) &= c^2(k/n)^{2\beta}(b_{(C2)}(x,y))^2+k^{-1}E(B^2(x,y)). \end{aligned}$$

Then the asymptotic variance and the optimal asymptotic mean squared error of $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$ are

$$\mathrm{avar}_{\lambda^{\mathrm{emp}}}(k,w)=k^{-1}(E\widetilde{B}(x,y))^2 \quad \text{and} \quad \mathrm{amse}_{\lambda^{\mathrm{emp}}}(k,w)=\mathrm{amse}_{\mathrm{emp}}(k_{\mathrm{emp}})\,.$$

In Figure 2, we plot the ratio of the variances of $\widehat{\lambda}(x,y;w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$ given by

$$\mathrm{ratio}_{\mathrm{var},\lambda} \quad = \quad \frac{E(B^2(x,y))}{(\lambda'(\alpha;x,y,\rho))^2\,\mathrm{avar}_\alpha(w_1)},$$

for $(x,y)=(\sqrt{2}\cos\phi,\sqrt{2}\sin\phi)$ against $\phi\in(0,\pi/2)$ for different pairs $(\alpha,\rho)\in\{1,5\}\times\{0.3,0.7\}$, which shows that the new estimator for $\lambda^X(x,y)$ has a smaller variance than the empirical estimator $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$.

4 Simulation Study

In this section we conduct a simulation study to compare $\widehat{\alpha}(k,w_1)$ with $\widehat{\alpha}(k,w_0)=\widetilde{\alpha}(1,1,k)$, and to compare $\widehat{\lambda}(x,y;k,w_1)$ with $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$ by drawing 1000 random samples with sample size $n=3000$ from an elliptical copula with $P(G>x)=\exp\{-x^{-\alpha}\}$, $x>0$.

For comparison of $\widehat{\alpha}(k, w_1)$ and $\widetilde{\alpha}(1,1,k)$, we plot the averages of $\widetilde{\alpha}(1,1,k)$, $\widehat{\alpha}(k, w_1)$ and corresponding mean squared errors in Figures 3 and 4. We observe that $\widehat{\alpha}(k, w_1)$ has a smaller mean squared error than $\widetilde{\alpha}(1,1,k)$ in most cases. Further, we plot $\widetilde{\alpha}(1,1,k)$ and $\widehat{\alpha}(k, w_1)$ based on a particular sample in Figure 7, which shows that $\widehat{\alpha}(k, w_1)$ is much smoother than $\widetilde{\alpha}(1,1,k)$ with respect to k. This is because $\widehat{\alpha}(k, w_1)$ employs more $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)'s$ and $\widetilde{\alpha}(1,1,k)$ only uses $\widehat{\lambda}^{\mathrm{emp}}(1,1;k)$. In summary, one may prefer $\widehat{\alpha}(k, w_1)$ to $\widetilde{\alpha}(1,1,k)$.

Next we compare the empirical estimator $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)$ with the new $\widehat{\lambda}(x, y; k, w_1)$. We plot the averages of $\widehat{\lambda}^{\mathrm{emp}}(1,1;k)$, $\widehat{\lambda}(1,1,k,w_1)$ and corresponding mean squared errors in Figures 5 and 6. We also plot estimators $\widehat{\lambda}^{\mathrm{emp}}(1,1;k)$ and $\widehat{\lambda}(1,1;k,w_1)$ based on a particular sample in Figure 8. Like the comparisons for estimators of α, we observe that $\widehat{\lambda}(1,1;k,w_1)$ has a slightly smaller mean squared error than $\widehat{\lambda}^{\mathrm{emp}}(1,1;k)$, but $\widehat{\lambda}(1,1;k,w_1)$ is much smoother than $\widehat{\lambda}^{\mathrm{emp}}(1,1;k)$ with respect to k. More improvement of $\widehat{\lambda}(x,y;k,w_1)$ over $\widehat{\lambda}^{\mathrm{emp}}(x,y;k,w_0)$ are found when x/y is away from one; see Figures 9 and 10.

Finally, we compare $\widehat{\lambda}(x,y;50,w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(x,y;50,w_0)$ for different x and y. It follows from Figure 5 that $k = 50$ is a reasonable choice. Again, we plot the averages of $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$, $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ for $0 \le \phi \le \pi/2$ and corresponding mean squared errors in Figures 11 and 12. Based on a particular sample, we also plot estimators $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ in Figure 13. From these figures, we observe that, when ϕ is away from $\pi/4$, $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ becomes much better than $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$.

In conclusion, with the help of an elliptical copula, we are able to estimate the tail dependence function more efficiently.

5 Elliptical Copula of Arbitrary Dimension

In this section we generalize our results in section 2 to the case, where the dimension $d \ge 2$ is arbitrary.

Theorem 5.1. *Assume that $X = (X_1, \ldots, X_d)^T$ has the same copula as the elliptical vector $Z = (Z_1, \ldots, Z_d)^T$, whose distribution is given in (1.2). W.l.o.g. assume that $AA^T = R$ is the correlation matrix of Z. Let $A_{i\,.}$ denote the i-th row of A and and let F_U denote the uniform distribution on $\mathcal{S}_d$. Then the tail copula of X is given by*

$$\begin{aligned} \lambda^X(x_1, \ldots, x_d) &:= \lim_{t\to 0} t^{-1} P\left(1 - F_1(X_1) < tx_1, \ldots, 1 - F_d(X_d) < tx_d\right) \\ &= \int\limits_{u \in \mathcal{S}_d, A_{1\,.}\,u>0, \ldots, A_{d\,.}\,u>0} \bigwedge_{i=1}^{d} x_i (A_{i\,.}\,u)^\alpha \, dF_U(u) \Big(\int\limits_{u \in \mathcal{S}_d, A_{1\,.}\,u>0} (A_{1\,.}\,u)^\alpha \, dF_U(u) \Big)^{-1}. \end{aligned} \tag{5.12}$$

Remark 5.2. (a) For $d = 2$ representation (5.12) coincides with (1.4). To see this write $u \in \mathcal{S}_2$ as $u = (\cos\phi, \sin\phi)^T$ for some $\phi \in (-\pi, \pi)$, $A_{1\cdot} = (1, 0)$ and $A_{2\cdot} = (\rho, \sqrt{1-\rho^2})$. Then, $Au = (\cos\phi, \rho\cos\phi + \sqrt{1-\rho^2}\sin\phi)^T = (\cos\phi, \sin(\phi + \arcsin\rho))^T$, giving the equivalence of (5.12) and (1.4).
(b) For $d \geq 3$ one can also use multivariate polar coordinates and obtain analogous representations. The expression, however, becomes much more complicated.

The estimation procedure in d dimensions is a simple extension of the two-dimensional case. Assume iid observations $X_i = (X_{i1}, \ldots, X_{id})^T$, $i = 1, \ldots, n$, with an elliptical copula. Then we can estimate ρ_{pq} via Kendall's τ and α_{pq} based on bivariate subvectors (X_{ip}, X_{iq}) for $1 \leq p, q \leq d$. Denote these estimators by $\widehat{\rho}_{pq}$ and (for any positive weight function w) $\widehat{\alpha}_{pq}(k, w)$, respectively. Then we estimate α and R by

$$\widehat{\alpha}(k, w) = \frac{1}{d(d-1)} \sum_{p \neq q} \widehat{\alpha}_{pq}(k, w) \quad \text{and} \quad \widehat{R} = (\widehat{\rho}_{pq})_{1 \leq p, q \leq d}.$$

For any decomposition $\widehat{A}\widehat{A}^T = \widehat{R}$, we obtain an estimator for A. This yields an estimator for $\lambda(x_1, \ldots, x_d)$ by replacing α and $A_{i\cdot}$ in (5.12) by $\widehat{\alpha}(k, w)$ and $\widehat{A}_{i\cdot}$, respectively. The asymptotic normality of this new estimator can be derived similarly as in Theorems 2.2 and 2.3.

In Figure 14 we give a three-dimensional example. We simulate a sample of length $n = 3\,000$ from an elliptical copula with $P(G > x) = \exp\{-x^{-\alpha}\}$, $x > 0$, and parameters $\rho_{12} = 0.3$, $\rho_{13} = 0.5$, $\rho_{23} = 0.7$ and $\alpha = 5$. In the upper row we plot the true tail copula $\lambda^X\left(\sqrt{3}\cos\phi_1, \sqrt{3}\sin\phi_1\cos\phi_2, \sqrt{3}\sin\phi_1\sin\phi_2\right)$, $\phi_1, \phi_2 \in (0, \pi/2)$, and each column corresponds to perspective, contour and grey-scale image plot of a λ^X, respectively. In the middle and lower row, we plot the corresponding estimators $\widehat{\lambda}(\ldots; 100, w_1)$ and $\widehat{\lambda}^{\text{emp}}(\ldots; 100)$, respectively. From this figure, we also observe that $\widehat{\lambda}$ becomes much better than $\widehat{\lambda}^{\text{emp}}$ in the three-dimensional case.

Next we apply our estimators to a three-dimensional real data set which consists of $n = 4\,903$ daily log returns of currency exchange rates of GBP, USD and CHF with respect to EURO between May 1985 and June 2004. As in Figure 14, we plot the perspective, contour and grey-scale image of $\widehat{\lambda}\left(\sqrt{3}\cos\phi_1, \sqrt{3}\sin\phi_1\cos\phi_2, \sqrt{3}\sin\phi_1\sin\phi_2;\, k, w_1\right)$ and $\widehat{\lambda}^{\text{emp}}(\ldots;\, k)$; see Figures 15, 16 and 17 for $k = 100$, $k = 150$ and $k = 200$, respectively. Comparing the contour plots (middle columns) of $\widehat{\lambda}$ and $\widehat{\lambda}^{\text{emp}}$, one may conclude that the assumption of an elliptical tail copula ist not an unrealistic restriction.

6 Proofs

Proof of Theorem 2.1. Define

$$c_0 = \int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \, d\phi, \qquad c_1 = \int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \ln(\cos\phi)\, d\phi,$$

$$\begin{aligned} D(\alpha, z) &= c_0 \int_z^{\pi/2} (\cos\phi)^\alpha \ln(\cos\phi)\, d\phi - c_1 \int_z^{\pi/2} (\cos\phi)^\alpha \, d\phi \quad \text{and} \\ C(\alpha, z) &= D(\alpha, z) + \left(\rho + \sqrt{1-\rho^2}\tan z\right)^{-\alpha} D(\alpha, -z + \arccos\rho). \end{aligned}$$

Then, by variable transformation, we obtain

$$\lambda(\alpha; x, y, \rho) = c_0^{-1} \left(x \int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2} (\cos\phi)^\alpha \, \mathrm{d}\phi + y \int_{g\left((x/y)^{-1/\alpha}\right)}^{\pi/2} (\cos\phi)^\alpha \, \mathrm{d}\phi \right)$$

and

$$\begin{aligned} \lambda'(\alpha; x, y, \rho) &:= \frac{\partial}{\partial\alpha} \lambda(\alpha; x, y, \rho) = c_0^{-2} \left[x D\left(\alpha, g\left((x/y)^{1/\alpha}\right)\right) + y D\left(\alpha, g\left((x/y)^{-1/\alpha}\right)\right)\right] \\ &= c_0^{-2} x C\left(\alpha, g\left((x/y)^{1/\alpha}\right)\right). \end{aligned}$$

Since

$$D_{0,1}(\alpha, z) := \frac{\partial}{\partial z} D(\alpha, z) = (\cos z)^\alpha \left(c_1 - c_0 \ln(\cos z)\right),$$

we can show that there exists $0 < z_0 < \pi/2$ such that

$$\begin{cases} D_{0,1}(\alpha, z) > 0, & \text{if } z \in (-\pi/2, -z_0), \\ D_{0,1}(\alpha, z) = 0, & \text{if } z = -z_0, \\ D_{0,1}(\alpha, z) < 0, & \text{if } z \in (-z_0, z_0), \\ D_{0,1}(\alpha, z) = 0, & \text{if } z = z_0, \\ D_{0,1}(\alpha, z) > 0, & \text{if } z \in (z_0, \pi/2). \end{cases}$$

Note that z_0 depends on α. Since $D(\alpha, 0) = \lim_{z \to \pm\pi/2} D(\alpha, z) = 0$, we have

$$\begin{cases} D(\alpha, z) > 0, & \text{if } z \in (-\pi/2, 0), \\ D(\alpha, z) < 0, & \text{if } z \in (0, \pi/2). \end{cases}$$

Hence, if $x/y \in \left[(\rho \vee 0)^{\alpha^*}, (\rho \vee 0)^{-\alpha^*}\right]$ for some $\alpha^* \in (0, \infty)$, then $C\left(\alpha, g\left((x/y)^{1/\alpha^*}\right)\right) < 0$ for all $\alpha > \alpha^*$. Since also $x/y \in [(\rho \vee 0)^\alpha, (\rho \vee 0)^{-\alpha}]$ holds for all $\alpha > \alpha^*$, we have $C\left(\alpha, g\left((x/y)^{1/\alpha}\right)\right) < 0$ for all $\alpha > \alpha^*$. Hence the theorem follows by choosing $\alpha^* = |\ln(x/y)/\ln(\rho \vee 0)|$.

Proof of Theorem 2.2. Using the same arguments as in Lemma 1 (Page 30) of Huang (1992) or Corollary 3.8 of Einmahl (1997), we can show that

$$\sup_{0<x,y<T}\left|\sqrt{k}\left(\widehat{\lambda}^{\mathrm{emp}}(x,y)-\lambda^X(x,y)\right)-b_{(C3)}b_{(C2)}(x,y)-\widetilde{B}(x,y)\right|=o_p(1) \tag{6.13}$$

as $n\to\infty$. Note that the above equation can also be shown in a way similar to Schmidt and Stadtmüller (2005) by taking the bias term into account. Since $\lambda(\alpha;x,y,\rho)$ in (1.4) is a continuous function of α, by invoking the delta method, the theorem follows from (6.13), $\widehat{\tau}-\tau=o_p(1/\sqrt{k})$ (see e.g. Hoeffding (1948)), $\sup_{\theta\in Q^*}|\lambda'(\alpha;\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,\rho)|<\infty$ and a Taylor expansion.

Proof of Theorem 2.3. It easily follows from (1.4) and Theorem 2.2.

Proof of Theorem 5.1. Since copulas are invariant under strictly increasing transformations, we can assume w.l.o.g that $AA^T=R$ is the correlation matrix. Therefore, the $Z_i\overset{\mathrm{d}}{=}RA_i\,.U$, $1\le i\le d$, have the same distribution, say F_Z. Hence

$$\begin{aligned}
&P\left(1-F_Z(Z_1)<tx_1,\ldots,1-F_Z(Z_d)<tx_d\right)\\
&=\int\limits_{u\in\mathcal{S}_d,A_1\,.\,u>0,\ldots,A_d\,.\,u>0} P\Big(G>\bigvee_{i=1}^{d}\frac{F_Z^{\leftarrow}(1-tx_i)}{A_i\,.\,u}\Big)\,dF_U(u),
\end{aligned} \tag{6.14}$$

where $F_Z^{\leftarrow}$ denotes the inverse function of F_Z. Since $P(G>\ \cdot)\in RV_{-\alpha}$ implies that $1-F_Z\in RV_{-\alpha}$, the inverse function $F_Z^{\leftarrow}$ is regularly varying in 0 with index $-1/\alpha$ (e.g. Resnick (1987), Proposition 0.8(v)). This implies

$$\lim_{t\to 0}\frac{P\left(G>F_Z^{\leftarrow}(1-tx_i)/(A_i\,.\,u)\right)}{P\left(G>F_Z^{\leftarrow}(1-t)\right)}=x_i(A_i\,.\,u)^{\alpha},\quad i=1,\ldots,d.$$

Now note that, for all $i=1,\ldots,d$,

$$\begin{aligned}
t &= P(Z_i>F_Z^{\leftarrow}(1-t)) \;=\; P(GA_i.U>F_Z^{\leftarrow}(1-t))\\
&= \int\limits_{u\in\mathcal{S}_d,A_i\,.\,u>0} P\left(G>\frac{F_Z^{\leftarrow}(1-t)}{A_i\,.\,u}\right)\,dF_U(u),
\end{aligned}$$

giving by means of Potter's bounds (e.g. see (1.20) in Geluk and de Haan (1987)),

$$\begin{aligned}
\lim_{t\to 0}\frac{t}{P\left(G>F_Z^{\leftarrow}(1-t)\right)} &= \lim_{t\to 0}\int\limits_{u\in\mathcal{S}_d,A_i\,.\,u>0}\frac{P(G>F_Z^{\leftarrow}(1-t)/(A_i\,.\,u))}{P\left(G>F_Z^{\leftarrow}(1-t)\right)}\,dF_U(u)\\
&= \int\limits_{u\in\mathcal{S}_d,A_i\,.\,u>0}(A_i\,.\,u)^{\alpha}\,dF_U(u)\quad\forall i=1,\ldots,d.
\end{aligned} \tag{6.15}$$

Applying the same method to (6.14) yields the proof.

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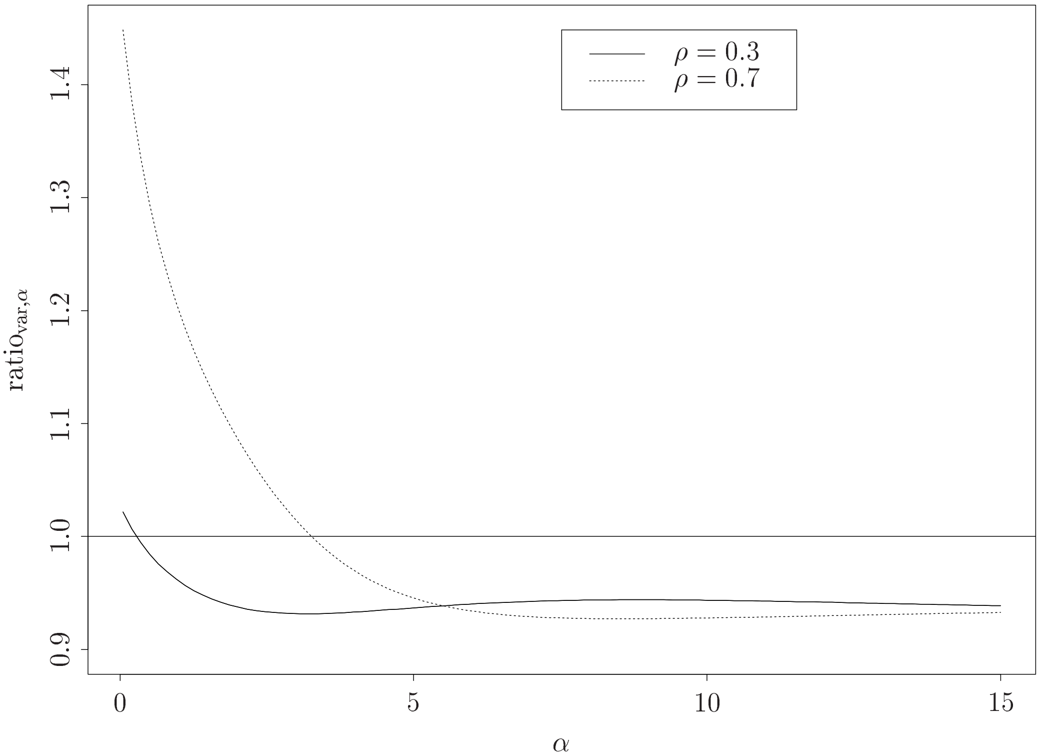

Figure 1: Theoretical ratios, ratio$_{\mathrm{var},\alpha}$, are plotted against α for $\rho = 0.3$ and 0.7.

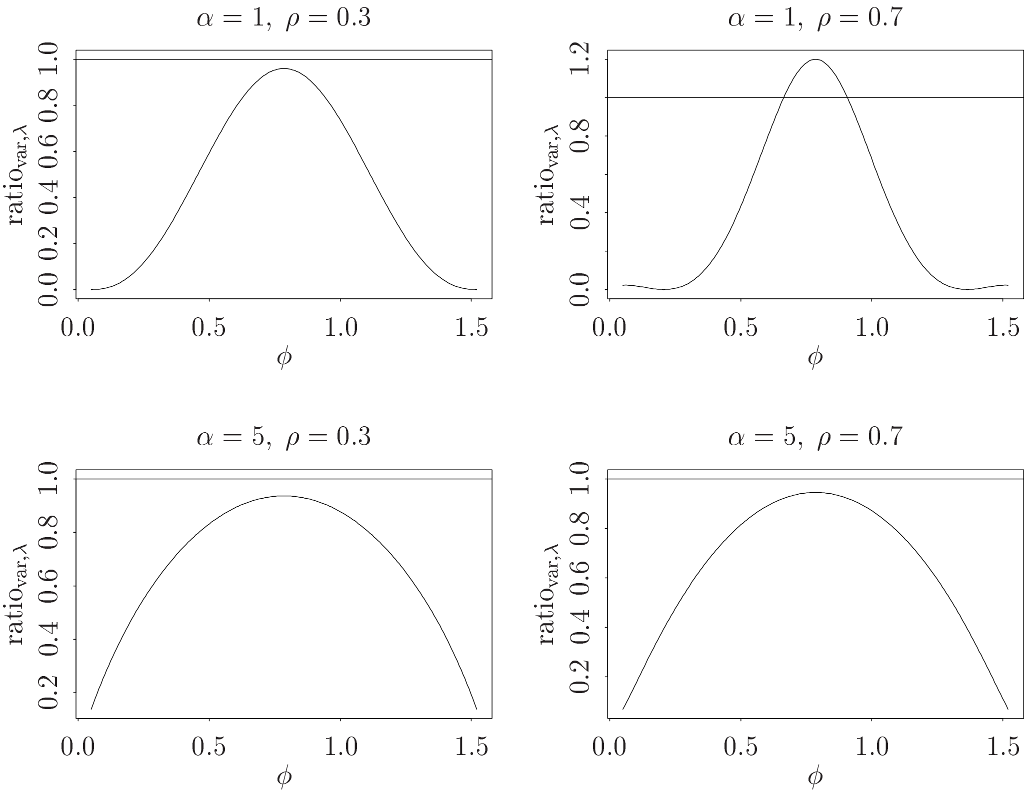

Figure 2: Theoretical ratios, ratio$_{\mathrm{var},\lambda}$, are plotted against $\phi \in (0, \pi/2)$ for $(\alpha, \rho) \in \{1, 5\} \times \{0.3, 0.7\}$.

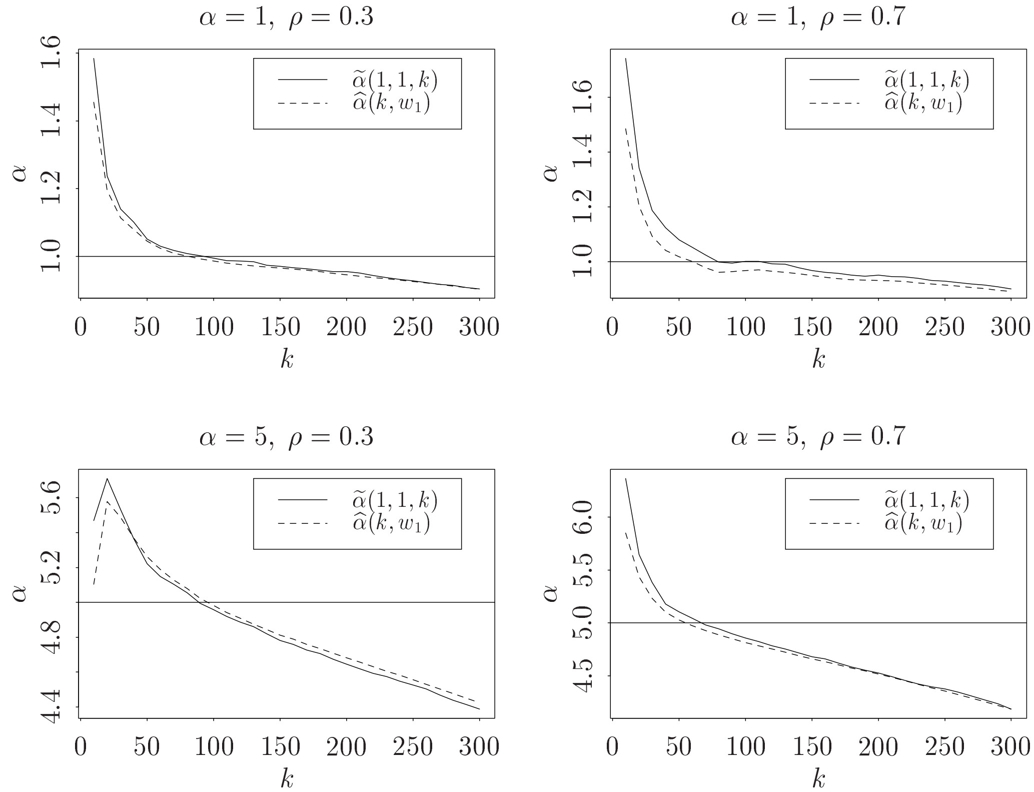

Figure 3: Averages of $\widetilde{\alpha}(1,1,k)$ and $\widehat{\alpha}(k,w_1)$ are plotted against $k = 10, 20, \ldots, 300$.

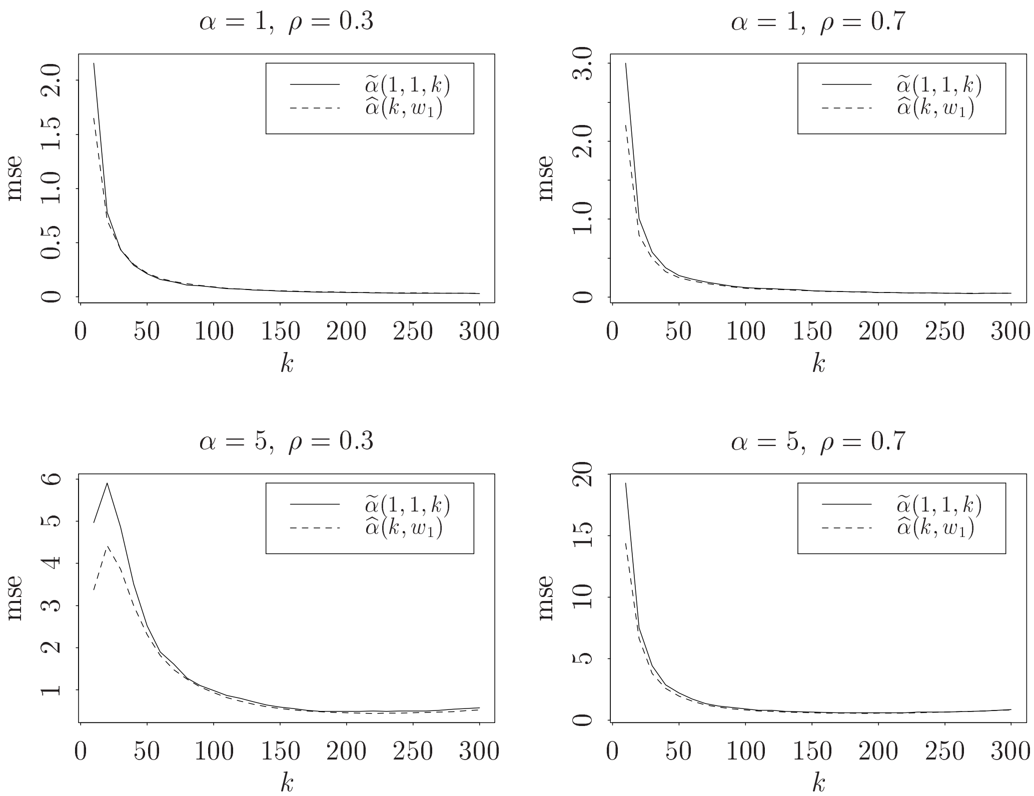

Figure 4: Estimated mean squared errors of estimators in Figure 3.

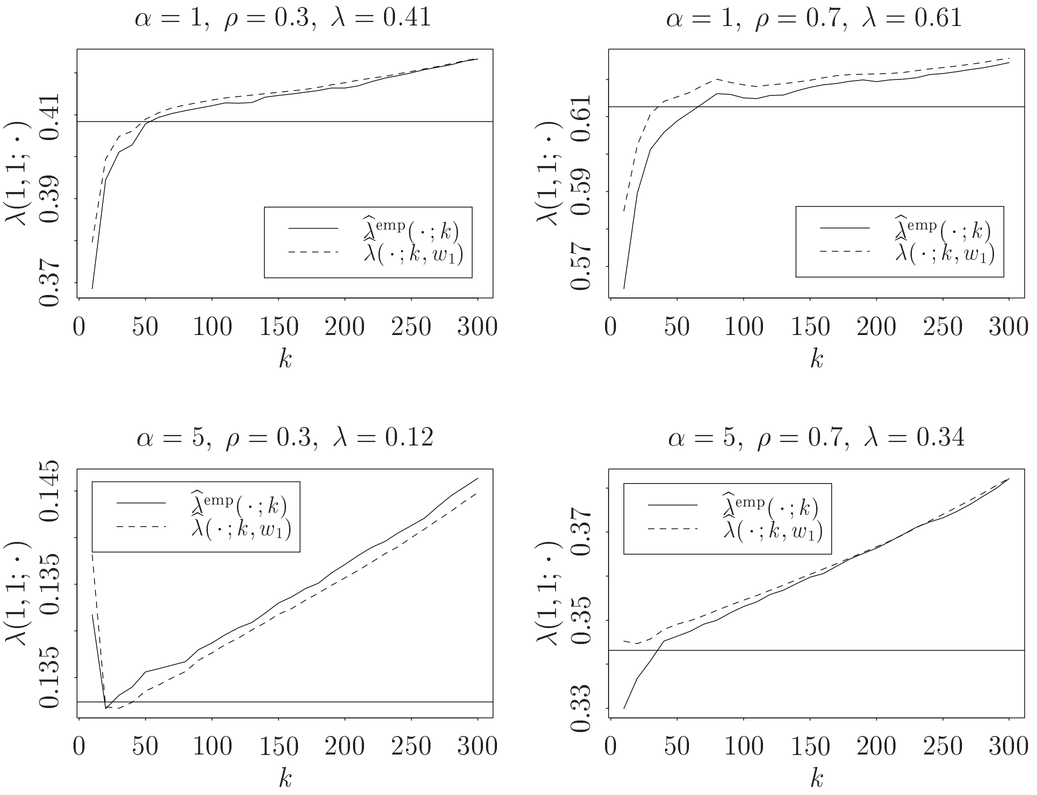

Figure 5: Averages of $\widehat{\lambda}^{\mathrm{emp}}(1,1;k)$ and $\widehat{\lambda}(1,1;k,w_1)$ are plotted against $k = 10, 20, \ldots, 300$.

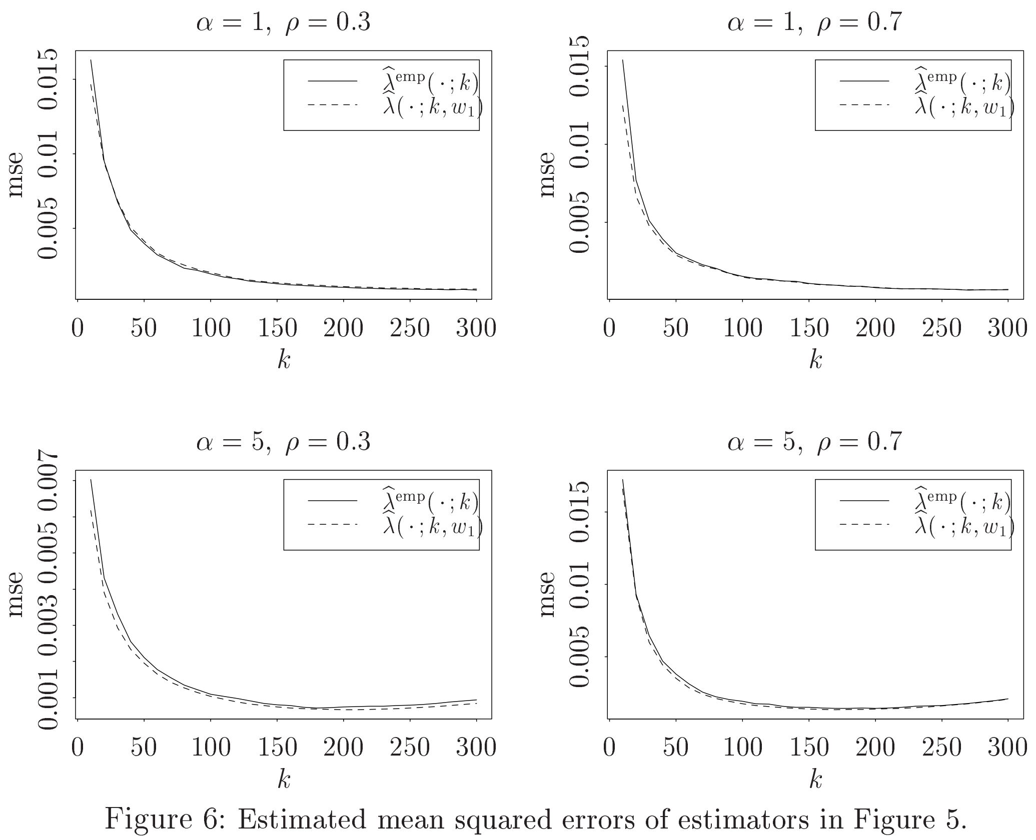

Figure 6: Estimated mean squared errors of estimators in Figure 5.

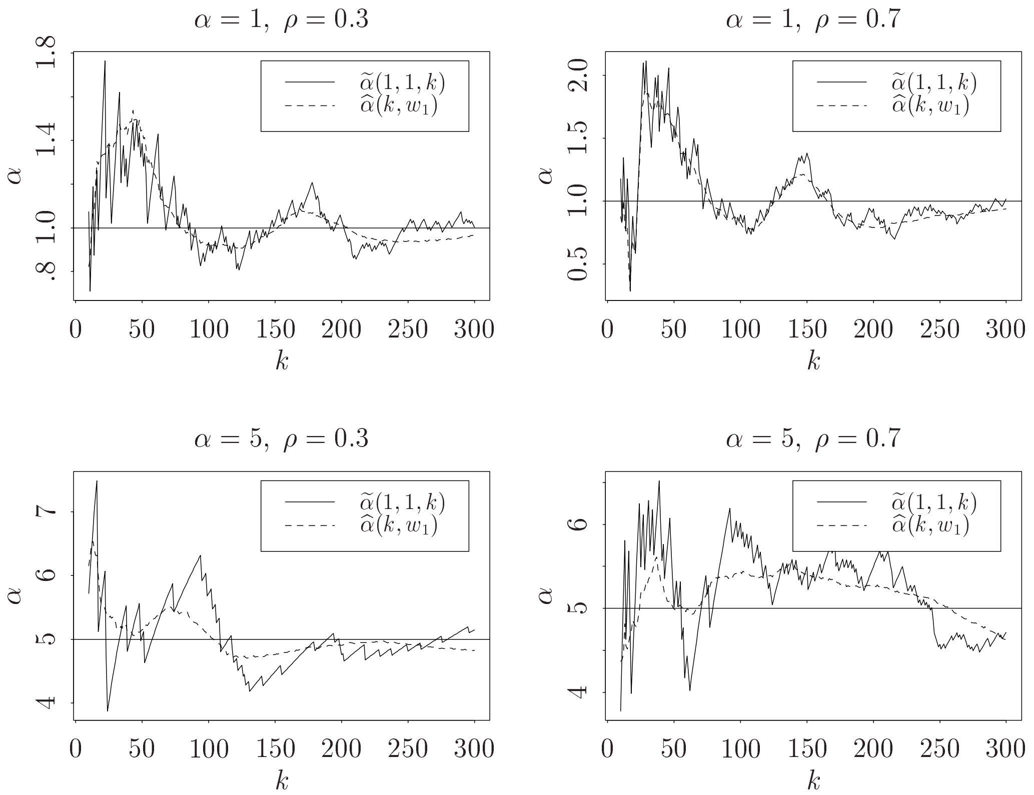

Figure 7: Estimators $\widetilde{\alpha}(1, 1, k)$ and $\widehat{\alpha}(k, w_1)$ based on a particular sample are plotted against $k = 10, 11, \ldots, 300$.

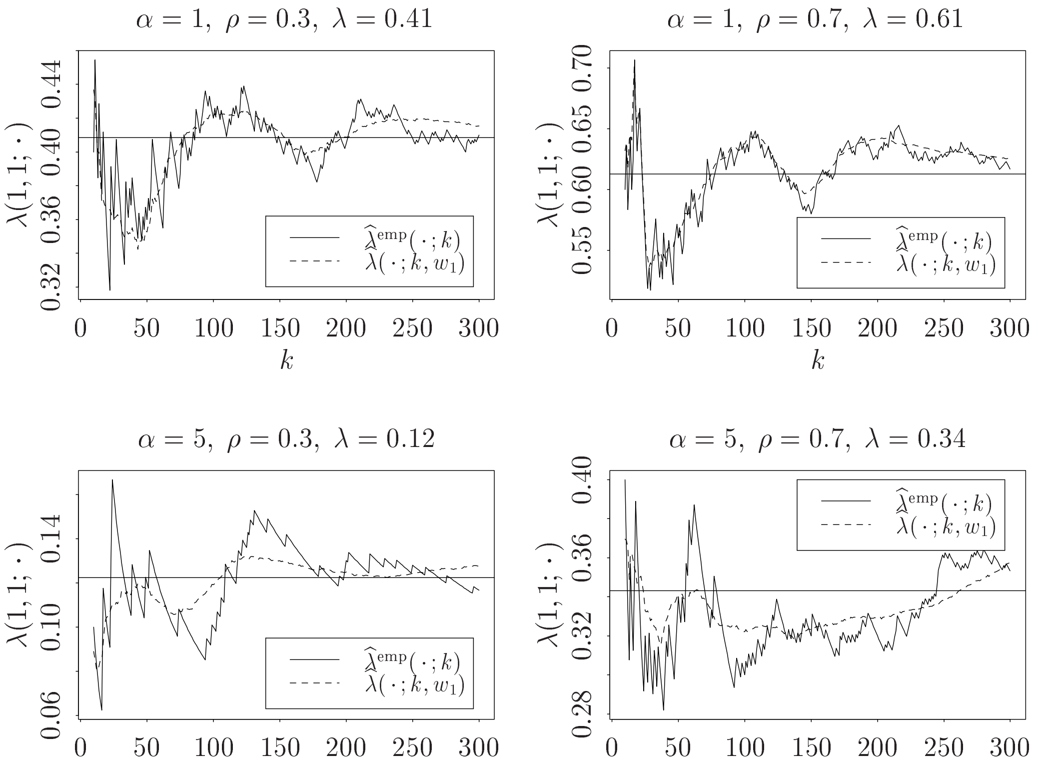

Figure 8: Estimators $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$ and $\widehat{\lambda}(1, 1; k, w_1)$ based on a particular sample are plotted against $k = 10, 11, \ldots, 300$.

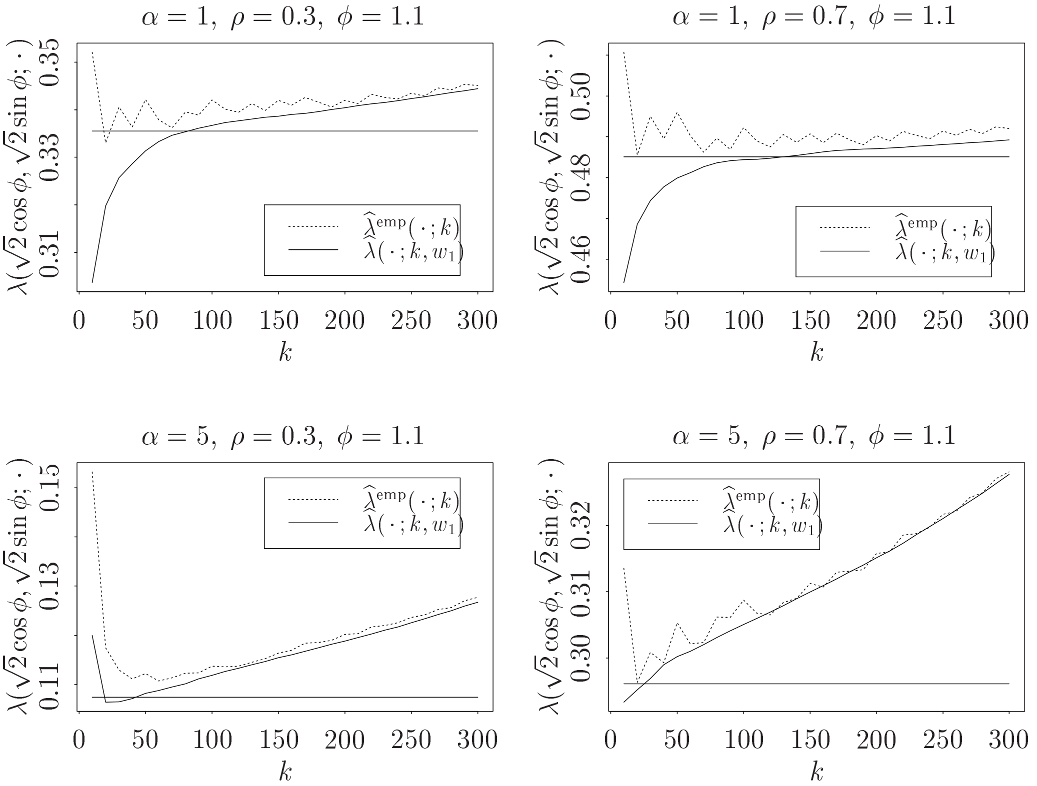

Figure 9: Averages of $\widehat{\lambda}^{\text{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; k)$ and $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; k, w_1)$ with $\phi = 1.1$ are plotted against $k = 10, 20, \ldots, 300$.

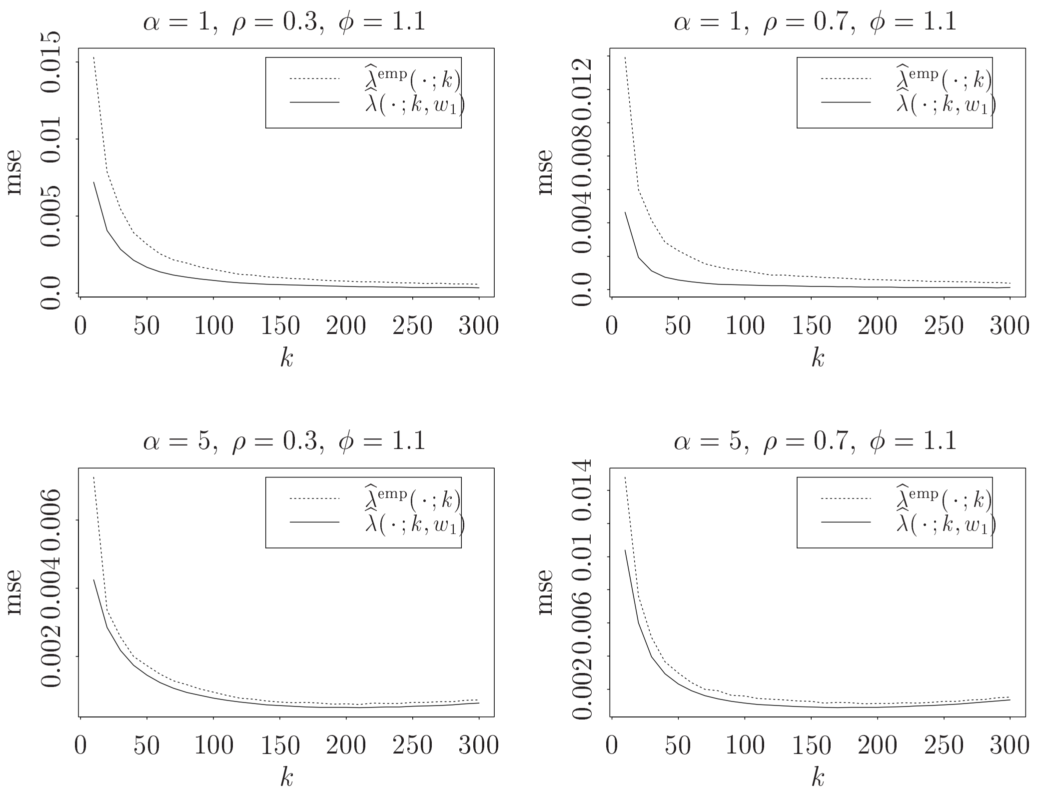

Figure 10: Estimated mean squared errors of estimators in Figure 9.

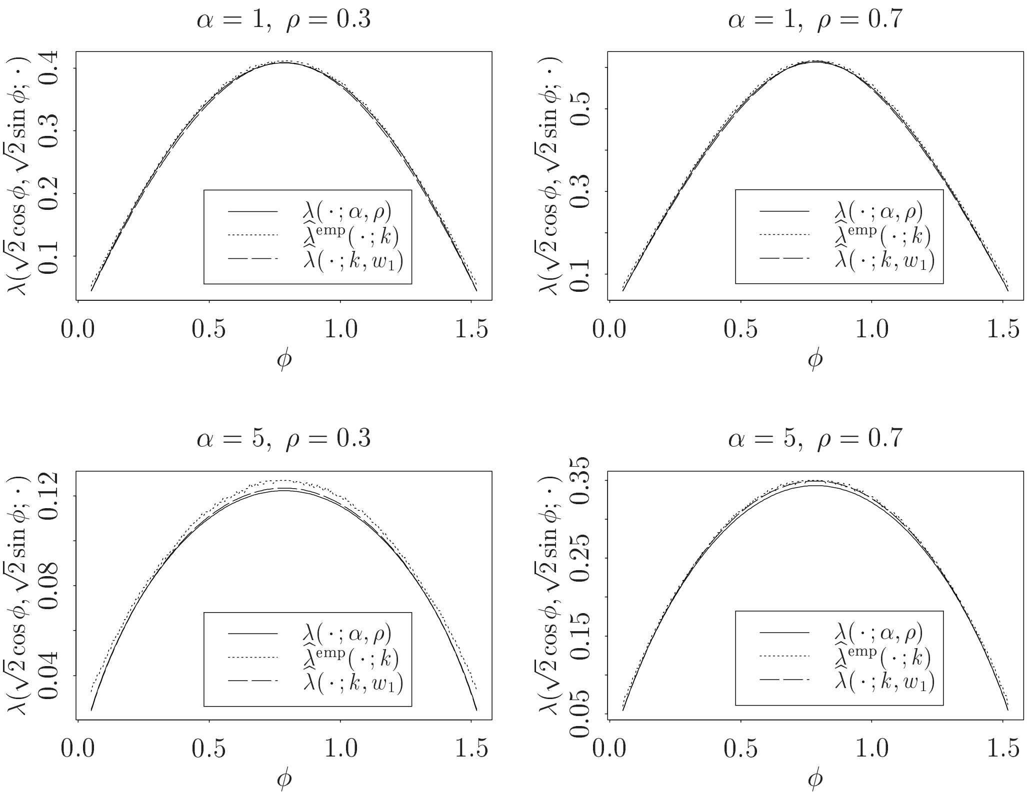

Figure 11: Averages of $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ and $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ are plotted against $\phi \in (0, \pi/2)$.

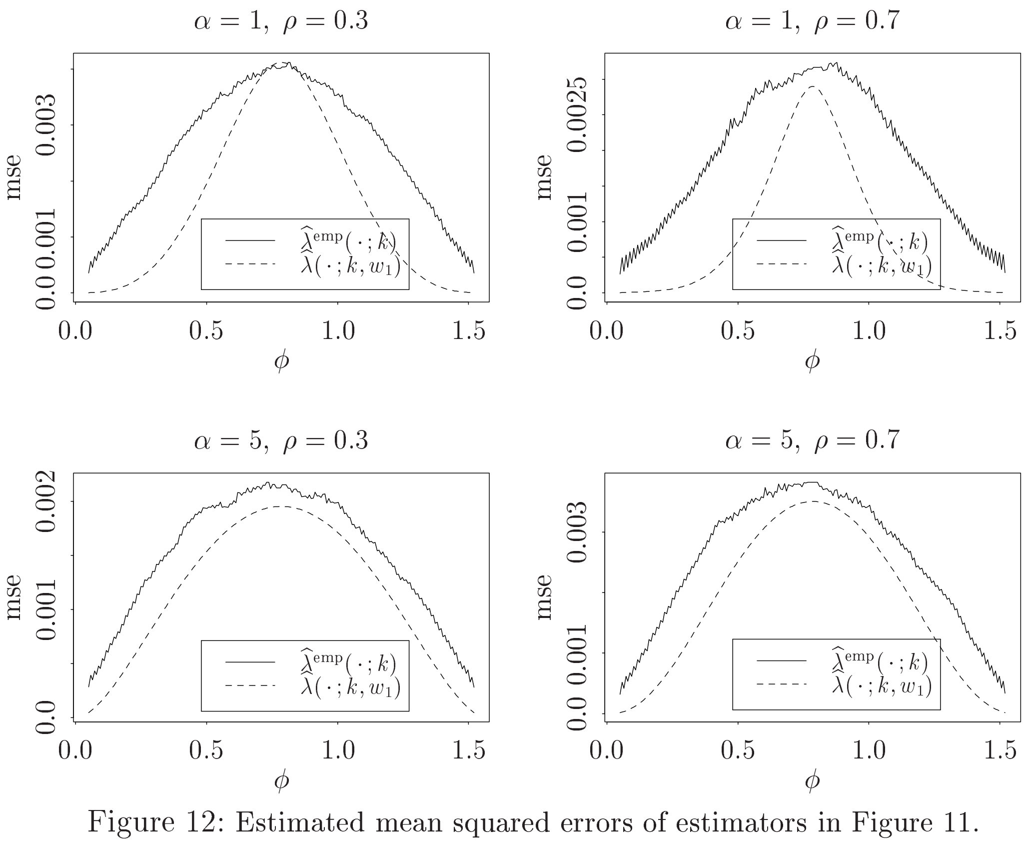

Figure 12: Estimated mean squared errors of estimators in Figure 11.

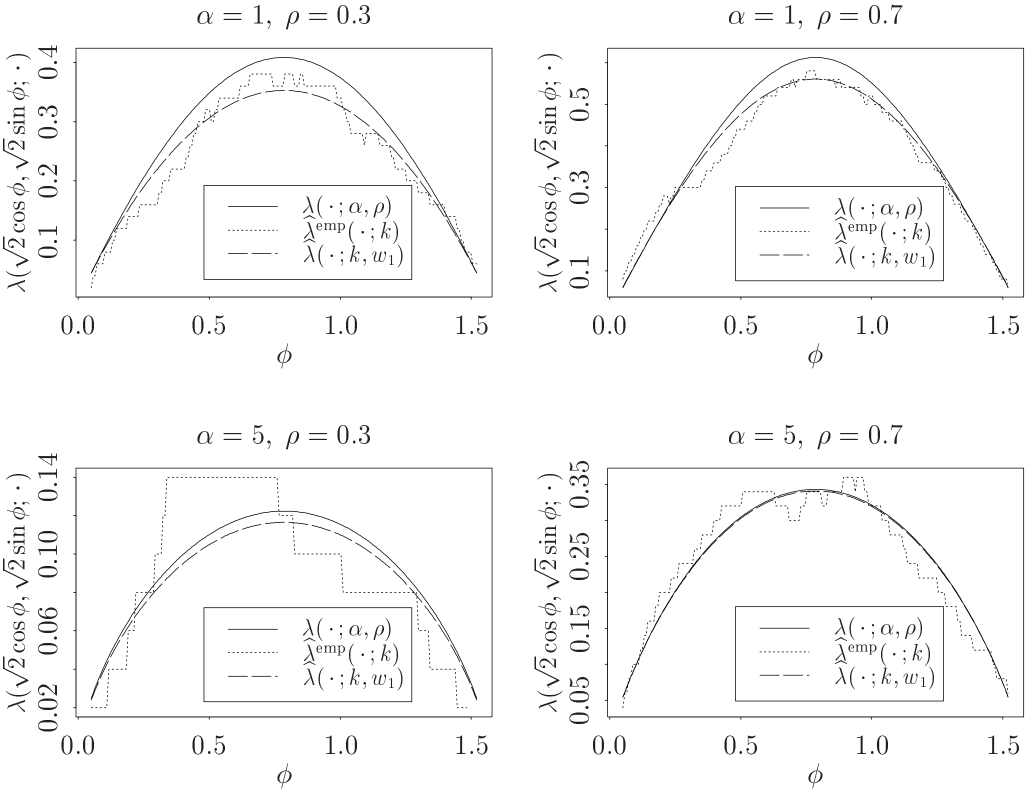

Figure 13: Estimators $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ and $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ based on a particular sample are plotted against $\phi \in (0, \pi/2)$.

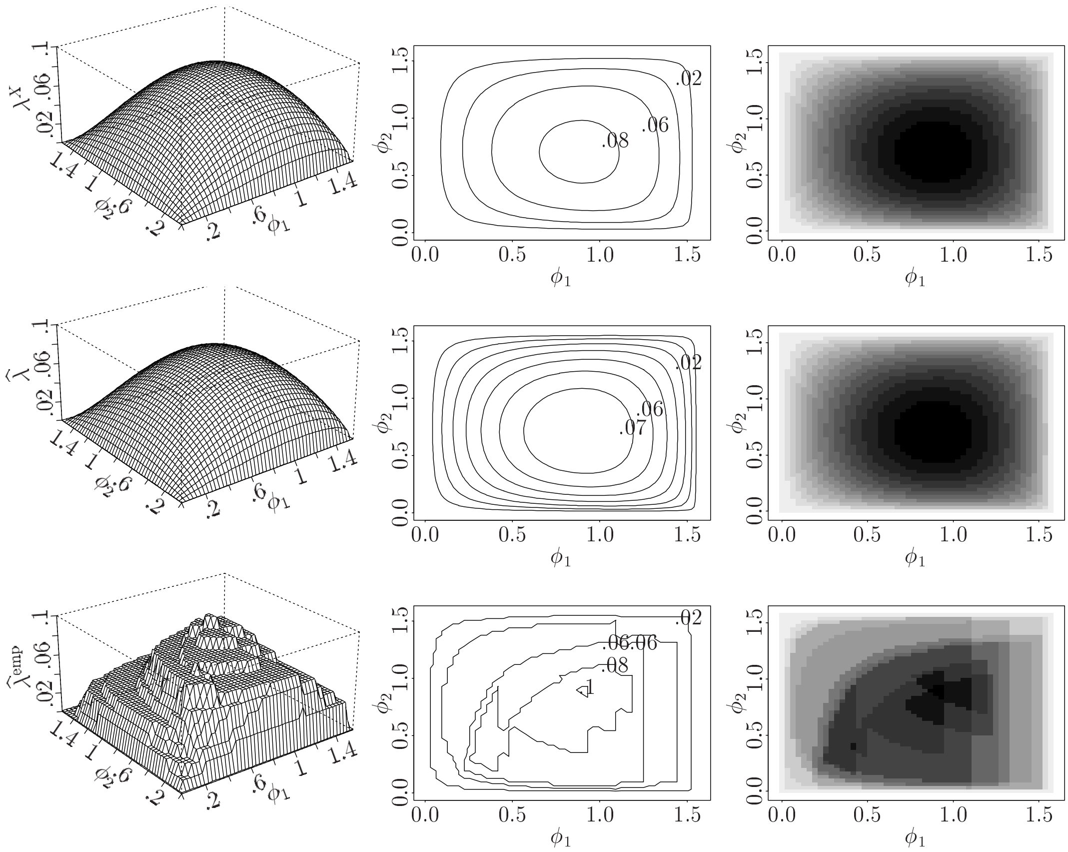

Figure 14: From left to right column: perspective, contour and grey-scale image plot of true $\lambda^X\left(\sqrt{3}\cos\phi_1, \sqrt{3}\sin\phi_1\cos\phi_2, \sqrt{3}\sin\phi_1\sin\phi_2\right)$ with parameters $\rho_{12} = 0.3$, $\rho_{13} = 0.5$, $\rho_{23} = 0.7$ and $\alpha = 5$ (first row) and corresponding estimators based on a particular sample, $\widehat{\lambda}(...; 100, w_1)$ (middle row) and $\widehat{\lambda}^{\mathrm{emp}}(...; 100)$ (lower row).

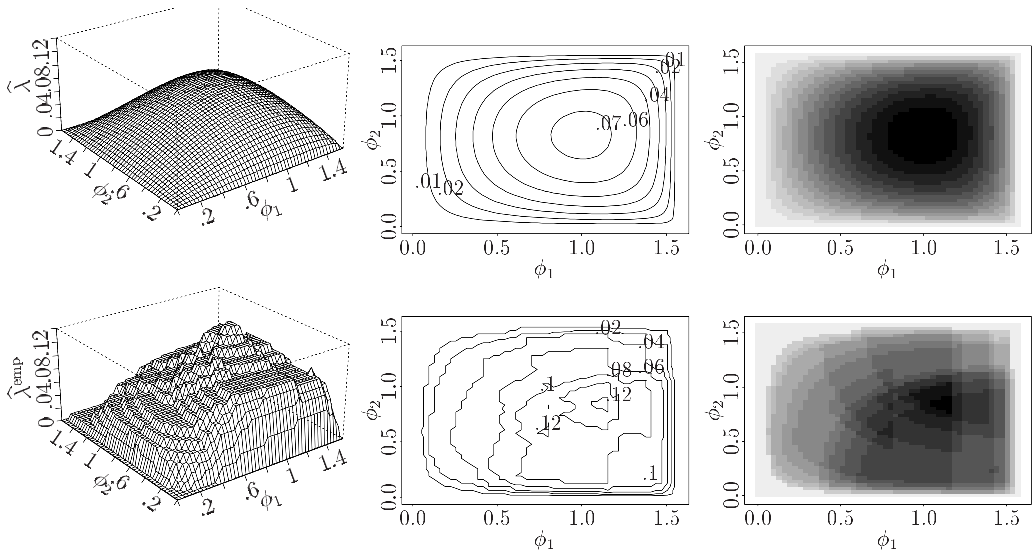

Figure 15: From left to right column: perspective, contour and grey-scale image plot of estimators $\widehat{\lambda}(...; 100, w_1)$ (upper row) and $\widehat{\lambda}^{\mathrm{emp}}(...; 100)$ (lower row) of currencies (GBP, USD, CHF).

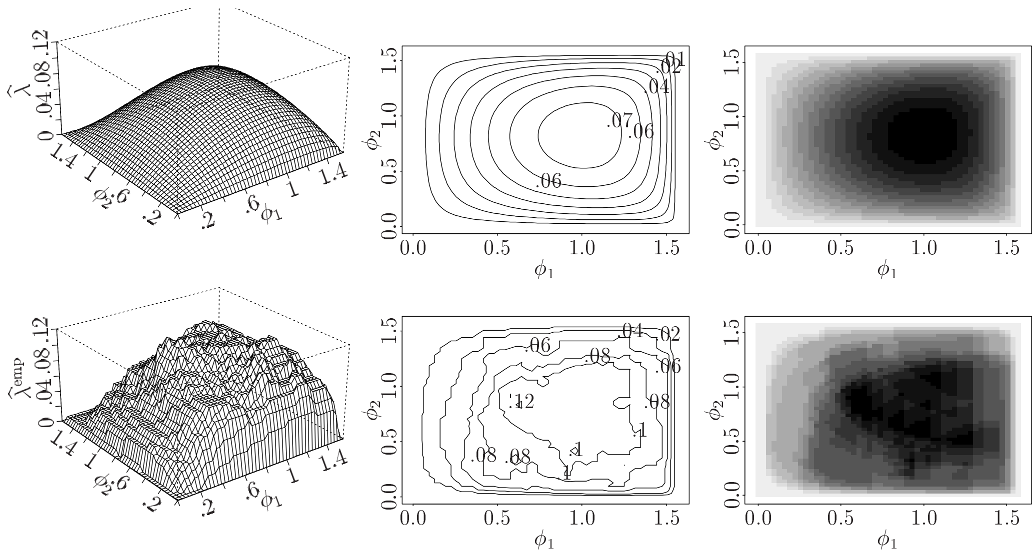

Figure 16: Same as Figure 15 but for $k = 150$.

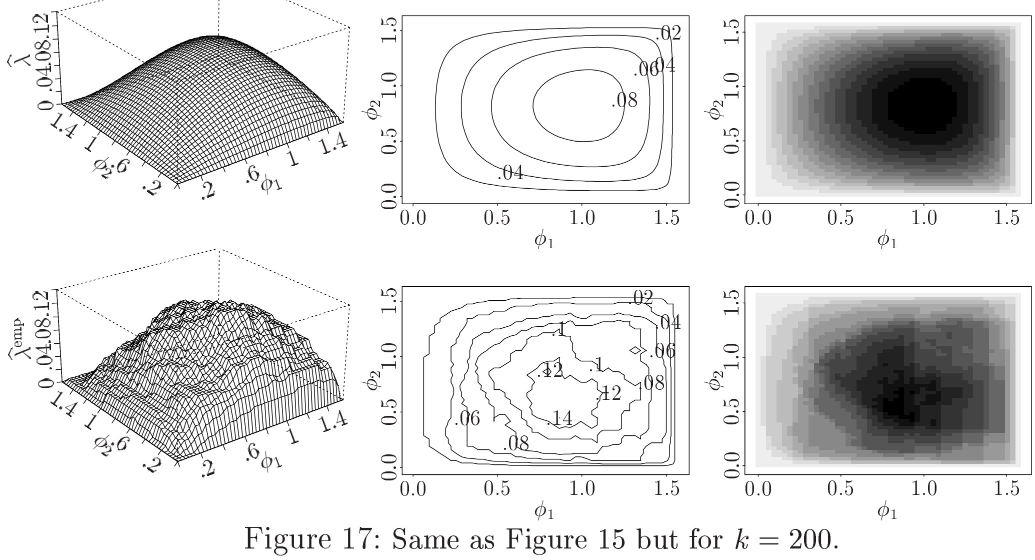

Figure 17: Same as Figure 15 but for $k = 200$.